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HITOSHI ISHII
SHIGEAKI KOIKE

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On ε-optimal controls for state constraint problems

by

Hitoshi ISHII[a,1], **Shigeaki KOIKE**[b,2]

[a] Department of Mathematics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan

[b] Department of Mathematics, Saitama University, Urawa, Saitama 338-8570, Japan

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In memory of Kazuo Horie

ABSTRACT. – We present a method of constructing ε-optimal controls in the feedback form for state constraint problems.

Our method is as follows: We first find feedback laws directly from the associated Hamilton–Jacobi–Bellman equation and an approximation of the value function by the inf-convolution. We then construct piece-wise constant controls so that corresponding cost functionals approximate the value function of state constraint problems. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous présentons une méthode de construction de contrôles ε-optimaux pour des problèmes avec contraintes d'état.

Notre méthode est la suivant : Premièrement, nous trouvons des lois en feedback directement à partir de l'équation de Hamilton–Jacobi–Bellman associée et d'une approximation de la fonction valuer par inf-convolution.

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Ensuite, nous construisons des contrôles constants par morceaux dont le coût approche la fonction valuer du problèmes avec contraintes d'état.

1. INTRODUCTION

Since the introduction, by Crandall and Lions in 1981, of the notion of viscosity solution, a notion of weak solution of partial differential equations, a matter of its importance has been the usefulness in justifying that the value function of an optimal control probldem is a weak solution of the associated Hamilton–Jacobi–Bellman (HJB for short) equations.

Although this characterization is important in itself and has many applications, it is still desirable that the notion of viscosity solution could be directly used to construct an optimal control like in the classical heuristic arguments which we present below. Given an optimal control problem, the classical heuristic arguments work rigorously only under a strong regularity assumption on the value function which we cannot usually expect. Indeed, as is well known, there are many optimal control problems which do not have any optimal control. In this viewpoint, it is natural and important to seek for an ε-optimal control for which, by definition, the cost functional differs at most by $\varepsilon > 0$ from the value function at the given point in the state space.

It was recent that Clarke, Ledyaev, Sontag and Subbotin [4] introduced a method of building an ε-optimal control in the feedback form for a given optimal control problem via the associated Hamilton–Jacobi equation in their study of feedback stabilization.

Our aim here is to extend the method due to Clarke, Ledyaev, Sontag and Subbotin to state constraint (SC for short) problems and thus to present a way of constructing an ε-optimal control in the feedback form for a general optimal control problem with state-constraint.

One of technical difficulties in this work may be explained as follows. The newly developed method by Clarke, Ledyaev, Sontag and Subbotin depends on approximation arguments mostly based on the techniques of inf-convolutions which give a convenient regularization of viscosity supersolutions. On the other hand, by the nature of SC problems, in order to design an ε-optimal control for SC problem, the state space should not be relaxed or replaced by a larger space. These are somewhat conflicting and, in order to solve this technical problem, our strategy is to replace first the state space by a smaller one and then to make an approximating argument, so that the state corresponding to the ε-optimal control obtained in our method is kept in the original state space.

We refer to [1] for a different method of finding ε-optimal controls by introducing the semidiscrete approximation. See [7] for the study of the SC problem by this approach.

Before studying the SC problem, in order to illustrate our strategy, we shall consider the problem in the whole space $\mathbf{R}^N$ since it is easier than the SC problem.

We are given functions f and g on $\mathbf{R}^N \times A$ which satisfy that

$$\begin{cases} f \in C(\mathbf{R}^N \times A, \mathbf{R}), \ g \in C(\mathbf{R}^N \times A, \mathbf{R}^N), \\ M_f := \sup_{a \in A} \|f(\cdot, a)\|_{L^\infty(\mathbf{R}^N)} < \infty, \\ M_g := \sup_{a \in A} \|g(\cdot, a)\|_{W^{1,\infty}(\mathbf{R}^N)} < \infty, \\ \sup_{a \in A} |f(x,a) - f(y,a)| \leqslant \omega_f(|x-y|) \quad (x, y \in \mathbf{R}^N), \end{cases} \tag{A1}$$

where $\omega_f : [0,\infty) \to [0,\infty)$ is continuous with $\omega_f(0) = 0$. Here, we let $A \subset \mathbf{R}^m$ (for some integer m) be a control set.

We denote by $\mathcal{A}$ the set of all measurable functions $\alpha : [0,\infty) \to A$. For any $\alpha \in \mathcal{A}$ and $x \in \mathbf{R}^N$, we denote by $X(\cdot\,; x, \alpha)$ the unique solution of

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = g(X(t), \alpha(t)) \quad (t > 0), \\ X(0) = x, \end{cases} \tag{1.1}$$

which is called the state starting from x with control α. For $a \in A$, we also write $X(\cdot\,; x, a)$ if $\alpha(t) = a$ for all $t \geqslant 0$.

Throughout this paper we deal with the following cost functional: for $\alpha \in \mathcal{A}$,

$$J(x, \alpha) = \int_0^\infty \mathrm{e}^{-t} f\big(X(t; x, \alpha), \alpha(t)\big)\, \mathrm{d}t.$$

Although our argument in this paper works for more general discount factor $\exp(-\int_0^t c(X(s; x, \alpha), \alpha(s))\,\mathrm{d}s)$, where $c : \mathbf{R}^N \times A \to \mathbf{R}$ is continuous and positive, in place of e^{-t}, for the sake of simplicity of the presentaion, we shall only treat the case when $c \equiv 1$ as above.

Next, we define the value function by

$$u(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha).$$

It is well known that the value function satisfies the following HJB equations in the viscosity sense:

$$u(x)+\sup_{a\in A}\{-\langle g(x,a),Du(x)\rangle - f(x,a)\}=0 \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

We shall recall an argument to find ε-optimal piece-wise constant controls for this unconstrained control problem assuming that u and Du are uniformly continuous.

(Step 1) Fix $\varepsilon>0$. For $x\in\mathbf{R}^N$, we choose $\hat\alpha(x)\in A$ such that

$$-\frac{\varepsilon}{2}\leqslant u(x)-\langle g(x,\hat\alpha(x)),Du(x)\rangle - f(x,\hat\alpha(x)).$$

(Step 2) Fix $x_0\in\mathbf{R}^N$. Choosing $a_0=\hat\alpha(x_0)\in A$, we let $X_0(\cdot)=X(\cdot\,;x_0,a_0)$ for a short period $\tau>0$. If τ is small enough, then we see that

$$-\varepsilon\leqslant u(X_0(t))-\langle g(X_0(t),a_0),Du(X_0(t))\rangle - f(X_0(t),a_0)$$
$$(t\in(0,\tau)).$$

Multiplying the above inequality by e^{-t}, we integrate the resulting inequality over $(0,\tau)$ to get

$$-\varepsilon(1-\mathrm{e}^{-\tau})\leqslant u(x_0)-\mathrm{e}^{-\tau}u(X_0(\tau))-\int_0^\tau \mathrm{e}^{-t}f(X_0(t),a_0)\,\mathrm{d}t.$$

(Step 3) Setting $x_1=X_0(\tau)$, we choose $a_1=\hat\alpha(x_1)\in A$. Again, solve (1.1) with $\alpha=a_1$ and $x=x_1$, and denote it by $X_1(\cdot)$. Inductively, we obtain $(a_k,x_k)=(\hat\alpha(x_k),X(\tau;x_{k-1},\hat\alpha(x_{k-1})))$ $(k\geqslant 2)$ such that

$$-\varepsilon\leqslant u(X_k(t))-\langle g(X_k(t),a_k),Du(X_k(t))\rangle - f(X_k(t),a_k)$$
$$(t\in[0,\tau)), \tag{(1.3)$_k$}$$

where $X_k(t)=X(t;x_k,\hat\alpha(x_k))$. Multiplying $((1.3)_k)$ by $\mathrm{e}^{-k\tau-t}$, and then integrating the resulting inequality over $(0,\tau)$, we take the summation over $k=0,1,2,\ldots$ to get

$$-\varepsilon\leqslant u(x_0)-\int_0^\infty \mathrm{e}^{-t}f(X(t;x_0,\alpha_\varepsilon),\alpha_\varepsilon(t))\,\mathrm{d}t,$$

where $\alpha_\varepsilon(t) = a_k$ for $t \in [k\tau, (k+1)\tau)$ $(k = 0, 1, 2, \ldots)$. Therefore, we have

$$\left|J(x_0, \alpha_\varepsilon) - u(x_0)\right| \leqslant \varepsilon.$$

We will follow this argument but we will have to use delicate tools which have been developed in the study of the viscosity solution theory (see Section 2) since we can not expect that Du is uniformly continuous. For instance, to make $(1.3)_k$ rigorous, we will need Proposition 2.3. We also refer to [2] and [1] for the general theory of viscosity solutions for HJB equations.

Moreover, since we deal with the SC problem, we will have to force the state (i.e., $X(\cdot\,; x_0, \alpha_\varepsilon)$ in the above argument) in the domain. For this purpose, we have to select suitable $\hat{\alpha}(x)$ when x is near the boundary of the domain.

This paper is organized as follows:

In the next section, we give some basic properties of the inf-convolution for the reader's convenience. In Section 3, we discuss several simple geometric properties, one of whose proofs is given in Appendix A. We discuss the SC problem for subdomains in Section 4. We show the main result and its proof in Section 5.

2. BASIC PROPERTIES OF INF-CONVOLUTIONS

We shall give various properties of the inf-convolution of functions.

For a compact subset F of $\mathbf{R}^N$, and for a bounded function $u : F \to \mathbf{R}$, we define the inf-convolution of u by

$$u_\lambda(x) = \inf_{y \in F}\left\{u(y) + \frac{|x-y|^2}{2\lambda}\right\} \quad \left(x \in \mathbf{R}^N\right).$$

For a function $f : F \to \mathbf{R}$, we shall denote by $D_F^- f(x)$ $(x \in F)$ the set of subdifferentials of f relative to F;

$$D_F^- f(x) = \bigl\{p \in \mathbf{R}^N \mid f(y) \geqslant f(x) + \langle p, y - x\rangle + \mathrm{o}\bigl(|y-x|\bigr) \\ \text{as } y \in F \to x\bigr\}.$$

Also, the set of superdifferentials of f is defined by $D_F^+ f(x) := -D_F^-(-f)(x)$ for $x \in F$.

Whenever $x \in \operatorname{int} F$, we shall simply write $D^\pm f(x)$ for $D_F^\pm f(x)$. Particularly, we will not write the subscript $\mathbf{R}^N$ if $F = \mathbf{R}^N$.

We will also use the following notation: for a function $f:\mathbf{R}^N\to\mathbf{R}$,

$$\overline{D}^- f(x) := \left\{ p\in\mathbf{R}^N \;\middle|\; \begin{array}{l}\text{There are } x_k\in\mathbf{R}^N \text{ and } p_k\in D^- f(x_k)\\ \text{such that } \lim_{k\to\infty}(x_k,p_k)=(x,p).\end{array}\right\}$$

We first give some well-known properties of the inf-convolution.

PROPOSITION 2.1. – (1) *The mapping* $x\to u_\lambda(x)-|x|^2/(2\lambda)$ *is concave*; $u_\lambda(\cdot)$ *is semiconcave.*

(2) $D^+u_\lambda(x)\neq\emptyset$ *for all* $x\in\mathbf{R}^N$ *and* $\{x\in\mathbf{R}^N \mid D^-u_\lambda(x)\neq\emptyset\}$ *is dense in* $\mathbf{R}^N$.

We next show some elementary properties for the reader's convenience.

PROPOSITION 2.2. – *Assume that* $u:F\to\mathbf{R}$ *is lower semicontinuous. If* $x\in\mathbf{R}^N$ *and* $x_\lambda\in F$ *satisfy that* $u_\lambda(x)=u(x_\lambda)+|x-x_\lambda|^2/(2\lambda)$, *then we have*

$$\frac{x-x_\lambda}{\lambda}\in D_F^- u(x_\lambda). \tag{2.1}$$

Furthermore, if $p\in\overline{D}^- u_\lambda(x)$ *for* $x\in\mathbf{R}^N$, *then there is* $x_\lambda\in F$ *such that*

$$u_\lambda(x)=u(x_\lambda)+\frac{|x-x_\lambda|^2}{2\lambda}\quad \textit{and}\quad p=\frac{x-x_\lambda}{\lambda}\in D_F^- u(x_\lambda).$$

Proof. – Since we have

$$u(x_\lambda)+\frac{|x-x_\lambda|^2}{2\lambda}\leqslant u(y)+\frac{|x-y|^2}{2\lambda}\quad (y\in F),$$

we easily see that $(x-x_\lambda)/\lambda\in D_F^- u(x_\lambda)$.

For $p\in\overline{D}^- u_\lambda(x)$, we choose $(x_k,p_k)\in F\times\mathbf{R}^N$ such that $\lim_{k\to\infty}(x_k,p_k)=(x,p)$ and $p_k\in D^-u_\lambda(x_k)$. We also choose $x_k^\lambda\in F$ so that

$$u_\lambda(x_k)=u\big(x_k^\lambda\big)+\frac{|x_k-x_k^\lambda|^2}{2\lambda}.$$

From the definition, we see that

$$\begin{aligned} u(z)+\frac{|y-z|^2}{2\lambda}\geqslant u_\lambda(y)\geqslant u\big(x_k^\lambda\big)+\frac{|x_k-x_k^\lambda|^2}{2\lambda}+\langle p_k,y-x_k\rangle\\ +\,\mathrm{o}(|y-x_k|)\end{aligned} \tag{2.2}$$

for any $z \in F$ and $y \in \mathbf{R}^N$.

Taking $z = x_k^\lambda$ and $y = x_k + \delta s$ for any $s \in S^{N-1}$ and $\delta > 0$ in (2.2), we have

$$\frac{|x_k + \delta s - x_k^\lambda|^2 - |x_k - x_k^\lambda|^2}{2\lambda\delta} \geqslant \langle p_k, s\rangle + \frac{\mathrm{o}(\delta)}{\delta}.$$

Thus, sending $\delta \to 0$ in the above, we have $p_k = (x_k - x_k^\lambda)/\lambda$. Hence, we find $x_\lambda \in F$ such that $p = (x - x_\lambda)/\lambda$.

Moreover, taking $y = x_k$ in (2.2), we have

$$u_\lambda(x_k) \geqslant u\big(x_k^\lambda\big) + \frac{|x_k - x_k^\lambda|^2}{2\lambda}.$$

Sending $k \to \infty$, from the lower semicontinuity of u, we conclude the second assertion. □

We next present a monotonicity type estimate for superdifferentials of the inf-convolution of functions.

PROPOSITION 2.3. – *For any $p \in D^+u_\lambda(x)$ and $q \in D^+u_\lambda(y)$ ($x, y \in \mathbf{R}^N$), we have*

$$\langle p - q, x - y\rangle \leqslant \frac{|x - y|^2}{\lambda}.$$

Proof. – Setting $v_\lambda(x) = u_\lambda(x) - |x|^2/(2\lambda)$, we note $D^+u_\lambda(x) - (x/\lambda) = D^+v_\lambda(x)$. The concavity of v_λ implies that

$$v_\lambda(y) \leqslant v_\lambda(x) + \left\langle p - \frac{x}{\lambda}, y - x\right\rangle \quad \text{and} \quad v_\lambda(x) \leqslant v_\lambda(y) + \left\langle q - \frac{y}{\lambda}, x - y\right\rangle.$$

Combining these inequalities, we conclude the assertion. □

We recall the following facts from [3].

LEMMA 2.4 ([3]). – *Assume that $u \in C(F, \mathbf{R})$.*

(1) *$\emptyset \neq \overline{D}^- u_\lambda(x) \subset D^+u_\lambda(x)$ for all $x \in \mathbf{R}^N$.*

(2) *Let $X : [0, T) \to \mathbf{R}^N$ be a Lipschitz continuous function. Then, we see that for almost all $t \in [0, T)$*

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(X(t)) = \left\langle \frac{\mathrm{d}X}{\mathrm{d}t}(t), p\right\rangle \quad \textit{provided } p \in D^+u_\lambda(X(t)).$$

3. SIMPLE GEOMETRIC PROPERTIES

Let $\Omega \subset \mathbf{R}^N$ be an open, bounded set.

We shall suppose the uniform exterior sphere condition for Ω:

$$\begin{cases} \text{There is } R > 0 \text{ satisfying the following:} \\ \text{For any } z \in \partial\Omega, \text{ there is } x \in \mathbf{R}^N \text{ for which } B(x, R) \cap \overline{\Omega} = \{z\}. \end{cases} \tag{A2}$$

Here and later, $B(x, r)$ denotes the standard closed ball with radius $r > 0$ and center $x \in \mathbf{R}^N$.

Our assumption on the vector fields $\{g(\cdot, a) \mid a \in A\}$ is as follows:

$$\begin{cases} \text{There is } \delta > 0 \text{ satisfying the following:} \\ \text{For each } z \in \partial\Omega, \text{ there is } a \in A \text{ such that } |g(x, a)| \geqslant \delta, \text{ and} \\ B(x + tg(x, a)/|g(x, a)|, \delta t) \subset \Omega \text{ for } 0 < t \leqslant \delta, \ x \in B(z, \delta) \cap \overline{\Omega}. \end{cases} \tag{A3}$$

We denote by $A(z)$ for $z \in \partial\Omega$ the set of all controls satisfying (A3).

For $\gamma \geqslant 0$, we shall define an open subset of Ω:

$$\Omega_\gamma = \{x \in \Omega \mid \operatorname{dist}(x, \Omega^c) > \gamma\}.$$

Notice that $\Omega_0 = \Omega$.

Under these assumptions, we will refer to $R > 0$ (in (A2)) and $\delta > 0$ (in (A3)) without mentioning where these come. We will also use the constant $r_0 > 0$ defined by

$$r_0 := \min\left\{\frac{\delta}{3}, \frac{\delta^2}{4M_g}\right\},$$

where M_g is a constant from (A1).

We introduce the set of generalized normal vectors at $z \in \partial\Omega_\gamma$ for $\gamma \in [0, r_0]$:

$$N_\gamma(z) = \{p \in \mathbf{R}^N \mid \langle p, x - z\rangle \leqslant o(|x - z|) \text{ as } \Omega_\gamma \ni x \to z\}.$$

We define the following set-valued mappings $T^0 : \overline{\Omega} \setminus \Omega_\gamma \to \partial\Omega$ and $T_\gamma : \overline{\Omega} \setminus \Omega_\gamma \to \partial\Omega_\gamma$: for $\gamma \in [0, r_0]$ and $x \in \overline{\Omega} \setminus \Omega_\gamma$,

$$T^0 x = \{z \in \partial\Omega \mid |z - x| = \operatorname{dist}(x, \Omega^c)\},$$
$$T_\gamma x = \{z \in \partial\Omega_\gamma \mid |z - x| = \operatorname{dist}(x, \Omega_\gamma)\}.$$

With these notations, we will write

$$A(T^0x) = \bigcup_{z\in T^0x} A(z).$$

3.1. Geometric properties of Ω_γ

We begin with the observation that (A3) together with (A1) implies the uniform interior cone property of Ω_γ for small $\gamma \geqslant 0$.

PROPOSITION 3.1. – *Assume that* (A1) *and* (A3) *hold. Let* $0 \leqslant \gamma \leqslant r_0$ *and* $z \in \partial\Omega_\gamma$. *Then, we have*

$$B\bigl(x+tg(x,a)/|g(x,a)|, t\delta/2\bigr) \subset \Omega_\gamma \quad \bigl(0<t\leqslant\delta,\ x\in B(z,\delta/3)\cap\overline{\Omega}_\gamma\bigr).$$

Proof. – Fix $x \in B(z,\delta/3)\cap\overline{\Omega}_\gamma$, and $\zeta \in B(0,\gamma)$. Let $y \in T^0z$. Observe that $x+\zeta = y+(x-z)+(z-y)+\zeta \in B(y,\delta)$, and that $x+\zeta \in \overline{\Omega}_\gamma + B(0,\gamma) \subset \overline{\Omega}$. That is,

$$x+\zeta \in B(y,\delta)\cap\overline{\Omega}.$$

Fix $a \in A(y)$ and set $\eta(x) = g(x,a)/|g(x,a)|$. Hence, by (A3), we have

$$x+\zeta+t\bigl(\eta(x+\zeta)+\xi\bigr) \in \Omega \quad \bigl(0<t\leqslant\delta,\ \zeta\in B(0,\gamma),\ \xi\in B(0,\delta)\bigr).$$

Set

$$\bar\eta = \eta(x+\zeta)-\eta(x).$$

Noting that $|\bar\eta| \leqslant 2M_g\gamma/\delta \leqslant \delta/2$, we have $\bar\eta + B(0,\delta/2) \subset B(0,\delta)$. Thus, we have

$$x+\zeta+t\bigl(\eta(x)+\xi\bigr) \in \Omega \quad \bigl(0<t\leqslant\delta,\ \zeta\in B(0,\gamma),\ \xi\in B(0,\delta/2)\bigr).$$

Therefore, $B(x+t\eta(x), t\delta/2) \subset \Omega_\gamma$ for $t \in (0,\delta]$ and $x \in B(z,\delta/3)\cap\overline{\Omega}_\gamma$. □

In the proof of Lemma 3.6, we will need the following estimate:

COROLLARY 3.2. – *Under the same assumptions as in Proposition* 3.1 *we have*

$$\bigl\langle g(x,a),\nu\bigr\rangle \leqslant -\frac{\delta^2}{2} \quad \bigl(x\in B(z,\delta/3)\cap\partial\Omega_\gamma,\ \nu\in N_\gamma(x)\cap S^{N-1}\bigr).$$

Proof. – Let $\eta(x) = g(x,a)/|g(x,a)|$. Since $x + t(\eta(x) + \xi) \in \Omega_\gamma$ for $\xi \in B(0, \delta/2)$ for small $t > 0$, we have

$$t\langle \eta(x) + \xi, \nu\rangle \leqslant o(t).$$

Dividing the above inequality by $t > 0$, we send $t \to 0$ to get

$$\langle \eta(x), \nu\rangle \leqslant -\langle \xi, \nu\rangle.$$

Therefore, taking $\xi = \delta\nu/2$, by (A3), we conclude the assertion. □

To prove Lemma 3.6, we will also need the fact that, under assumptions (A1)–(A3), we can take a special sphere outside of Ω_γ ($\gamma \geqslant 0$), which touches $\partial\Omega_\gamma$. For the reader's convenience, we give its proof in Appendix A. As will be seen in Lemma 6.3, to verify that the uniform exterior shpere condition holds for Ω_γ, we only need to suppose (A2).

PROPOSITION 3.3. – *Assume that* A1), (A2) *and* (A3) *hold. Let* $0 < \gamma \leqslant r_0$, $x \in \partial\Omega_\gamma$ *and* $\nu \in N_\gamma(x) \cap S^{N-1}$. *Then, we have*

$$B(x + R\delta\nu/2, R\delta/2) \subset (\Omega_\gamma)^c.$$

To estimate the distance from $x \in \overline{\Omega} \setminus \Omega_\gamma$ to Ω_γ, we give the next proposition:

PROPOSITION 3.4. – *Assume that* (A3) *holds. Let* $0 < \gamma \leqslant \delta^2$. *Then, we have*

$$\operatorname{dist}(x, \Omega_\gamma) \leqslant \left(1 + \frac{1}{\delta}\right)\gamma \quad (x \in \overline{\Omega} \setminus \Omega_\gamma).$$

Remark. – We note that (A3) impose the Lipschitz continuity of $\partial\Omega$ (see, e.g., [5]). We also note that the above assertion fails for general domains. For instance, if Ω has a cusp, then the above estimate might fail.

Proof. – Fix $x \in \overline{\Omega} \setminus \Omega_\gamma$. Let $z \in T^0x$. Notice that $|z - x| \leqslant \gamma$. Let $\eta(z) \in S^{N-1}$ from (A3) for this $z \in \partial\Omega$. Then, we have

$$B(z + t\eta(z), \delta t) \subset \Omega \quad (0 < t \leqslant \delta).$$

Choosing $\tau \in (0, \delta]$ so that $\gamma = \delta\tau$, we have

$$B(z + \tau\eta(z), \gamma) \subset \Omega.$$

This implies that $z + \tau\eta(z) \in \Omega_\gamma$. Hence,

$$\operatorname{dist}(x, \Omega_\gamma) \leqslant |z + \tau\eta(z) - x| \leqslant |z - x| + \tau \leqslant \gamma + \tau = \left(1 + \frac{1}{\delta}\right)\gamma. \qquad \square$$

3.2. Estimates on the subdifferentials of u_λ

In this subsection, we use the notation: For $\gamma \geqslant 0$, let $u : \overline{\Omega}_\gamma \to \mathbf{R}$ be continuous with a modulus of continuity ω_u. For $\lambda > 0$, we define $u_\lambda : \mathbf{R}^N \to \mathbf{R}$ by

$$u_\lambda(x) = \inf_{y \in \overline{\Omega}_\gamma} \left(u(y) + \frac{|x - y|^2}{2\lambda} \right).$$

For $x \in \overline{\Omega}$, we choose $x_\lambda \in \overline{\Omega}_\lambda$ such that

$$u_\lambda(x) = u(x_\lambda) + \frac{|x - x_\lambda|^2}{2\lambda}.$$

Also, we fix $\varepsilon \in (0, 1]$.

To show that u_λ is a viscosity supersolution of an approximate HJB equation in Lemma 4.1 provided that u is a viscosity supersolution of a HJB equation, we will use the following observation. We note that the same idea can be found in Section 3 of [4].

LEMMA 3.5. – *Assume that* (A3) *holds. Let $\gamma \leqslant \sqrt{\lambda\varepsilon}$. Then, there are $\lambda_1 = \lambda_1(\omega_u, \varepsilon, \delta, \sup_{\Omega_\gamma} |u|) > 0$ and $C_1 = C_1(\delta) > 0$ such that*

$$\frac{|x - x_\lambda|^2}{2\lambda} < C_1\varepsilon \quad (0 < \lambda \leqslant \lambda_1).$$

Proof. – Let $z \in T_\gamma x \subset \partial\Omega_\gamma$. From the definition of u_λ, we have

$$\frac{|x - x_\lambda|^2}{2\lambda} \leqslant \frac{|x - z|^2}{2\lambda} + \omega_u(|x_\lambda - z|). \tag{3.1}$$

By Proposition 3.4, we have

$$|x - x_\lambda|^2 \leqslant \left(1 + \frac{1}{\delta}\right)^2 \gamma^2 + \lambda M,$$

where $M := 4\sup_{\Omega_\gamma}|u|$. Hence,

$$|x-x_\lambda|^2 \leqslant \left[\varepsilon\left(1+\frac{1}{\delta}\right)^2 + M\right]\lambda,$$

and so,

$$|x_\lambda - z|^2 \leqslant 2\left(|x_\lambda - x|^2 + |x-z|^2\right) \leqslant 2\left[2\left(1+\frac{1}{\delta}\right)^2 + M\right]\lambda.$$

Thus, setting $C = \sqrt{2[2(1+\frac{1}{\delta})^2 + M]}$, by (3.1), we have

$$\frac{|x-x_\lambda|^2}{2\lambda} \leqslant \frac{|x-z|^2}{2\lambda} + \omega_u(C\sqrt{\lambda}) \leqslant \left(1+\frac{1}{\delta}\right)^2\frac{\varepsilon}{2} + \omega_u(C\sqrt{\lambda}). \quad (3.2)$$

Choose $\lambda_1 > 0$ such that

$$\omega_u(C\sqrt{\lambda}) \leqslant \varepsilon \quad (0 < \lambda \leqslant \lambda_1),$$

to conclude that

$$\frac{|x-x_\lambda|^2}{2\lambda} < C_1\varepsilon := \left[\left(1+\frac{1}{\delta}\right)^2 + 1\right]\varepsilon \quad (0<\lambda\leqslant\lambda_1). \qquad \square$$

The following lemma gives an essential estimate on superdifferentials of the inf-convolution u_λ.

LEMMA 3.6. – *Assume that* (A1), (A2) *and* (A3) *hold. Let* $x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}$, $z \in T_\gamma x \subset \partial\Omega_\gamma$, *and* $0 < \theta < 1$. *Then, there is* $\lambda_2 = \lambda_2(\omega_u, \varepsilon, \delta, \sup_{\Omega_\gamma}|u|, \theta) \in (0, r_0^2]$ *such that*

$$\left\langle \frac{x-z}{|x-z|}, \frac{x-x_\lambda}{|x-x_\lambda|}\right\rangle \geqslant \theta \quad (0<\gamma^2\leqslant\lambda\leqslant\lambda_2). \quad (3.3)$$

Moreover, let $\gamma^2 = \varepsilon\lambda$. *For any* $M_1 > 0$, *there is* $\lambda_3 = \lambda_3(\omega_u, \varepsilon, \delta, \sup_{\Omega_\gamma}|u|, M_1, M_g) \in (0, \lambda_2]$ *such that*

$$-\left\langle g(x,a), \frac{x-x_\lambda}{\lambda}\right\rangle \geqslant M_1 \quad (a\in A(T^0x),\ 0<\lambda\leqslant\lambda_3). \quad (3.4)$$

Proof. – Recall that, from Proposition 3.4,

$$\frac{\gamma}{2} \leqslant |x-z| \leqslant \left(1+\frac{1}{\delta}\right)\gamma. \tag{3.5}$$

By (3.2), we have

$$\frac{|x-x_\lambda|^2}{2\lambda} \leqslant \frac{|x-z|^2}{2\lambda} + \omega_u(C\sqrt{\lambda}),$$

where

$$C = \sqrt{2\left[2\left(1+\frac{1}{\delta}\right)^2 + 4\sup_{\Omega_\gamma}|u|\right]}.$$

Set $r = |x-z|$. We may assume by choosing λ small enough that $r < \rho := R\delta/2$.

Setting $\nu = (x-z)/r$, we observe that $\nu \in N_\gamma(z) \cap S^{N-1}$. Thus, in view of Proposition 3.3, we have

$$B(z+\rho\nu, \rho) \subset (\Omega_\gamma)^c.$$

Hence, setting $\xi = z + \rho\nu$, we have

$$\begin{aligned}\rho^2 \leqslant |x_\lambda - \xi|^2 &= |x_\lambda - x|^2 + |x-\xi|^2 + 2\langle x-\xi, x_\lambda - x\rangle \\ &\leqslant r^2 + 2\lambda\omega_u(C\sqrt{\lambda}) + (\rho-r)^2 + 2(\rho-r)\langle \nu, x - x_\lambda\rangle.\end{aligned}$$

Thus, we have

$$\langle \nu, x - x_\lambda\rangle \geqslant \frac{r\rho - r^2 - \lambda\omega_u(C\sqrt{\lambda})}{\rho - r}.$$

Then, we observe that

$$\begin{aligned}\left\langle \nu, \frac{x-x_\lambda}{|x-x_\lambda|}\right\rangle &\geqslant \frac{(\rho-r)r - \lambda\omega_u(C\sqrt{\lambda})}{(\rho-r)|x-x_\lambda|} \\ &\geqslant \frac{(\rho-r)r - \lambda\omega_u(C\sqrt{\lambda})}{(\rho-r)r(1+\frac{2\lambda}{r^2}\omega_u(C\sqrt{\lambda}))^{1/2}} \\ &\geqslant \frac{(\rho-r) - \frac{\lambda}{r}\omega_u(C\sqrt{\lambda})}{(\rho-r)(1+\frac{2\lambda}{r^2}\omega_u(C\sqrt{\lambda}))^{1/2}}.\end{aligned}$$

Since $\lambda/r = \gamma^2/(r\varepsilon)$ and $\lambda/r^2 = \gamma^2/(r^2\varepsilon)$, by (3.5), λ/r and λ/r^2 are bounded. Therefore, we can choose $\lambda_2 > 0$ so that if $0 < \lambda \leqslant \lambda_2$, then the right hand side of the above is greater than the given θ.

We assume henceforth that λ satisfies the condition described above.

According to (3.3), we have

$$\left| \frac{x - x_\lambda}{|x - x_\lambda|} - \left\langle \nu, \frac{x - x_\lambda}{|x - x_\lambda|} \right\rangle \nu \right|^2 \leqslant 1 - \theta^2.$$

Fix $a \in A(T^0 x)$. By Corollary 3.2, we see that

$$\langle g(x,a), \nu \rangle \leqslant -\frac{\delta^2}{2}.$$

Writing

$$g(x,a) = \alpha\nu + v \quad \text{and} \quad \frac{x - x_\lambda}{|x - x_\lambda|} = \beta\nu + w,$$

where $\alpha = \langle g(x,a), \nu \rangle$ and $\beta = \langle (x - x_\lambda)/|x - x_\lambda|, \nu \rangle$, we have

$$\langle v, \nu \rangle = \langle w, \nu \rangle = 0,$$

$$\alpha \leqslant -\frac{\delta^2}{2}, \qquad \beta \geqslant \theta, \qquad |v| \leqslant M_g, \qquad |w|^2 = 1 - \beta^2,$$

$$-\left\langle g(x,a), \frac{x - x_\lambda}{|x - x_\lambda|} \right\rangle = -\alpha\beta - \langle v, w \rangle \geqslant \frac{\delta^2 \beta}{2} - M_g\left(1 - \beta^2\right)^{1/2}.$$

If we suppose that θ is close enough to 1 such that $\delta\beta/2 - M_g(1 - \beta^2)^{1/2} \geqslant \delta^2/4$, then we observe that

$$-\left\langle g(x,a), \frac{x - x_\lambda}{\lambda} \right\rangle \geqslant \frac{\delta^2 |x - x_\lambda|}{4\lambda}.$$

Hence, we have

$$-\left\langle g(x,a), \frac{x - x_\lambda}{\lambda} \right\rangle \geqslant \frac{\delta^2 \gamma}{8\lambda} = \frac{\delta^2}{8} \sqrt{\frac{\varepsilon}{\lambda}}.$$

Therefore, there is $\lambda_3 > 0$ such that

$$-\left\langle g(x,a), \frac{x - x_\lambda}{\lambda} \right\rangle \geqslant M_1 \quad (0 < \lambda \leqslant \lambda_3). \qquad \square$$

3.3. A property on behavior of states

In this subsection, we observe that the state $X(\cdot\,; x, a)$ for $a \in A(T^0 x)$ does not move closer to the boundary for a short period.

PROPOSITION 3.7. – *Assume that* (A1), (A2) *and* (A3) *hold. Then, there is* $t_0 > 0$ *such that*

$$X(t; x, a) \in \overline{\Omega}_\gamma \quad \big(0 \leqslant \gamma \leqslant r_0,\ x \in \partial\Omega_\gamma,\ a \in A\big(T^0 x\big),\ t \in [0, t_0]\big).$$

Proof. – Fix $x \in \partial\Omega_\gamma$ and $a \in A(T^0 x)$. Write $X(\cdot) = X(\cdot\,; x, a)$ simply.

In view of Proposition 3.1, it suffices to show that there is $t_0 > 0$ such that

$$X(t) \in \bigcup_{0 \leqslant r \leqslant \delta} B\big(x + r\eta(x), r\delta/2\big) \quad \big(t \in (0, t_0]\big), \tag{3.6}$$

where $\eta(x) = g(x, a)/|g(x, a)|$.

We note that

$$\big|X(t) - x - r\eta(x)\big| \leqslant \int_0^t \left| g\big(X(s), a\big) - \frac{r}{t}\eta(x) \right| \mathrm{d}s.$$

For any $t > 0$, choosing $r = t|g(x, a)|$, we have

$$\begin{aligned} \big|X(t) - x - r\eta(x)\big| &\leqslant \int_0^t \big|g(X(s), a) - g(x, a)\big| \,\mathrm{d}s \\ &\leqslant \frac{M_g^2 t^2}{2} = \frac{M_g^2 t}{2|g(x, a)|} r. \end{aligned} \tag{3.7}$$

Setting $t_0 = \min\{\delta^2/M_g^2, \delta/M_g\}$, we see that $r = t|g(x, a)| \leqslant \delta$ for $t \in (0, t_0]$. Moreover, since the right-hand side of (3.7) is estimated from above by $\delta r/2$, (3.6) is valid. □

4. SC PROBLEMS FOR SUBDOMAINS

In this section, we always suppose that (A1), (A2) and (A3) hold, and that $0 \leqslant \gamma \leqslant r_0$.

We shall introduce the value function of the SC problem for $\overline{\Omega}_\gamma$. For this purpose, we define

$$\mathcal{A}_\gamma(x) = \big\{\alpha \in \mathcal{A} \mid X(t; x, \alpha) \in \overline{\Omega}_\gamma \text{ for } t \geqslant 0\big\} \quad \big(x \in \overline{\Omega}_\gamma\big).$$

We introduce the notation: for $z \in \partial\Omega_\gamma$,

$$A_\gamma(z) = \left\{ a \in A \,\middle|\, \begin{array}{l} \text{There is } s > 0 \text{ such that } X(t;x,a) \in \overline{\Omega}_\gamma \\ \text{for } t \in [0,s] \text{ and } x \in B(z,s) \cap \overline{\Omega}_\gamma. \end{array} \right\}$$

We note that Proposition 3.7 implies that $\emptyset \neq A(T^0 x) \subset A_\gamma(x)$ for $x \in \overline{\Omega}_\gamma$ provided $\gamma \in [0, r_0]$. Thus, we see that $\mathcal{A}_\gamma(x) \neq \emptyset$ for $\gamma \in [0, r_0]$ and $x \in \overline{\Omega}_\gamma$.

We shall consider the HJB equation:

$$u(x) + \sup_{a \in A}\{-\langle g(x,a), Du(x)\rangle - f(x,a)\} = 0. \tag{4.1}$$

In order to study the SC problem for $\overline{\Omega}_\gamma$, we adapt the following definition of viscosity solutions of (4.1) in $\overline{\Omega}_\gamma$ as in [5]:

DEFINITION. – *We call $u : \overline{\Omega}_\gamma \to \mathbf{R}$ a viscosity subsolution (respectively, supersolution) of* (4.1) *in $\overline{\Omega}_\gamma$ if*

$$u^*(x) + \sup_{a \in A_\gamma(x)} \{-\langle g(x,a), p\rangle - f(x,a)\} \leqslant 0$$
$$\textit{for all } x \in \overline{\Omega}_\gamma,\ p \in D^+_{\overline{\Omega}_\gamma} u^*(x)$$

(*respectively,*

$$u_*(x) + \sup_{a \in A}\{-\langle g(x,a), p\rangle - f(x,a)\} \geqslant 0$$
$$\textit{for all } x \in \overline{\Omega}_\gamma,\ p \in D^-_{\overline{\Omega}_\gamma} u_*(x)).$$

We call $u : \overline{\Omega}_\gamma \to \mathbf{R}$ a viscosity solution of (4.1) *in $\overline{\Omega}_\gamma$ if it is both a viscosity sub- and supersolution of* (4.1) *in $\overline{\Omega}_\gamma$.*

Here, the superscript and subscript $*$, respectively, denote the upper and lower semicontinuous envelopes of the function. See [1] or [2] for these definitions.

We now denote by $u^\gamma : \overline{\Omega}_\gamma \to \mathbf{R}$ the value function of the SC problem for $\overline{\Omega}_\gamma$:

$$u^\gamma(x) = \inf_{\alpha \in \mathcal{A}_\gamma(x)} \int_0^\infty e^{-t} f\big(X(t;x,\alpha), \alpha(t)\big)\, dt.$$

It is known in [5] for example that the DPP holds for u_γ ($\gamma \geqslant 0$): for $s > 0$,

$$u^\gamma(x) = \inf_{\alpha\in\mathcal{A}_\gamma(x)}\left(\int_0^s \mathrm{e}^{-t} f\big(X(t;x,\alpha),\alpha(t)\big)\,\mathrm{d}t + \mathrm{e}^{-s}u^\gamma\big(X(s;x,\alpha)\big)\right). \tag{4.2}$$

It is also known in [5] that u^γ is a viscosity solution of (4.1) in $\overline{\Omega}_\gamma$ in the above sense. Furthermore, the comparison principle in [5] implies the continuity of u^γ on $\overline{\Omega}_\gamma$.

We begin with the following observation which will be needed in Section 5:

LEMMA 4.1. – *For $\varepsilon \in (0,1]$ and $\lambda > 0$, we set $\gamma^2 = \varepsilon\lambda$. Let $u \in C(\overline{\Omega}_\gamma, \mathbf{R})$ be a viscosity supersolution of (4.1) in $\overline{\Omega}_\gamma$. Set $u_\lambda(x) = \inf\{u(y) + |x-y|^2/(2\lambda) \mid y \in \overline{\Omega}_\gamma\}$. Then, there are constants $\lambda_4 = \lambda_4(\omega_f, \omega_u, \delta, \varepsilon, \sup_{\Omega_\gamma}|u|) > 0$ and $C_2 = C_2(\delta, M_g) > 0$ such that*

$$-C_2\varepsilon \leqslant u_\lambda(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\}$$
$$\big(0<\lambda\leqslant\lambda_4, x\in\overline{\Omega}, p\in \overline{D}^- u_\lambda(x)\big).$$

Proof. – We note that it is enough to show the assertion for $p \in D^- u_\lambda(x)$.

Choosing $x_\lambda \in \overline{\Omega}_\gamma$ such that $u_\lambda(x) = u(x_\lambda) + |x - x_\lambda|^2/(2\lambda)$, by Proposition 2.2, we see that $p = (x - x_\lambda)/\lambda \in D^-_{\overline{\Omega}_\gamma} u(x_\lambda)$. Thus, from the definition, we have

$$0 \leqslant u(x_\lambda) + \sup_{a\in A}\{-\langle g(x_\lambda,a), p\rangle - f(x_\lambda,a)\}.$$

Hence, we have

$$0 \leqslant u_\lambda(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\} + M_g\lambda|p|^2 + \omega_f\big(|x - x_\lambda|\big). \tag{4.3}$$

In view of Lemma 3.5, there are $\lambda_1 = \lambda_1(\omega_u, \varepsilon, \delta, \sup_{\Omega_\gamma}|u|) > 0$ and $C_1 = C_1(\delta) > 0$ such that

$$\lambda|p|^2 = \frac{|x - x_\lambda|^2}{\lambda} \leqslant 2C_1\varepsilon \quad (0<\lambda\leqslant\lambda_1).$$

Choose $\lambda_4 = \lambda_4(\lambda_1, C_1, \omega_f) \in (0, \lambda_1]$ so that

$$\omega_f(|x - x_\lambda|) \leqslant \omega_f(\sqrt{2C_1\lambda}) \leqslant \varepsilon \quad (0 < \lambda \leqslant \lambda_4).$$

Then, fixing

$$C_2 = 2C_1 M_g + 1,$$

for $0 < \lambda \leqslant \lambda_4$, by (4.3), we have

$$-C_2\varepsilon \leqslant u_\lambda(x) + \sup_{a\in A}\{-\langle g(x,a), p\rangle - f(x,a)\}. \qquad \square$$

We shall present a convergence result of u^γ to u^0 as $\gamma \to 0$. For this purpose, we first extend u^γ on $\overline{\Omega}_\gamma$ into $\overline{\Omega}$ by

$$\bar{u}^\gamma(x) = \begin{cases} u^\gamma(x) & \text{for } x \in \overline{\Omega}_\gamma, \\ -\infty & \text{for } x \in \overline{\Omega} \setminus \overline{\Omega}_\gamma. \end{cases}$$

We denote the relaxed limit supremum of $\bar{u}^\gamma$ at $x \in \overline{\Omega}$ by

$$\begin{aligned} v(x) &= \overline{\lim_{\gamma\to 0}}^{*} \bar{u}^\gamma(x) \\ &:= \lim_{\gamma\to 0} \sup\{\bar{u}^r(y) \mid y \in B(x,\gamma) \cap \overline{\Omega},\ 0 < r \leqslant \gamma\}. \end{aligned}$$

Notice that v is upper semicontinuous in $\overline{\Omega}$, and that $|v(x)| \leqslant M_f$ for $x \in \overline{\Omega}$.

LEMMA 4.2. – *For $x \in \overline{\Omega}$ and $p \in D^+v(x)$, we have*

$$v(x) + \sup_{a\in A(x)} \{-\langle g(x,a), p\rangle - f(x,a)\} \leqslant 0.$$

Remark. – This assertion is slightly weaker than that of the definition of viscosity subsolutions of (4.1) in $\overline{\Omega}$ since the supremum in the above is taken over $A(x)$ in place of $A_0(x)$.

Proof. – Because of stability of viscosity subsolutions, it suffices to prove that if $\phi \in C^1$ satisfies that $v(x) = \phi(x)$ for $x \in \partial\Omega$ and that $v \leqslant \phi$ in $\overline{\Omega}$, then we have

$$0 \geqslant \phi(x) + \sup_{a\in A(x)} \{-\langle g(x,a), D\phi(x)\rangle - f(x,a)\}.$$

Suppose that this inequality fails; there are $\theta > 0$ and $a \in A(x)$ such that

$$\theta \leqslant \phi(x) - \langle g(x,a), D\phi(x)\rangle - f(x,a).$$

We select $\gamma_k \in (0, r_0]$ and $x_k \in \overline{\Omega}_{\gamma_k}$ such that

$$\lim_{k\to\infty}(x_k, \gamma_k) = (x, 0), \qquad |\phi(x) - \phi(x_k)| < \frac{1}{k},$$
$$v(x) \leqslant u^{\gamma_k}(x_k) + \frac{1}{k}.$$

Let $\hat{\gamma}_k = \operatorname{dist}(x_k, \Omega^c) \geqslant \gamma_k$. We note that $\hat{\gamma}_k \to 0$ as $k \to \infty$. Thus, we may suppose that $A(x) \subset A(T^0 x_k)$. Set $X_k(\cdot) = X(\cdot\,; x_k, a)$ for simplicity. By Proposition 3.7, we can find $t_0 > 0$ (independent of k) such that

$$X_k(t) \in \overline{\Omega}_{\hat{\gamma}_k} \quad \bigl(t \in [0, t_0]\bigr). \tag{4.4}$$

For large k, we may also suppose

$$\frac{\theta}{2} \leqslant \phi(x_k) - \langle g(x_k, a), D\phi(x_k)\rangle - f(x_k, a).$$

Furthermore, we can take smaller $t_0 > 0$ if necessary to get

$$\frac{\theta}{4} \leqslant \phi\bigl(X_k(t)\bigr) - \bigl\langle g\bigl(X_k(t), a\bigr), D\phi\bigl(X_k(t)\bigr)\bigr\rangle - f\bigl(X_k(t), a\bigr)$$
$$\bigl(t \in [0, t_0]\bigr). \tag{4.5}$$

Multiplying (4.5) by e^{-t}, we take the integration over $(0, t_0)$ to get

$$\frac{\theta}{4}\bigl(1 - \mathrm{e}^{-t_0}\bigr) \leqslant \phi(x_k) - \phi\bigl(X_k(t_0)\bigr) - \int_0^{t_0} \mathrm{e}^{-t} f\bigl(X_k(t), a\bigr)\,\mathrm{d}t$$
$$\leqslant \frac{2}{k} + u^{\gamma_k}(x_k) - \phi\bigl(X_k(t_0)\bigr) - \int_0^{t_0} \mathrm{e}^{-t} f\bigl(X_k(t), a\bigr)\,\mathrm{d}t.$$

Hence, for a fixed k, we have

$$\frac{\theta}{8}\bigl(1 - \mathrm{e}^{-t_0}\bigr) \leqslant u^{\gamma_k}(x_k) - \phi\bigl(X_k(t_0)\bigr) - \int_0^{t_0} \mathrm{e}^{-t} f\bigl(X_k(t), a\bigr)\,\mathrm{d}t.$$

By the DPP (4.2) for u^{γ_k} with $s = t_0$, we have

$$\frac{\theta}{8}\left(1 - \mathrm{e}^{-t_0}\right) \leqslant u^{\gamma_k}\left(X_k(t_0)\right) - \phi\left(X_k(t_0)\right).$$

Since we may suppose that $X_k(t_0)$ converges to a point $\bar{z} \in \overline{\Omega}$ as $k \to \infty$ (by taking a subsequence if necessary), taking the lim sup (as $k \to \infty$) in the above, we have

$$\frac{\theta}{8}\left(1 - \mathrm{e}^{-t_0}\right) \leqslant v(\bar{z}) - \phi(\bar{z}),$$

which is a contradiction. □

Now we state our convergence result.

THEOREM 4.3. – *For any $\varepsilon > 0$, there is $\gamma(\varepsilon) > 0$ such that if $0 < r \leqslant \gamma(\varepsilon)$, then*

$$\left|u^r(x) - u^0(x)\right| < \varepsilon \quad (x \in \overline{\Omega}_r). \tag{4.6}$$

Proof. – In view of Lemma 4.2, by the comparison result in [5] (or [6]), we obtain that

$$v(x) \leqslant u^0(x) \quad (x \in \overline{\Omega}). \tag{4.7}$$

We remark that although we used a slightly different $A(x)$ (for $x \in \partial\Omega$) from that of Lemma 4.2 to construct "test functions" in [5], we can construct test functions having the same properties as in [5] by using $A(x)$ in Lemma 4.2.

Since $0 \leqslant u^r - u^0$ holds in $\overline{\Omega}_r$ for any $r > 0$, (4.7) implies (4.6). Indeed, otherwise, there exists $\varepsilon_0 > 0$, $r_k > 0$ with $\lim_{k\to\infty} r_k = 0$, and $x_k \in \overline{\Omega}_{r_k}$ with $\lim_{k\to\infty} x_k = z$ for some $z \in \overline{\Omega}$ such that

$$u^{r_k}(x_k) \geqslant u^0(x_k) + \varepsilon_0.$$

Taking the relaxed limit supremum in the above as $k \to \infty$, we get a contradiction to (4.7). □

5. MAIN RESULT

We shall write u for the value function u^0 of the SC problem for $\overline{\Omega}$:

$$u(x) = \inf_{\alpha \in \mathcal{A}_0(x)} \int_0^\infty \mathrm{e}^{-t} f\left(X(t; x, \alpha), \alpha(t)\right) \mathrm{d}t.$$

DEFINITION. – *For $\varepsilon > 0$, we call $\alpha \in \mathcal{A}$ an ε-optimal control of the SC problem for $\overline{\Omega}$ at $x \in \overline{\Omega}$ if $\alpha \in \mathcal{A}_0(x)$ and*

$$0 \leqslant \int_0^\infty e^{-t} f\big(X(t;x,\alpha),\alpha(t)\big)\,dt - u(x) < \varepsilon.$$

We notice that the first inequality always holds if $\alpha \in \mathcal{A}_0(x)$.
Our main result is as follows:

THEOREM 5.1. – *Assume that* (A1), (A2) *and* (A3) *hold. Let $u \in C(\overline{\Omega}, \mathbf{R})$ be the value function of the SC problem for $\overline{\Omega}$, and $\varepsilon > 0$. Then, there exist a constant $\hat{\tau} \in (0, t_0]$, and a mapping $\hat{\alpha}_\varepsilon : x \in \overline{\Omega} \to \hat{\alpha}_\varepsilon(x) \in A$ such that if for any $x \in \overline{\Omega}$ we set*

$$\alpha_\varepsilon(t) = \hat{\alpha}_\varepsilon(x_k) \quad \textit{for } t \in \big[k\hat{\tau}, (k+1)\hat{\tau}\big) \ (k = 0, 1, 2, \ldots),$$

where

$$\begin{cases} x_0 = x, \\ x_{k+1} = X(\hat{\tau}; x_k, \hat{\alpha}_\varepsilon(x_k)), \end{cases}$$

then α_ε is an ε-optimal control of the SC problem for $\overline{\Omega}$ at x.

Proof of Theorem 5.1. –
Step 1: Construction of $\hat{\alpha}_\varepsilon$ and choice of $\hat{\tau}$.
First of all, by Theorem 4.2, we can choose $\gamma_1 \in (0, r_0]$ so that

$$0 \leqslant u^\gamma(x) - u(x) < \frac{\varepsilon}{4} \quad \big(x \in \overline{\Omega}_\gamma,\ \gamma \in (0, \gamma_1]\big). \tag{5.1}$$

In what follows, we always fix $\lambda = \lambda(\gamma) := \gamma^2/\varepsilon$.
For $\lambda = \gamma^2/\varepsilon \in (0, \gamma_1^2/\varepsilon]$, we define

$$u_\lambda^\gamma = u_{\lambda(\gamma)}^\gamma(x) = \inf_{y \in \overline{\Omega}_\gamma} \left(u^\gamma(y) + \frac{|x-y|^2}{2\lambda} \right).$$

For any $x \in \overline{\Omega}$, we choose $x_\lambda \in \overline{\Omega}_\gamma$ so that $u_\lambda^\gamma(x) = u^\gamma(x_\lambda) + |x - x_\lambda|^2/(2\lambda)$. Then, by Lemma 3.5, there is $\lambda_1 \in (0, \gamma_1^2/\varepsilon]$ such that

$$\frac{|x - x_\lambda|^2}{2\lambda} < \frac{\varepsilon}{4} \quad (0 < \lambda \leqslant \lambda_1). \tag{5.2}$$

Taking smaller $\lambda_1 > 0$, the choice of which depends only on the modulus of continuity of u, we may suppose that

$$\omega_u\big(|x - x_\lambda|\big) < \frac{\varepsilon}{4} \quad (0 < \lambda \leqslant \lambda_1). \tag{5.3}$$

From the definition, it is easy to see that

$$\sup_{x \in \overline{\Omega}_\gamma} \big|u^\gamma(x)\big| \leqslant M_f \quad (\gamma \geqslant 0).$$

Thus, setting $M_1 := 2M_f$, by Lemma 3.6 together with Proposition 2.2, we find $\lambda_3 \in (0, \lambda_1]$ such that if $\lambda \in (0, \lambda_3]$, $x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}$, $p \in \overline{D}^- u_\lambda^\gamma(x)$ and $a \in A(T^0 x)$, then we have

$$0 \leqslant u_\lambda^\gamma(x) - \big\langle g(x,a), p\big\rangle - f(x,a). \tag{5.4}$$

Furthermore, by Lemma 4.1, we find $\lambda_4 \in (0, \lambda_3]$ such that if $\lambda \in (0, \lambda_4]$, $x \in \overline{\Omega}$ and $p \in \overline{D}^- u_\lambda^\gamma(x)$, then

$$-\frac{\varepsilon}{16} \leqslant u_\lambda^\gamma(x) + \sup_{a \in A}\big\{-\langle g(x,a), p\rangle - f(x,a)\big\}. \tag{5.5}$$

In what follows, we fix $\hat{\tau} := \gamma^3 = (\varepsilon\lambda)^{3/2} = \varepsilon\lambda\gamma$.
We claim that if

$$-\frac{\varepsilon}{8} \leqslant u_\lambda^\gamma(x) - \big\langle g(x,a), p\big\rangle - f(x,a)$$

for $a \in A$, $x \in \overline{\Omega}$ and $p \in \overline{D}^- u_\lambda^\gamma(x)$, and if $X(t) := X(t; x, a) \in \overline{\Omega}$ for $0 \leqslant t \leqslant \hat{\tau}$, then we have

$$-\frac{\varepsilon}{4}\big(1 - \mathrm{e}^{-\hat{\tau}}\big) \leqslant u_\lambda^\gamma(x) - \mathrm{e}^{-\hat{\tau}} u_\lambda^\gamma\big(X(\hat{\tau})\big) - \int_0^{\hat{\tau}} \mathrm{e}^{-t} f\big(X(t), \hat{\alpha}_\varepsilon(x)\big)\,\mathrm{d}t. \tag{5.6}$$

To prove this claim, we first observe that

$$\big|X(t) - x\big| \leqslant t M_g.$$

Thus, we may easily have

$$u_\lambda^\gamma(x) \leqslant u_\lambda^\gamma\big(X(t)\big) + \frac{t M_g d_0}{\lambda}, \tag{5.7}$$

where $d_0 := \sup\{|x-y| \mid x, y \in \Omega\}$.

Taking smaller t_0 if necessary, we may suppose that

$$\omega_f(|X(t)-x|) < \frac{\varepsilon}{16} \quad (t \in [0, t_0]). \tag{5.8}$$

Moreover, we have

$$\left|\frac{X(t)-x}{t} - g(X(t),a)\right| \leqslant \frac{1}{t}\int_0^t |g(X(s),a) - g(X(t),a)|\,\mathrm{d}s \leqslant \frac{tM_g^2}{2}. \tag{5.9}$$

For the sake of simplicity, we shall use the symbol C_0 to denote various positive constants depending only on M_g, M_f and d_0.

Since $\lambda|p| \leqslant C_0$ and $\lambda|p(t)| \leqslant C_0$ for $p \in \overline{D}^- u_\lambda^\gamma(x)$ and $p(t) \in \overline{D}^- u_\lambda^\gamma(X(t))$ by (2.1), (5.9) implies that

$$-\langle g(x,a), p\rangle + \langle g(X(t),a), p(t)\rangle \leqslant \frac{tC_0}{\lambda} + \left\langle \frac{X(t)-x}{t},\ p(t)-p\right\rangle.$$

Hence, by noting Lemma 2.4(1), Proposition 2.3 together with (5.6) yields that

$$-\langle g(x,a), p\rangle \leqslant -\langle g(X(t),a), p(t)\rangle + \frac{tC_0}{\lambda}.$$

By the above inequality with (5.7) and (5.8), we see that

$$-\frac{C_0\hat{\tau}}{\lambda} - \frac{3\varepsilon}{16} \leqslant u_\lambda^\gamma(X(t)) - \langle g(X(t),a), p(t)\rangle - f(X(t),a) \quad (t \in [0, \hat{\tau}]).$$

Since we may suppose that $0 < \gamma \leqslant \min\{t_0^{1/3}, 1/(16C_0)\}$, recalling $\hat{\tau} = \varepsilon\lambda\gamma$, multiplying the above inequality by e^{-t} and then, integrating the resulting inequality over $(0, \hat{\tau})$, in view of Lemma 2.4(2), we get (5.6).

Next, because of the choice of $\hat{\tau}$, we may suppose that

$$X(t; x, a) \in \overline{\Omega} \quad (x \in \overline{\Omega}_{\frac{\gamma}{2}},\ a \in A,\ t \in [0, \hat{\tau}]). \tag{5.10}$$

We now fix $\lambda = \lambda_4$. Thus, $\gamma = \sqrt{\lambda_4\varepsilon}$ and $\hat{\tau} = (\lambda_4\varepsilon)^{3/2}$.

We shall define the mapping $\hat{\alpha}_\varepsilon : \overline{\Omega} \to A$ in the following manner:

$$\hat{\alpha}_\varepsilon(x) = \begin{cases} a \in A(T^0 x), & \text{provided } x \in \overline{\Omega} \setminus \overline{\Omega}_{\gamma/2}, \\ a \in A, & \text{for which } -\varepsilon/8 \leqslant u_\lambda^\gamma(x) - \langle g(x,a), p\rangle \\ & -f(x,a), \text{ holds for } p \in \overline{D}^- u_\lambda^\gamma(x), \\ & \text{provided } x \in \overline{\Omega}_{\gamma/2}. \end{cases}$$

Step 2: Verification.

In view of (5.4) and (5.5), we observe that if $x \in \overline{\Omega}$ and $p \in \overline{D}^- u_\lambda^\gamma(x)$, then we have

$$-\frac{\varepsilon}{8} \leqslant u_\lambda^\gamma(x) - \langle g(x, \hat{\alpha}_\varepsilon(x)), p\rangle - f(x, \hat{\alpha}_\varepsilon(x)). \tag{5.11}$$

Furthermore, (5.10) and Proposition 3.7 yield that

$$X(t; x, \hat{\alpha}_\varepsilon(x)) \in \overline{\Omega} \quad (x \in \overline{\Omega},\ t \in [0, \hat{\tau}]). \tag{5.12}$$

Now we shall verify that $\alpha_\varepsilon \in \mathcal{A}$ defined in Theorem 5.1 satisfies our assertions.

Recall that $x_0 = x$, $t_0 = 0$, $x_{k+1} = X(\hat{\tau}; x_k, \hat{\alpha}_\varepsilon(x_k))$, and $\alpha_\varepsilon(t) = \hat{\alpha}_\varepsilon(x_k)$ for $t \in [k\hat{\tau}, (k+1)\hat{\tau})$ $(k = 0, 1, 2, \ldots)$. Due to (5.12), it is obvious to see that $\alpha_\varepsilon \in \mathcal{A}_0(x)$.

By (5.11) with x_k for $k \geqslant 0$, our claim in Step 1 yields that

$$-\frac{\varepsilon}{4}(1 - \mathrm{e}^{-\hat{\tau}}) \leqslant u_\lambda^\gamma(x_k) - \mathrm{e}^{-\hat{\tau}} u_\lambda^\gamma(X(\hat{\tau}; x_k, \hat{\alpha}_\varepsilon(x_k))) - \int_0^{\hat{\tau}} \mathrm{e}^{-t} f(X(t; x_k, \hat{\alpha}_\varepsilon(x_k)), \hat{\alpha}_\varepsilon(x_k))\, \mathrm{d}t.$$

Multiplying the above inequality by $\mathrm{e}^{-k\hat{\tau}}$ and then, taking the summation over $k = 0, 1, 2, \ldots$, from the definition of α_ε, we have

$$-\frac{\varepsilon}{4} \leqslant u_\lambda^\gamma(x_0) - \int_0^\infty \mathrm{e}^{-t} f(X(t; x_0, \alpha_\varepsilon), \alpha_\varepsilon(t))\, \mathrm{d}t. \tag{5.13}$$

Let $x_\lambda \in \overline{\Omega}_\gamma$ satisfiy that $u_\lambda^\gamma(x_0) = u^\gamma(x_\lambda) + |x_0 - x_\lambda|^2/(2\lambda)$. Then, by (5.2), we have

$$u_\lambda^\gamma(x_0) < u^\gamma(x_\lambda) + \frac{\varepsilon}{4}.$$

Hence, by (5.1) and (5.3), we have

$$u_\lambda^\gamma(x_0) < u(x_0) + \frac{3\varepsilon}{4}.$$

This together with (5.13) yields that

$$-\varepsilon < u(x_0) - \int_0^\infty e^{-t} f\big(X(t; x_0, \alpha_\varepsilon), \alpha_\varepsilon(t)\big)\,dt \leqslant 0. \qquad \square$$

APPENDIX A

In order to prove Proposition 3.3, we will need the following lemmas: Let $P \subset S^{N-1}$ and define

$$K = \operatorname{co} P.$$

Let $p \in K$.

LEMMA A.1. – *We have*

$$B\big(p, |p|\big) \subset \bigcup_{q\in P} B(q, 1).$$

Proof. – For $p \in K$, we set $p = \sum_{i=1}^n \lambda_i q_i$, where

$$\lambda_i \geqslant 0, \quad \sum_{i=1}^n \lambda_i = 1, \quad q_i \in P.$$

Fix $y \in B(0, |p|)$. We want to show that there is $i \in \{1, \ldots, n\}$ such that $|p + y - q_i| \leqslant 1$, which implies that

$$p + y \in B(q_i, 1) \subset \bigcup_{q\in P} B(q, 1).$$

We may assume that

$$|p + y - q_1| \leqslant |p + y - q_2| \leqslant \cdots \leqslant |p + y - q_n|.$$

We shall prove that $|p + y - q_1| \leqslant 1$. From this chain of inequalities we get

$$\langle q_1, p + y\rangle \geqslant \langle q_2, p + y\rangle \geqslant \cdots \geqslant \langle q_n, p + y\rangle.$$

Note that

$$\langle q_1 - q_i, p\rangle \geqslant \langle q_i - q_1, y\rangle \quad (i = 2, \dots, n),$$

and that

$$\begin{aligned} p - q_1 &= \left(1 - \sum_{i=2}^{n} \lambda_i\right) q_1 + \lambda_2 q_2 + \cdots + \lambda_n q_n - q_1 \\ &= \lambda_2(q_2 - q_1) + \cdots + \lambda_n(q_n - q_1). \end{aligned}$$

We compute that

$$\begin{aligned} |p + y - q_1|^2 &= |p - q_1|^2 + 2\langle y, p - q_1\rangle + |y|^2 \\ &\leqslant |p - q_1|^2 + 2\lambda_2\langle y, q_2 - q_1\rangle + 2\lambda_3\langle y, q_3 - q_1\rangle \\ &\quad + \cdots + 2\lambda_n\langle y, q_n - q_1\rangle + |p|^2 \\ &\leqslant |p - q_1|^2 - 2\lambda_2\langle p, q_2 - q_1\rangle - 2\lambda_3\langle p, q_3 - q_1\rangle \\ &\quad - \cdots - 2\lambda_n\langle p, q_n - q_1\rangle + |p|^2 \\ &\leqslant |p - q_1|^2 - 2\langle p, p - q_1\rangle + |p|^2 \\ &\leqslant |p - q_1 - p|^2 = |q_1|^2 = 1, \end{aligned}$$

and finish the proof. □

Let $\gamma > 0$ and $z \in \partial\Omega_\gamma$. For simplicity, we set

$$N(z) = N_\gamma(z) := \{p \in \mathbf{R}^N \mid \langle p, x - z\rangle \leqslant o(|x - z|) \text{ as } \Omega_\gamma \ni x \to z\}.$$

We remark that $N(z)$ is closed.

We also define

$$N_T(z) = \{tp \mid t \geqslant 0,\ p \in S^{N-1} \text{ such that } z + rp \in \partial\Omega\}.$$

LEMMA A.2. – $N(z) = \overline{\mathrm{co}}\, N_T(z)$.

Proof. – We first prove that

$$\mathrm{co}\, N_T(z) \subset N(z).$$

Note that $N(z)$ is a convex set.

Let $p \in N_T(z) \cap S^{N-1}$. By the definition of Ω_γ, we have

$$\operatorname{int} B(z + \gamma p, \gamma) \subset \Omega_\gamma^c.$$

Namely,

$$\Omega_\gamma \subset \left(\operatorname{int} B(z+\gamma p,\gamma)\right)^c.$$

Thus, if $y \in \Omega_\gamma$, then

$$\gamma^2 \leqslant \left|y-(z+\gamma p)\right|^2 = |y-z|^2 - 2\gamma\langle p, y-z\rangle + \gamma^2,$$

and hence,

$$\langle p, y-z\rangle \leqslant \frac{1}{2\gamma}|y-z|^2.$$

Thus we see that $p \in N(z)$ and moreover that $\operatorname{co} N_T(z) \subset N(z)$.

Next we prove that

$$N(z) \subset \overline{\operatorname{co}}\, N_T(z).$$

We argue by contradiction. We assume for notational simplicity that $z = 0$. Suppose that there were a point $p \in S^{N-1} \cap N(0)$ such that $p \notin \overline{\operatorname{co}}\, N_T(0)$.

Choose a convex conic neighborhood V of $\overline{\operatorname{co}}\, N_T(0)$ so that $p \notin \overline{V}$. By the Hahn–Banach theorem, there is a vector $n \in S^{N-1}$ such that

$$\langle n, p\rangle > 0, \qquad \langle n, q\rangle \leqslant 0 \quad (q \in \overline{V}).$$

According to the definition of $N_T(0)$, we see that

$$B(0,\gamma) \cap \Omega^c = \left\{\gamma p \mid p \in N_T(0) \cap S^{N-1}\right\}.$$

Therefore, by continuity, there is $\eta > 0$ such that

$$B(0,\gamma+\eta) \cap V^c \cap \Omega^c = \emptyset.$$

We want to prove that for $t < \eta$,

$$B(tn,\gamma) \subset \Omega. \tag{A.1}$$

To see this, let $q \in B(tn,\gamma)$. If $q \notin V$, then we have $q \in \Omega$ since

$$q \in B(0,\gamma+\eta) \cap V^c.$$

It remains to consider the case when $q \in V$. Since $\langle n, q\rangle \leqslant 0$, we have

$$\gamma^2 \geqslant |q-tn|^2 = |q|^2 - 2t\langle n, q\rangle + t^2 \geqslant |q|^2 + t^2.$$

Hence,

$$|q|^2 \leqslant \gamma^2 - t^2 < \gamma^2.$$

Since $0 \in \partial\Omega_\gamma$ and $q \in \operatorname{int} B(0,\gamma)$, we see that $q \in \Omega$ and that (A.1) holds.

In view of (A.1), we see that if $0 \leqslant t \leqslant \eta$, then

$$tn \in \Omega_\gamma.$$

Hence, since $p \in N(0)$, we have

$$t\langle n, p\rangle \leqslant \mathrm{o}(t) \quad \text{as } t \to 0.$$

This yields that $\langle n, p\rangle \leqslant 0$, which contradicts our choice of $\boldsymbol{n}$. □

We next show that the uniform exterior shpere condition hlods for Ω_γ.

LEMMA A.3. – *Assume that* (A2) *holds. Let* $z \in \partial\Omega_\gamma$ *and* $p \in N_T(z) \cap S^{N-1}$. *Then, we have*

$$B\big(z + (R+\gamma)p, R+\gamma\big) \cap \overline{\Omega}_\gamma = \{z\}.$$

Proof. – Fix $z \in \partial\Omega_\gamma$ and $p \in N_T(z) \cap S^{N-1}$. We have $y := z + \gamma p \in \partial\Omega$. By assumption (A2) there is $x \in \mathbf{R}^N$ such that

$$B(x, R) \cap \overline{\Omega} = \{y\}.$$

We claim that

$$x = z + (R+\gamma)p,$$

and

$$B(x, R+\gamma) \cap \overline{\Omega}_\gamma = \{z\}.$$

Indeed, since

$$|z-y| = \gamma, \qquad |y-x| = R, \qquad B(z,\gamma) \cap B(x,R) = \{y\},$$

we see that $x = z + (R+\gamma)p$. It is immediate to see that $z \in B(x, R+\gamma)$. Suppose for a moment that there were a point $\xi \in B(x, R+\gamma) \cap \overline{\Omega}_\gamma$ such that $\xi \neq z$. Then $B(\xi,\gamma) \subset \overline{\Omega}$. In particular, $\eta := \xi + \gamma(x-\xi)/(R+\gamma) \in \overline{\Omega}$. It follows that $\eta = x + R(\xi - x)/(R+\gamma) \in B(x, R)$. Hence, $\eta \in B(x,R) \cap \overline{\Omega}$. Therefore, we have $\eta = y$. This is a contradiction. □

Now we shall present a proof of Proposition 3.3.

Proof of Proposition 3.3. – Fix any $0 < \gamma \leqslant r_0$ and $x \in \partial\Omega_\gamma$. According to Lemma A.3 we have

$$B\big(x + (R+\gamma)p, R+\gamma\big) \cap \overline{\Omega}_\gamma = \{x\} \quad \big(p \in N_T(x) \cap S^{N-1}\big),$$

which implies that

$$\bigcup_{p\in P} B(x + Rp, R) \subset (\Omega_\gamma)^c.$$

Here and henceforth we write $P = N_T(x) \cap S^{N-1}$. We write $K = \operatorname{co} P$ as well. Using Lemma A.1, for any $p \in K$, we see that

$$B\big(x + Rp, R|p|\big) = x + RB\big(p, |p|\big) \subset x + R \bigcup_{q\in P} B(q, 1)$$
$$= \bigcup_{q\in P} B(x + Rq, R) \subset (\Omega_\gamma)^c.$$

It is well known that

$$\bigcup_{t\geqslant 0} tK = \operatorname{co} N_T(x) \quad \big(\subset N(x)\big).$$

Let $a \in A(T^0 x)$. By Proposition 3.1, we have

$$B(x + t\eta, \delta t/2) \subset \Omega_\gamma \quad (0 < t \leqslant \delta),$$

where $\eta = g(x,a)/|g(x,a)|$. By the definition of $N(x)$, we have

$$t\langle \eta + \xi, p\rangle \leqslant \mathrm{o}(t) \quad \text{as } t \to 0 \quad \big(p \in N(x),\ \xi \in B(0, \delta/2)\big),$$

from which we get

$$\langle \eta, p\rangle \leqslant -\delta|p|/2 \quad \big(p \in N(x)\big).$$

This yields that

$$|p| \geqslant \delta/2 \quad (p \in K). \tag{A.2}$$

Indeed, for $p = \sum_{i=1}^n \lambda_i q_i$ with $\lambda_i > 0$, $q_i \in P$ satisfying $\sum_{i=1}^n \lambda_i = 1$, we have

$$-|p| \leqslant \langle \eta, p\rangle = \sum_{i=1}^n \lambda_i \langle \eta, q_i\rangle \leqslant -\frac{\delta}{2}\sum_{i=1}^n \lambda_i |q_i| = -\frac{\delta}{2},$$

and hence,

$$|p| \geqslant \delta/2,$$

which ensures that (A.2) holds. Thus we see that if $\nu \in N(x) \cap S^{N-1}$, then $\nu = tp$ for some $p \in K$ and $t > 0$ satisfying $|p| \geqslant \delta/2$, and

$$B(x + R\delta\nu/2, R\delta/2) \subset B(x + Rp, R|p|) \subset (\Omega_\gamma)^c.$$

This completes the proof. □

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