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A nonlinear oblique derivative boundary value problem for the heat equation Part 2: Singular self-similar solutions

by

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ABSTRACT. – This paper continues the study started in [12]. In the upper half-plane, consider the elliptic equation $-U^{\varepsilon}_{xx} - \varepsilon U^{\varepsilon}_{zz} - \frac{1}{2}(xU_x + zU_z) = 0$, submitted to the nonlinear oblique derivative boundary condition $U_x = UU_z$ on the axis $x = 0$. The solution of this problem appears to be the self-similar solution of the heat equation with the same boundary condition. As ε goes to 0, the function U^{ε} converges to the non trivial solution U of the corresponding degenerate problem. Moreover there exists $z_0 > 0$ such that U vanishes for $z \geq z_0$, is C^{∞} on $]0, z_0[\times\mathbb{R}_+$, is continuous on the boundary $x = 0$ and is discontinuous on the half-axis $\{z = 0, x > 0\}$. © Elsevier, Paris

Key words: Nonlinear oblique derivative condition, degenerate elliptic problems, self-similar solution.

RÉSUMÉ. – Cet article poursuit l'étude commencée dans [12]. Soit, dans le demi-plan supérieur, l'équation elliptique $-U^{\varepsilon}_{xx} - \varepsilon U^{\varepsilon}_{zz} - \frac{1}{2}(xU_x + zU_z) = 0$, soumise à la condition aux limites à dérivée oblique non linéaire $U_x = UU_z$ sur l'axe $x = 0$. La solution de ce problème apparaît comme la solution autosemblable de l'équation de la chaleur soumise à la même condition aux limites. Lorsque ε tend vers 0, la fonction U^ε converge vers la solution U du problème dégénéré correspondant. De plus il existe un réel $z_0 > 0$ tel que U s'annule pour $z \geq z_0$, est C^∞ sur $]0, z_0[\times\mathbb{R}_+$, est continue sur la frontière $x = 0$ et discontinue sur le demi-axe $\{z = 0,\ x > 0\}$. © Elsevier, Paris

1. INTRODUCTION AND MAIN RESULTS

This paper continues the study initiated in [12]. Let us first briefly recall the problem dealt with and the main results obtained in [12].

We consider a nonlinear oblique derivative boundary condition for the heat equation, in the half-plane $\mathbb{R}^2_+ = \{(Z, X) \in \mathbb{R} \times \mathbb{R}_+\}$:

$$\text{(1.1.NH)} \qquad \begin{cases} B_t - \Delta B = 0 & (\mathbb{R}^2_+) \\ B_X - KBB_Z = 0 & (X = 0) \\ B(t, -\infty, X) = 1, \quad B(t, +\infty, X) = 0 & \\ B(0, Z, X) = B_0(Z, X). & \end{cases}$$

The above system arises in plasma physics (see [11] for the modeling), and describes the diffusive propagation of a magnetic field in a uniform plasma, in presence of a perfectly conductive electrode which is placed on the axis $X = 0$. The non-homogeneous condition at $Z \to -\infty$ stands for a source of magnetic field.

In some realistic physical situations, the parameter K turns out to be very large [4]. The aim of this part is to let $K \to +\infty$ in these equations, thanks to an adequate scaling. Introduce the small parameter $\varepsilon = 1/K^2$ and let us define the new variables

$$Z' := \frac{Z}{K}, \quad X' := X;$$

since we will only work in these variables we drop the primes at once. Equation (1.1.NH) becomes

$$\text{(1.2.NH)} \qquad \begin{cases} B_t - B_{XX} - \varepsilon B_{ZZ} = 0 & (\mathbb{R}^2_+) \\ B_X - BB_Z = 0 & (X = 0) \\ B(t, -\infty, X) = 1, B(t, +\infty, X) = 0 & \\ B(0, Z, X) = B_0(Z, X). & \end{cases}$$

This paper is devoted to the behaviour of the self-similar solutions of (1.2.NH) as $\varepsilon \to 0$. Recall that self-similar solutions of (1.2.NH) are steady solutions in the variables

$$\tau = \mathrm{Log}(t+1), \quad z = \frac{Z}{\sqrt{t+1}}, \quad x = \frac{X}{\sqrt{t+1}}.$$

Hence the self-similar problem associated to (1.2.NH) reads

$$\text{(1.3.NH)} \qquad \begin{cases} -U^\varepsilon_{xx} - \varepsilon U^\varepsilon_{zz} - \dfrac{1}{2}(zU^\varepsilon_z + xU^\varepsilon_x) = 0 & (\mathbb{R}^2_+) \\ U^\varepsilon_x = U^\varepsilon U^\varepsilon_z & (x=0) \\ U^\varepsilon(-\infty, x) = 1, U^\varepsilon(+\infty, x) = 0. \end{cases}$$

This system becomes degenerate as $\varepsilon \to 0$. Hence classical existence and smoothness results [2] for elliptic equations cannot be applied directly. Nevertheless, the scheme used in [12] to prove the C^∞ regularity of the self-similar solution is robust enough with respect to ε and will be adapted here.

Let formally $\varepsilon \to 0$ in (1.3NH). The degenerate self-similar problem writes

$$\text{(1.4.NH)} \qquad \begin{cases} -U_{xx} - \dfrac{1}{2}(zU_z + xU_x) = 0 & (\mathbb{R}^2_+) \\ U_x = UU_z & \text{(x=0)} \\ U(-\infty, x) = 1, U(+\infty, x) = 0. \end{cases}$$

We denote by $\psi(z)$ the solution of

$$-\psi''(z) - \frac{1}{2}z\psi'(z) = 0, \qquad \psi(-\infty) = 1,\ \psi(+\infty) = 1.$$

Let us set $u := U - \psi$; this is the solution of the associated homogeneous problem. The starting point of our study is the following result, proved in [12]:

THEOREM A.1.1 (Self-similar problem with $\epsilon > 0$). – *There exists a unique solution* $U \in C^\infty(\mathbb{R}^2_+)$ *of* (1.3NH). *Moreover we have the following properties:*

- $\exists C > 0$ *such that* $0 \le U(z,x) - \psi(\frac{z}{\varepsilon}) \le C \exp\left[-\frac{1}{8}\left(\frac{z^2}{\varepsilon^2} + x^2\right)\right]$;
- U *is decreasing with respect to* z *and* x.

Such a result may be classically obtained by a topological degree argument combined with strong enough *a priori* estimates, as in [9]. We presented in [12] an alternative method, based on estimates of $(u_z)^2$ and

$(u_z)^3$ at the boundary, which will turn out to be suitable in the present context.

The main result of this paper is the following existence and uniqueness theorem:

THEOREM 1.1 (Convergence to the solution of the degenerate problem). – *(i) As* $\varepsilon \longrightarrow 0$, *and after extraction of a subsequence, the solution* U^ε *of* (1.3NH) *converges in* $L^1_{loc}(\mathbb{R}^2_+)$ *strong and a.e. to a weak solution* U *of* (1.4NH).

(ii) There exists $z_0 > 0$ *such that this function* U *verifies*

$$U \equiv 1 \quad \textit{on } \mathbb{R}^*_- \times \mathbb{R}_+, \tag{1.5}$$

$$U \equiv 0 \quad \textit{on } [z_0, +\infty[\times\mathbb{R}_+, \tag{1.6}$$

$$U \textit{ is discontinuous along the axis } z = 0,\ x > 0, \tag{1.7}$$

$$U \in C^\infty(]0, z_0[\times\mathbb{R}_+), \tag{1.8}$$

$$\textit{the trace of } U \text{ on } \{x = 0\} \textit{ is continuous.} \tag{1.9}$$

This result illustrates the rapid penetration of the magnetic field at the electrode: we have $U > 0$ on a nontrivial portion of the axis $\{x = 0,\ z > 0\}$, whereas the magnetic field does not penetrate on the part $\{z > 0\}$ of the cathode, i.e when $x = +\infty$.

Because we do not know *a priori* what regularity property is satisfied by the weak solutions of (1.4NH), uniqueness is not completely trivial. A relevant definition of weak solution may be the following one: a function $U(z, x)$ is an *entropy solution* of (1.4NH) if, besides satisfying the minimal smoothness assumptions so that a weak formulation makes sense, has a BV trace at $\{x = 0\}$, whose z-derivative is bounded from above. The solution constructed in Theorem 1.1 is trivially an entropy solution.

Armed with this definition we are able to prove the following result:

THEOREM 1.2 (Uniqueness). – *There is a unique entropy solution to Problem (1.4NH). As a consequence, the whole sequence* $(U^\varepsilon)_{\varepsilon>0}$ *converges to* U.

The reason why this theorem holds is that the function $V(z,x) = \int_z^{+\infty} U(z',x)\,dz$ is a viscosity solution, in the sense given in Crandall-Ishii-Lions [5], to the problem

$$
(1.10) \qquad \begin{cases} -V_{xx} - \frac{1}{2}(zV_z + xV_x) + \frac{1}{2}V = 0 & (\mathbb{R}^2_+) \\ V_x = -\frac{1}{2}V_z^2 & (x=0) \\ V(z,x) \sim -z \text{ as } z \to -\infty, U(+\infty, x) = 0. \end{cases}
$$

A uniqueness result for the above problem will be obtained in a straightforward way. Uniqueness of entropy solutions in this framework is anything but surprising if one thinks about nonlinear conservation laws, from which we have obviously borrowed the terminology: for a given function $f(x) \in L^1(\mathbb{R})$, a function $u(x)$ is an entropy solution of $\lambda u + (u^2)' = f(x)$ if and only if $v(x) = \int_{-\infty}^{x} u(y)\,dy$ is a viscosity solution of the Hamilton-Jacobi equation $\lambda v + (v')^2 = \int_{-\infty}^{x} f(y)\,dy$. In this context, an entropy solution is precisely a BV solution with bounded from above x-derivative; see Lions-Souganidis [10] for more details.

As a final introductory remark, we point out that the results of papers 1 and 2 remain valid if the boundary condition is replaced by $B_x = f(B)_z$, where f is a C^1 nonnegative nondecreasing function.

The paper is organized as follows. The second section is mainly devoted to the convergence property (i), namely stated in Proposition 2.1 below, with (1.5) and (1.6). Lemma 2.2 implies (1.7). Next, the third section is devoted to the proof of the regularity of the solution: (1.8) is stated in Proposition 3.4 and (1.9) is stated in Proposition 3.5. Section 4 is devoted to uniqueness of entropy solutions to (1.4NH). Finally, in the last section we show some numerical simulations that enable to visualize the function U and its different properties.

2. NON-TRIVIAL WEAK SOLUTIONS

First recall several notations used in [12]. If u is a function defined on $\mathbb{R}^2_+$, γu denotes its trace on the boundary $\{x=0\}$, when it is well defined; as soon as no confusion is possible, we shall also denote by u this trace. The notation $\int_{\mathbb{R}}$ will stand for an integral calculated on the boundary and C

will denote a generic positive constant independent of ε. For $M > 0$, we denote by Ξ_M the strip

$$\Xi_M = \{(z, x) \in \mathbb{R} \times [0, M]\}.$$

Finally, we will, as is classical, denote by $C_0^\infty(\mathbb{R}_+^2)$ the set of all compactly supported C^∞ funtions from $\mathbb{R}_+^2$ to $\mathbb{R}$.

Consider the solution U^ε of (1.3NH) and set $u^\varepsilon(z,x) = U^\varepsilon(z,x) - \psi^\varepsilon(z)$, where

$$\psi^\varepsilon(z) = \psi\left(\frac{z}{\sqrt{\varepsilon}}\right) = \frac{1}{2\sqrt{\pi}} \int_{z/\sqrt{\varepsilon}}^{+\infty} e^{-\frac{\sigma^2}{4}} d\sigma.$$

The homogeneous problem associated to (1.3NH) writes

$$\text{(1.3H)} \qquad \begin{cases} -u_{xx}^\varepsilon - \varepsilon u_{zz}^\varepsilon - \dfrac{1}{2}(z u_z^\varepsilon + x u_x^\varepsilon) = 0 & (\mathbb{R}_+^2) \\ u_x^\varepsilon = U^\varepsilon U_z^\varepsilon = (u^\varepsilon + \psi^\varepsilon)(u^\varepsilon + \psi^\varepsilon)_z & (x = 0) \\ u^\varepsilon(z,x) \to 0 \text{ as } |z,x| \to +\infty. \end{cases}$$

Remark that $\psi^\varepsilon \to 1 - H$ a.e. in $\mathbb{R}_+^2$, where H denotes the Heaviside function. Hence one can define similarly an homogeneous problem associated to (1.4H), denoted (1.4NH), its solution being $u = U - 1 + H$.

PROPOSITION 2.1. – *Let u^ε be the solution of* (1.3H). *Then u^ε converges -up to a subsequence- to a weak solution of* (1.3H), *$u \in L^2(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$. We have $u \in BV(\Xi_M)$, for every $M > 0$, $u_x \in L^2(\mathbb{R}_+^2)$, $\gamma u \in BV(\mathbb{R})$ and the convergence holds in the following sense:*

$$\text{(2.1)} \qquad u^\varepsilon \to u \textit{ in } L^2(\mathbb{R}_+^2) \textit{ weak, in } L^1_{loc}(\mathbb{R}_+^2) \textit{ strong and a.e. in } \mathbb{R}_+^2,$$

$$\text{(2.2)} \qquad u_x^\varepsilon \to u_x \textit{ in } L^2(\mathbb{R}_+^2) \textit{ weak and } u_z^\varepsilon \to u_z \textit{ in } \mathcal{M}_b(\Xi_M) \textit{ weak } *,$$

$$\text{(2.3)} \qquad \gamma u^\varepsilon \to \gamma u \textit{ in } L^1_{loc}(\mathbb{R}) \textit{ strong and a.e. in } \mathbb{R},$$

$$\text{(2.4)} \qquad \gamma u_z^\varepsilon \to \gamma u_z \textit{ in } \mathcal{M}_b(\mathbb{R}) \textit{ weak } *,$$

where $\mathcal{M}_b(\Xi_M)$ and $\mathcal{M}_b(\mathbb{R})$ denote respectively the spaces of bounded measures on Ξ_M and $\mathbb{R}$. Moreover we have

$$\text{(2.5)} \qquad \forall \alpha > 0, \quad u^\varepsilon \to 0 \quad \text{uniformly on }]-\infty, -\alpha] \times \mathbb{R}_+,$$

$$u^\varepsilon \to 0 \quad \text{uniformly on } [4\sqrt{\pi}, +\infty[\times \mathbb{R}_+. \tag{2.6}$$

Proof. – In [12] we remarked that ψ^ε is a sub-solution of (1.3H) and we constructed a super-solution for this problem, denoted $\psi^\varepsilon + \Lambda^\varepsilon$: we have

$$\psi^\varepsilon(z) \le U^\varepsilon(z,x) \le \psi^\varepsilon(z) + \Lambda^\varepsilon(z,x). \tag{2.7}$$

In the rescaled variables, the function Λ^ε writes

$$\Lambda^\varepsilon(z,x) = \begin{cases} 2\psi(x)(1-\psi^\varepsilon(z)) & \text{for } z \le 0 \\ 2\psi(x)(1-\psi^\varepsilon(z) - a^\varepsilon z) & \text{for } 0 \le z \le 4\sqrt{\pi} \\ 2\psi(x)\psi^\varepsilon(z - 4\sqrt{\pi} + \delta^\varepsilon) & \text{for } z \ge 4\sqrt{\pi}, \end{cases}$$

where $\delta^\varepsilon = \sqrt{-2\varepsilon \mathrm{Log}(\varepsilon/4)}$ and $a^\varepsilon = \frac{1}{4\sqrt{\pi}}\big(1 - \psi^\varepsilon(4\sqrt{\pi}) - \psi^\varepsilon(\delta^\varepsilon)\big)$. Estimate (2.7) enables to obtain some informations on the behaviour of U^ε. Indeed, as $\varepsilon \to 0$, the lower bound ψ^ε converges to $1 - H$ uniformly on each $]-\infty, -\alpha] \cup [\alpha, +\infty[$, $\alpha > 0$. Moreover, the upper bound $\psi^\varepsilon + \Lambda^\varepsilon$ converges to a function $\overline{\Phi}$ uniformly on the same interval, this limit $\overline{\Phi}$ being defined by

$$\overline{\Phi}(z,x) = \begin{cases} 1 & \text{for } z \le 0, \\ 2\psi(x)\left(1 - \dfrac{z}{4\sqrt{\pi}}\right) & \text{for } 0 \le z \le 4\sqrt{\pi}, \\ 0 & \text{for } z \ge 4\sqrt{\pi}. \end{cases}$$

Consequently we deduce (2.5) and (2.6).

Next, (2.7) and the exponential decay of the function Λ^ε at the infinity, uniform with respect to ε, enable to infer

$$\|u^\varepsilon\|_{L^1(\mathbb{R}^2_+)} + \|\gamma u^\varepsilon\|_{L^1(\mathbb{R})} \le C.$$

By Theorem A.1.1, U^ε is decreasing along z and x. Thus, since $u^\varepsilon(z,x) = U^\varepsilon(z,x) - \psi^\varepsilon(z)$, u^ε is decreasing with respect to x and vanishes as $x \to +\infty$. Therefore we have

$$\int_{\mathbb{R}^2_+} |u^\varepsilon_x| = -\int_{\mathbb{R}^2_+} u^\varepsilon_x = \int_{\mathbb{R}} \gamma u^\varepsilon = \|\gamma u^\varepsilon\|_{L^1(\mathbb{R})} \le C.$$

Recall that $\Xi^M = \{(z,x) \in \mathbb{R} \times [0,M]\}$, for $M > 0$. We have

$$\int_{\Xi_M} |u^\varepsilon_z| \le \int_{\Xi_M} |U^\varepsilon_z| + \int_{\Xi_M} |\psi^\varepsilon_z| = -\int_{\Xi_M} U^\varepsilon_z - \int_{\Xi_M} \psi^\varepsilon_z = 2M.$$

and

$$\int_{\mathbb{R}} |\gamma u_z^\varepsilon| \leq \int_{\mathbb{R}} |\gamma U_z^\varepsilon| + \int_{\mathbb{R}} |\psi_z^\varepsilon| = 2.$$

Consequently

$$\forall M > 0, \quad \|u^\varepsilon\|_{W^{1,1}(\Xi_M)} + \|\gamma u^\varepsilon\|_{W^{1,1}(\mathbb{R})} \leq C. \tag{2.8}$$

Hence, by compactness and trace theorems ([6], Chapter 5), there exists $u \in BV(\Xi_M)$ verifying $\gamma u \in BV(\mathbb{R})$ and such that, after extraction of a subsequence, we have (2.3), (2.4), and the L^1, $\mathcal{M}_b$, a.e., convergences stated in (2.1) and (2.2). Remark that since $0 \leq U^\varepsilon \leq 1$, we also have, after another extraction,

$$\gamma U^\varepsilon \rightharpoonup \gamma U \text{ in } L^\infty(\mathbb{R}) \text{ weak}*, \tag{2.9}$$

where $U = u + 1 - H$.

Next, multiply (1.3H) by u^ε, integrate it over $\mathbb{R}_+^2$ then integrate by parts. We get the energy estimate

$$\int_{\mathbb{R}_+^2} (u_x^\varepsilon)^2 + \varepsilon \int_{\mathbb{R}_+^2} (u_z^\varepsilon)^2 + \frac{1}{2} \int_{\mathbb{R}_+^2} (u^\varepsilon)^2 = - \int_{\mathbb{R}} U^\varepsilon U_z^\varepsilon u^\varepsilon$$
$$\leq - \int_{\mathbb{R}} U^\varepsilon U_z^\varepsilon = \frac{1}{2},$$

since we have $0 \leq u^\varepsilon \leq 1$ and $U_z^\varepsilon \leq 0$ (Theorem A.1.1). Hence

$$\|u_x^\varepsilon\|_{L^2(\mathbb{R}_+^2)} + \|u^\varepsilon\|_{L^2(\mathbb{R}_+^2)} + \sqrt{\varepsilon}\|u_z^\varepsilon\|_{L^2(\mathbb{R}_+^2)} \leq C. \tag{2.10}$$

This completes the proofs of (2.1) and (2.2).

It remains to see in what sense u is solution of the limiting model (1.4H). For that it suffices to write a weak formulation of (1.3H). For all $\varphi \in C_0^\infty(\mathbb{R}_+^2)$ we have

(2.11)
$$\int_{\mathbb{R}_+^2} u_x^\varepsilon \varphi_x - \varepsilon \int_{\mathbb{R}_+^2} u^\varepsilon \varphi_{zz} + \frac{1}{2} \int_{\mathbb{R}_+^2} u^\varepsilon [(z\varphi)_z + (x\varphi)_x] - \int_{\mathbb{R}} \frac{(U^\varepsilon)^2}{2} \varphi_z = 0.$$

Properties (2.1) and (2.2) allow us to pass to the limit in the three linear terms. To treat the nonlinear one, we write it $\frac{1}{2}\int_{\mathbb{R}} (\gamma U^\varepsilon)(\gamma U^\varepsilon \varphi_z)$. Since φ is compactly supported, by (2.3) we have $\gamma U^\varepsilon \varphi_z \to \gamma U \varphi_z$ in $L^1(\mathbb{R})$ strong.

Hence we can pass to the limit in this term, thanks to (2.9). The asymptotic problem writes finally

(2.12H)

$$\forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^2}) \quad \int_{\mathbb{R}_+^2} u_x \varphi_x + \frac{1}{2} \int_{\mathbb{R}_+^2} u[(z\varphi)_z + (x\varphi)_x] - \int_{\mathbb{R}} \frac{U^2}{2} \varphi_z = 0.$$

Remark that it is equivalent to write this weak formulation for the function U, which is non homogeneous at the infinity:

(2.12NH)

$$\forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^2}) \quad \int_{\mathbb{R}_+^2} U_x \varphi_x + \frac{1}{2} \int_{\mathbb{R}_+^2} U[(z\varphi)_z + (x\varphi)_x] - \int_{\mathbb{R}} \frac{U^2}{2} \varphi_z = 0.$$

The following lemma shows that the solution of the limiting system (1.4H) is non trivial, i.e. that U is not equal to $1 - H$:

LEMMA 2.2. – *The limiting function U decreases with respect to z and x, is discontinuous on the axis $\{0\} \times \mathbb{R}_+^*$ and $u = U - 1 + H$ verifies*

$$\int_{\mathbb{R}_+^2} u = \frac{1}{2}. \tag{2.13}$$

Proof. – The monotonicity properties are consequences of Theorem A.1.1 and Proposition 2.1. The discontinuity of U on $\{0\} \times \mathbb{R}_+^*$ is immediate and comes from (2.7) and the properties of the limiting sub-/super-solutions $1 - H$ and $\overline{\Phi}$.

Consider now the solution u^ε of (1.3H). Straightforward calculations give

$$\begin{cases} -\displaystyle\int_{\mathbb{R}_+^2} u_{zz}^\varepsilon = 0 \\ -\displaystyle\int_{\mathbb{R}_+^2} u_{xx}^\varepsilon = \int_{\mathbb{R}} u_x^\varepsilon = \int_{\mathbb{R}} U^\varepsilon U_z^\varepsilon = -\frac{1}{2} \\ -\displaystyle\frac{1}{2} \int_{\mathbb{R}_+^2} (z u_z^\varepsilon + x u_x^\varepsilon) = \int_{\mathbb{R}_+^2} u^\varepsilon. \end{cases}$$

Hence (1.3H) implies

$$\int_{\mathbb{R}_+^2} u^\varepsilon = \frac{1}{2}.$$

Thanks to (2.7) and the decay properties of Λ^ε, for every $\delta > 0$, there exists a compact subset $\mathcal{K}_\delta \subset \overline{\mathbb{R}_+^2}$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\int_{\mathbb{R}_+^2 \setminus \mathcal{K}_\delta} u^\varepsilon \leq \delta.$$

Thus

$$\frac{1}{2} - \delta \leq \int_{\mathcal{K}_\delta} u^\varepsilon \leq \frac{1}{2}.$$

By (2.1) we can pass to the limit in this integral as $\varepsilon \longrightarrow 0$, δ being fixed; then we let $\delta \to 0$ to obtain (2.13). □

Set $z_0 = \sup\{z > 0 : \gamma U > 0 \text{ a.e. on } [0, z]\}$. This real number is well defined thanks to (2.6) and Lemma 2.2, and verifies $0 < z_0 \leq 4\sqrt{\pi}$.

3. REGULARITY OF THE SOLUTION

3.1. Smoothness on $]0, z_0[\times[0, +\infty[$

To prove the smoothness of U on $]0, z_0[\times[0, +\infty[$, we mainly follow the scheme of [12] and alternatively obtain interior and boundary estimates for U^ε and its derivatives. These estimates pass to the limit U, after extraction of subsequences from U^ε. The main difference with [12] is that the equation (1.4H) inside $\mathbb{R}^2_+$ is degenerate and the interior Agmon-Douglis-Nirenberg estimates [2] cannot be applied in this context.

Let $\delta > 0$ be fixed such that $\delta < z_0$. The function U^ε being non-increasing along z, and thanks to the a.e. convergence of γU^ε and to the above definition of z_0, we can find $\eta > 0$ and $\varepsilon_\delta > 0$ such that

$$\forall \varepsilon < \varepsilon_\delta \quad \forall z \in [0, z_0 - \delta] \quad U^\varepsilon(z, 0) > \eta. \tag{3.1}$$

If δ is small enough, these constant real numbers η and ε_δ being fixed, we define a cut-off function in z, $\chi_1(z, x) = \chi_1(z)$, such that $0 \leq \chi_1(z) \leq z$, $\chi_1 \in C^\infty$ on $]0, +\infty[$ and

$$\begin{cases} \chi_1(z) \equiv 0 \text{ for } z \leq 0 \text{ or } z \geq z_0 - \delta \\ \chi_1(z) \equiv z \text{ for } 0 \leq z \leq z_0 - 2\delta. \end{cases} \tag{3.2}$$

If M is a positive constant, we also define a C^∞ cut-off function in x, $\chi_2(z, x) = \chi_2(x)$, such that $0 \leq \chi_2(x) \leq 1$ and

$$\begin{cases} \chi_2(x) \equiv 1 \text{ for } 0 \leq x \leq M \\ \chi_2 \text{ is decreasing for } M \leq x \leq M + 1 \\ \chi_2(x) \equiv 0 \text{ for } x \geq M + 1. \end{cases} \tag{3.3}$$

The constants M and δ will be once and for all understood to be large -resp. small- enough for our purpose. We have

$$0 \leq \chi_1(z) \leq z_0, \quad 0 \leq \chi_2(x) \leq 1, \quad |\chi_1'| \leq C_\delta, \quad 0 \leq -\chi_2' \leq C. \tag{3.4}$$

Denote $\omega = [0, z_0 - 2\delta]$ and $\Omega = [0, z_0 - 2\delta] \times [0, M]$. We have $\chi_1 \equiv z$ on ω and $\chi_1 \chi_2 \equiv z$ on Ω.

In the following lemmas, we will obtain different estimates for u^ε, γU^ε and their derivatives on ω and Ω. For that we use several test functions in the weak formulation (2.11) of (1.3H). These test functions will take the form

$$\varphi = \chi_1^{k_1}\, \chi_2^{k_1} \left(\partial_z^{k_3} u^\varepsilon\right)^{k_4} \left(\partial_x^{k_5} u^\varepsilon\right)^{k_6}.$$

Because of the cut-off function χ_1, these functions φ may not be C^1 along the axis $\{0\} \times \mathbb{R}^+$. Nevertheless they are regular enough, as (2.11) will be written instead

$$\int_{\mathbb{R}^2_+} u_x^\varepsilon \varphi_x + \varepsilon \int_{\mathbb{R}^2_+} u_z^\varepsilon \varphi_z - \frac{1}{2} \int_{\mathbb{R}^2_+} \left[z u_z^\varepsilon + x u_x^\varepsilon\right] \varphi - \int_{\mathbb{R}} \frac{(U^\varepsilon)^2}{2} \varphi_z = 0. \tag{3.5}$$

In the sequel, $\mathcal{O}(1)$ or $C(M, \delta)$ denote quantities which can depend on δ and M but are uniformly bounded with respect to ε.

LEMMA 3.1. – *There exists $\varepsilon_\delta > 0$ such that, for $\varepsilon < \varepsilon_\delta$, we have*

$$\|\sqrt{z}\, \gamma U_z^\varepsilon\|_{L^2(\omega)} + \|z u_z^\varepsilon\|_{L^2(\Omega)} \leq C(M, \delta). \tag{3.6}$$

$$\|u_{xx}^\varepsilon\|_{L^2(\Omega)} \leq C(M, \delta). \tag{3.7}$$

Proof. – Let us first do the following remark. To estimate $\|\gamma U_z^\varepsilon\|_{L^2(\mathbb{R})}$ in the regular case studied in [12], it was sufficient to plug the test function $\varphi = u_z^\varepsilon$ in the weak formulation. Here, it is not so simple, since we do not have an $L^2(\mathbb{R}^2_+)$ estimate of u_z^ε independent of ε. Nevertheless, thanks to a suitable test function, we shall obtain these two estimates by the same time; they are stated in (3.6).

Setting

$$\varphi_1 = \chi_1\, \chi_2^2\, u_z^\varepsilon, \quad \varphi_2 = \chi_1\, \chi_2^2\, u_x^\varepsilon,$$

the idea of the proof is to take the test function $\varphi = \alpha\varphi_2 - \beta\varphi_1$ in (3.5), if α and β are positive real numbers that will be made precise later.

• We first consider only φ_1 in (3.5); we treat separately the different terms of this expression, using (2.10), $U_z \leq 0$, $\psi^{\varepsilon\prime} \leq 0$ and the properties of the cut-off functions χ_1 and χ_2 (3.2), (3.3), (3.4). The first term is

$$\begin{aligned} \int_{\mathbb{R}^2_+} u_x^\varepsilon (\varphi_1)_x &= \int_{\mathbb{R}^2_+} \chi_1 \chi_2^2\, u_x^\varepsilon\, u_{zx}^\varepsilon + 2 \int_{\mathbb{R}^2_+} \chi_1 \chi_2 \chi_2'\, u_x^\varepsilon\, u_z^\varepsilon \\ &= -\frac{1}{2} \int_{\mathbb{R}^2_+} (u_x^\varepsilon)^2 \chi_1' \chi_2^2 + 2 \int_{\mathbb{R}^2_+} \chi_1 \chi_2 \chi_2'\, u_x^\varepsilon\, u_z^\varepsilon; \end{aligned}$$

the first integral in the right hand side is $\mathcal{O}(1)$ thanks to (2.10). For the other terms of (3.5) we write

$$\varepsilon \int_{\mathbb{R}^2_+} u^\varepsilon_z (\varphi_1)_z = \frac{\varepsilon}{2} \int_{\mathbb{R}^2_+} (u^\varepsilon_z)^2 \, \chi_1' \chi_2^2 = \mathcal{O}(1),$$

$$\begin{aligned} - \int_{\mathbb{R}} \frac{(U^\varepsilon)^2}{2} (\varphi_1)_z &= \int_{\mathbb{R}} U^\varepsilon U^\varepsilon_z \, \chi_1 \, (U^\varepsilon_z - \psi^{\varepsilon\prime}) \\ &\leq \int_{\mathbb{R}} \chi_1 \, U^\varepsilon (U^\varepsilon_z)^2. \end{aligned}$$

$$-\frac{1}{2} \int_{\mathbb{R}^2_+} (x \, u^\varepsilon_x + z u^\varepsilon_z) \varphi_1 = -\frac{1}{2} \int_{\mathbb{R}^2_+} x \, \chi_1 \chi_2^2 \, u^\varepsilon_x u^\varepsilon_z - \frac{1}{2} \int_{\mathbb{R}^2_+} z \, \chi_1 \chi_2^2 \, (u^\varepsilon_z)^2.$$

Plugging these estimates in (3.5), we obtain finally

$$\text{(3.8)} \quad \begin{aligned} \int_{\mathbb{R}} -\chi_1 U^\varepsilon (U^\varepsilon_z)^2 + \frac{1}{2} \int_{\mathbb{R}^2_+} z \chi_1 \chi_2^2 \, (u^\varepsilon_z)^2 \\ \leq \int_{\mathbb{R}^2_+} \chi_1 \left(2\chi_2 \chi_2' - \frac{1}{2} x \chi_2^2 \right) u^\varepsilon_x u^\varepsilon_z + \mathcal{O}(1). \end{aligned}$$

- Next, with the test function φ_2, (3.5) can also be written

$$\text{(3.9)} \quad - \int_{\mathbb{R}^2_+} u^\varepsilon_{xx} \varphi_2 + \varepsilon \int_{\mathbb{R}^2_+} u^\varepsilon_z (\varphi_2)_z - \frac{1}{2} \int_{\mathbb{R}^2_+} x u^\varepsilon_x \varphi_2 - \frac{1}{2} \int_{\mathbb{R}^2_+} z u^\varepsilon_z \varphi_2 = 0.$$

We estimate the different terms as follows:

$$\begin{aligned} - \int_{\mathbb{R}^2_+} u^\varepsilon_{xx} \varphi_2 &= -\frac{1}{2} \int_{\mathbb{R}^2_+} ({u^\varepsilon_x}^2)_x \, \chi_1 \chi_2^2 \\ &= \frac{1}{2} \int_{\mathbb{R}} (u^\varepsilon_x)^2 \, \chi_1 + \int_{\mathbb{R}^2_+} (u^\varepsilon_x)^2 \chi_1 \chi_2 \chi_2' \\ &= \frac{1}{2} \int_{\mathbb{R}} (U^\varepsilon)^2 (U^\varepsilon_z)^2 \, \chi_1 + \mathcal{O}(1). \end{aligned}$$

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^2_+} u^\varepsilon_z (\varphi_2)_z &= \varepsilon \int_{\mathbb{R}^2_+} \chi_1' \chi_2^2 \, u^\varepsilon_z \, u^\varepsilon_x + \varepsilon \int_{\mathbb{R}^2_+} \chi_1 \chi_2^2 \, u^\varepsilon_z \, u^\varepsilon_{zx} \\ &= \varepsilon \int_{\mathbb{R}^2_+} \chi_1' \chi_2^2 \, u^\varepsilon_z \, u^\varepsilon_x - \varepsilon \int_{\mathbb{R}^2_+} (u^\varepsilon_z)^2 \, \chi_1 \chi_2 \chi_2' - \frac{\varepsilon}{2} \int_{\mathbb{R}} (u^\varepsilon_z)^2 \, \chi_1. \end{aligned}$$

In the right hand side of this equality, by (2.10), the two integrals calculated over $\mathbb{R}^2_+$ are $\mathcal{O}(1)$. The boundary integral can also be written

$$-\frac{\varepsilon}{2}\int_{\mathbb{R}}(u^\varepsilon_z)^2\chi_1 = -\frac{\varepsilon}{2}\int_{\mathbb{R}}(U^\varepsilon_z)^2\chi_1 + \frac{\varepsilon}{2}\int_{\mathbb{R}}\chi_1\psi^{\varepsilon\prime}(2U^\varepsilon_z - \psi^{\varepsilon\prime}).$$

To estimate the second term of the right hand side, it suffices to remark that

$$(3.10)\qquad 0 \le -\varepsilon\psi^{\varepsilon\prime} = \frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}\exp\left(-\frac{z^2}{4\varepsilon}\right) \le \frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}$$

and

$$\int_{\mathbb{R}}|\chi_1(2U^\varepsilon_z - \psi^{\varepsilon\prime})| \le -z_0\int_{\mathbb{R}}\left(2U^\varepsilon_z + \psi^{\varepsilon\prime}\right) = 3z_0.$$

Therefore we have

$$\varepsilon\int_{\mathbb{R}^2_+}u^\varepsilon_z(\varphi_2)_z = -\frac{\varepsilon}{2}\int_{\mathbb{R}}(U^\varepsilon_z)^2\chi_1 + \mathcal{O}(\sqrt{\varepsilon}).$$

The last term of (3.9) that we can estimate directly is

$$-\frac{1}{2}\int_{\mathbb{R}^2_+}xu^\varepsilon_x\varphi_2 = -\frac{1}{2}\int_{\mathbb{R}^2_+}x(u^\varepsilon_x)^2\chi_1\chi_2^2 \ge -\frac{M+1}{2}z_0\int_{\mathbb{R}^2_+}(u^\varepsilon_x)^2 = \mathcal{O}(1).$$

Finally (3.9) reads

$$(3.11)\quad \frac{1}{2}\int_{\mathbb{R}}(U^\varepsilon)^2(U^\varepsilon_z)^2\chi_1 - \frac{\varepsilon}{2}\int_{\mathbb{R}}(U^\varepsilon_z)^2\chi_1 = \mathcal{O}(1) + \frac{1}{2}\int_{\mathbb{R}^2_+}z\chi_1\chi_2^2u^\varepsilon_z u^\varepsilon_x.$$

• Let now α and β be two positive real numbers. The test function $\alpha\varphi_2 - \beta\varphi_1$ in (3.5) gives in fact the linear combination $\{\beta$ (3.8)+ α (3.11)$\}$:

(3.12)

$$\int_{\mathbb{R}}\left(\frac{\alpha}{2}U^{\varepsilon 2} - \beta U^\varepsilon - \frac{\alpha\varepsilon}{2}\right)\chi_1 U^{\varepsilon 2}_z + \frac{\beta}{2}\int_{\mathbb{R}^2_+}z\chi_1\chi_2^2(u^\varepsilon_z)^2 \le \mathcal{O}(1) + \mathcal{I}_1,$$

where

$$\mathcal{I}_1 = \frac{1}{2}\int_{\mathbb{R}^2_+}\left(\alpha z\chi_2^2 - \beta x\chi_2^2 + 4\beta\chi_2\chi_2'\right)\chi_1 u^\varepsilon_z u^\varepsilon_x.$$

By (3.1) and (3.2), we have $U^\varepsilon(z,0) > \eta$ on the support of χ_1. Hence, for α large enough and $\varepsilon < \eta^2$, we have

$$\left(\frac{\alpha}{2}U^{\varepsilon 2} - \beta U^\varepsilon - \frac{\alpha\varepsilon}{2}\right) > \eta.$$

Let us fix such an α; there holds

$$\eta\int_0^{z_0-2\delta}(\gamma U_z^\varepsilon)^2\,\chi_1 < \int_{\mathbb{R}}\left(\frac{\alpha}{2}U^{\varepsilon 2}-\beta U^\varepsilon-\frac{\alpha\varepsilon}{2}\right)\chi_1\,(\gamma U_z^\varepsilon)^2. \tag{3.13}$$

To estimate the integral $\mathcal{I}_1$, we use the inequality

$$\forall(a,b),\quad \forall A>0,\quad |ab|\le \frac{a^2}{2A}+\frac{Ab^2}{2}.$$

Hence, if we write, for $x\le M+1$,

$$\begin{aligned}
&\left|\left(\alpha z\chi_1\chi_2^2-\beta x\chi_1\chi_2^2+4\beta\chi_1\chi_2\chi_2'\right)u_z^\varepsilon u_x^\varepsilon\right|\le \alpha z\chi_1\chi_2^2\left[\frac{(u_z^\varepsilon)^2}{2A}+\frac{A(u_x^\varepsilon)^2}{2}\right]\\
&\quad+\beta(M+1)\chi_2^2\left[\frac{\chi_1^2(u_z^\varepsilon)^2}{2A}+\frac{A(u_x^\varepsilon)^2}{2}\right]+4\beta{\chi_2'}^2\left[\frac{\chi_1^2\chi_2^2(u_z^\varepsilon)^2}{2A}+\frac{A(u_x^\varepsilon)^2}{2}\right]\\
&\quad\le C_0\frac{z\chi_1\chi_2^2(u_z^\varepsilon)^2}{2A}+C_1A(u_x^\varepsilon)^2,
\end{aligned}$$

and setting $A=C_0/\beta$, we get

$$\begin{aligned}
\mathcal{I}_1&\le\frac{\beta}{4}\int_{\mathbb{R}_+^2}z\chi_1\chi_2^2\,(u_z^\varepsilon)^2+\frac{C_1A}{2}\int_{\mathbb{R}_+^2}(u_x^\varepsilon)^2\\
&\le\frac{\beta}{4}\int_{\mathbb{R}_+^2}z\chi_1\chi_2^2\,(u_z^\varepsilon)^2+\mathcal{O}(1).
\end{aligned}$$

Therefore (3.12) yields

$$\eta\int_0^{z_0-2\delta}(\gamma U_z^\varepsilon)^2\,\chi_1+\frac{\beta}{4}\int_{\mathbb{R}_+^2}z\chi_1\chi_2^2\,(u_z^\varepsilon)^2=\mathcal{O}(1).$$

Next by (3.2) it comes

$$\eta\int_\omega z\,(\gamma U_z^\varepsilon)^2+\frac{\beta}{4}\int_\Omega z^2\,(u_z^\varepsilon)^2=\mathcal{O}(1);$$

(3.6) is proved. Remark that this estimate works because we consider only the $z\ge 0$.

The estimate (3.7) is immediate and comes directly from (1.3), thanks to (2.10) and (3.6), and since on Ω we have $0\le x\le M$. □

LEMMA 3.2. – *There exists $\varepsilon_\delta>0$ such that for $\varepsilon<\varepsilon_\delta$ we have*

$$\|z^{1/3}\gamma U_z^\varepsilon\|_{L^3(\omega)}\le C(M,\delta) \tag{3.14}$$

Proof. – To prove this lemma we will show the following preliminary estimate:

$$\int_\Omega z\Big(|u_x^\varepsilon|^3 + \big|(u_x^\varepsilon)^2\, u_z^\varepsilon\big| + \varepsilon\big|u_x^\varepsilon\,(u_z^\varepsilon)^2\big| + \varepsilon|u_z^\varepsilon|^3\Big) = \mathcal{O}(1). \tag{3.15}$$

For that, we use the test function

$$\varphi = \chi_1\chi_2\, u^\varepsilon(u_x^\varepsilon + u_z^\varepsilon)$$

in (3.5). Straightforward computations lead to

$$\int_{\mathbb{R}_+^2} \chi_1\chi_2\,\big[(u_x^\varepsilon)^3 + (u_x^\varepsilon)^2\, u_z^\varepsilon + \varepsilon u_x^\varepsilon\,(u_z^\varepsilon)^2 + \varepsilon(u_z^\varepsilon)^3\big] = \mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4, \tag{3.16}$$

where

$$\mathcal{I}_2 = \int_{\mathbb{R}_+^2} \big\{\chi_1'\chi_2\, u^\varepsilon\big[(u_x^\varepsilon)^2 - 2\varepsilon u_x^\varepsilon u_z^\varepsilon - \varepsilon(u_z^\varepsilon)^2\big] + \chi_1\chi_2'\, u^\varepsilon\big[-(u_x^\varepsilon)^2 + \varepsilon(u_z^\varepsilon)^2\big]\big\}$$

$$\mathcal{I}_3 = \int_{\mathbb{R}_+^2} \big[-2\chi_1\chi_2'\, u^\varepsilon u_x^\varepsilon u_z^\varepsilon \;+\; \chi_1\chi_2\, u^\varepsilon\,(x u_x^\varepsilon + z u_z^\varepsilon)(u_x^\varepsilon + u_z^\varepsilon)\big]$$

$$\mathcal{I}_4 = \int_{\mathbb{R}} \chi_1\, u^\varepsilon\big[-(u_x^\varepsilon)^2 - 2u_x^\varepsilon\, u_z^\varepsilon + \varepsilon(u_z^\varepsilon)^2\big].$$

The term $\mathcal{I}_2$ is $\mathcal{O}(1)$ thanks to (2.10), (3.4) and $0 \le u^\varepsilon \le 1$. For the integral $\mathcal{I}_3$ we recall moreover that $|\chi_1(z)| \le |z|$ and use (3.6) to get $\mathcal{I}_3 = \mathcal{O}(1)$. Next, since in fact we need a lower bound for these terms, for $\mathcal{I}_4$ it suffices to write

$$\begin{aligned}
\mathcal{I}_4 &= \int_{\mathbb{R}} \chi_1\, u^\varepsilon\big[-(U^\varepsilon U_z^\varepsilon)^2 - 2U^\varepsilon U_z^\varepsilon(U_z^\varepsilon - \psi^{\varepsilon\prime}) + \varepsilon(u_z^\varepsilon)^2\big]\\
&\ge -\int_{\mathbb{R}} \chi_1\, u^\varepsilon(U_z^\varepsilon)^2(U^{\varepsilon 2} + 2U^\varepsilon)\\
&\ge -C\int_{\mathbb{R}} z(U_z^\varepsilon)^2 = \mathcal{O}(1),
\end{aligned}$$

thanks to (3.6). Hence (3.16) reads

$$\int_{\mathbb{R}_+^2} \chi_1\chi_2\,\big[(u_x^\varepsilon)^3 + (u_x^\varepsilon)^2\, u_z^\varepsilon + \varepsilon u_x^\varepsilon\,(u_z^\varepsilon)^2 + \varepsilon(u_z^\varepsilon)^3\big] \ge \mathcal{O}(1).$$

The four terms of the left hand side can in fact be estimated separately thanks to sign considerations, if we come back to the non-homogeneous functions $U_z^\varepsilon = u_z^\varepsilon + \psi^{\varepsilon\prime} \le 0$ and $U_x^\varepsilon = u_x^\varepsilon \le 0$. As previously in Lemma 3.1, we only have to take care of the fact that $\psi^{\varepsilon\prime}$ is not bounded in L^∞. Remark that

$$\forall t \ge 0 \quad t\, e^{-t^2} \le 1/\sqrt{2e},$$

thus

$$\forall z \quad |\chi_1 \psi^{\varepsilon\prime}(z)| = -\chi_1 \psi^{\varepsilon\prime}(z) \le \frac{z}{2\sqrt{\varepsilon\pi}} \exp\left(-\frac{z^2}{4\varepsilon}\right) \le \frac{1}{\sqrt{2e\pi}}. \tag{3.17}$$

Therefore, from (2.10), (3.10) and (3.17), we deduce the estimates

$$\begin{cases} \displaystyle\int_{\mathbb{R}_+^2} \chi_1\chi_2\, (u_x^\varepsilon)^3 = \int_{\mathbb{R}_+^2} \chi_1\chi_2\, (U_x^\varepsilon)^3, \\ \displaystyle\int_{\mathbb{R}_+^2} \chi_1\chi_2\, (u_x^\varepsilon)^2\, u_z^\varepsilon = \int_{\mathbb{R}_+^2} \chi_1\chi_2\, (U_x^\varepsilon)^2\, U_z^\varepsilon + \mathcal{O}(1), \\ \displaystyle\int_{\mathbb{R}_+^2} \chi_1\chi_2\, \varepsilon\, u_x^\varepsilon\, (u_z^\varepsilon)^2 = \int_{\mathbb{R}_+^2} \chi_1\chi_2\, \varepsilon\, U_x^\varepsilon\, (U_z^\varepsilon)^2 + \mathcal{O}(\sqrt{\varepsilon}), \\ \displaystyle\int_{\mathbb{R}_+^2} \chi_1\chi_2\, \varepsilon\, (u_z^\varepsilon)^3 = \int_{\mathbb{R}_+^2} \chi_1\chi_2\, \varepsilon\, (U_z^\varepsilon)^3 + \mathcal{O}(1), \end{cases}$$

which finally imply (3.15).

The same kind of calculations (but easier), which we shall not develop here, can lead to the same kind of estimate as (3.15), in which we replace the z under the integral by an ε. We only state it here; it will be useful in the sequel of the proof:

$$\varepsilon \int_\Omega \left(\left|(u_x^\varepsilon)^2\, u_z^\varepsilon\right| + \varepsilon |u_z^\varepsilon|^3 \right) = \mathcal{O}(1). \tag{3.18}$$

To prove (3.14), consider now in (3.5) the test function

$$\varphi = \chi_1\chi_2 \left[(u_x^\varepsilon)^2 - \varepsilon (u_z^\varepsilon)^2 \right].$$

After some calculations we obtain

$$\begin{aligned} &-\int_{\mathbb{R}} \chi_1 U^\varepsilon \left(\frac{(U^\varepsilon)^2}{3} - \varepsilon \right) (U_z^\varepsilon)^3 = \\ &\quad \int_{\mathbb{R}_+^2} \left(\frac{1}{3}\chi_1\chi_2' - \frac{1}{2} x \chi_1\chi_2 \right) (u_x^\varepsilon)^3 + \int_{\mathbb{R}_+^2} \left(\varepsilon \chi_1'\chi_2 - \frac{1}{2} z \chi_1\chi_2 \right) (u_x^\varepsilon)^2\, u_z^\varepsilon \\ &\quad + \int_{\mathbb{R}_+^2} \left(-\chi_1\chi_2' + \frac{1}{2} x \chi_1\chi_2 \right) \varepsilon u_x^\varepsilon\, (u_z^\varepsilon)^2 \\ &\quad + \int_{\mathbb{R}_+^2} \left(-\frac{\varepsilon}{3}\chi_1'\chi_2 + \frac{1}{2} z \chi_1\chi_2 \right) \varepsilon (u_z^\varepsilon)^3 \\ &\quad + \int_{\mathbb{R}} \varepsilon U^\varepsilon \chi_1 \left(2 (U_z^\varepsilon)^2 \psi^{\varepsilon\prime} - U_z^\varepsilon (\psi^{\varepsilon\prime})^2 \right). \end{aligned}$$

In the right hand side of this equality, the integrals $\int_{\mathbb{R}^2_+}$ can be estimated thanks to (3.15) and (3.18), and the boundary integral $\int_{\mathbb{R}}$ can be estimated thanks to (3.6), (3.10), (3.17) and $U_z^\varepsilon \leq 0$. For the left hand side, by (3.1) and (3.2) it suffices to take $\varepsilon < \eta^2/6$ to obtain

$$0 \leq -\int_\omega \chi_1 (U_z^\varepsilon)^3 \leq -\frac{6}{\eta^3}\int_{\mathbb{R}} \chi_1 U^\varepsilon \left(\frac{(U^\varepsilon)^2}{3} - \varepsilon\right)(U_z^\varepsilon)^3 = \mathcal{O}(1),$$

thus (3.14) is proved. □

LEMMA 3.3. – *For ε small enough we have*

$$\|z u_{zx}^\varepsilon\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|z u_{zz}^\varepsilon\|_{L^2(\Omega)} \leq C(M,\delta), \tag{3.19}$$

$$\|z\sqrt{z}\,\gamma U_{zz}^\varepsilon\|_{L^2(\omega)} + \|z^2\, u_{zz}^\varepsilon\|_{L^2(\Omega)} \leq C(M,\delta). \tag{3.20}$$

Denote $V^\varepsilon = U_z^\varepsilon$ and $v^\varepsilon = u_z^\varepsilon = V^\varepsilon - \psi^{\varepsilon\prime}$. We have

$$\text{(3.21H)} \qquad \begin{cases} -v_{xx}^\varepsilon - \varepsilon v_{zz}^\varepsilon - \dfrac{1}{2}(z v_z^\varepsilon + x v_x^\varepsilon) - \dfrac{1}{2} v^\varepsilon = 0 & (\mathbb{R}^2_+) \\ v_x^\varepsilon = U^\varepsilon V_z^\varepsilon + V^{\varepsilon 2} & (x=0) \\ v^\varepsilon(z,x) \to 0 \text{ as } |z,x| \to +\infty. \end{cases}$$

To prove the lemma we will again use a weak formulation of this system and choose different test functions. Recall that by (3.2) and for every integer $k > 1$ there holds

$$\begin{cases} \|\chi_1^k\|_{W^{k,\infty}(\mathbb{R})} \leq C_\delta, \quad 0 \leq \dfrac{\chi_1^k}{|z|^k} \leq 1, \\ \chi_1^k(z) \equiv z^k \text{ on } [0, z_0 - \delta], \\ \left\|\chi_1^k \dfrac{d^k}{dz^k}\psi^\varepsilon\right\|_{L^\infty(\mathbb{R})} \leq C. \end{cases} \tag{3.22}$$

With the test function $\varphi = \chi_1^2 \chi_2\, v^\varepsilon$, we obtain

$$\int_{\mathbb{R}^2_+} \chi_1^2 \chi_2\, (v_x^\varepsilon)^2 + \varepsilon \int_{\mathbb{R}^2_+} \chi_1^2 \chi_2\, (v_z^\varepsilon)^2 = \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7 + \mathcal{I}_8,$$

where

$$\begin{cases} \mathcal{I}_5 = \dfrac{\varepsilon}{2}\displaystyle\int_{\mathbb{R}^2_+} (u^\varepsilon_z)^2 (\chi_1^2)'' \chi_2 \\ \mathcal{I}_6 = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2_+} \left[\frac{\chi_1^2}{z^2}\chi_2'' - \chi_1' \frac{\chi_1}{z}\chi_2 - \frac{1}{2}x\frac{\chi_1^2}{z^2}\chi_2'\right](zu^\varepsilon_z)^2 \\ \qquad + \displaystyle\int_{\mathbb{R}} \left[\chi_1' \frac{\chi_1}{z} U^\varepsilon\right](\sqrt{z}U^\varepsilon_z)^2 \\ \mathcal{I}_7 = \displaystyle\int_{\mathbb{R}} \left[-2U^\varepsilon \chi_1\chi_1'\psi^{\varepsilon\prime} - U^\varepsilon \chi_1^2 \psi^{\varepsilon\prime\prime}\right] U^\varepsilon_z \\ \mathcal{I}_8 = -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}} \chi_1^2 \,(U^\varepsilon_z)^3. \end{cases}$$

These four terms are $\mathcal{O}(1)$ thanks to

- (2.10) for $\mathcal{I}_5$,
- (3.22) and (3.6) for $\mathcal{I}_6$,
- (3.17), (3.22), $0 \le U^\varepsilon \le 1$ and $\int_{\mathbb{R}} |U^\varepsilon_z| = 1$ for $\mathcal{I}_7$,
- (3.14) for $\mathcal{I}_8$.

This proves (3.19). To show (3.20), similarly to (3.6), we use the test function

$$\varphi = \chi_1^3 \chi_2^2 \,(\alpha v^\varepsilon_x - \beta v^\varepsilon_z).$$

The calculations are very close to those of Lemma 3.1; we use (3.6), (3.10), (3.19) and (3.22) to obtain

$$\int_{\mathbb{R}} \left(\frac{\alpha}{2}U^{\varepsilon 2} - \beta U^\varepsilon - \frac{\alpha\varepsilon}{2}\right)\chi_1^3 V_z^{\varepsilon 2} + \frac{\beta}{2}\int_{\mathbb{R}^2_+} z\chi_1^3\chi_2^2 \,(v^\varepsilon_z)^2 \le \mathcal{O}(1) + \mathcal{I}_1',$$

where

$$\mathcal{I}_1' = \frac{1}{2}\int_{\mathbb{R}^2_+} \left(\alpha z \chi_2^2 - \beta x \chi_2^2 + 4\beta\chi_2\chi_2'\right)\chi_1^3 \, v^\varepsilon_z \, v^\varepsilon_x.$$

We just notice that the exponent of χ_1 in φ has been chosen to estimate, thanks to (3.19), the following term that appears in the calculations (in the right hand side):

$$-\frac{3\beta}{2}\int_{\mathbb{R}^2_+} \chi_1^2\chi_1'\chi_2^2\left[(v^\varepsilon_x)^2 - \varepsilon(v^\varepsilon_z)^2\right].$$

To conclude, it suffices to take the same α and β as for (3.12), then to estimate $\mathcal{I}_1'$ as $\mathcal{I}_1$. □

From this lemma and Sobolev embeddings we deduce

$$\left\|z\sqrt{z}\gamma U\right\|_{C^{1,\alpha}(\omega)} + \left\|z^2 u\right\|_{C^{0,\beta}(\Omega)}, \tag{3.23}$$

for $0 \leq \alpha < 1/2$ and $0 \leq \beta < 1$. Following the same scheme, with appropriate cut-off functions $\chi_1^{k_1}\chi_2^2$, we could prove the continuity on ω of functions of the form $z^{k_2}\partial_z^m(\gamma U)\partial_x^p(\gamma U)$ and the continuity on Ω of functions of the form $z^{k_3}\partial_z^m U \partial_x^p U$. Nevertheless, for the sake of simplicity, we will not consider in the sequel the behaviour of the self-similar solution U near the axis $\{z = 0\}$ and restrict our study on

$$\omega_\delta = [\delta, z_0 - \delta], \quad \Omega_\delta = \omega_\delta \times [0, M].$$

We define another cut-off function $0 \leq \chi_3(z) \leq 1$ such that

$$\begin{cases} \text{Supp } (\chi_3) = \omega_\delta, \\ \chi_3(z) \equiv 1 \text{ for } z \in \omega_{2\delta}, \\ \chi_3 \in C^\infty(\mathbb{R}). \end{cases}$$

PROPOSITION 3.4. – *We have $U \in C^\infty(]0, z_0[\times[0, +\infty[)$.*

Proof. – Setting

$$U_{(n)}^\varepsilon = \partial_z^n U^\varepsilon, \quad \psi_{(n)}^\varepsilon = \partial_z^n \psi^\varepsilon, \quad u_{(n)}^\varepsilon = U_{(n)}^\varepsilon - \psi_{(n)}^\varepsilon,$$

we will prove recurrently, for every integer n, the property

$$(\mathcal{P}_n) \begin{cases} (i) & \|\partial_x u_{(n)}^\varepsilon\|_{L^2(\Omega_\delta)} \leq C(n, \delta, M) \\ (ii) & \|\partial_z u_{(n)}^\varepsilon\|_{L^2(\Omega_\delta)} \leq C(n, \delta, M) \\ (iii) & \|\partial_z \gamma U_{(n)}^\varepsilon\|_{L^2(\omega_\delta)} \leq C(n, \delta, M) \end{cases}$$

By (2.10) and Lemma 3.1, we already have $(\mathcal{P}_0)$ and by Lemma 3.3, we have $(\mathcal{P}_1)$. We now assume that $(\mathcal{P}_k)$ holds for $0 \leq k \leq n-1$, $\delta > 0$ and that $n \geq 2$. The function $u_{(n)}^\varepsilon$ is solution of the system

(3.24H)

$$\begin{cases} -\partial_{xx} u_{(n)}^\varepsilon - \varepsilon \partial_{zz} u_{(n)}^\varepsilon - \dfrac{1}{2}(z\partial_z u_{(n)}^\varepsilon + x\partial_x u_{(n)}^\varepsilon) - \dfrac{n}{2} u_{(n)}^\varepsilon = 0 & (\mathbb{R}_+^2) \\ \partial_x u_{(n)}^\varepsilon = \partial_z^n (U^\varepsilon U_{(1)}^\varepsilon) = \displaystyle\sum_{0 \leq k \leq [\frac{n+1}{2}]} a_{n,k}\, U_{(k)}^\varepsilon\, U_{(n+1-k)}^\varepsilon & (x = 0). \end{cases}$$

In this system we do not need the values of the coefficients $a_{n,k}$; we will only use further $a_{n,0} = 1$. Multiply (3.24H) by $\varphi = \chi_3(z)\chi_2(x)u_{(n)}^\varepsilon(z, x)$ and integrate over $\mathbb{R}_+^2$. After some calculations we obtain

$$\int_{\mathbb{R}_+^2} \chi_3\chi_2\, (\partial_x u_{(n)}^\varepsilon)^2 + \varepsilon \int_{\mathbb{R}_+^2} \chi_3\chi_2\, (\partial_z u_{(n)}^\varepsilon)^2 = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4$$

where

$$\begin{cases} \mathcal{J}_1 = \frac{1}{4}\int_{\mathbb{R}^2_+} [(2\varepsilon\chi_3'' - z\chi_3')\chi_2 + \chi_3(2\chi_2'' - x\chi_2') + 2(n-1)\chi_3\chi_2]\,(u^\varepsilon_{(n)})^2 \\ \mathcal{J}_2 = -\sum_{1\le k\le[\frac{n+1}{2}]} a_{n,k}\int_{\mathbb{R}} \chi_3\, U^\varepsilon_{(k)}\, U^\varepsilon_{(n+1-k)}\, U^\varepsilon_{(n)} \\ \mathcal{J}_3 = \sum_{1\le k\le[\frac{n+1}{2}]} a_{n,k}\int_{\mathbb{R}} \chi_3\, U^\varepsilon_{(k)}\, U^\varepsilon_{(n+1-k)}\, \psi^\varepsilon_{(n)} \\ \mathcal{J}_4 = -\int_{\mathbb{R}} \chi_3\, U^\varepsilon\, U^\varepsilon_{(n+1)}\, U^\varepsilon_{(n)} + \int_{\mathbb{R}} \chi_3\, U^\varepsilon\, U^\varepsilon_{(n+1)}\, \psi^\varepsilon_{(n)}. \end{cases}$$

Since $u^\varepsilon_{(n)} = \partial_z u^\varepsilon_{(n-1)}$, Property $(\mathcal{P}_{n-1})(ii)$ implies $\mathcal{J}_1 = \mathcal{O}(1)$. Next, Property $(\mathcal{P}_{n-1})(iii)$ writes

$$\|\gamma U^\varepsilon_{(n)}\|_{L^2(\omega_\delta)} \le C. \tag{3.25}$$

If $k \ge 1$ then by Property $(\mathcal{P}_{n-k})(iii)$ we also have

$$\forall k\ge 1 \quad \|\gamma U^\varepsilon_{(n+1-k)}\|_{L^2(\omega_\delta)} \le C. \tag{3.26}$$

Moreover, $k \le [\frac{n+1}{2}]$ and $n \ge 2$ imply $1 \le k \le n-1$; thus by $(\mathcal{P}_k)(iii)$, $(\mathcal{P}_{k-1})(iii)$ and Sobolev embeddings we have

$$\forall k\le n-1 \quad \|\gamma U^\varepsilon_{(k)}\|_{L^\infty(\omega_\delta)} \le C. \tag{3.27}$$

From these three estimates (3.25), (3.26) and (3.27) we infer directly $\mathcal{J}_2 = \mathcal{O}(1)$. Remark now that $\psi^\varepsilon_{(n)}$ writes

$$\psi^\varepsilon_{(n)}(z) = \frac{Q(z,\frac{1}{\varepsilon})}{\sqrt{\varepsilon}}\exp(-z^2/4\varepsilon),$$

where Q is a polynomial; thus, since $z \ge \delta$ on ω_δ,

$$\|\psi^\varepsilon_{(n)}\|_{L^\infty(\omega_\delta)} \le C. \tag{3.28}$$

(3.26), (3.27) and (3.28) imply $\mathcal{J}_3 = \mathcal{O}(1)$. Finally, we integrate by parts the two terms of $\mathcal{J}_4$; thanks to (3.25), (3.27) and (3.28) we have

$$\begin{aligned} \mathcal{J}_4 &= \frac{1}{2}\int_{\mathbb{R}} (\chi_3 U^\varepsilon_{(1)} + \chi_3' U^\varepsilon)\,(U^\varepsilon_{(n)})^2 \\ &\quad - \int_{\mathbb{R}} (\chi_3' U^\varepsilon \psi^\varepsilon_{(n)} + \chi_3 U^\varepsilon_{(1)}\psi^\varepsilon_{(n)} + \chi_3 U^\varepsilon \psi^\varepsilon_{(n+1)}) U^\varepsilon_{(n)} \\ &= \mathcal{O}(1). \end{aligned}$$

Therefore we get

$$\int_{\Omega_{2\delta}} (\partial_x u^\varepsilon_{(n)})^2 + \varepsilon \int_{\Omega_{2\delta}} (\partial_z u^\varepsilon_{(n)})^2 \le C(n,\delta,M); \tag{3.29}$$

Property $(\mathcal{P}_{n+1})(i)$ is proved.

With a change of δ, we can now consider (3.29) with the integrals calculated on Ω_δ instead of $\Omega_{2\delta}$. Let the test function

$$\varphi = \chi_3\chi_2^2\,(\alpha\partial_x u^\varepsilon_{(n)} - \beta\partial_z u^\varepsilon_{(n)}).$$

Thanks to $(\mathcal{P}_{n-1})$ and (3.29), similarly to the proofs of (3.6) and (3.20), straightforward calculations lead to

$$\int_{\mathbb{R}} \chi_3\Big[\frac{\alpha}{2}(\partial_x u^\varepsilon_{(n)})^2 - \beta(\partial_x u^\varepsilon_{(n)})(\partial_z u^\varepsilon_{(n)}) - \frac{\alpha\varepsilon}{2}(\partial_z u^\varepsilon_{(n)})^2\Big] + \frac{\beta}{2}\int_{\mathbb{R}^2_+} z\chi_3\chi_2^2\,(\partial_z u^\varepsilon_{(n)})^2 \le \mathcal{O}(1) + \mathcal{I}_1'', \tag{3.30}$$

where

$$\mathcal{I}_1'' = \frac{1}{2}\int_{\mathbb{R}^2_+} \chi_3\,\big(\alpha z\chi_2^2 - \beta x\chi_2^2 + 4\beta\chi_2\chi_2'\big)\partial_z u^\varepsilon_{(n)}\,\partial_x u^\varepsilon_{(n)}.$$

Let us treat the boundary term. Replacing $(\partial_x u^\varepsilon_{(n)})^2$ by its expression given by (3.24H) leads to estimating terms of the form

$$\mathcal{I}(k,k') = \int_{\mathbb{R}} \chi_3\,U^\varepsilon_{(k)}U^\varepsilon_{(k')}U^\varepsilon_{(n+1-k)}U^\varepsilon_{(n+1-k')} \quad \text{with } 0 \le k \le k' \le \left[\frac{n+1}{2}\right].$$

If $k \ge 1$ and $k' \ge 1$, by (3.26) and (3.27) we have $\mathcal{I}(k,k') = \mathcal{O}(1)$. If $k = 0$ and $k' = 1$ then an integration by parts yields

$$\begin{aligned}\mathcal{I}(0,1) = &-\frac{1}{2}\int_{\mathbb{R}} \chi_3\,U^\varepsilon U^\varepsilon_{(2)}\Big(U^\varepsilon_{(n)}\Big)^2 \\ &-\frac{1}{2}\int_{\mathbb{R}} \chi_3\,U^\varepsilon_{(1)}U^\varepsilon_{(1)}\Big(U^\varepsilon_{(n)}\Big)^2 - \frac{1}{2}\int_{\mathbb{R}} \chi_3'\,U^\varepsilon U^\varepsilon_{(1)}\Big(U^\varepsilon_{(n)}\Big)^2.\end{aligned}$$

We already know that $U^\varepsilon_{(n)} \in L^2(\omega_\delta)$ and the functions U^ε, $U^\varepsilon_{(1)}$, χ_3 and χ_3' are bounded in $L^\infty(\omega_\delta)$. Hence the second and the third terms of $\mathcal{I}(0,1)$ are $\mathcal{O}(1)$. For the first one, two cases have to be considered. If $n > 2$ then, by (3.27), $U^\varepsilon_{(2)}$ is also in $L^\infty(\omega_\delta)$. If $n = 2$ the term to estimate writes

$$\left|\int_{\mathbb{R}} \chi_3 U^\varepsilon\Big(U^\varepsilon_{(2)}\Big)^3\right| \le C\|\chi_3 U^\varepsilon U^\varepsilon_{(2)}\|_{L^\infty(\omega_\delta)}\,\|U^\varepsilon_{(2)}\|^2_{L^2(\omega_\delta)}.$$

Thanks to

$$\|U^\varepsilon_{(2)}\|_{L^2(\omega_\delta)} = \mathcal{O}(1)$$

and Gagliardo-Nirenberg inequality [1], we have thus, for every $\sigma > 0$,

$$\begin{aligned}\left|\int_{\mathbb{R}} \chi_3 U^\varepsilon \left(U^\varepsilon_{(2)}\right)^3\right| &\le \sigma \int_{\mathbb{R}} \chi_3^2\, U^{\varepsilon 2}\left(\partial_z U^\varepsilon_{(2)}\right)^2 + C_\sigma \\ &\le \sigma \int_{\mathbb{R}} \chi_3\, U^{\varepsilon 2}\left(\partial_z U^\varepsilon_{(2)}\right)^2 + C_\sigma,\end{aligned}$$

since $0 \le \chi_3 \le 1$. Finally we add $\mathcal{I}(0,0)$ and obtain

$$\int_{\mathbb{R}} \chi_3 \frac{\alpha}{2}(\partial_x u^\varepsilon_{(n)})^2 \ge \int_{\mathbb{R}} \chi_3 \frac{\alpha}{2}\Big(1 - \frac{\sigma}{2}\Big) U^{\varepsilon 2}\left(\partial_z U^\varepsilon_{(n)}\right)^2 + \mathcal{O}(1).$$

Set $\sigma = 1$. The other boundary integrals of (3.30) are easier to estimate since they contain lower order terms. We do not detail the calculations and (3.30) writes

$$\text{(3.31)}\quad \int_{\mathbb{R}} \chi_3 \Big[\frac{\alpha}{4} U^{\varepsilon 2} - \beta U^\varepsilon - \frac{\alpha\varepsilon}{2}\Big]\left(\partial_z U^\varepsilon_{(n)}\right)^2 + \frac{\beta}{2}\int_{\mathbb{R}^2_+} z\chi_3\chi_2^2\,(\partial_z u^\varepsilon_{(n)})^2 \le \mathcal{O}(1) + \mathcal{I}''_1.$$

We conclude exactly as for (3.12), and obtain $(\mathcal{P}_{n+1})(ii)$ and $(\mathcal{P}_{n+1})(iii)$. Therefore, recurrently, $(\mathcal{P}_n)$ holds for every integer $n \ge 0$.

To end the proof of the lemma, thanks to Sobolev embeddings it is sufficient to show that

$$\text{(3.32)}\qquad \forall (p,m) \in \mathbb{N}^2 \quad \|\partial_x^p \partial_z^m u^\varepsilon\|_{L^2(\Omega_\delta)} \le C(m,p,\delta,M).$$

Recurrently it is easy to see that, for all (p,m), we have

$$\partial_x^p \partial_z^m u^\varepsilon = \sum_{k=0}^{p+m} Q_{p,m,\varepsilon}(z,x) u^\varepsilon_{(k)} + \sum_{k=0}^{p+m} R_{p,m}(z,x,\varepsilon)\partial_x u^\varepsilon_{(k)},$$

where $Q_{p,m}$ and $R_{p,m}$ are polynomial in z, x and ε. This formula comes after successive derivations of (3.24H): a term $\partial_x^2 \partial_z^m u^\varepsilon$ can be replaced by terms of order 0 or 1 in ∂_x. Therefore (3.32) can be deduced from the properties $(\mathcal{P}_n)$. □

3.2. Continuity at the boundary

In this section we consider the trace $\gamma U(z)$ of the self-similar solution at the boundary. By Proposition 2.1, we have $\gamma U \equiv 0$ on $]-\infty, 0] \cup [z_0, +\infty[$ and, by Proposition 3.4, this function is C^∞ on $]0, z_0[$. Since γU is decreasing, one can define

$$\gamma U(0+) = \lim_{z \to 0+} \gamma U(z), \quad \gamma U(z_0-) = \lim_{z \to z_0-} \gamma U(z).$$

PROPOSITION 3.5. – *We have* $\gamma U \in C^0(\mathbb{R})$.

Proof. – Consider again the weak formulation (2.12H) of (1.4H), with appropriate test functions. Remark that (2.12H) can also be written in a weaker formulation that does not take in account $u_x \in L^2(\mathbb{R}^2_+)$:

(3.33)
$$\forall \varphi \in C_0^\infty(\overline{\mathbb{R}^2_+}) - \int_{\mathbb{R}^2_+} u\varphi_{xx} + \frac{1}{2}\int_{\mathbb{R}^2_+} u[(z\varphi)_z + (x\varphi)_x] - \int_{\mathbb{R}} \frac{U^2}{2}\varphi_z - \int_{\mathbb{R}} u\varphi_x = 0.$$

Let M denote an arbitrary large positive constant, and define $\chi_2 = \chi_2(x)$ by (3.3). Next, let $\varphi_0 = \varphi_0(z) \in C_0^\infty(\mathbb{R})$ be such that

$$\begin{cases} \varphi_0(z) = 0 \text{ for } z \leq -M - 1 \text{ or } z \geq z_0/2 \\ \varphi_0(z) = 1 \text{ for } -M \leq z \leq z_0/3. \end{cases}$$

Let $0 < \eta < z_0/3$ and define the following three subsets of $\mathbb{R}^2_+$:

$$\begin{cases} \mathcal{D}_1 =]-\infty, 0] \times \mathbb{R}_+ \\ \mathcal{D}_2 = [0, \eta] \times \mathbb{R}_+ \\ \mathcal{D}_3 = [\eta, +\infty[\times \mathbb{R}_+. \end{cases}$$

Plug now the test function $\varphi(z, x) = \varphi_0(z)\chi_2(x)$ in (3.33). We split the different integrals which appears in this formulation as follows

$$\int_{\mathbb{R}^2_+} = \int_{\mathcal{D}_1} + \int_{\mathcal{D}_2} + \int_{\mathcal{D}_3} \quad \text{and} \quad \int_{\mathbb{R}} = \int_{-\infty}^{0} + \int_{0}^{\eta} + \int_{\eta}^{+\infty}.$$

Recall that, otherwise mentioned, a one-dimensional integral denotes an integral calculated on the axis $\{x = 0\}$:

$$\int_a^b v := \int_a^b (\gamma v) dz.$$

Denoting by $[a, b]$ an interval of $\mathbb{R}$ (bounded or not), we introduce the notation

$$A(u, \varphi, [a, b]) = -\int_{[a,b]\times\mathbb{R}_+} u\varphi_{xx} + \frac{1}{2}\int_{[a,b]\times\mathbb{R}_+} u[(z\varphi)_z + (x\varphi)_x]$$
$$-\int_a^b \frac{U^2}{2}\varphi_z - \int_a^b u\varphi_x.$$

Hence, Equation (3.33) writes

$$(3.34) \qquad A(u, \varphi,]-\infty, 0]) + A(u, \varphi, [0, \eta]) + A(u, \varphi, [\eta, +\infty[) = 0.$$

• On $\mathcal{D}_1$, $u \equiv 0$ and $U \equiv 1$, thus

$$A(u, \varphi,]-\infty, 0]) = -\int_{-\infty}^0 \frac{1}{2}\varphi_z = -\frac{1}{2}.$$

• On $\mathcal{D}_2$, $0 \leq u \leq 1$ and $U \equiv u$:

$$|A(u, \varphi, [0, \eta])| \leq C(M+1)\eta.$$

• On $\mathcal{D}_3 \cap \operatorname{Supp}(\varphi)$, by Proposition 3.4, $u \equiv U$ is smooth; hence we can integrate the different terms of $A(u, \varphi, [\eta, +\infty[)$ by parts:

$$A(u, \varphi, [\eta, +\infty[) = \int_{[\eta,+\infty[\times\mathbb{R}_+} \left(-U_{xx} - \frac{1}{2}(zU_z + xU_x)\right)\varphi$$
$$+ \int_\eta^{+\infty} (UU_z - U_x)\varphi$$
$$+ \frac{1}{2}U^2(\eta, 0) - \frac{1}{2}\int_0^{+\infty} \eta U(\eta, x)\varphi(\eta, x)dx.$$

Moreover U is a strong solution of (1.4NH). This implies

$$A(u, \varphi, [\eta, +\infty[) = \frac{1}{2}U^2(\eta, 0) + \mathcal{O}(\eta).$$

Finally (3.34) reads

$$\frac{1}{2}U^2(\eta, 0) + \mathcal{O}(\eta) - \frac{1}{2} = 0;$$

thus $\gamma U(0+) = 1 = \gamma U(0-)$.

This proof can be adapted easily to show $\gamma U(z_0-) = 0 = \gamma U(z_0+)$. Consider indeed the test function $\varphi(z, x) = \varphi_1(z)\chi_2(x)$ in (3.33), where $\varphi_1 \in C_0^\infty(\mathbb{R})$ is defined by

$$\begin{cases} \varphi_1(z) = 0 \text{ for } z \leq z_0/3 \text{ or } z \geq z_0 + M + 1 \\ \varphi_1(z) = 1 \text{ for } z_0/2 \leq z \leq z_0 + M. \end{cases}$$

It suffices to split $\mathbb{R}^2_+$ as follows

$$\mathbb{R}^2_+ =]-\infty, z_0-\eta] \times \mathbb{R}_+ \bigcup [z_0-\eta, z_0] \times \mathbb{R}_+ \bigcup [z_0, +\infty[\times \mathbb{R}_+,$$

then to remark that $U \equiv 0$ on $]z_0, +\infty[\times \mathbb{R}_+$ and U is smooth on $]-\infty, z_0-\eta] \times \mathbb{R}_+ \cap \mathrm{Supp}(\varphi)$: we can thus proceed as above. □

4. UNIQUENESS OF THE SOLUTION

This part is devoted to the proof of Theorem 1.2, i.e. the uniqueness of an entropy solution to Problem (1.4H). Let us say that a function $u(x, z)$ is an entropy solution to (1.4NH) if and only if it satisfies the two following properties.

• (E1) $u \in BV(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$, $\gamma u \in L^\infty(\mathbb{R})$, $u_x \in L^2(\mathbb{R}^2_+)$; morever it is a solution of the weak formulation

(4.1)
$$\forall \varphi \in C_0^\infty(\mathbb{R}^2_+), \quad \int_{\mathbb{R}^2_+} u_x\varphi_x - \frac{1}{2}\int_{\mathbb{R}^2_+} \big((z\varphi)_z + (x\varphi)_x\big)u - \frac{1}{2}\int_{\mathbb{R}} (\gamma u)^2 \varphi_z = 0.$$

• (E2) There holds, uniformly in $x \in \mathbb{R}_+$:

$$\lim_{z \to -\infty} u(z, x) = 1, \qquad \lim_{z \to +\infty} u(z, x) = 0.$$

• (E3) The boundary measure u_z is locally bounded from above.

A few remarks are in order. First of all, notice that the above assumptions are trivially satisfied by the solution U that we constructed in the previous parts of this paper. Second, let us notice that the regularity properties of Assumption (E1) are the minimal ones so that the weak formulation (4.1) makes sense. Let us recall that, by virtue of Chap. 5, Section 2 of [6], the fact that u belongs to $BV(\mathbb{R}^2_+)$ ensures the existence of of an L^1 trace; because we wish the term u^2 to make sense, we require u to be in $L^\infty(\mathbb{R})$, which is not a very stringent assumption. Finally, Assumption (E3) implies (i) that γu belongs to $BV_{loc}(\mathbb{R})$, and (ii) that γu has lateral limits $\gamma u^-(z)$ and $\gamma u^+(z)$; moreover, if z_0 is a point of discontinuity of γu we have $\gamma u_-(z_0) > \gamma u^+(z_0)$.

The integrated version of Problem (1.4) reads, at least formally:

$$(4.2) \qquad \begin{cases} -v_{xx} - \dfrac{1}{2}(zv_z + xv_x) + \dfrac{1}{2}v = 0 & (\mathbb{R}^2_+) \\ v_x = -\dfrac{1}{2}v_z^2 & (x = 0) \\ v(z, x) \sim -z \text{ as } z \to -\infty, v(+\infty, x) = 0. \end{cases}$$

Most of the results that will be needed are gathered in the Crandall-Ishii-Lions 'User's guide' [5]. Let us recall the definition of a viscosity solution to (4.2). If $p = (p_1, p_2)$ denotes a vector of $\mathbb{R}^2$, and $M = (m_{ij})_{1 \le i,j \le 2}$, let $\underline{H}(z, x, v, p, M)$ and $\overline{H}(z, x, v, p, M)$ denote the following hamiltonians:

$$(4.3) \quad \begin{aligned} \underline{H}(z, x, v, p, M) &= \min\left(-m_{11} - \frac{1}{2}(zp_1 + xp_2 - v), -p_2 - \frac{p_1^2}{2}\right) \\ \overline{H}(z, x, v, p, M) &= \max\left(-m_{11} - \frac{1}{2}(zp_1 + xp_2 - v), -p_2 - \frac{p_1^2}{2}\right) \end{aligned}$$

A viscosity supersolution to (4.2) will here be a uniformly continuous function $\overline{v}(z, x)$ such that, for all $\phi \in C_0^\infty(\mathbb{R}^2+)$, there holds

$$\overline{H}(z_0, x_0, v(z_0, x_0), D\phi(x_0, z_0), D^2\phi(x_0, z_0)) \ge 0$$

at any local minimum point (z_0, x_0) of $v - \phi$. Similarly, a viscosity subsolution to (4.2) will be a uniformly continuous function $\underline{v}(z, x)$ such that, for all $\phi \in C_0^\infty(\mathbb{R}^2+)$, there holds

$$\underline{H}(z_0, x_0, v(z_0, x_0), D\phi(x_0, z_0), D^2\phi(x_0, z_0)) \le 0$$

at any local maximum point (z_0, x_0) of $v - \phi$. A viscosity solution to (4.2) will be a uniformly continuous function such that the two following properties hold:

- (V1) there holds, uniformly in $x \in \mathbb{R}_+$:

$$v(z, x) \sim -z \text{ as } z \to -\infty, \qquad \lim_{z \to +\infty} v(z, x) = 0.$$

- (V2) v is a both a viscosity subsolution and a viscosity supersolution to (4.2).

Once again, this definition calls for a one remark. Due to the unboundedness of the coefficients and the particular form of the hamiltonian, it is necessary to prescribe in a precise manner the growth of the solution at $z = -\infty$. Indeed, for any $\alpha > 0$, Problem (1.4NH) has a solution tending to α as $z \to -\infty$, producing a viscosity solution to (4.2) behaving like $-\alpha z$ at $z = -\infty$. This is to be compared with the situation presented in [5], Section 5.D.

To prove Theorem 1.2, we proceed in two steps: first, we prove a uniqueness result for Problem (4.2); then we prove that, for any entropy solution $u(z, x)$ of (1.4NH), the function $v(z, x) = \int_z^{+\infty} u(z', x)\, dz'$ is a viscosity solution to (4.2). Although the uniqueness result below is, striclty speaking, not contained in the literature, the proof that we are going to give is by now extremely classical. Let us concentrate on the first part of our programme; the most general result in this direction is due to Barles [3]; our situation does not completely fit in this context due (i) to the unboundedness of the solution and of the hamiltonian, (ii) to the quadratic growth in the boundary conditions. However, as is pointed out in Section 5.A of [5], things are considerably simpler when smooth sub and supersolutions are known. This is the case here; as a matter of fact there holds

LEMMA 4.1. – *Let $U(z,x)$ be the solution constructed in Theorem 1.1. Then $V(z,x) = \int_z^{+\infty} U(z',x)\,dz'$ is a viscosity solution to problem (4.2). Moreover, it is an admissble test function in this problem.*

Proof. – Let us first recall that $V(z,0) = \int_z^{+\infty} U(z',0)\,dz'$ is C^1 and Lipschitz over $\mathbb{R}$, and the function $zV_z(z,x) = -zU(z,x)$ belongs to $C(\mathbb{R}_+,X) \cap C(\mathbb{R}_-,X)$ where

$$X = \{f(x) \in UC(\mathbb{R}_+), \qquad f(x)\exp\left(\frac{x^2}{16}\right) \in UC(\mathbb{R}_+)\}$$

and $UC(\mathbb{R}_\pm)$ denotes the space of all bounded uniformly continuous functions on $\mathbb{R}_\pm$. Recall now that the operator L, defined from

$$D(L) = \{v(x) \in X,\ v(0) = 0, \qquad -v'' - \frac{1}{2}xv' \in X\}$$

to X, and with the expression

$$Lv = -v'' - \frac{1}{2}(xv' - v)$$

is an isomorphism. Hence we have $V \in C^1(\mathbb{R}_+, D(L)) \cap C^1(\mathbb{R}_-, D(L))$, and the result of the lemma follows immediately:

• if (z_0, x_0) is a maximum (resp. minimum) of $V - \phi$, then

- if $x_0 \neq 0$, then $V_x = \phi_x$, $V_{xx} \leq \phi_{xx}$ at (z_0, x_0) (resp. $V_{xx} \geq \phi_{xx}$); because $z\phi_z$ vanishes at the points of discontinuity of V_z, this is sufficient to get the result;

- if $x_0 = 0$, because $\gamma V \in C^1(\mathbb{R})$, then $V_z = \phi_z$ and $-V_x \geq -\phi_x$ (resp. $-V_x \geq -\phi_x$). This is once again sufficient.

• Because $V \in C^1(\mathbb{R}_+, D(L)) \cap C^1(\mathbb{R}_-, D(L))$, V is an admissible test function, for $zV_z = 0$ at the points of discontinuity of V_z. □

This preliminary result leads us to the

PROPOSITION 4.2. – *Problem (4.2) has a unique viscosity solution.*

Proof. – Let v be a viscosity solution to Problem (4.2). To prove that $v = V$ we proceed in two steps: first, we make precise the behaviour of v as $z \to -\infty$, then we use - a slight modification of - V as a test function.

1. Let us prove that $z^- - \|v(.,0)\|_{L^\infty(\mathbb{R}_+)} \leq v(z,x) \leq z^- + \|v(.,0)\|_{L^\infty(\mathbb{R}_+)}$ for $z \leq 0$. Only the left inequality will be proved, the other being similar. To prove it we shall show that, for every $\alpha < 1$, there holds $\alpha z^- - \|v(.,0)\|_{L^\infty(\mathbb{R}_+)} \leq v(z,x)$. On the portion of the boundary, $\{z \geq 0,\ x = 0\}$ this inequality is trivially true. If it were not so somewhere else, there would exist, because $v(z,x) \sim z^-$ as $z \to -\infty$ uniformly in x, a global strictly negative minimum (z_0, x_0) for $v - \alpha z^- - \|v(.,0)\|_{L^\infty(\mathbb{R}_+)}$. We point out that, for the same reasons as in Lemma 4.1, z^- is an admissible test function. Therefore we have, at that point:

$$\max\left(-\frac{\alpha}{2}, \frac{\alpha z_0 + v(z_0,x_0)}{2}\right) \geq 0;$$

this is impossible.

2. Let us now prove that $v = V$; for this one needs to prove that $v \geq V$, then $v \leq V$. Let us prove the first inequality, the other being once again similar. Assume the result to be false; there exists $z_1 > 0$ and $x_1 > 0$ such that

$$(v - V)(z_1, x_1) \leq -2\|v - V\|_{L^\infty([z_0,+\infty[\times\mathbb{R}_+)}$$

Therefore, due to Step 1, there exists a global strictly negative minimum to the function $v - V + \eta \mathrm{Log}(1 + z_1 - z)$ on $]-\infty, z_1] \times \mathbb{R}_+$, provided $\eta > 0$ is small enough. Let us assume $\eta > 0$ to be indeed small enough, and in any case stricly less than $-\min_{\mathbb{R}^2_+}(v - V) > 0$. We have, if $x_0 > 0$:

$$\frac{\eta z_0}{2(1 + z_1 - z_0)} + \frac{1}{2}(v - V)(z_1, x_1) \geq 0.$$

This is impossible; as a consequence $x_0 = 0$. But then this point is also a minimum of the function $v - V + \eta \mathrm{Log}(1 + z_1 - z) + \varepsilon x$, for any $\varepsilon > 0$. This time we get, using the boundary condition: $-\varepsilon \geq 0$, an impossibility. □

To end the proof of Theorem 1.2, it remains to prove the

PROPOSITION 4.3. – *Let $u(z,x)$ be an entropy solution to Problem (1.4H). Then $v(z,x) = \int_z^{+\infty} u(z',x)\,dz'$ is a viscosity solution to (4.2).*

Proof. – Let $\rho(z)$ be a C^∞ function, supported in $[-\frac{1}{2}, \frac{1}{2}]$, satisfying $0 \leq \rho \leq 1$, and with unit total mass. Let ρ_ε be the classical mollifier $\rho_\varepsilon(z) = \frac{1}{\varepsilon}\rho(\frac{z}{\varepsilon})$. Let us denote $v^\varepsilon(z,x) = \rho_\varepsilon *_z v$; we have, in the classical sense:

$$\left(-\partial_{xx} - \frac{1}{2}(x\partial_x + z\partial_z + 1)\right)v^\varepsilon = o(1) \quad (\mathbb{R}^2_+)$$
$$v^\varepsilon_x = -\frac{1}{2}\rho_\varepsilon * (v^\varepsilon_z)^2 \quad (x = 0)$$

with the same conditions at $-\infty$ as v. Moreover, the properties of ρ imply $\rho_\varepsilon * (v^\varepsilon_z)^2 \geq (\rho_\varepsilon * v^\varepsilon_z)^2$; as a consequence v^ε is a viscosity supersolution to 4.2. Let us notice that $v^\varepsilon_z = -\rho_\varepsilon * u$ is uniformly bounded, by assumption. Therefore γv^ε_x is bounded, hence - see the proof of Lemma 4.1 - v^ε_x is uniformly bounded. Hence $(v^\varepsilon)_\varepsilon$ is uniformly Lipschitz, as is v, and the sequence $(v^\varepsilon)_\varepsilon$ converges uniformly to v. Hence v is a viscosity supersolution to (4.2).

To see that it is a subsolution, let, for the last time, (z_0, x_0) be a point of maximum for $v - \phi$. There exists a sequence $(z_\varepsilon, x_\varepsilon)$ of minima of $v^\varepsilon - \phi$, tending to $(z_\varepsilon, x_\varepsilon)$. The only nontrivial case is when $x_\varepsilon = 0$ for all $\varepsilon > 0$; if z_0 is a point of continuity of $\gamma v_z = -\gamma u$, we have $-\phi_x \leq -v^\varepsilon_x$ at $(z_\varepsilon, 0)$. Now, if z_0 is a point of continuity for γu, we have $\lim_{\varepsilon \to 0} \gamma(\rho_\varepsilon * u)(z_\varepsilon) = \gamma u(z_0)$; moreover we have $\gamma u(z_0) = \phi_z(z_0, 0)$. As a consequence, we have

$$-\phi_x(z_0, 0) \leq \frac{1}{2}\phi_z(z_0, 0)^2.$$

To conclude, we notice that z_0 cannot be a point of discontinuity of γu: if it were so, we would have

$$\phi_z(z_0, 0) \leq \gamma v_z(z_0^-) = -\gamma u(z_0^-) < -\gamma u(z_0^+) = \gamma v_z(z_0^+) \leq \phi_z(z_0, 0).$$

This is clearly impossible, and the proof of the proposition is over. □

To conclude this section, let us point out that we have not restricted the smoothness assumption for the only sake of reaching the best level of generality. We indeed have in mind numerical approximations of this problem, and the convergence proofs do not rely on fine regularity properties of the solution: what makes things work is precisely the entropy condition $\partial_z \gamma u \leq$ Constant, even in the fully diffusive case - i.e. $\varepsilon = O(1)$: see [11].

5. NUMERICAL RESULTS

We present here some numerical simulations which show different properties of the solutions of the parabolic and elliptic systems that we have studied in this paper and in its first part [12]. We have computed an approximation of the solution of the parabolic problem (1.1NH), i.e., the heat equation

$$B_t - B_{XX} - \varepsilon B_{ZZ} = 0 \quad \text{in the rectangle } (Z, X) \in [0, L] \times [0, l],$$

submitted to the following boundary conditions:

$$\begin{cases} B_X = KB\,B_Z & \text{on } [0, L] \times \{0\}, \\ B_X = 0 & \text{on } [0, L] \times \{l\}, \\ B = 1 & \text{on } \{0\} \times [0, l], \\ B = 0 & \text{on } \{L\} \times [0, l]. \end{cases}$$

Remark that we keep here both parameters ε and K for numerical reasons. The parameter ε of the theorical study corresponds here to ε/K^2.

Scheme. – Let us describe briefly our numerical scheme. It is detailed in [11], where a stability and convergence study of a one-order version of this scheme is performed. We use an explicit finite difference scheme. A time iteration from t_k to t_{k+1} is made in two steps: first a prediction on $[t_k, t_{k+1/2}]$, then a correction on $[t_k, t_{k+1}]$ using the previously calculated values of B at $t_{k+1/2}$.

For the space discretization, we take a special care to the z variable. The scheme for the heat equation is the most simple that we can take, with a five points Laplacian, the difficulty lying in the boundary condition on $[0, L] \times \{0\}$. It can be seen as a Burgers equation, thus discretized with a nonlinear hyperbolic method [8]: we take a second order TVD slope-limiter scheme.

Numerical results

• Convergence to the self-similar solution.

To observe the long-time behaviour of the solution, we represent the results of the computations in the self-similar variables $z = \dfrac{Z}{K\sqrt{t+1}}$, $x = \dfrac{X}{\sqrt{t+1}}$, at different time steps. We set $B^{self}(t, z, x) = B(t, Z, X) = B(t, K\sqrt{t+1}\,z, \sqrt{t+1}\,x)$. This function is the numerically calculated solution expressed in self-similar variables:

it is supposed to converge to the self-similar solution U of (1.4NH), as $t \to \infty$. The parameters of the computation are the following:

L	l	ε	K	Δz	Δx	Δt
7	3	10^{-3}	10	2.3×10^{-2}	1.7×10^{-2}	5×10^{-6}

On Figure 1, we have represented the curve at the boundary $\{x = 0\}$, $\gamma B^{self}(t, z)$, at time steps 5000, 25000, 50000, 75000 and 100000. We have also represented the inital data $B_0(Z, 0)$, in non-rescaled variables.

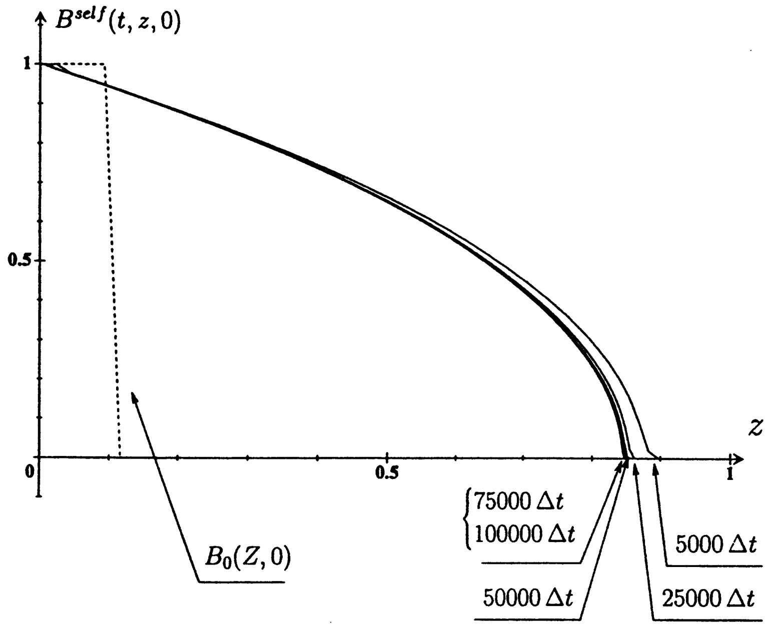

Fig. 1. - Time-asymptotic behaviour of B^{self} at the boundary

After 50000 iterations, one cannot distinguish the different curves: the curve $\gamma B^{self}(t, z)$ seems to converge numerically to the curve $\gamma U(z)$. This phenomenon is of course also observed on the whole surface $B^{self}(t, z, x)$.

• Properties of the limiting solution $U(z, x)$

Let us now consider the surface obtained at the last iteration, at time step 100000. Let $t_1 = 100000\,\Delta t$. This function $B^{self}(t_1, z, x)$ is an

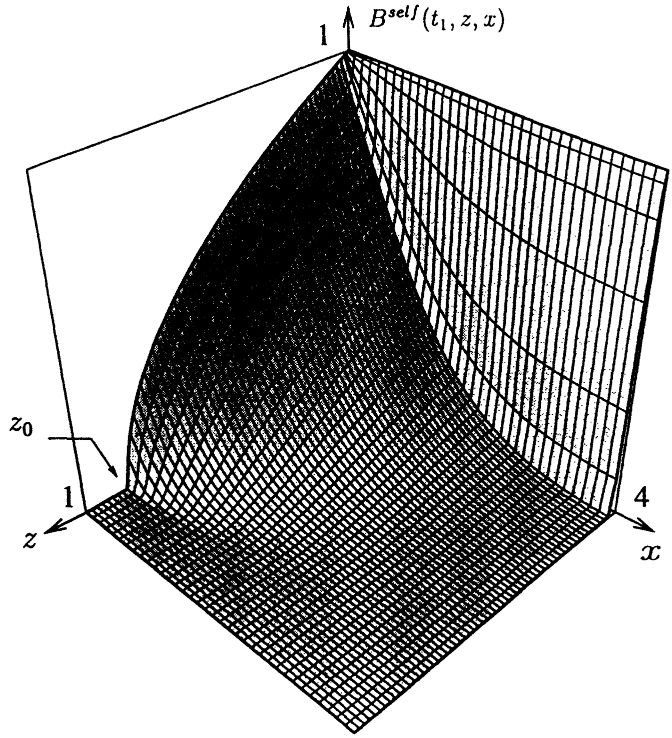

Fig. 2. - $B^{self}(t, z, x)$ at time step 100000

approximation of $U(z, x)$ and is represented on the figure below, in the quarter-space $(z, x) \in (\mathbb{R}_+)^2$.

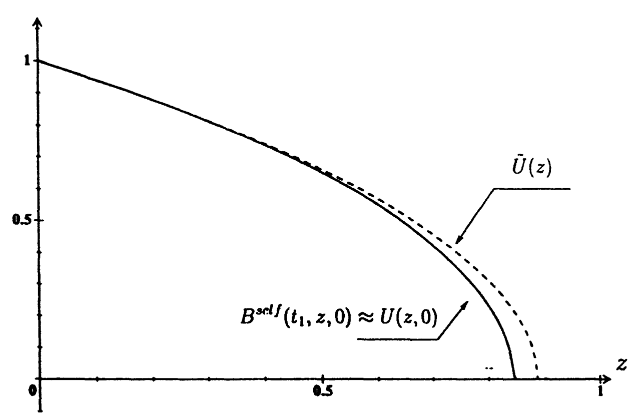

Fig. 3. - Approximations of the self-similar solution U

We can observe on this figure the following properties:

- U is discontinuous on the axis $\{0\} \times \mathbb{R}_+^*$, i.e. the magnetic field B penetrates into the domain only through the point $(0, 0)$. This property was stated in Lemma 2.2.
- $U(z, x) = 0$ for $z > z_0$. This property was stated in Proposition 2.1.
- U is continuous at the boundary $\{x = 0\}$ as it was stated in Proposition 3.5.

• Approximation of the curve at the boundary

Let us now focus on the curve $U(z, 0) \approx B^{self}(t^1, z, 0)$ at the boundary, and specially on its derivative. It seems numerically that $U_z(z_0, 0) = -\infty$. We are not able to prove it yet. Moreover, we notice that, on $[0, z_0]$ the curve γU is very close to the curve given by $\tilde{U}(z) = \sqrt{1 - \frac{2z}{\sqrt{\pi}}}$. This curve $\tilde{U}(z)$ and the curve $B^{self}(t, z, 0)$, computed for $t = t_1 = 100000\,\Delta t$, are represented on Figure 3. The function $\tilde{U}$ has the same behaviour as γU but does not coincide with this function. Moreover, these two curves have the same slope at $z = 0$, which is $1/\sqrt{\pi}$. This value corresponds to the value predicted in [7] (see comments on this fact in [11]). It is possible to prove it rigourously: indeed, with the aid of Lemmas 3.1 and 4.1, we see that $U \in C((\mathbb{R}_+)_z, H^2((\mathbb{R}_+)_x))$. Hence γU_x has a limit as $z \to 0^+$, and so has $\gamma U_z = \frac{\gamma U_x}{\gamma U}$.

6. CONCLUSION OF PART 2

The results obtained in this paper have allowed us to quantify the effects of the nonlinear boundary condition $B_X = KBB_Z$ on the diffusion of the magnetic field B due to the Laplacian. Without this boundary condition (i.e. when $K = 0$), the field penetrates as the function $\psi\left(\frac{Z}{\sqrt{t}}, \frac{X}{\sqrt{t}}\right)$.

In self-similar variables, we have performed a rescaling in z which has two effects, asymptotically, as $K \to +\infty$:

- it suppresses the diffusive penetration along the z direction, since $\psi^\varepsilon \to 1 - H$. Indeed, nothing penetrates into the domain $\{z > 0, x \geq 0\}$ through the axis $\{z = 0, x > 0\}$ anymore and thefunction U is discontinuous on this half-line.

- nevertheless, a propagation of the magnetic field still occurs inside thisdomain, since U is not reduced to the function $1 - H$. This propagation occurs at the boundary $\{x = 0\}$, and starts at the point $(0, 0)$.

This penetration appears thus as the effect of the boundary condition. If we come back to the non rescaled self-similar variables, the self-similar field is close to the function $U\left(\frac{z}{K}, x\right)$. Hence, in the initial evolution variables, the field penetrates similarly to the function $U\left(\frac{Z}{\sqrt{K^2 t}}, \frac{X}{\sqrt{t}}\right)$. Therefore, the time scale of the penetration along the boundary is K^2 times faster than the diffusive one due to the Laplacian. That is the reason why it is called a *rapid penetration at the boundary*.

A third and final paper will deal with the singular Cauchy Problem and the actual convergence of the unsteady solution to the self-similar solution described in this paper.

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