ANNALES DE L'I. H. P., SECTION C

P.-L. LIONS
N. MASMOUDI

On a free boundary barotropic model

Annales de l'I. H. P., section C, tome 16, nº 3 (1999), p. 373-410
<http://www.numdam.org/item?id=AIHPC_1999__16_3_373_0>

© Gauthier-Villars, 1999, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section C » (http://www.elsevier.com/locate/anihpc) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 16, n° 3, 1999, p. 373–410 *Analyse non linéaire*

On a free boundary barotropic model

by

P.-L. LIONS

CEREMADE-URA CNRS 749 Université Paris Dauphine,
place de Lattre de Tassigny, 75775 Paris cedex 16, France

and

N. MASMOUDI*

DMI, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris, France

ABSTRACT. – We prove stability (or compactness) and existence results for a free boundary model of a barotropic compressible fluid. Then, we construct weak solutions as asymptotic limits of the isentropic compressible Navier-Stokes equations as γ goes to ∞. © Elsevier, Paris

RÉSUMÉ. – Nous montrons l'existence de solutions faibles globales pour un modèle de fluide compressible barotrope. Ces solutions sont construites à partir d'une limite asymptotique des solutions de Navier-Stokes isentropique compressible. © Elsevier, Paris

1. INTRODUCTION

We consider the following system of equations, written in $(0,\infty)\times\Omega$, where $\Omega = \mathbb{R}^N$ or $\Omega = \mathbf{T}^N$,

$$(1)\qquad \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0 \text{ in } (0,T)\times\Omega,\ 0\le\rho\le 1 \text{ in } (0,T)\times\Omega,$$

* CEREMADE-URA CNRS 749 Université Paris Dauphine, place de Lattre de Tassigny, 75775 Paris cedex 16, France.

AMS Classifications: 76 N 10, 76 T 05.

$$(2)\quad \frac{\partial \rho u}{\partial t} + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - \xi \nabla \mathrm{div} u + \nabla \pi = \rho f \text{ in } (0,T) \times \Omega,$$

$$(3)\quad \mathrm{div} u = 0 \text{ a.e. on } \Big\{\rho = 1\Big\},$$

$$(4)\quad \pi = p(\rho) \text{ a.e. on } \Big\{\rho < 1\Big\},\ \pi \geq p(1) \text{ a.e. on } \Big\{\rho = 1\Big\}$$

where $T \in (0,\infty)$, $\mu > 0$, $\mu + \xi > 0$, $p(\rho)$ is a nondecreasing positive continuous function and $f = f(t,x)$ is a given function corresponding to the volumic force terms (for instance we can assume that $f \in L^1(0,T;L^2)^N$).

The unknowns (ρ, u, π) correspond respectively to the density of the fluid which is a nonnegative function, the velocity which is a vector-valued function in $\mathbb{R}^N$ and the pressure. The system must be complemented with initial conditions, namely

$$(5)\quad \rho u\Big|_{t=0} = m^0\ ,\ \rho\Big|_{t=0} = \rho^0,$$

where $1 \geq \rho^0 \geq 0$ a.e. , $\rho^0 \in L^1(\Omega)$, $m^0 \in L^2(\Omega)$, $m^0 = 0$ a.e. on $\{\rho^0 = 0\}$, $\rho^0 \not\equiv 0$, and $\rho^0 |u^0|^2 \in L^1$, denoting by $u^0 = \dfrac{m^0}{\rho^0}$ on $\{\rho^0 > 0\}$, $u^0 = 0$ on $\{\rho^0 = 0\}$. In the case of $\mathbf{T}^N$, we also impose that $\fint \rho^0 = M < 1$, otherwise the above system reduces formally to the incompressible Navier-Stokes equations in which case we do not know whether the (hydrostatic) pressure is bounded from below.

One of the motivations to study this free boundary problems is the study of fluids with imbedded domains (large bubbles) filled with gaz : standard models involve a threshold on the pressure beyond which one has the incompressible Navier-Stokes equations for the fluid and below which one has a compressible model for the gaz. Another motivation is the possibility to study a compressible-like system which includes the incompressible case as particular case ($\rho \equiv 1$).

Now we are going to define precisely the weak solutions (solutions à la Leray) we will use. We look for solutions satisfying

$$(6)\quad \rho \in L^\infty(0,T;\mathrm{L}^\infty \cap L^1(\Omega)) \cap C(0,T;L^p) \text{ for any } 1 \leq p < \infty$$

$$(7)\quad \nabla u \in L^2(0,T,L^2) \text{ and } u \in L^2(0,T;H^1(B)),$$

where $B = \mathbf{T}^N$ if $\Omega = \mathbf{T}^N$ and B is any ball in $\mathbb{R}^N$ if $\Omega = \mathbb{R}^N$, in this second case we also impose that $u \in L^2(0,T,L^{2N/N-2}(\mathbb{R}^N))$, if in addition $N \geq 3$.

We also require

$$\rho|u|^2 \in L^\infty(0,\infty;L^1) \text{ and } \rho u \in L^\infty(0,\infty;L^2) \tag{8}$$

Finally we impose that

$$\pi \in \mathcal{M}((0,T)\times\Omega), \tag{9}$$

where $\mathcal{M}((0,T)\times\Omega)$ is the space of bounded measures on $(0,T)\times\Omega$.

Next, equations (1), (2) must be satisfied in the distributional sense. This can be written using a weak formulation, namely we require that the following identities hold for all $\phi \in C^\infty([0,\infty)\times\Omega)$ and for all $\Phi \in C^\infty([0,\infty)\times\Omega)^N$ compactly supported in $[0,\infty)\times\Omega$ (i.e. vanishing identically for t large enough)

$$-\int_0^\infty dt\int_\Omega \rho\partial_t\phi - \int_\Omega \rho^0\phi(0) - \int_0^\infty dt\int_\Omega \rho u.\nabla\phi = 0, \tag{10}$$

$$\begin{aligned}(11)\quad & -\int_0^\infty dt\int_\Omega \rho u.\partial_t\Phi - \int_\Omega m^0.\Phi(0) - \int_0^\infty dt\int_\Omega \rho(u.\nabla\Phi).u + \\ & + \int_0^\infty dt\Big\{\int_\Omega \mu Du.D\Phi + \xi \mathrm{div}u\mathrm{div}\Phi\Big\} - \pi\ \mathrm{div}\Phi = \int_0^\infty dt\int_\Omega \rho f.\Phi.\end{aligned}$$

We want to point out (and we will come back to this issue later on) that the weak formulation (10) for ρ contains the initial condition $\rho(0)=\rho^0$ since ρ is assumed to be continous in time with values in L^2. However (11) does not yield that $\rho u(0) = m^0$. In fact π is a mesure and all we can deduce in terms of continuity concerns the divergence-free part of ρu, namely $P(\rho u)$, where $P(v) = v - \nabla(\Delta)^{-1}\nabla.v$. Hence if Φ is divergence-free $div(\Phi)=0$ then (11) becomes

$$(12)\quad \begin{cases} -\displaystyle\int_0^\infty dt\int_\Omega \rho u.\partial_t\Phi - \int_\Omega m^0.\Phi(0) - \int_0^\infty dt\int_\Omega \rho(u.\nabla\Phi).u + \\ + \displaystyle\int_0^\infty dt\int_\Omega \mu Du.D\Phi = \int_0^\infty dt\int_\Omega \rho f.\Phi \end{cases}$$

which yields that $P(\rho u)(0) = P(m^0)$.

On the other hand equation (3) and the condition $0 \le \rho \le 1$ must be understood in the sense of almost everywhere defined functions.

Notice then that condition (4) does not make sense since π is not assumed to be a function (defined almost every where). However this condition can be rewritten as follows

$$\rho(\pi - p(\rho)) = \pi - p(\rho) \geq 0. \tag{13}$$

We will show that this condition makes sense under the above requirements and equation (4) should be understood in the sense of (13). In fact we will show the following regularity theorem

THEOREM 1.1. – *If (ρ, u, π) satisfy (1), (2) and the above requirements hold then condition (13) makes sense and we have*

$$\pi \in \mathcal{M}(0, T; L^1(\Omega)), \tag{14}$$

$$\rho u \in BV([0, \infty); L^2(\Omega) - w), \tag{15}$$

$$P(\rho u) \in C([0, \infty); L^2(\Omega) - w). \tag{16}$$

In this result, $BV([0, \infty); L^2 - w)$ and $C([0, \infty); L^2 - w)$ are respectively the space of bounded variation and continous functions on $[0, \infty)$ with value in a bounded set of $L^2(\Omega)$ equipped with the weak topology.

Now we are able to state our main existence result. As is customary when dealing with global weak solutions of partial differential equations (and due to the weak convergences in the approximating systems) the global weak solution we are going to construct will satisfy in addition the following energy inequalities

$$E(t) + \int_0^t D(s)ds \leq E^0 + \int_0^t \int_\Omega \rho u.f \quad \text{a.e. } t, \tag{17}$$

$$\frac{d\,E}{dt} + D \leq \int_\Omega \rho u.f \quad \text{in } \mathcal{D}'(0, \infty) \tag{18}$$

where $E(t) = \int_\Omega \frac{1}{2}\rho|u|^2(t)$, $D(t) = \int_\Omega \mu|Du|^2(t) + \xi\ (div\ u)^2(t)$ and $E^0 = \int_\Omega \frac{1}{2}\rho^0|u^0|^2$. Moreover, π will be bounded in $\mathcal{M}((0, T) \times \Omega)$ by a constant depending on the initial data and on f,

$$||\pi||_{\mathcal{M}((0,T)\times\Omega)} \leq C(\rho^0, m^0, f)$$

THEOREM 1.2. – *Besides the conditions on the initial data already given above we assume that $p(\rho) = 0$ and that f satisfies the following condition*

$$(19)\quad \begin{cases} f \in L^1(0,T;L^2(\Omega)) + L^2(0,T;L^r(\Omega)) \quad \text{if} \quad \Omega = \mathbf{T}^2, \\ f \in L^1(0,T;L^2 + L^\infty(\Omega)) + L^2(0,T;L^r(B)) \quad \text{if} \quad \Omega = \mathbb{R}^2, \\ f \in L^1(0,T;L^2 + L^\infty(\Omega)) + L^2(0,T;L^{\frac{2N}{N+2}}(\Omega)) \quad \text{if} \quad N \geq 3, \end{cases}$$

where $1 < r \leq 2$ and B is a bounded ball in $\mathbb{R}^2$, then there exists a solution (ρ, u, π) for the system $(1) - (4)$ satisfying the above requirement and the energy inequality.

In the next sections, we shall give precise conditions on $p(\rho)$, and state more general results. Besides, the requirements on ρ can be weakened (in the case of $\mathbb{R}^N$), in fact we can take a fluid of infinite mass. For instance we can impose the following condition

$$(20)\qquad \rho^0 - \overline{\rho} \in L^\infty(0,\infty;L^1(\mathbb{R}^n)),$$

for some constant $\overline{\rho}$, such that $0 < \overline{\rho} < 1$ (or more generally for some reference function $\overline{\rho}$ in $\mathbb{R}^N$ satisfying $0 \leq \overline{\rho} \leq 1$), we will come back to this issue in details in section 5.

A priori estimates will be derived in section 2, In section 3 we study the compactness of sequences of solutions satisfying the requirement above, then the existence results will be proved in section 4, using the convergence of solutions of the compressible isentropique Navier-Stokes equations as γ goes to the infinity (where the pressure is given by $p(\rho) = \rho^\gamma$). Finally in section 6, we give two convergence results to the incompressible Navier-Stokes system.

2. A PRIORI ESTIMATES

We are going to show here that the notion of weak solutions we have defined above is a natural one by showing some a priori bounds. First, we can notice that the conditions (1) and (3) are compatible, in fact we have the following lemma

LEMMA 2.1. – *Let $u \in L^2(0,T;H^1_{loc})$ and $\rho \in L^2_{loc}$ satisfying*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \text{ in } (0,T) \times \Omega, \quad \rho(0) = \rho^0$$

then the following two assertions are equivalent

1. *div* $u = 0$ *a.e. on* $\{\rho \geq 1\}$ *and* $0 \leq \rho^0 \leq 1$,
2. $0 \leq \rho \leq 1$.

Proof. – Let us begin by the first implication (1) $\rightarrow$ 2)), using the regularization lemma stated in [7] (lemma 2.3 p 43), we get

$$\frac{\partial \beta(\rho)}{\partial t} + \operatorname{div}(\beta(\rho)u) = (\beta(\rho) - \rho\beta'(\rho)) \operatorname{div} u, \tag{21}$$

for any C^1 function β from $\mathbb{R}$ to $\mathbb{R}$ such that $|\beta(t)| \leq C + C\ t$. Next let β be defined by

$$\begin{cases} \beta(t) = 0 & \text{if} \quad t \leq 0, \\ \beta(t) = t & \text{if} \quad 0 \leq t \leq 1, \\ \beta(t) = 1 & \text{if} \quad 1 \leq t, \end{cases} \tag{22}$$

then we get

$$\frac{\partial \beta(\rho)}{\partial t} + \operatorname{div}(\beta(\rho)u) = 1_{\{\rho \geq 1\}} \operatorname{div}(u) = 0, \tag{23}$$

in fact taking any sequence of C^1 functions β_n such that $\beta_n(t) = \beta(t)$ on $]-\infty, -\frac{1}{n}[\ \cup\ ]\frac{1}{n}, 1 - \frac{1}{n}[\ \cup\ ]1 + \frac{1}{n}, +\infty[$ and $|\beta'(t)| \leq C$ uniformly in n, we can rewrite (21) with β replaced by β_n, then $\beta_n(\rho)$ converges pointwise and in L^2 to $\beta(\rho)$. Moreover $(\beta_n(\rho) - \rho\beta_n'(\rho)) \operatorname{div} u$ (which is bounded in L^2_{loc}) converges pointwise to 0, and we can then recover (23). In addition we have $\beta(\rho)(t = 0) = \rho^0$; and setting $d = \beta(\rho) - \rho$, we see that d solves the same equation as ρ and $\beta(\rho)$ and that $d(0) = 0$. Applying the regularization lemma another time, we see that $|d|$ also solves the same equation and since $\int_\Omega |d|(t) = \int_\Omega |d|(0)$ for any t, we get that $d = 0$ and hence $\beta(\rho) = \rho$ which yields 2).

Now, we turn out to the proof of the second implication. Since $0 \leq \rho \leq 1$, we see that ρ is bounded and then (21) holds for any C^1 function β. Writing it for $\beta(\rho) = \rho^k$, for any integer k, we get

$$\frac{\partial \rho^k}{\partial t} + \operatorname{div}(\rho^k u) = (1 - k)\rho^k \operatorname{div} u. \tag{24}$$

Since $0 \leq \rho^k \leq 1$, we see that $\partial_t \rho^k$ is bounded in $W^{-1,\infty}((0,T) \times \Omega)$, in addition $div(\rho^k u)$ is bounded in $L^\infty(0,T; H^{-1}_{loc}(\Omega))$, for $|\rho^k u| \leq |\rho u| \in L^\infty(L^2_{loc})$. This yields that $k\rho^k \operatorname{div} u$ is a bounded distribution (in H^{-1}_{loc} for instance). Letting k go to the infinity, we get

$$\rho^k \operatorname{div} u \underset{k}{\rightharpoonup} 0 \quad \text{in } \mathcal{D}'. \tag{25}$$

Besides, we have

$$\rho^k \to 1_{\{\rho=1\}} \quad \text{a.e,}$$

so we get

$$\rho^k \mathrm{div} u \to 1_{\{\rho=1\}} \mathrm{div} u \quad \text{a.e,}$$

and since $|\rho^k \ \mathrm{div} u| \leq \mathrm{div} u \in L^2_{loc}((0,T) \times \Omega)$, we get that $div\ u = 0$ a.e. on $\{\rho = 1\}$. This completes the proof of the lemma. □

Next, we concentrate on the estimates we can derive from the equations. As is often the case when dealing with weak solutions, a priori bounds are obtained by making formal computations on the equation. Of course, those computations are rigorous if we assume that (ρ, u, π) are smooth enough.

Since $0 \leq \rho \leq 1$ and $\int \rho(t) = \int \rho^0$, we see that $\rho \in L^\infty(0,T; \mathrm{L}^\infty \cap L^1(\Omega))$. Then, the continuity of ρ in $L^p(\Omega) - w$ is deduced from the bound on $\partial_t \rho$ in $L^\infty(H^{-1})$, using the appendix C of [7]. The continuity of ρ in $L^p(\Omega)$ (for $1 \leq p < \infty$) is then a simple consequence of the conservation of the mass and of the L^∞ bound. Indeed, taking $\beta(\rho) = \sqrt{\beta}$ in (21), we get

$$\frac{\partial \sqrt{\rho}}{\partial t} + \mathrm{div}(\sqrt{\rho} u) = \frac{1}{2}\sqrt{\rho}\ \mathrm{div} u. \tag{26}$$

In fact, we must approximate $\sqrt{\rho}$ by the following C^1 functions $\beta_\varepsilon = \sqrt{\rho + \varepsilon}$ and then observe that

$$\beta_\varepsilon(\rho) - \rho\beta'_\varepsilon(\rho) = \sqrt{\rho+\varepsilon} - \frac{\rho}{2\sqrt{\rho+\varepsilon}} \to \frac{1}{2}\sqrt{\rho} \ \text{ in } L^2.$$

We deduce then that $\sqrt{\rho} \in C([0,T]; L^2 - w)$ and since

$$\int_\Omega (\sqrt{\rho})^2(t) = \int \rho(t) = M$$

is independent of t we see that $\sqrt{\rho} \in C([0,T]; L^2)$, hence $\rho \in C([0,T]; L^1)$ and then we conclude using the L^∞ bound on ρ.

Then, multiplying the momentum equation (2) by u and using (1), we get (at least formally),

$$\begin{cases} \partial_t \left[\rho \dfrac{|u|^2}{2}\right] + \mathrm{div}\left(\rho u \dfrac{|u|^2}{2} + \pi u\right) - \pi \mathrm{div} u - \mu \Delta \dfrac{|u|^2}{2} + \\ -\xi \mathrm{div}(u \mathrm{div} u) + \mu (Du)^2 + \xi (\mathrm{div} u)^2 = \rho f.u, \end{cases} \tag{27}$$

integrating over Ω and using the fact that $\pi \mathrm{div} u = 0$ (since $\mathrm{div} u = 0$ a.e. on $\left\{\rho = 1\right\}$ and $\pi = 0$ a.e. on $\left\{\rho < 1\right\}$), we get the following energy identity

$$\frac{\partial}{\partial t}\int_\Omega \rho\frac{|u|^2}{2} + \int_\Omega \mu(Du)^2 + \xi(\mathrm{div} u)^2 = \int_\Omega \rho f.u, \tag{28}$$

and then integrating over $(0,t)$, we obtain

$$\int_\Omega \rho\frac{|u|^2}{2}(t) + \int_0^t\int_\Omega \mu(Du)^2 + \xi(\mathrm{div} u)^2 = \int_\Omega \rho\frac{|u|^2}{2}(0) + \int_0^t\int_\Omega \rho f.u. \tag{29}$$

Since $\mu + \xi > 0$, we obtain (7) integrating by parts and using the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_\Omega (div\ u)^2 &= \int_\Omega \partial_i u_i \partial_j u_j = \int_\Omega \partial_j u_i \partial_i u_j \\ &\leq \int_\Omega \sum_{i,j}(\partial_i u_j)^2 = \int_\Omega (Du)^2.\end{aligned} \tag{30}$$

Hence, there exists $\nu > 0 (= inf(\mu, \mu+\xi))$ such that we have for almost all t the following inequality where we have assumed in addition that $f \in L^1((0,T;L^2)$,

$$\int_\Omega \rho\frac{|u|^2}{2}(t) + \int_0^t\int_\Omega \nu(Du)^2 \leq \int_\Omega \rho^0\frac{|u^0|^2}{2} + \int_0^t ||\sqrt{\rho}||_{L^\infty}||f||_{L^2}||\sqrt{\rho}u||_{L^2}. \tag{31}$$

Using Gronwall's inequality and noticing that $||\sqrt{\rho}||_{L^\infty} \leq 1$, we deduce the first part of (8) and the bound on Du in $L^2(0,T;L^2(\Omega))$. Then we get a bound on ρu in $L^\infty(0,T;L^2)$, since $\rho \in L^\infty$.

Next, we give the necessary changes to handle force terms satisfying (force). Let us set $f = f_1 + f_2 + f_3$ such that $f_1 \in L^1(0,T;L^2(\Omega))$, $f_2 \in L^1(0,T;L^\infty(\Omega))$ and $f_3 \in L^2(0,T;X)$, where $X = L^r(\Omega)$ if $\Omega = \mathbf{T}^2$, $X = L^r(B)$ if $\Omega = \mathbb{R}^2$ and $X = L^{\frac{2N}{N+2}}(\Omega)$ if $N \geq 3$. Then, we see that f_1 can be treated as in the case when $f = f_1 \in L^1(0,T;L^2)$, the second term f_2 (which can be taken null in the periodic case) is treated as follows,

$$\left|\int_\Omega \rho f_2.u\right| \leq ||\rho||_{L^1}^{1/2}||f_2||_{L^\infty}||\sqrt{\rho}u||_{L^2}. \tag{32}$$

And we use to conclude the conservation of the mass, namely $||\rho||_{L^1} = M$.

For the third term, we need to use the bound on Du in L^2.

If $\Omega = \mathbf{T}^2$, then we have

$$||u - \fint u||_{L^{r'}} \leq C||Du||_{L^2},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and hence

$$\left|\int_\Omega \rho f_3 . \left(u - \fint u\right)\right| \leq C||\rho||_{L^\infty}||f_3||_{L^r}||Du||_{L^2}, \tag{33}$$
$$\leq \frac{C}{\varepsilon}||f_3||^2_{L^r} + \varepsilon||Du||^2_{L^2}.$$

On the other hand, we have

$$M \fint u = \int \rho u - \int \rho\left(u - \fint u\right)$$

and since $M > 0$, we deduce that for almost all t

$$\left|\fint u\right| \leq ||\sqrt{\rho}u||_{L^2} + C||\rho||_{L^2}||Du||_{L^2}$$

which yields

$$\left|\int_\Omega \rho f_3 . \left(\fint u\right)\right| \leq C||f_3||_{L^1}(||\sqrt{\rho}u||_{L^2} + ||Du||_{L^2}),$$

and finally we deduce

$$\left|\int_\Omega \rho f_3 . u\right| \leq \frac{C}{\varepsilon}||f_3||^2_{L^r} + ||\sqrt{\rho}u||^2_{L^2} + \varepsilon||Du||^2_{L^2}.$$

Then, taking ε small enough ($\varepsilon \leq \nu/2$), we can absorb the second term of the right hand side in $\nu||Du||^2_{L^2}$. Applying Gronwall's inequality, we conclude easily as before.

If $\Omega = \mathbb{R}^2$, then f_3 is assumed to be bounded in $L^2(0,T;L^r(B))$ for some fixed ball B, hence we have

$$\int_\Omega \rho f_3 . u = \int_B \rho f_3 . u = \int_{B_R} \rho f_3 . \left(u - \fint_{B_R} u\right) + \int_{B_R} \rho f_3 . \fint_{B_R} u,$$

where B_R is a big enough ball (the radius R will be chosen later on). Next we know, using the classical Sobolev inequalities, that a bound on Dh in $L^2(B_1)$ yields a bound on $h - \fint h$ in $L^2(L^q(B_1))$ for all $2 \leq q < +\infty$

$$||h - \fint_{B_1} h||_{L^q(B_1)} \leq C_q||Dh||_{L^2(B_1)},$$

then by a scaling argument we get that

$$||h - \fint_{B_R} h||_{L^q(B_R)} \leq C_q R^{2/q} ||Dh||_{L^2(B_R)}.$$

In fact, let h be defined on B_1 and h_R defined on B_R by $h_R(x) = h\left(\frac{x}{R}\right)$, then we have

$$\begin{aligned}\int_{B_R} \left|h_R - \fint_{B_R} h_R\right|^q dx &= \int_{B_1} \left|h - \fint_{B_1} h\right|^q R^2 dy \\ &\leq C_q^q R^2 ||Dh||^q_{L^2(B_1)},\end{aligned}$$

$$\begin{aligned}\int_{B_R} |D(h_R)|^2 \, dx &= \frac{1}{R^2} \int_{B_R} \left|Dh\left(\frac{x}{R}\right)\right|^2 dx \\ &= \int_{B_1} |Dh(y)|^2 \, dy\end{aligned}$$

We then get as in the periodic case,

$$\begin{aligned}\left|\int_{B_R} \rho f_3 . \left(u - \fint_{B_R} u\right)\right| &\leq C||\rho||_{L^\infty} ||f_3||_{L^r} ||u - \fint_{B_R} u||_{L^{r'}} \\ &\leq C R^{2/r'} ||\rho||_{L^\infty} ||f_3||_{L^r} ||Du||_{L^2} \\ &\leq \frac{C}{\varepsilon} R^{4/r'} ||\rho||^2_{L^\infty} ||f_3||^2_{L^r} + \varepsilon ||Du||_{L^2},\end{aligned}$$

where $\frac{1}{r'} + \frac{1}{r} = 1$ (in the sequel we shall assume that $r' \geq 4$, which means that $r \leq 4/3$, this assumption can be made without loss of generality since B is bounded). In order to complete the estimates we need a bound on $\fint_{B_R} u$. For this we shall use the fact that the fluid does not flow very fast to infinity, indeed arguing as in [6] we can take a cut-off function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi = 0$ if $|x| \geq 2$, $\phi = 1$ if $|x| \leq 1$, $0 \leq \phi \leq 1$, then we multiply (1) by $\phi\left(\frac{x}{R}\right)$ and integrate by parts in x to find

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^2} \rho \phi\left(\frac{x}{R}\right) dx &= \int_{\mathbb{R}^2} \frac{1}{R} \rho u . \nabla \phi\left(\frac{x}{R}\right) dx \\ &\geq -\frac{1}{R} C ||\nabla \phi||_{L^\infty} ||\sqrt{\rho}||_{L^2} ||\sqrt{\rho} u||_{L^2},\end{aligned}$$

then taking R big enough, we get for all t

$$\int_{B_R} \rho \, dx \geq \frac{M}{2} - \frac{C\,M}{R} \int_0^t ||\sqrt{\rho} u||_{L^2},$$

hence, as long as

$$\sup_{0\le t\le T} ||\sqrt{\rho}u||_{L^2} \le \frac{R}{4CT}, \tag{34}$$

we have

$$\inf_{0\le t\le T} \int_{B_R} \rho\, dx \ge \frac{M}{4}$$

and we will show that this holds true if R is chosen big enough. In the sequel, we assume that (34) holds, hence computing as in the periodic case, we get

$$\begin{aligned}
\frac{M}{4} \fint_{B_R} u &\le \int_{B_R} \rho u - \int_{B_R} \rho\Big(u - \fint_{B_R} u\Big) \\
&\le ||\sqrt{\rho}||_{L^2}||\sqrt{\rho}u||_{L^2} + ||\rho||_{L^q}||u - \fint_{B_R} u||_{L^{q'}} \\
&\le ||\sqrt{\rho}||_{L^2}||\sqrt{\rho}u||_{L^2} + ||\rho||_{L^q} R^{2/q'}||Du||_{L^2} \\
&\le ||\sqrt{\rho}||_{L^2}||\sqrt{\rho}u||_{L^2} + \frac{C}{\varepsilon}||\rho||^2_{L^q} R^{4/q'} + \varepsilon||Du||^2_{L^2},
\end{aligned}$$

where $\frac{1}{q'} + \frac{1}{q} = 1$, $q' > 4$ (for instance we can take $q' = r'$). Hence, the second term can be estimated as follows

$$\begin{aligned}
\left|\int_{B_R} \rho f_3 . \fint_{B_R} u\right| &\le ||\rho||_{L^\infty} \frac{4}{M}[||\sqrt{\rho}||_{L^2}||\sqrt{\rho}u||_{L^2} \\
&\qquad + ||\rho||_{L^r} R^{2/r'}||Du||_{L^2}\Big]||f_3||_{L^1} \\
&\le C||\sqrt{\rho}u||^2_{L^2} + CR^{4/r'}||f_3||^2_{L^r(B)} + \varepsilon||Du||^2_{L^2}.
\end{aligned}$$

Summing up the two estimates, we get

$$\left|\int_\Omega \rho f_3 . u\right| \le \frac{C}{\varepsilon} R^{4/r'}||f_3||^2_{L^r} + ||\sqrt{\rho}u||^2_{L^2} + \varepsilon||Du||^2_{L^2} \tag{35}$$

where C is a constant that does not depend on R and where (35) is true as long as (34) holds. Taking ε small enough ($\varepsilon \le \nu/2$), we can absorb the second term of the right hand side in $\nu||Du||^2_{L^2}$. Then, taking R large enough we conclude easily.

If $N \ge 3$ then we have

$$||u||_{L^{\frac{2N}{N-2}}} \le C||Du||_{L^2}$$

and hence

$$(36)\qquad \left|\int_\Omega \rho f_3.u\right| \le C||f_3||_{L^{\frac{2N}{N+2}}}||u||_{L^{\frac{2N}{N-2}}} \le \frac{C}{\varepsilon}||f_3||^2_{L^{\frac{2N}{N+2}}} + \varepsilon||Du||^2_{L^2},$$

and we conclude easily. □

Next, in order to deduce the L^1 bound on π (we assume that $\pi \in L^1$ since π will be the weak limit of a bounded sequence in L^1), we apply the operator $(-\Delta)^{-1}\mathrm{div}$ to the momentum equation (this idea was used by P.-L. Lions in the case of compressible Navier-Stokes equations). We begin by the case $\Omega = \mathbb{R}^N$,

$$(37)\qquad \begin{cases} \pi = \partial_t(-\Delta)^{-1}\mathrm{div}(\rho u) - R_iR_j(\rho u_iu_j)+ \\ \\ +(\mu+\xi)div(u) + (-\Delta)^{-1}\mathrm{div}(\rho f), \end{cases}$$

where R_i is the Riesz transform ($R_i = \Delta^{-1}\partial_i$) and where we must fulfill the summation over i and j. Here we have used that $\Delta^{-1}[\Delta(\mathrm{div}u)] = \mathrm{div}u$ and that $\Delta^{-1}[\Delta\pi] = \pi$, since we assume that $\mathrm{div}u \in L^2(\mathbb{R}^N)$ a.e. on t and that $\pi \in L^1(\mathbb{R}^N)$ a.e. on t. In order to obtain bounds which depend only on the initial data, we can integrate this identity and use the fact that $\pi \ge 0$, but since we have only an L^2 bound on $div\ u$, we cannot integrate on the whole space. To recover this, we use the fact that $\rho\pi = \pi$ and multiply (37) by ρ,

$$(38)\qquad \begin{cases} \pi = \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] + \mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)]+ \\ \\ +\rho u_iR_iR_j(\rho u_j) - \rho R_iR_j(\rho u_iu_j)+ \\ \\ +(\mu+\xi)\rho div(u) + \rho(-\Delta)^{-1}\mathrm{div}(\rho f). \end{cases}$$

Integrating (38) over $\mathbb{R}^N$, we get a bound in L^1 that depends only on the initial data. We are going to explain how we treat the six terms that occur in the right hand side of (38). First we have

(39)

$$\left|\int_0^T\int_{\mathbb{R}^N} \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]\right| \le 2||\rho(-\Delta)^{-1}\mathrm{div}(\rho u)||_{L^\infty(0,T;L^1(\Omega))},$$

then using that $\rho u \in L^\infty(L^{3/2})$, we get $(-\Delta)^{-1}\mathrm{div}(\rho u) \in L^\infty(L^q)$, with $\dfrac{1}{q} = \dfrac{2}{3} - \dfrac{1}{N} > 0$, and then we conclude by using that $\rho \in L^\infty(L^{q'})$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$,

$$\left|\int_0^T \int_{\mathbb{R}^N} \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]\right| \leq 2||\rho||_{L^\infty(L^{q'})}||\rho u||_{L^\infty(L^{3/2})}.$$

Notice that we cannot use the continuity of ρu in $L^p - w$ $(1 \leq p \leq 2)$ at this stage (since $\pi \in \mathcal{M}$) and that we must explain the meaning of (39). Let ϕ_n be a sequence of $C_c^\infty(0,T)$ that converges to $1_{[0,T]}$ in $L^1(\mathbb{R})$ (and in $L^q(\mathbb{R})$ for all $1 \leq q \leq \infty$) and such that ϕ_n is nondecreasing on $[0, \frac{1}{n}]$, increasing on $[T - \frac{1}{n}, T]$ and $\phi_n(t) = 1$ on $[\frac{1}{n}, T - \frac{1}{n}]$. Then we get

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^N} \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]\phi_n(t)\right| &= \left|\int_0^T\int_{\mathbb{R}^N} [\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]\partial_t\phi_n(t)\right| \\ &\leq C||\partial_t\phi_n(t)||_{L^1(0,T)} \\ &\leq C\left[\int_0^{1/n} \partial_t\phi_n(t) - \int_{T-1/n}^T \partial_t\phi_n(t)\right] \\ &= 2||\rho(-\Delta)^{-1}\mathrm{div}(\rho u)||_{L^\infty(0,T;L^1(\Omega))}.\end{aligned}$$

where $C = ||\rho(-\Delta)^{-1}\mathrm{div}(\rho u)||_{L^\infty(0,T;L^1(\Omega))}$.

Next, we explain why the integration over $\mathbb{R}^N$ of the second term vanishes by using a cut-off function, let $\phi \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, $\phi = 1$ on B_1, $\phi = 0$ on $\mathbb{R}^N - B_2$, then for any $R \in (1, +\infty)$, we have, setting $U = \rho u(-\Delta)^{-1}\mathrm{div}(\rho u)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} div\ U\ \phi\left(\frac{x}{R}\right)\ dx\right| &= \left|\int_{\mathbb{R}^N} U.\frac{1}{R}\nabla\phi\left(\frac{x}{R}\right)\ dx\right| \\ &\leq ||U||_{L^1(\mathbb{R}^N)}\ \frac{1}{R}||\nabla\phi||_{L^\infty(\mathbb{R}^N)},\end{aligned} \tag{40}$$

and we conclude by letting R tend to $+\infty$. We only need to show that $U \in L^\infty(0,T;L^1)$, which can be deduced from the bound we have on ρu in $L^\infty(0,T;L^q)$, where $\frac{1}{q} + (\frac{1}{q} - \frac{1}{N}) = 1$ $(1 < q \leq 2)$.

The third and the fourth terms are handled similarly if $N \geq 3$. In fact, since $\rho|u|^2 \in L^1(0,T;L^s)$ $\left(s = \frac{N}{N-2} > 1\right)$ and since the Riesz transforms are bounded in L^p for $1 < p < +\infty$, we see that $R_iR_j(\rho u_iu_j) \in L^1(0,T;L^s)$ and then using the fact that $\rho \in L^\infty(0,T;L^{s'})$, we see that $\rho R_iR_j(\rho u_iu_j) \in L^1$. The third term is treated in the same way

and is in fact simpler. However if $N = 2$, the fourth term cannot be treated by this method since we no longer know that $\rho|u|^2 \in L^1(0,T;L^s)$ for some $s > 1$. Nevertheless, since $Du \in L^2(0,T;\mathbb{R}^2)$, we see (as in P.-L. Lions [6]) that $u \in L^2(0,T;BMO)$. In fact for any cube Q in $\mathbb{R}^2$, we have

$$\begin{aligned}\int_Q \left| u - \fint_Q u \right| dx &\leq |Q|^{1/2} \left(\int_Q \left| u - \fint_Q u \right|^2 dx \right)^{1/2} \\ &\leq C|Q| \left(\int_Q |Du|^2 \, dx \right)^{1/2} \\ &\leq C|Q| \int_{\mathbb{R}^2} |Du|^2 \, dx.\end{aligned}$$

Hence using the Coifman-Rochberg-Weiss commutator theorem [3], we get that $[u, R_i R_j]$ is bounded in L^p for $1 < p < +\infty$, so we have the following estimate

$$\begin{aligned}&||R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j)||_{L^2(0,T;L^2)} \\ &\quad \leq C||u||_{L^2(0,T;BMO)}||\rho u||_{L^\infty(0,T;L^2)}.\end{aligned}$$

Next, we use that $\rho \in L^2(0,T;L^2)$ to deduce the desired L^1 bound.

For the fifth, we have the following straightforward computation

$$\left| \int_0^T \int_{\mathbb{R}^N} \rho \; div \; u \right| \leq ||\rho||_{L^2(0,T;L^2)}||div \; u||_{L^2(0,T;L^2)}. \tag{41}$$

Finally, for the sixth term we have merely

$$\left| \int_0^T \int_{\mathbb{R}^N} \right| \rho(-\Delta)^{-1}\mathrm{div}(\rho f)| \leq ||\rho||_{L^\infty(L^{q'})}||\rho f||_{L^1(L^r)}$$

where $\frac{1}{q'} + \frac{1}{r} - \frac{1}{N} = 1$ since $\rho f \in L^1(L^r)$ and ($||\rho f||_{L^1(L^r)} \leq C||f||_{L^1(L^2)+L^2(L^r)}$ with $r < 2$).

Next, we explain the changes in the above argument we must perform in the periodic case. Now, (37) is replaced by

$$\begin{cases} \pi - \fint \pi = \partial_t(-\Delta)^{-1}\mathrm{div}(\rho u) - R_i R_j(\rho u_i u_j) + \\ \qquad\qquad + (\mu + \xi) div(u) + (-\Delta)^{-1}\mathrm{div}(\rho f). \end{cases} \tag{42}$$

Before integrating over $\mathbf{T}^N$, we multiply (as in the case of $\mathbb{R}^N$) by ρ. Notice however that we do so for different reasons than in the whole space case.

In fact, since $\mathbf{T}^N$ is a bounded domain $div(u) \in L^1((0,T)\times\Omega)$ and we can integrate (37) without any problem, but this integration gives no estimates on π, since the integral of $\pi - \fint \pi$ vanishes. This is why we multiply by ρ,

$$(43)\quad \begin{cases} \pi - \rho \fint \pi = \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] + \mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)] + \\ \qquad +\rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j) + \\ \qquad +\rho(\mu+\xi)div(u) + \rho(-\Delta)^{-1}\mathrm{div}(\rho f). \end{cases}$$

And integrating (43) over $\mathbf{T}^N$, we get

$$(44)\qquad (1-M)\int_0^T \int_{\mathbf{T}^N} \pi \le C(\rho^0, m^0)$$

in fact the estimates here are simpler than in the whole space case, and since $M < 1$, we get the desired bound.

Now, we are going to prove theorem 1.1. From the bounds on ρ, ρu, u, π and in the case $N \ge 3$ or $\Omega = \mathbf{T}^2$ we deduce that,

$$(45)\quad \begin{cases} \rho \in C([0,T]; L^p) \cap C^1([0,T]; H^{-1}), \\ \pi \in W^{-1,\infty}(H^1) + L^1(L^{N/(N-2)} \cap L^\alpha(L^\beta) + L^2(L^2) + L^1(L^q), \end{cases}$$

where we have used (37) to deduce the bounds on π and where $1 < \alpha, \beta < \infty$ and $\dfrac{1}{\beta} = \dfrac{1}{\alpha} r + (1 - \dfrac{1}{\alpha})$, $r = \dfrac{N-2}{N}$ if $N \ge 3$ and $1 < r < \infty$ if $N = 2$, $\dfrac{1}{q} = \dfrac{1}{r} - \dfrac{1}{N}$.

Then, π can be rewritten as $\pi = \partial_t h + \pi_2 + \pi_3 + \pi_4$ where $h \in L^\infty(H^1)$. The terms $\rho\pi_2$, $\rho\pi_3$, $\rho\pi_4$ are well-defined. To give a meaning to $\rho\partial_t h$, we use that $\rho = div_x g$ where $g \in C^1(0,T;L^2)^N$. Hence, we can write (and define $\rho\pi_1$ as follows)

$$\begin{aligned} \rho\pi_1 &= div_x g \partial_t h \\ &= div_x \partial_t[gh] - div_x[h\partial_t g] - \partial_t[g div_x h] + \partial_t div_x h \end{aligned}$$

Next, in order to show that $\pi \in \mathcal{M}(0,T;L^1(\Omega))$, we only need to prove that

$$\lim_{|\omega|\to 0} \int_0^T \int_\omega \pi = 0,$$

which can be deduced easily from (45).

Since $\pi \in \mathcal{M}$, it is easy to see that $\rho u \in BV(0,T;H^{-m})$ for m large enough. Then, since $\rho u \in L^\infty(0,T;L^2)$, we deduce that $\rho u \in BV(0,T;L^2 - w)$. We recall here that the values taken by ρu on $[0,T]$ belong to a fixed ball B_R of $L^2(\Omega)$ and that we can equip B_R with a "weak topology" distance d. Then $\phi \in BV([0,T];(B_R,d))$ if and only if the following supremum is finite

$$||\phi||_{BV(0,T;L^2-w)} = \sup_{\substack{n-1\in I\!N \\ 0=t_0<t_1<\ldots<t_n=T}} \sum_{i=0}^{n} d(\phi(t_{i+1},\phi(t_i)).$$

Finally to deduce the continuity of $P(\rho u)$, we apply the operator P to the momentum equation and get

$$\frac{P(\partial \rho u)}{\partial t} + P(\mathrm{div}(\rho u \otimes u)) - \mu \Delta P(u) = P(\rho f) \ .$$

Then, the continuity in the weak topology is deduced from the following bound on $\dfrac{\partial P(\rho u)}{\partial t}$ and the Appendix C of [7]. In deed, we have

$$\frac{\partial P(\rho u)}{\partial t} \in L^\infty(0,T;W^{-1,1}) + L^2(0,T;H^{-1}) + L^1(0,T;L^2) + L^2(0,T;L^r).$$

3. COMPACTNESS

In this section, we are concerned with a sequence of weak solutions (ρ_n, u_n, π_n) of (1)–(4), with $f = 0$ for simplicity. This sequence is assumed to exist even though we have not proved yet existence results for (1) - (4). The sequence (ρ_n, u_n, π_n) satisfies, uniformly in n, the a priori estimates derived in the previous section and the following initial conditions

$$\rho_n u_n \Big|_{t=0} = m_n^0 \ , \ \rho_n \Big|_{t=0} = \rho_n^0, \tag{46}$$

where $0 \le \rho_n^0 \le 1$ a.e. , ρ_n^0 is bounded in $L^1(\Omega)$, m_n^0 is bounded in $L^2(\Omega)$, $m_n^0 = 0$ a.e. on $\{\rho_n^0 = 0\}$, $\rho \not\equiv 0$ and $\rho_n^0 |u_n^0|^2$ is bounded in L^1, denoting by $u_n^0 = \dfrac{m_n^0}{\rho_n^0}$ on $\{\rho_n^0 > 0\}$, $u_n^0 = 0$ on $\{\rho_n^0 = 0\}$. In the case of $\mathbf{T}^N$, we also assume that $\fint \rho_n^0 = M$, for some fixed M (or some M_n, $0 < M_n < 1$ such that $M_n \to M$), with $0 < M < 1$.

Without loss of generality, extracting subsequences if necessary also denoted by (ρ_n, u_n, π_n), we can assume that (ρ_n, u_n, π_n) converge to

some (ρ, u, π) in the sense of distributions. More precisely, we can assume that $\rho_n \rightharpoonup \rho$ weakly in $L^p((0,T) \times \Omega)$ for any $1 < p < +\infty$, $\rho \in L^\infty(0,T, L^1 \cap L^\infty(\Omega))$ and $0 \leq \rho \leq 1$, $u_n \rightharpoonup u$ weakly in $L^2(0,T; H^1(B))$, where $B = \mathbf{T}^N$ if $\Omega = \mathbf{T}^N$ and B is any ball in $\mathbb{R}^N$ if $\Omega = \mathbb{R}^N$, $D\, u_n \rightharpoonup D\, u$ weakly in $L^2(0,T; L^2(\Omega))$. Finally, we may assume that $\pi_n \rightharpoonup \pi$ weakly in $\mathcal{M}((0,T) \times \Omega)$, where π is a positive measure.

THEOREM 3.1. – *Under the above assumptions, we have*

$$\rho_n u_n \rightharpoonup \rho u \quad \textit{weakly star in} \quad L^\infty(0,T; L^2(\Omega))$$

$$\rho_n u_n \otimes u_n \rightharpoonup \rho u \otimes u \quad \textit{weakly in} \quad L^\alpha(0,T; L^\beta(\Omega)),$$

where $1 < \alpha, \beta < \infty$ *and* $\dfrac{1}{\beta} = \dfrac{1}{\alpha} r + (1 - \dfrac{1}{\alpha})$, $r = \dfrac{N-2}{N}$ *if* $N \geq 3$ *and* $1 < r < \infty$ *if* $N = 2$, *(if* $\Omega = \mathbb{R}^2$, *the convergence holds locally in space)*

Remark. – In general (ρ, u, π) is not a solution of (1)–(4), in fact condition (4) does not hold in all times. The homogeneisation example given in [6] for the case of the compressible Navier-Stokes equations can be adapted to our system. In the following theorem, we give a sufficient condition on the initial data, for (ρ, u, π) to be a solution.

THEOREM 3.2. – *If we assume in addition that* ρ_n^0 *converges to* ρ^0 *in* $L^1(\Omega)$, *then* (ρ, u, π) *is a solution of (1)–(4) and in addition the following strong convergences hold*

$$\rho_n \to \rho \quad \textit{in} \quad C(0,T; L^p(\Omega)) \textit{ for any } 1 \leq p < +\infty,$$

$$\rho_n u_n \to \rho u \quad \textit{in} \quad L^p(0,T; L^r(\Omega)) \textit{ for any } 1 \leq p < +\infty,\ 1 \leq r < 2,$$

$$\rho_n u_n \otimes u_n \to \rho u \otimes u \quad \textit{in} \quad L^p(0,T; L^1(\Omega)) \textit{ for any } 1 \leq p < +\infty,$$

this last convergence holds locally if $\Omega = \mathbb{R}^2$.

Remark. – Using the bound on Du_n in L^2, we see that the strong convergence of $\rho_n u_n$ also holds in $L^p(0,T; L^r(\Omega))$ for any $p > 2$ and $1 \leq r < \dfrac{2\,p\,N}{p\,N - 4}$ if $\Omega \neq \mathbb{R}^2$. If $\Omega = \mathbb{R}^2$ then this convergence is local in space.

For $\rho_n u_n \otimes u_n$ the convergence also holds in $L^p(0,T; L^q(\Omega))$ (locally if $\Omega = \mathbb{R}^2$) for any $p \geq 2$ and $1 < q < \dfrac{p\,N}{p\,N - 2}$.

Proof. – The proof of the two theorems is very close to the proof given in [6]. In fact theorem 3.1 is deduced from the following compensated-compactness lemma,

LEMMA 3.3. – *Let g^n, h^n converge weakly to g, h respectively in $L^{p_1}(0,t;L^{p_2})$, $L^{q_1}(0,t;L^{q_2})$, where $1 \leq p_1,\ p_2 \leq +\infty$, $\frac{1}{p_1}+\frac{1}{q_1} = \frac{1}{p_2}+\frac{1}{q_2} = 1$. We assume in addition that*

$$\left\{\begin{array}{ll} \dfrac{\partial g^n}{\partial t} & \textit{is bounded in} \quad \mathcal{M}(0,T;W^{-m,1}(\Omega)) \\ \textit{for some} \ \ m \geq 0 \ \ \textit{independent of} \ \ n & \end{array}\right. \tag{47}$$

$$||h^n||_{L^1(0,T;H^s)} \quad \textit{is bounded for some} \ \ s > 0. \tag{48}$$

Then, $g^n h^n$ converges to gh in $\mathcal{D}'$.

Next, let us observe that ρ_n is bounded in $L^\infty((0,T)\times\Omega)$ and converges weakly to ρ in $L^{p_1}(0,T;L^{p_2}(\Omega))$ for any $1 \leq p_1,\ p_2 < +\infty$ and that u_n converges weakly to u in $L^2(0,T;L^\beta)$ where $\beta = \dfrac{N}{N-2}$ if $N \geq 3$ and $2 \leq \beta < +\infty$ if $N = 2$ (this holds locally in space if $\Omega = \mathbb{R}^2$). We can take $p_1 = 2$ and p_2 such that $\dfrac{1}{p_2}+\dfrac{1}{\beta} = 1$. In order to apply the lemma 3.3 ($g^n = \rho_n$, $h^n = u_n$), we need some compactness in time and in space which are straightforward since we know that $\dfrac{\partial \rho_n}{\partial t}$ is bounded in $L^\infty(0,T;H^{-1})$ and that u_n is bounded in $L^2(0,T;H^1)$ (locally if $\Omega = \mathbb{R}^N$). Hence, using the preceding lemma we obtain the weak convergence of $\rho_n u_n$ to ρu (in the sense of distributions).

Then, we apply the lemma to the following couple ($g^n = \rho_n u_n$, $h^n = u_n$). In order to do so, we observe that $\rho_n u_n$ converges weakly to ρu in $L^{p_1}(0,T;L^{p_2}\Omega)$ for any $1 < p_1 < +\infty$, $1 < p_2 \leq 2$ (in fact convergence in the sense of distribution is equivalent to weak convergence in these spaces). We can take $p_1 = 2$ and p_2 such that $\dfrac{1}{p_2}+\dfrac{1}{\beta} = 1$. Next we have in view of (2) the following bound

$$\frac{\partial \rho_n u_n}{\partial t} \text{ bounded in } L^\infty(0,T;W^{-1,1}) + L^2(0,T;H^{-1}) + \mathcal{M}(0,T;W^{-1-\varepsilon,1}),$$

for any $\varepsilon > 0$. Hence, applying lemma 3.3 once more, we deduce the weak convergence of $\rho_n u_n \otimes u_n$ to $\rho u \otimes u$ in the sense of distributions and hence in the spaces $L^\alpha(0,T;L^\beta(\Omega))$.

In conclusion, these weak convergences show that the limit (ρ, u, π) satisfy the equation (1), (2), (3). In fact, it is obvious for the first

and the second equation, for the third one we apply lemma 2.1 since $u \in L^2(0,T;H^1_{loc})$ and that $0 \leq \rho \leq 1$. However in general (4) does not hold and we will see below that if the ρ_n^0 converges to ρ^0 then (4) holds and (ρ, u, π) is a solution of the initial system.

Now, we turn to the proof of theorem 3.2. The idea of the proof relies on the use of some compactifying commutators. This idea was used by P.-L. Lions in [6]. In what follows we will give a sketch of the proof and omit the problems related to the justification of the computation. We refer to [6] for the missing justifications.

Taking $\beta(\rho) = \rho\ log\rho$ in (21), we get

$$\frac{\partial \rho\ log(\rho)}{\partial t} + \mathrm{div}(\rho\ log(\rho)u) = -\rho\ \mathrm{div}u. \tag{49}$$

This equality follows from approximating $\rho\ log(\rho)$ by the following C^1 functions $\beta_\varepsilon = \rho\ log(\rho + \varepsilon)$ and observing that

$$\beta_\varepsilon(\rho) - \rho\beta'_\varepsilon(\rho) = -\rho^2(\rho+\varepsilon)^{-1} \to \rho \ \text{ in } L^2.$$

Next, we observe that this equality also holds for ρ_n and that extracting subsequences if necessary we can assume that $\rho_n\ log\rho_n$ converges weakly to $\overline{s}$ in $L^p((0,T)\times\Omega)$ for $1 < p < +\infty$. In fact, since $0 \leq \rho_n \leq 1$, we see that for any $p > 1$, there exists a constant C_p (independent of n) such that for all n, we have $|\rho_n\ log\rho_n|^p \leq C_p\rho_n$. Then, passing to the limit in the equation satisfied by $\rho_n\ log\rho_n$, we obtain

$$\frac{\partial \overline{s}}{\partial t} + \mathrm{div}(\overline{s}u) = -\overline{\rho\ \mathrm{div}u}, \tag{50}$$

where $\overline{\rho\ \mathrm{div}u}$ denotes the weak limit of $\rho_n\ \mathrm{div}u_n$. In fact we can apply lemma 3.3 to the pair ($g^n = \rho_n\ log\rho_n$, $h^n = u_n$), which yields the weak convergence of $\rho_n\ log\rho_n u_n$ to $\overline{s}u$. We see then that $-\rho_n\ \mathrm{div}u_n$ converges weakly to the first hand side of (50). Let s denote $\rho\ log(\rho)$, then we get

$$\frac{\partial}{\partial t}(\overline{s} - s) + \mathrm{div}[(\overline{s} - s)u] = -\overline{\rho\ \mathrm{div}u} + \rho\ \mathrm{div}u. \tag{51}$$

Next, using the momentum equation we shall show that the second hand side of (51) is actually equal to $\rho\pi - \pi$. Hence, integrating in x, we deduce

$$\frac{\partial}{\partial t}\int_\Omega (\overline{s} - s)\ dx \leq 0.$$

Let us then notice that, by standard convexity considerations, we obtain that $\overline{s} \geq s$ a.e., and since $\overline{s} - s|_{t=0} = 0$ we conclude that $\overline{s} = s$ for almost all t in $(0, T)$. We also deduce at the same time that $\rho\pi = \pi$.

Now, we should only compute the right hand side of (51). We begin by the case of the whole space $\mathbb{R}^N$, with $N \geq 3$ and then explain the changes that must be done in the other cases. Taking the divergence of (2), applying $(-\Delta)^{-1}$ and multiplying by ρ, we get the equation (38) written for (ρ_n, u_n, π_n). Next passing to the limit and taking subsequences if necessary we get

$$(52) \qquad \begin{cases} \pi - (\mu+\xi)\overline{\rho div(u)} = \partial_t[\overline{\rho(-\Delta)^{-1}\mathrm{div}(\rho u)}] \\ \qquad\qquad\qquad\qquad +\mathrm{div}[\overline{\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)}] \\ \qquad\qquad\qquad\qquad +\overline{\rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j)} \end{cases}$$

where the $\overline{A}$ denotes the weak limit of A_n. Next, we apply the same computations for (ρ, u, π) and we deduce

$$(53) \qquad \begin{cases} \rho\pi - (\mu+\xi)\rho div(u) = \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ \qquad\qquad\qquad\qquad +\mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ \qquad\qquad\qquad\qquad +\rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j). \end{cases}$$

Let us notice that we cannot use the equation $\rho\pi = \pi$. Next arguing as in [6], we show that the second terms in the right hand side of (52) and (53) coincide. In fact, using lemma 3.3, we see that $\rho_n(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$ weakly converges to $\rho(-\Delta)^{-1}\mathrm{div}(\rho u)$ and that $\rho_n u_n(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$ weakly converges to $\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)$. The last term can be written as $\rho[u_i, R_i R_j]\rho u_j$, (where $[u_i, R_i R_j]$ denotes the commutator of u_i and $R_i R_j$). Using the general results on commutator of this type of Bajanski and R. Coifman [1] and R. Coifman and Y. Meyer [2], we deduce that $[u_{n,i}, R_i R_j]\rho u_{n,j}$ is bounded in $L^1(0, T; W^{1,q})$, where $\dfrac{1}{q} = \dfrac{1}{2} + \dfrac{1}{\beta} < 1$, since $u \in L^2(0, T; H^1)$ (locally if $\Omega = \mathbb{R}^N$) and $\rho u \in L^2(0, T; L^\beta)$, where $\beta = \dfrac{2N}{N-2} > 2$ if $N \geq 3$ and $2 < \beta < +\infty$ if $N = 2$ (locally if $\Omega = \mathbb{R}^2$). Lemma 3.3 applies to this case too and yields the weak convergence of $\rho_n[u_{n,i}, R_i R_j]\rho u_{n,j}$ to $\rho[u_i, R_i R_j]\rho u_j$. Hence, we obtain

$$-\overline{\rho div(u)} + \rho div(u) = \frac{1}{\mu+\xi}[\rho\pi - \pi].$$

Now, we concentrate on the proof (see also [6]) of the strong convergences stated in theorem 3.2. From the equality $s = \overline{s}$ and the fact that $\rho\, log\rho$ is strictly convex, we deduce easily the strong convergence

of ρ_n to ρ in $L^p((0,T)\times\Omega)$, using Young measures. Hence, to get the convergence of ρ_n to ρ in $C([0,T];L^p(\Omega))$ for $1\le p<+\infty$, we only need to show this for $p=1$. Then using appendix C of [7], we see that ρ_n and $\sqrt{\rho_n}$ are relatively compact in $C([0,T];L^2-w)$ and converge in this space respectively to $\sqrt{\rho}$ and to ρ. In particular we have $\rho(0)=\rho^0$. Next to get the strong convergence of $\sqrt{\rho_n}$ to $\sqrt{\rho}$ in $C([0,T];L^2)$ we notice that for any sequence t_n of $[0,T]$ converging to t we have

$$\int_\Omega \sqrt{\rho_n}^2(t_n)=\int_\Omega \rho_n^0\to\int_\Omega \rho^0=\int_\Omega \sqrt{\rho}^2(t).$$

Hence ρ_n converges in $C([0,T];L^1)$ to ρ.

Next, to get the convergence of $\rho_n u_n$ to ρu in $L^p(0,T;L^r)$. We use the bound of $\rho_n u_n$ in $L^\infty(0,T;L^2)$ as well as the following strong convergences

$$\sqrt{\rho_n}\to\sqrt{\rho}\quad\text{in}\quad L^p((0,T)\times\Omega),$$

$$\sqrt{\rho_n}u_n\to\sqrt{\rho}u\quad\text{in}\quad L^2((0,T)\times\Omega).$$

For this last convergence we use that $\rho_n|u_n|^2$ converges weakly to $\rho|u|^2$ and that $\sqrt{\rho_n}u_n$ converges weakly to $\sqrt{\rho}u$ (this can be deduced from lemma comp in the same way as the weak convergence of $\rho_n u_n$).

Moreover, using the bound on Du_n in L^2 we deduce a bound on ρu in $L^p(0,T;L^r)$ for any $p>2$ and $2\le r\le \dfrac{2\,p\,N}{p\,N-4}$ if $N\ge 3$ ($2\le r<\dfrac{2\,p\,N}{p\,N-4}$ if $N=2$ and the bound is local in space if $\Omega=\mathbb{R}^2$). This yields the convergence stated in the remark.

The convergence of $\rho_n u_n\otimes u_n$ to $\rho u\otimes u$ is deduced easily from the convergence of $\sqrt{\rho_n}u_n$ to $\sqrt{\rho}u$ in $L^p(0,T;L^r)$ for some couple (p,r) with $p>2$ and $r>2$.

4. EXISTENCE RESULTS

In this section we are going to prove the existence of weak solutions (ρ,u,π). It is classical (see for instance [6]) to deduce the existence of solutions, using the compactness results already shown, via a regularization (or some layers of regularization) of the equations or via a time discretization, which uses stationary problems. Nevertheless we are not going to use this classical method but we are going to show a more general convergence result and deduce the existence of solutions to our system as

a consequence of that result. Besides, one of our motivations to study the system (1)–(4) is the following convergence result concerning solutions of the compressible Navier-Stokes equations as γ tend to the infinity.

Let γ_n be a sequence of nonnegative real numbers that goes to infinity. Let (ρ_n, u_n) be a sequence of solutions of the isentropic compressible Navier-Stokes equations

$$(54) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \ , \quad \rho \geq 0, \\ \dfrac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \mu_n \Delta u - \xi_n \nabla \operatorname{div} u + a \nabla \rho^{\gamma_n} = 0 \ . \end{cases}$$

where $\mu_n > 0$ and $\mu_n + \xi_n > 0$, μ_n and ξ_n tend respectively to μ and ξ as n goes to the infinity, with $\mu > 0$ and $\mu + \xi > 0$ (in the sequel, we assume for simplicity that $\mu_n = \mu$ and $\xi_n = \xi$). Global weak solutions of the above system have been shown to exist by P.-L. Lions ([8], [9]), if we assume in addition that $\gamma_n > \dfrac{N}{2}$ if $N \geq 4$, $\gamma_n \geq \dfrac{3}{2}$ if $N = 2$ and $\gamma_n \geq \dfrac{9}{5}$ if $N = 3$. These assumptions are true for n large enough. The sequence (ρ_n, u_n) satisfies in addition the following initial conditions and the following bounds,

$$(55) \qquad \rho_n u_n \Big|_{t=0} = m_n^0 \ , \ \rho_n \Big|_{t=0} = \rho_n^0,$$

where $0 \leq \rho_n^0$ a.e., ρ_n^0 is bounded in $L^1(\Omega)$ and $\rho_n^0 \in L^{\gamma_n}$ with $\int (\rho_n^0)^{\gamma_n} \leq C\gamma_n$ for some fixed C, $m_n^0 \in L^{2\gamma_n/(\gamma_n+1)}(\Omega)$, and $\rho_n^0 |u_n^0|^2$ is bounded in L^1, denoting by $u_n^0 = \dfrac{m_n^0}{\rho_n^0}$ on $\{\rho_n^0 > 0\}$, $u_n^0 = 0$ on $\{\rho_n^0 = 0\}$. In the case of $\mathbf{T}^N$, we also assume that $\fint \rho_n^0 = M_n$, for some M_n such that $0 < M_n \leq M < 1$ and $M_n \to M$. Furthermore, we assume that $\rho_n^0 u_n^0$ converges weakly in L^2 to some m^0 and that ρ_n^0 converges weakly in L^1 to some ρ^0. Our last requirement concerns the following energy bounds we impose on the sequence of solutions we consider,

$$(56) \ E_n(t) + \int_0^t D_n(s) ds \leq E_n^0 \quad \text{a.e. } t, \ \frac{dE_n}{dt} + D_n \leq 0 \quad \text{in } \mathcal{D}'(0, \infty)$$

where $E_n(t) = \int \frac{1}{2} \rho_n |u_n|^2(t) + \dfrac{a}{\gamma_n - 1} (\rho_n)^{\gamma_n}(t)$, $D_n(t) = \int \mu |Du_n|^2(t)$ $+ \xi \ (\operatorname{div} u_n)^2(t)$ and $E_n^0 = \int \frac{1}{2} \rho_n^0 |u_n^0|^2 + \dfrac{a}{\gamma_n - 1} (\rho_n^0)^{\gamma_n}$.

We recall that the results in [6] yield the existence of solutions (ρ_n, u_n) satisfying the above requirements if we have $\gamma_n > \frac{N}{2}$ if $N \geq 4$, $\gamma_n \geq \frac{3}{2}$ if $N = 2$ and $\gamma_n \geq \frac{9}{5}$ if $N = 3$. We wish to mention an additional estimate which is available but that, however, we shall not use in this proof. Indeed, the proof made in P.-L. Lions [6] (Chapter 7, section 7.1) yields the following bound for all $T \in (0, \infty)$

$$\int_0^T dt \int \rho_n^{\gamma_n + \theta_n} \leq C\gamma_n \quad \text{where } \theta_n = \frac{2}{N}\gamma_n - 1 . \tag{57}$$

Unfortunately, this estimate is not uniform in n. Instead, we shall use another estimate which can be derived as (57) was in [6] and which is uniform in n, namely an L^1 bound for $(\rho_n)^{\gamma_n}$.

Without loss of generality, extracting subsequences if necessary, we can assume that (ρ_n, u_n) converges weakly to (ρ, u). More precisely we can assume that $\rho_n \rightharpoonup \rho$ weakly in $L^p((0,T) \times \Omega)$ for any $1 \leq p \leq \infty$ and that $\rho \in L^\infty(0,T; L^p)$ (in fact we will show that ρ actually satisfies $0 \leq \rho \leq 1$), $u_n \rightharpoonup u$ weakly in $L^2(0,T; H^1_{loc})$.

THEOREM 4.1. – *Under the above conditions, we have* $0 \leq \rho \leq 1$ *and*

$$(\rho_n - 1)_+ \to 0 \quad \textit{in } L^\infty(0,T; L^p) \textit{ for any } 1 \leq p < +\infty.$$

Moreover, $(\rho_n)^{\gamma_n}$ *is bounded in* L^1 *(for* n *such that* $\gamma_n \geq N$*). Then extracting subsequences again, there exists* $\pi \in \mathcal{M}((0,T) \times \Omega)$ *such that*

$$(\rho_n)^{\gamma_n} \underset{n}{\rightharpoonup} \pi. \tag{58}$$

If in addition ρ_n^0 *converges in* L^1 *to* ρ^0 *then* (ρ, u, π) *is a solution of (1)-(4) and the following strong convergences hold*

$$\rho_n \to \rho \quad \textit{in} \quad C(0,T; L^p(\Omega)) \textit{ for any } 1 \leq p < +\infty$$

$$\rho_n u_n \to \rho u \quad \textit{in} \quad L^p(0,T; L^q(\Omega)) \textit{ for any } 1 \leq p < +\infty,\ 1 \leq q < 2$$

$$\rho_n u_n \otimes u_n \to \rho u \otimes u \quad \textit{in} \quad L^p(0,T; L^1(\Omega)) \textit{ for any } 1 \leq p < +\infty.$$

Remark 1. – The limit (ρ, u, π) satisfies in addition the following energy bounds which is to be compared with (17)

$$\begin{cases} E(t) + \displaystyle\int_0^t D(s)ds \leq E^0 + \liminf_n \frac{(\rho_n^0)^{\gamma_n}}{\gamma_n} & \text{a.e. } t, \\ \dfrac{d\,E}{dt} + D \leq 0 \quad \text{in } \mathcal{D}'(0,\infty) \end{cases} \tag{59}$$

where E and D are defined as in (17).

Remark 2. – Theorem 4.1 yields immediately the existence theorem existence. In fact we can take, for instance, for any $n > sup(\frac{N}{2}, 2)$, $\gamma_n = n$, $\rho_n^0 = \rho^0$ and $m_n^0 = m^0$, we see then that

$$\liminf_n \frac{(\rho_n^0)^{\gamma_n}}{\gamma_n} = 0$$

and hence (17) is satisfied.

Proof. – The proof is divided in three steps.

Step 1. – From the energy conservation and the mass conservation, we deduce that for any $1 < p < \infty$, we have for n such that $\gamma_n > p$,

$$\begin{aligned} ||\rho_n||_{L^\infty(0,T;L^p)} &\leq ||\rho_n||^{\theta_n}_{L^\infty(0,T;L^1)} ||\rho_n||^{1-\theta_n}_{L^\infty(0,T;L^{\gamma_n})} \\ &\leq M_n^{\theta_n} (C\gamma_n)^{(1-\theta_n)/\gamma_n} \end{aligned}$$

where we have used Holder's inequality and where θ_n is given for any n by $\frac{1}{p} = \theta_n + \frac{1-\theta_n}{\gamma_n}$. Then, letting n go to infinity, we deduce that $\theta_n \to \frac{1}{p}$ and that

$$||\rho||_{L^\infty(0,T;L^p)} \leq \liminf_{n\to\infty} ||\rho_n||_{L^\infty(0,T;L^p)} \leq M^{\frac{1}{p}}.$$

Hence, letting p go to infinity, we obtain

$$||\rho||_{L^\infty(0,T;L^\infty)} \leq \liminf_{p\to\infty} ||\rho||_{L^\infty(0,T;L^p)} \leq 1.$$

We next introduce $\phi_n = (\rho_n - 1)_+$. We are going to show that ϕ_n goes to 0 uniformly in t in all L^p spaces. In fact, from the energy conservation that we have for any t, we deduce

$$\int_\Omega (1+\phi_n)^{\gamma_n} 1_{\{\phi_n > 0\}} \leq \int_\Omega (\rho_n)^{\gamma_n} \leq C\gamma_n.$$

Next, for any $p > 1$, there exists a constant a (for instance we can take $a_p = \frac{1}{2p!}$ if p is an integer) such that for any k large enough the following inequality holds,

$$(1+x)^k \geq 1 + a_p\, k^p x^p,$$

for any nonnegative x. To see that we can consider the case where p is an integer and then make an "interpolation". In fact if $p \in I\!N$, we can take $f(x) = (1+x)^k - a_p\ k^p x^p$ and notice that a sufficient condition for f to be nonnegative on $\mathbb{R}^+$ is that $f^{(p)}(x) = k(k-1)...(k-p+1)(1+x)^{k-p} - a_p\ p!k^p$ is nonnegative in $\mathbb{R}^+$ and we can see that this holds true for k large enough if $a = \dfrac{1}{2p!}$ for instance. Next, if $p < q \leq p+1$ with $p \in I\!N$, we have $(1+x)^k \geq 1+a_p\ (kx)^p$ and $(1+x)^k \geq 1+a_{p+1}\ (kx)^{p+1}$ for k large enough. Then, we notice that for all $x \in \mathbb{R}^+$ we have $(kx)^q \leq sup((kx)^p, (kx)^{p+1})$, hence we can take $a_q = inf(a_p, a_{p+1}) = a_{p+1}$.

Hence, we have for n large enough

$$\int_\Omega \phi_n^p \leq \frac{C\ a_p}{(\gamma_n)^{p-1}},$$

which yields the convergence of $(\rho_n - 1)_+$ to 0. Let us notice that this convergence is obviously "very fast".

Step 2. – Now, we turn to the proof of the L^1 bound on $(\rho_n)^{\gamma_n}$. We begin by treating the whole space case and then explain the necessary modifications we in the periodic case. Applying as in [6], the operator $(-\Delta)^{-1}\ div$ to (54), we obtain

(60)

$$(\rho_n)^{\gamma_n} = \partial_t(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) - R_i R_j(\rho(u_n)_i(u_n)_j) + (\mu+\xi)div(u_n).$$

Then multiplying (60) by ρ_n, we deduce (we omit the indices n in the right hand side for the sake of clarity),

$$\begin{cases} (\rho_n)^{\gamma_n+1} = \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] + \mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ \qquad\qquad +\rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j) + (\mu+\xi)\rho div(u), \end{cases}$$

This manipulation and essentially the multiplication of ρ_n by $\partial_t(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$ should be justified. This can be justified in the same way as for the product of ρ by π. Then, integrating (61) over $(0,T) \times \mathbb{R}^N$, we see that $(\rho_n)^{\gamma_n+1}$ is bounded in $L^1((0,T) \times \mathbb{R}^N)$, uniformly in n. In fact, the proof is the same as the proof of the bound of π in $\mathcal{M}((0,T) \times \mathbb{R}^N)$, so we are only going to mention the changes that must be performed. We also refer to [6] (Chapter 7, section 7.1) for the proof of (57). First, we notice that we do not have a L^∞ bound on ρ_n. However, since $\int(\rho_n)^{\gamma_n}(t) \leq C\gamma_n$ for a.e. t, we see that there exists

a constant C_1 $\left(\text{for instance } C_1 = exp\left(\frac{C}{e}\right)\right)$, such that for any n, we have $||\rho_n||_{L^\infty(0,T;L^{\gamma_n})} \leq C_1$, since

$$\sup_{\gamma>0}(C\gamma)^{1/\gamma} = exp\left(\frac{C}{e}\right).$$

Next, we remark that the norm of $\rho_n u_n$ in $L^\infty(0,T;L^{2\gamma_n/(\gamma_n+1)} \cap L^1)$ is independent of n (we use here the fact that the norm of $\sqrt{\rho_n}$ in $L^\infty(L^{2\gamma_n} \cap L^2)$ is bounded independently of n and that $\sqrt{\rho_n}u_n$ is bounded in $L^\infty(L^2)$). Hence, we may write

$$\left|\int_0^T \int_{\mathbb{R}^N} \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]\right| \leq 2||\rho||_{L^\infty(L^{q'})}||\rho u||_{L^\infty(L^r)},$$

with $\left(\frac{1}{r} - \frac{1}{N}\right) + \frac{1}{q'} = 1$, $1 \leq r \leq \frac{2\gamma_n}{\gamma_n+1}$ and $1 \leq q' \leq \gamma_n$ (such a choice is possible if $\gamma_n \geq N$ for instance).

For the second term, we only need to show that $U = \rho_n u_n(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$ belongs to $L^\infty(L^1)$ for all n, which can be deduced from the following bound

$$||U||_{L^\infty(L^1)} \leq ||\rho_n u_n||^2_{L^\infty(L^q)},$$

with $q = \frac{2N}{N+1} \leq \frac{2\gamma_n}{\gamma_n+1}$, since $\gamma_n \geq N$.

For the third and the fourth term, we must distinguish two cases, namely $N \geq 3$ and $N = 2$. In the first case, we have $\rho|u|^2 \in L^1(L^s)$, with $\frac{1}{s} = \frac{N-2}{N} + \frac{1}{\gamma_n} < 1$. Therefore $R_iR_j(\rho u_i u_j) \in L^1(L^s)$ and hence $\rho R_iR_j(\rho u_i u_j) \in L^1(L^1)$, since $\frac{N-2}{N} + 2\frac{1}{\gamma_n} \leq 1$. For the third term we recall the fact that $\rho_n u_n, R_iR_j(\rho(u_n)_i(u_n)_j) \in L^2(L^r \cap L^1)$, where $\frac{1}{r} = \frac{N-2}{2N} + \frac{1}{\gamma_n} \geq 2$. The case $N = 2$ is treated using the bound on u_n in $L^2(0,T;BMO)$ and on $\rho_n u_n$ in $L^2(L^{3/2})$ (if $\gamma_n \geq 3$), which yield

$$||[u, R_iR_j](\rho u_j)||_{L^1(0,T;L^{3/2})} \leq C||u||_{L^2(0,T;BMO)}||\rho u||_{L^2(0,T;L^{3/2})}$$

and, multiplying by ρ, we deduce

$$\begin{aligned}&||\rho[u, R_iR_j](\rho u_j)||_{L^1(0,T;L^1)}\\ &\quad\leq C||\rho||_{L^\infty(0,T;L^3)}||u||_{L^2(0,T;BMO)}||\rho u||_{L^2(0,T;L^{3/2})}.\end{aligned}$$

Finally for the fifth term, we have the following straightforward computation

$$\left|\int_0^T \int_{\mathbb{R}^N} \rho_n \; div \; u_n\right| \leq ||\rho_n||_{L^2(0,T;L^2)} ||div \; u_n||_{L^2(0,T;L^2)}. \tag{61}$$

where we use the fact that ρ_n is bounded in $L^2(0,T;L^2)$ since $\gamma_n \geq N \geq 2$.

Using this bound on $(\rho_n)^{\gamma_n+1}$ and the fact that $\rho \in L^\infty(0,T;L^1)$, we deduce the desired bound

$$\int_0^T \int_{\mathbb{R}^N} (\rho_n)^{\gamma_n} \leq \int_0^T \int_{\mathbb{R}^N} (\rho_n)^{\gamma_n+1} + \rho_n.$$

Hence extracting subsequences again, there exists $\pi \in \mathcal{M}((0,T) \times \Omega)$ such that

$$(\rho_n)^{\gamma_n} \underset{n}{\rightharpoonup} \pi. \tag{62}$$

Step 3. – Finally, we show that (ρ, u, π) is actually a solution of the initial system and that the strong convergences hold. First, we observe that $\dfrac{\partial \rho_n}{\partial t}$ is bounded in $L^\infty(0,T;W^{-1,1})$ and that u_n is bounded in $L^2(0,T;H^1)$. Then, using the compactness lemma 3.3, we get that $\rho_n u_n$ converges weakly to ρu. On the other hand, using the bound on $(\rho_n)^{\gamma_n}$ in L^1, we deduce that $\dfrac{\partial \rho_n u_n}{\partial t}$ is bounded in $L^1(0,T;W^{-1,1})$ and hence $\rho_n u_n \otimes u_n \rightharpoonup \rho u \otimes u$. The only point that should be proved is the relation $\rho\pi = \pi$. Using the same notations as in the previous section ($s = \rho \; log(\rho)$ and $\overline{s} = \overline{\rho} \; log(\rho)$), we get

$$\frac{\partial}{\partial t}(\overline{s} - s) + \mathrm{div}[(\overline{s} - s)u] = -\overline{\rho \; \mathrm{div} u} + \rho \; \mathrm{div} u. \tag{63}$$

Then applying the operator $(-\Delta)^{-1}\mathrm{div}$ to the momentum equation, multiplying by ρ_n and passing to the limit (extracting subsequences if necessary), we obtain

$$\begin{cases} \overline{(\rho_n)^{\gamma_n+1}} - (\mu+\xi)\overline{\rho div(u)} = \partial_t \overline{[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]} \\ \qquad\qquad + \mathrm{div}\overline{[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)]} \\ \qquad\qquad + \overline{\rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j)} \end{cases} \tag{64}$$

Changing the order of the multiplication by ρ and the passage to the weak limit, we obtain

$$\begin{cases} \rho\pi - (\mu+\xi)\rho div(u) = \partial_t[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ \qquad\qquad + \mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ \qquad\qquad + \rho u_i R_i R_j(\rho u_j) - \rho R_i R_j(\rho u_i u_j). \end{cases} \tag{65}$$

The same computations as in the last section show that the second hand sides of (64) and (65) are equal. Hence, we get

$$-\overline{\rho div(u)} + \rho div(u) = \frac{1}{\mu+\xi}\Big[\rho\pi - \overline{(\rho_n)^{\gamma_n+1}}\Big].$$

Reporting this in (63), we get

$$\frac{\partial}{\partial t}(\overline{s}-s) + \operatorname{div}[(\overline{s}-s)u] = \frac{1}{\mu+\xi}\Big[\rho\pi - \overline{(\rho_n)^{\gamma_n+1}}\Big]. \tag{66}$$

Next, we notice that $\rho\pi = \overline{\rho(\rho_n)^{\gamma_n}} \leq \overline{(\rho_n)^{\gamma_n+1}}$. Indeed we have

$$\begin{aligned}\overline{(\rho_n)^{\gamma_n+1}} - \overline{\rho_n}\,\overline{(\rho_n)^{\gamma_n}} &= \overline{((\rho_n)^{\gamma_n})(\rho_n-\rho)} \\ &= \overline{\Big((\rho_n)^{\gamma_n} - (\rho)^{\gamma_n}\Big)\Big(\rho_n-\rho\Big)} \\ &\geq 0\end{aligned}$$

where we have used that

$$(\rho)^{\gamma_n} \to 1_{\{\rho=1\}}$$

almost everywhere and in $L^p((0,T)\times\Omega)$ for $1 \leq p < \infty$ which yields the following weak convergence

$$(\rho)^{\gamma_n}\Big(\rho_n - \rho\Big) \rightharpoonup 0$$

Next, integrating (66) in x, we get

$$\frac{\partial}{\partial t}\int_\Omega (\overline{s}-s)\,dx \leq 0.$$

Then, since $\overline{s} - s|_{t=0} = 0$ and $s \leq \overline{s}$, we see that $s = \overline{s}$. Therefore, we obtain that

$$\rho\pi = \overline{(\rho_n)^{\gamma_n+1}}$$

Next, we see that for any $\varepsilon > 0$ there exists n_0 such that for $n \geq n_0$ and $x \geq 0$, we have

$$x^{\gamma_n+1} \geq x^{\gamma_n} - \varepsilon.$$

Applying this inequality to ρ_n and passing to the weak limit, we get

$$\overline{(\rho_n)^{\gamma_n+1}} \geq \pi - \varepsilon.$$

Then, letting ε go to 0, we get

$$\rho\pi \geq \pi.$$

Next, using that $0 \leq \rho \leq 1$, we obtain

$$\rho\pi \leq \rho.$$

However, since the product $\rho\pi$ is not defined almost everywhere, we must explain the above inequality. We denote by $\omega_k = k^{N+1}\omega(k.)$ a smooothing sequence in both variables t and x, where $\omega \in C^\infty(\mathbb{R}^{N+1})$, $\omega \geq 0$, $\fint_{\mathbb{R}^{N+1}} \omega = 1$, $\mathrm{Supp}(\omega) \in B_1(\mathbb{R}^{N+1})$. Then, we denote by $\rho_k = \rho * \omega_k$ (resp $\pi_k = \pi * \omega_k$) a sequence of nonnegatif smooth functions converging to ρ (resp π)

$$\begin{cases} \rho_k \to \rho & \text{in } C([0,T]; L^p) \cap C^1([0,T]; H^{-1}), \\ \pi_k \to \pi & \text{in } W^{-1,2}(H^1) + L^1(L^q), \end{cases} \tag{67}$$

for some $q > 1$ and p such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Hence writing $(\rho - 1)\pi$ as

$$(\rho - 1)\pi = (\rho_k - 1)\pi_k + (\rho - \rho_k)\pi_k + (\rho - 1)(\pi - \pi_k)$$

we conclude by letting k go to the infinity.

Finally, we deduce that

$$\rho\pi = \rho.$$

Therefore, (ρ, u, π) is a solution of (1)-(4). The strong convergences stated in the theorem are then deduced easily as in the proof of the compactness theorem.

5. GENERAL PRESSURE LAW AND INFINITE MASS

In this section, we discuss two related issues. The first one is the case where we include a pressure ($\pi = p(\rho)$ on $\{\rho < 1\}$), the fluid is assumed to behave as a general barotropic fluid as long as its density is smaller than 1. We also study the case where we no longer assume that the total mass is finite in the case of $\mathbb{R}^N$.

We begin with the case of a general pressure law fluid. The pressure law p is assumed to be a continuous nondecreasing function on $[0,\infty)$ vanishing at 0. For instance, we can take $p(\rho) = a\rho^\gamma$. Next, let q be defined (up to a linear function) by $\frac{d}{dt}\left(\frac{q(t)}{t}\right) = \frac{p(t)}{t^2}$ for $t > 0$. Notice then that multiplying the moment equation by u, the term $\pi \operatorname{div} u$ does not vanish and is equal to $p(\rho) \operatorname{div} u$, since $\operatorname{div} u = 0$ on $\{\rho = 1\}$. Then, using (1), we get taking $\beta(\rho) = q(\rho)$ in (21)

$$\frac{\partial q(\rho)}{\partial t} + \operatorname{div}(q(\rho)u) = -p(\rho) \operatorname{div} u, \tag{68}$$

since $q(\rho) - \rho q'(\rho) = -p(\rho)$. This equation need some justifications if $q(t)$ is not C^1 ($q \in C^1([0,\infty))$ if and only if $\int_0^1 \frac{p(t)}{t^2} < +\infty$) and we only have to approximate q by $q_\delta(\rho) = \rho \int_c^{\rho+\delta} \frac{p(t)}{t^2}$... Now, integrating (27), we obtain (at least formally)

$$\frac{\partial}{\partial t}\int_\Omega \left[\rho\frac{|u|^2}{2} + q(\rho)\right] + \int_\Omega \mu(Du)^2 + \xi(\operatorname{div} u)^2 = \int_\Omega \rho u.f. \tag{69}$$

Next since q is bounded from below, we see that we deduce the same a priori bounds in the periodic case. We also notice that, in this case, we can make the computations directly on the term $p(\rho)\operatorname{div} u$, since $p(\rho)$ is bounded and that $||p(\rho)\operatorname{div} u||_{L^1} \leq \frac{C}{\varepsilon} + \varepsilon||\operatorname{div} u||_{L^2}$, for any $\varepsilon > 0$.

In the whole space case, this bound from below is not sufficient to derive the desired bounds. We thus begin by the case when we can take q such that $q \geq 0$. In fact if $\int \frac{p(t)}{t^2} < +\infty$ then we can take $q(\rho) = \rho \int_0^\rho \frac{p(t)}{t^2}$ and hence $q(\rho) \geq 0$. We then obtain (we omit here the force term f)

$$\int_\Omega \rho\frac{|u|^2}{2}(T) + \int_0^T \int_\Omega \mu(Du)^2 + \xi(\operatorname{div} u)^2 = \int_\Omega \rho\frac{|u|^2}{2}(0) + q(\rho^0) \tag{70}$$

and the same bounds follow. However if $\int \frac{p(t)}{t^2} = +\infty$, we can no longer take q such that $q \geq 0$ and hence we cannot obtain such bounds directly. For instance, if $p(\rho) = a\rho^\gamma$ with $0 < \gamma < 1$, then $q(\rho) = -\frac{a}{1-\gamma}\rho^\gamma + C\rho$ and we can check easily that

$$\sup_{\int \rho = M} \rho^\gamma = +\infty.$$

One idea to recover some bounds is to introduce (as in P.-L. Lions [6]) a reference function $\overline{\rho}$, such that $0 \leq \overline{\rho} \leq 1$, $\overline{\rho} \in L^1$ and $\nabla(q'(\overline{\rho})) \in L^\infty + L^2(\mathbb{R}^N)$. Another case, where we can recover bounds is the case when the force f is potential, namely $f = -\nabla V$ for some V. In fact if the force is attractive enough we see that the fluid cannot disperse widely. We study below a case where we can recover some bounds, we assume that $V = V_+ - V_-$, with $V_+,\ V_- \ \geq 0$ and

$$(71) \qquad \begin{cases} p(\rho) = a\rho^\gamma \quad \text{with} \quad 0 < \gamma < 1, \\ V_- \in L^1 + L^\infty, \\ \displaystyle\int_{\mathbb{R}^N} (V_+ + 1)^{-\frac{\gamma}{1-\gamma}} < +\infty \\ f = -\nabla\ V + f', \quad \text{where } f' \text{ satisfies (19).} \end{cases}$$

In this case the energy estimates yield

$$(72) \quad \frac{\partial}{\partial t}\int_\Omega \left[\rho\frac{|u|^2}{2} + q(\rho) + \rho V\right] + \int_\Omega \mu(Du)^2 + \xi(\mathrm{div} u)^2 = \int_\Omega \rho u.f'.$$

To recover some bounds, we must show that

$$\inf_{\int \rho = M,\ 0\leq\rho\leq 1} \int \rho V + q(\rho) > -\infty.$$

In fact, we have

$$\int \rho V_- \leq (1+M)||V_-||_{L^1+L^\infty}$$

and the infimum of $\rho V_+ - \frac{a}{1-\gamma}\rho^\gamma$ under the constraint $\int \rho = M$ is reached for

$$\overline{\rho} = \left(\frac{1-\gamma}{a}V_+ + C\right)^{-\frac{1}{1-\gamma}}$$

where C is such that $\int \overline{\rho} = M$. Hence, for almost all t, we have

$$\int \rho V_+ + q(\rho) \geq \int \overline{\rho} V_+ + q(\overline{\rho}) \geq -\infty$$

and we recover the desired bounds as above.

We next turn to the study of another topic, namely the case of infinite mass in the whole space case. We require ρ^0 to satisfy the following conditions

$$(73) \qquad \begin{cases} \rho - \overline{\rho} \in L^2(\mathbb{R}^N) \quad \text{where } \overline{\rho} \text{ satisfies} \\ 0 \leq \overline{\rho} \leq 1, \quad meas\{\overline{\rho} = 1\} \text{ is finite,} \\ \nabla(q'(\overline{\rho})) \in L^\infty + L^2(\mathbb{R}^N) \quad \text{and} \quad \mathrm{q}''(\overline{\rho}) \geq \mathrm{c}, \end{cases}$$

for some constant $c > 0$. Notice that we impose no integrability conditions on $\overline{\rho}$. In this case the energy estimates are the same as in the case where we only include a pressure. In fact, since ρ and $\overline{\rho}$ are bounded and $q''(\overline{\rho}) \geq c$, we can find two constants c_1 and c_2, such that we have

$$c_1(\rho - \overline{\rho})^2 \geq q(\rho) - q(\overline{\rho}) - q'(\overline{\rho})(\rho - \overline{\rho}) \geq c_2(\rho - \overline{\rho})^2.$$

We begin our investigations by the following simpler case ($\overline{\rho}$ is a constant such that $0 \leq \overline{\rho} < 1$). The energy estimates then yields the following inequality

(74)

$$\int_\Omega \rho \frac{|u|^2}{2} + c_2(\rho - \overline{\rho})^2(T) + \int_0^T \int_\Omega \nu(Du)^2 \leq \int_\Omega \rho \frac{|u|^2}{2}(0) + c_1(\rho^0 - \overline{\rho})^2$$

Next in order to deduce a bound for π in L^1, we multiply (37) by $\rho - \overline{\rho}$, and we obtain

$$\begin{aligned}(1 - \overline{\rho})\pi &= \partial_t[(\rho - \overline{\rho})(-\Delta)^{-1}\mathrm{div}(\rho u)] + \mathrm{div}[\rho u(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ &\quad + \rho u_i R_i R_j(\rho u_j) - (\rho - \overline{\rho})R_i R_j(\rho u_i u_j) + (\mu + \xi)(\rho - \overline{\rho})div(u).\end{aligned}$$

Then we deduce a bound on π in L^1 integrating this equality. In fact, all the terms in the right-hand side are treated as in section 2, replacing ρ by $\rho - \overline{\rho}$ (recall that $\rho - \overline{\rho} \in L^\infty(0, T; L^2(\mathbb{R}^N))$). The case $N = 2$ does not create further difficulty and we treat the term $(\rho - \overline{\rho})R_i R_j(\rho u_i u_j)$ as follows

$$(\rho - \overline{\rho})R_i R_j(\rho u_i u_j) = (\rho - \overline{\rho})[R_i R_j, u_i](\rho u_j) + (\rho - \overline{\rho})u_i R_i R_j(\rho u_j).$$

In the general case, namely when $\overline{\rho}$ is no longer assumed to be a constant, the energy estimates yield the same bounds as above. In fact, for the extra term, we have

$$\left| \int \rho u \nabla(q'(\overline{\rho})) \right| \leq ||\rho u||_{L^1 \cap L^2} ||\nabla(q'(\overline{\rho}))||_{L^\infty + L^2}.$$

Next, in order to obtain the estimate on π, we use the existence of an $\alpha > 0$ such that $meas\{\overline{\rho} > 1 - \alpha\}$ is finite. In fact

$$\cap_{\alpha > 0}\{\overline{\rho} > 1 - \alpha\} = \{\overline{\rho} = 1\}.$$

Using this α, we see that

$$\overline{\rho} - inf(\overline{\rho}, 1 - \alpha) \in L^2,$$

then multiplying (37) by $\rho - inf(\overline{\rho}, 1 - \alpha)$ instead of $\rho - \overline{\rho}$, we deduce

$$\alpha \int_{\mathbb{R}^N} \pi \leq \int [1 - inf(\overline{\rho}, 1 - \alpha)]\pi,$$

which yields the same L^1 bound since $\rho - inf(\overline{\rho}, 1 - \alpha) \in L^\infty(L^2)$.

6. CONVERGENCE TO THE INCOMPRESSIBLE NAVIER-STOKES SYSTEM

In this section, we discuss two results concerning the convergence to the incompressible Navier-Stokes system in the case of two space dimensions (N=2). These two results are still true, for $N \geq 3$, on any interval of time $(0, T)$ for which we know that there exists a strong solution u (which is unique) for the incompressible Navier-Stokes equation.

The first issue concerns the limit $M \to 1$ (in the periodic case). Let M_n be a sequence of real numbers ($0 < M_n < 1$) converging to 1 and (ρ_n, u_n, π_n) a sequence of solutions of (1) - (4) in $\mathbf{T}^2$ satisfying the energy inequality. Let us explain the heuristics which lead to the incompressible Navier-Stokes equation. Indeed since $\rho_n \leq 1$ and $\fint \rho_n(t) = M_n \to 1$ we deduce that $\rho \to 1$. Next, we conclude from (1) that the weak limit of u_n satisfies the divergence-free condition : $div\ u = 0$. Hence, we expect that (ρ_n, u_n, π_n) converges, in some sense, to $(1, u, \pi)$, where (u, π) is the weak solution of the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla \pi = 0. \tag{75}$$

We shall show that this heuristic derivation is basically correct if we impose in addition some extra conditions on the initial data. We assume that (ρ_n, u_n, π_n) satisfy the following initial conditions

$$\rho_n u_n \Big|_{t=0} = m_n^0 \ , \ \rho_n \Big|_{t=0} = \rho_n^0, \tag{76}$$

where $0 \leq \rho_n^0 \leq 1$ a.e., $\fint \rho_n^0 = M_n$, and $\sqrt{\rho_n^0} u_n^0$ converges strongly in L^2 to a divergence-free vector u^0.

THEOREM 6.1. – *Under the above assumptions, ρ_n converges to 1 in $C([0,T]; L^p(\Omega))$ for $1 \leq p < +\infty$, $\sqrt{\rho_n}\ u_n$ converges to u in $L^\infty(0,T; L^2(\Omega))$ and Du_n converges to Du in $L^2(0,T; L^2(\Omega))$ for all $T \in (0, \infty)$ where u is the unique solution of the incompressible Navier-Stokes equations corresponding to the initial condition u^0.*

Proof. – The proof of this theorem is similar to a proof introduced in [11]. First it is easy to see that ρ_n converges to 1 in $C([0,T]; L^p)$ (for $1 \leq p < +\infty$). Next, we can assume, extracting subsequences if necessary, that u_n converges weakly in $L^2(0,T; H^1(\Omega))$ to some v. Then, applying lemma 3.3, we deduce that $\sqrt{\rho_n} u_n$ and $\rho_n u_n$ converge weakly-star to v in $L^\infty(0,T; L^2(\Omega))$ and hence v is divergence free. At this point, we cannot

deduce, as before, the weak convergence of $\rho_n u_n \otimes u_n$ to $v \otimes v$ since we have not a uniform bound on π_n and hence no compactness in time for $\rho_n u_n$.

Let $u_0^\delta \in C_0^\infty(\Omega)$ be such that : $\mathrm{div} u_0^\delta = 0$ in Ω, $u_0^\delta \to u^0$ in $L^2(\Omega)$ as δ goes to 0_+. We denote by u^δ the solution of the incompressible Navier-Stokes equations corresponding to the initial condition u_0^δ. As is well-known, u^δ is smooth on $([0,\infty) \times \Omega)$ and u^δ converges to u in $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ (and thus in $L^4(\Omega \times (0,T))$ for all $T \in (0,\infty)$. The energy inequality for u_n yields for almost all t

$$\int_\Omega \frac{1}{2}\rho_n |u_n|^2(t) + \int_0^t \int_\Omega \mu |Du_n|^2 + \xi(div\ u_n)^2 \leq \int_\Omega \frac{1}{2}\rho_n^0 |u_n^0|^2. \tag{77}$$

Then using the conservation of energy for u^δ, we get for all t

$$\int_\Omega \frac{1}{2}|u^\delta|^2(t) + \int_0^t \int_\Omega \mu |Du^\delta|^2(s)ds = \int_\Omega \frac{1}{2}|u_0^\delta|^2 \tag{78}$$

In addition, since u^δ is divergence-free and $P(\rho_n u_n)$ is continuous in time, we get for all t

$$\begin{cases} \displaystyle\int_\Omega \rho_n u_n . u^\delta(t) - \int_0^t \int_\Omega \rho_n u_n . \partial_t u^\delta + \rho_n u_n u_n . \nabla u^\delta + \\ \displaystyle +2\mu \int_0^t \int_\Omega \nabla u_n . \nabla u^\delta + \mu \int_0^t \int_\Omega u_n \Delta u^\delta = \int_\Omega \rho_n^0 u_n^0 . u_0^\delta. \end{cases} \tag{79}$$

Combining (77), (78) and (79), we get

$$\begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\sqrt{\rho_n}u_n - u^\delta|^2(t) + \int_0^t \int_\Omega \mu |D(u_n - u^\delta)|^2 + \xi |\mathrm{div}(u_n - u^\delta)|^2 \leq \\ \displaystyle + \int_\Omega (\sqrt{\rho_n} - 1)\sqrt{\rho_n} u_n . u^\delta(t) + \\ \displaystyle - \int_0^t \int_\Omega \rho_n u_n [\partial_t u^\delta + u^\delta . \nabla u^\delta - \mu \Delta u^\delta] - \mu \int_0^t \int_\Omega (\rho_n - 1) u_n . \Delta u^\delta + \\ \displaystyle + \int_0^t \int_\Omega \Big[(u_n - u^\delta) + u^\delta + (\rho_n - 1)u_n\Big][(u_n - u^\delta).\nabla u^\delta] + \\ \displaystyle + \int_\Omega \frac{1}{2}|\sqrt{\rho_n^0}u_n^0 - u_0^\delta|^2 - \int_\Omega (\sqrt{\rho_n^0} - 1)\sqrt{\rho_n^0} u_n^0 . u_0^\delta \end{cases} \tag{80}$$

Using the Gagliardo-Nirenberg inequality, we get for almost all t

$$\begin{aligned} &\left| \int_\Omega (u_n - u^\delta)[(u_n - u^\delta).\nabla u^\delta] \right| \leq \\ &\leq ||u_n - u^\delta||_{L^4}^2 ||\nabla u^\delta||_{L^2} \\ &\leq ||u_n - u^\delta||_{L^2} ||D(u_n - u^\delta)||_{L^2} ||\nabla u^\delta||_{L^2} \\ &\leq \Big[||\sqrt{\rho_n}u_n - u^\delta||_{L^2} + ||(\sqrt{\rho_n} - 1)u_n||_{L^2} \Big] ||D(u_n - u^\delta)||_{L^2} ||\nabla u^\delta||_{L^2} \end{aligned}$$

and then, by the Cauchy-Schwarz inequality, we get

$$||\sqrt{\rho_n}u_n - u^\delta||_{L^2}||D(u_n - u^\delta)||_{L^2}||\nabla u^\delta||_{L^2} \leq$$
$$\leq \frac{\nu}{2}||D(u_n - u^\delta)||^2_{L^2} + \frac{1}{2\nu}||\sqrt{\rho_n}u_n - u^\delta||^2_{L^2}||\nabla u^\delta||^2_{L^2}$$

where $\nu = inf(\mu, \mu + \xi)$.

Next, let $A^\delta_n(t)$ be defined by

$$(81)\begin{cases} A^\delta_n(t) = \Bigg| \int_\Omega (\sqrt{\rho_n} - 1)\sqrt{\rho_n}u_n.u^\delta(t) + \\ - \int_0^t \int_\Omega \rho_n u_n[\partial_t u^\delta + u^\delta.\nabla u^\delta - \mu\Delta u^\delta] - \mu \int_0^t \int_\Omega (\rho_n - 1)u_n.\Delta u^\delta + \\ + \int_0^t \int_\Omega \Big[u^\delta + (\rho_n - 1)u_n\Big][(u_n - u^\delta).\nabla u^\delta] - \int_\Omega (\sqrt{\rho^0_n} - 1)\sqrt{\rho^0_n}u^0_n.u^\delta_0 + \\ + ||(\sqrt{\rho_n} - 1)u_n||_{L^2}||D(u_n - u^\delta)||_{L^2}||\nabla u^\delta||_{L^2}\Bigg| \end{cases}$$

Then, we see easily that for all δ, $A^\delta_n(t)$ converges to 0 for almost all t, uniformly in t and that $\int A^\delta_n(t)\ dt$ goes to 0 when n goes to ∞. In deed for the terms containing the factor $(\rho_n - 1)$ or $(\sqrt{\rho_n} - 1)$, we use that

$$||\sqrt{\rho_n} - 1||_{L^\infty(0,T;L^p)} \leq ||\rho_n - 1||_{L^\infty(0,T;L^p)} \underset{n}{\rightarrow} 0$$

and for the two others, we use the weak convergence of $\rho_n u_n$ to v which is divergence-free and the weak convergence of $u_n - u^\delta$ to $v - u^\delta$ and then notice that

$$\int_0^t \int_\Omega v[\partial_t u^\delta + u^\delta.\nabla u^\delta - \mu\Delta u^\delta] = \int_0^t \int_\Omega v.\nabla\pi = 0$$

$$\int_0^t \int_\Omega u^\delta[(v - u^\delta).\nabla u^\delta] = -\int_0^t \int_\Omega \frac{|u^\delta|^2}{2} div(v - u^\delta) = 0.$$

The fact that the convergence is uniform in time for these two terms can be deduced from the following elementray lemma

LEMMA 6.2. – *If f_n is bounded in $L^\alpha(0,T)$ with $\alpha > 1$ and f_n converges weakly to 0, then*

$$\int_0^t f_n \rightarrow 0$$

uniformly in t.

Hence, (80) yields for almost all t

$$(82)\quad \begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\sqrt{\rho_n}u_n - u^\delta|^2(t) + \int_0^t \int_\Omega \frac{\nu}{2}|D(u_n - u^\delta)|^2 \leq \\ \displaystyle A_n^\delta(t) + \int_\Omega \frac{1}{2}|\sqrt{\rho_n^0}u_n^0 - u_0^\delta|^2 + C\int_0^t \int_\Omega ||\sqrt{\rho_n}u_n - u^\delta||_{L^2}^2 ||\nabla u^\delta||_{L^2}^2. \end{cases}$$

By Grönwall's inequality, we deduce that we have for almost all $t \in (0, \infty)$

$$(83)\quad \begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\sqrt{\rho_n}u_n - u^\delta|^2(t) \leq \\ \displaystyle \leq \Big[\int_\Omega \frac{1}{2}|\sqrt{\rho_n^0}u_n^0 - u_0^\delta|^2 + \sup_{0\leq s\leq T} A_n^\delta(s)\Big] e^{C\int_0^t ||\nabla u^\delta(s)||_{L^2}^2} \end{cases}$$

Then, letting n go to infinity, we obtain

$$\begin{aligned} \Big\| v - u^\delta(t)\Big\|_{L^\infty(0,T;L^2(\Omega)} &\leq \overline{\lim_n} \Big\|\sqrt{\rho_n}u_n - u^\delta\Big\|_{L^\infty(0,T;L^2(\Omega)} \\ &\leq C_0 \Big\|u^0 - u_0^\delta\Big\|_{L^2(\Omega)} \end{aligned}$$

where

$$C_0 = \sup_{0<\delta<1} e^{C\int_0^T ||\nabla u^\delta(s)||_{L^2}^2} < +\infty.$$

Then, letting δ go to infinity, we recover the uniform convergence in t of $\sqrt{\rho_n}u_n$ to u, since

$$\begin{aligned} &\overline{\lim_n} \Big\|\sqrt{\rho_n}u_n - u\Big\|_{L^\infty(0,T;L^2)} \\ &\leq \lim_\delta \Big[C_0 \Big\|u^0 - u_0^\delta\Big\|_{L^2(\Omega)} + \Big\|u^\delta - u\Big\|_{L^\infty(0,T;L^2)}\Big] = 0 \end{aligned}$$

Going back to (82), we get that Du_n converges in L^2 to Du and that u_n converges to u in $L^2(0,T;H^1(\Omega)$. Finally, we also get that $\nabla\pi_n$ converges weakly to $\nabla\pi$ in $H^{-1}((0,T)\times\Omega)$ for instance.

Remark. – It is worth noticing that the above proof shows that, for $N \geq 2$, the result is still true on any interval of time $(0,T)$ for which we know that there exists a solution u (which is in fact unique) of the incompressible Navier-Stokes equations corresponding to u^0 which satisfies :

$$Du \in L^1(0,T;L^\infty(\Omega)) + L^2(0,T;L^N(\Omega)).$$

The second issue, we are going to study concerns the convergence of solutions of the compressible Navier-Stokes system (54) as γ_n goes to infinity in the case where $\fint \rho_n^0 = M \geq 1$. Let (ρ_n, u_n) be a sequence of solutions of (54) satisfying in addition (55) and (56) and such that $\fint \rho_n^0 = M \geq 1$, $\fint (\rho_n^0)^{\gamma_n} \leq M^{\gamma_n} + C_n \gamma_n$ for some constant C_n, such that C_n converges to 0 and $\sqrt{\rho_n^0} u_n^0$ converges strongly in L^2 to a divergence-free vector u^0.

THEOREM 6.3. – *Under the above assumptions, ρ_n converges to M in $C([0,T]; L^p(\Omega))$ for $1 \leq p < +\infty$, $\sqrt{\rho_n}\ u_n$ converges to u in $L^\infty(0,T; L^2(\Omega))$ and Du_n converges to Du in $L^2(0,T; L^2(\Omega))$ for all $T \in (0,\infty)$ where u is the unique solution of the incompressible Navier-Stokes equations corresponding to the initial condition u^0 and where the viscosity μ is replaced by μ/M.*

Proof. – The proof of this theorem is the same as the previous one. The only point we must show is how we obtain uniform bounds. In fact the energy inequality (56) reads for almost all t

$$(84)\quad \begin{cases} \displaystyle\int \frac{1}{2}\rho_n |u_n|^2(t) + \frac{a}{\gamma_n - 1}(\rho_n)^{\gamma_n}(t) + \int_0^t \int \mu |Du_n|^2 + \xi (div u_n)^2 \leq \\ \displaystyle\leq \int \frac{1}{2}\rho_n^0 |u_n^0|^2 + \frac{a}{\gamma_n - 1}(\rho_n^0)^{\gamma_n} \leq \int \frac{1}{2}\rho_n^0 |u_n^0|^2 + \frac{a}{\gamma_n - 1}|\Omega| M^{\gamma_n} + C \end{cases}$$

Hence, to get uniform bounds, we notice that we deduce from Jensen's inequality that we have for almost all t

$$\int_\Omega (\rho_n)^{\gamma_n}(t) \geq |\Omega| \Big[\int_\Omega \rho_n\Big]^{\gamma_n}(t) = |\Omega| M^{\gamma_n}.$$

Arguing as above, we can extract subsequences that converge weakly. Next, to see that $\rho \equiv M$ and that ρ_n converges strongly to M we use that for all p $(1 < p < +\infty)$ and for n large enough $(\gamma_n > p)$, we have

$$\begin{aligned} ||\rho_n||_{L^\infty(0,T;L^p)} &\leq ||\rho_n||^{\theta_n}_{L^\infty(0,T;L^1)} ||\rho_n||^{1-\theta_n}_{L^\infty(0,T;L^{\gamma_n})} \\ &\leq M_n^{\theta_n} (C\gamma_n + |\Omega| M^{\gamma_n})^{(1-\theta_n)/\gamma_n} \end{aligned}$$

where θ_n is given for any n by $\dfrac{1}{p} = \theta_n + \dfrac{1-\theta_n}{\gamma_n}$. Then, letting n go to infinity, we deduce that

$$||\rho||_{L^\infty(0,T;L^p)} \leq \liminf_{n\to\infty} ||\rho_n||_{L^\infty(0,T;L^p)} \leq M.$$

Then, since $\fint \rho(t) = M$ for all t, we get, using the Jensen inequality (or letting p go to infinity) that $\rho \equiv M$. We also deduce the convergence of ρ_n to M in $L^p(0,T;L^p)$. Next, to get the convergence in $L^\infty(0,T;L^p)$, we write that

$$\rho_n^{\gamma_n} \geq M^{\gamma_n}\left[1 + \gamma_n \frac{(\rho_n - M)_+}{M} + \frac{\gamma_n(\gamma_n - 1)}{2}\left(\frac{(\rho_n - M)_+}{M}\right)^2\right]. \tag{85}$$

and deduce easily from the bounds we have on $\rho_n^{\gamma_n}$ that $(\rho_n - M)_+$ converges to 0 in $L^\infty(0,T;L^2)$ and then in $L^\infty(0,T;L^1)$. Using that $\fint \rho_n = M$, we get that $(\rho - M)_-$ goes to 0 in $L^\infty(0,T;L^1)$. Finally, to conclude we remark that we can use the expansion of $\rho_n^{\gamma_n}$ up to the order p as in (85) and deduce that $(\rho_n - M)_+$ tends to 0 in $L^\infty(0,T;L^p)$. Then, we see that

$$\int |\rho_n - M|^p \leq \int (\rho_n - M)_+^p + M^{p-1}(M - \rho_n)_+. \tag{86}$$

And letting n go to infinity, we conclude easily.

Eventually, we see that the proof of the previous theorem can be adapted to this case. The study of more general initial data will be considered in a forthcoming paper.

REFERENCES

[1] BAJSANSKI and R. COIFMAN, in *Proc. Symp. Pure Math.*, 10, AMS.

[2] R. COIFMAN and Y. MEYER, On commutators of singular integrals and bilinear singular integrals., *Trans. Amer. Math. Soc.*, Vol. **212**, 1975, pp. 315-331.

[3] R. COIFMAN, R. ROCHBERG and G. WEISS, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.*, Vol. **2**, 103, 1976, no. 3, pp. 611-635.

[4] J. LERAY, Étude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pure Appl.*, Vol. **12**, 1933, pp. 1-82.

[5] J. LERAY, Essais sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, Vol. **63**, 1934, pp. 193-248.

[6] P.-L. LIONS, Mathematical Topics in Fluid Mechanics. Vol. **2**, Compressible Models, Clarendon Press, Oxford Science Publications, Oxford, 1998.

[7] P.-L. LIONS, Mathematical Topics in Fluid Mechanics. Vol. **1**, Incompressible Models, Clarendon Press, Oxford Science Publications, Oxford, 1996.

[8] P.-L. LIONS, Existence globale de solutions pour les équations de Navier-Stokes compressibles isentropiques, *C. R. Acad. Sci. Paris*, Vol. **316**, 1993, pp. 1335-1340.

[9] P.-L. LIONS, Compacité des solutions des équations de Navier-Stokes compressibles isentropiques, *C. R. Acad. Sci. Paris*, Vol. **317**, 1993, pp. 115-120.

[10] P.-L. LIONS, Limites incompressible et acoustique pour des fluides visqueux, compressibles et isentropiques, *C. R. Acad. Sci. Paris*, Vol. **317**, 1993, pp. 1197-1202.

[11] P.-L. LIONS and N. MASMOUDI, Incompressible limit of a viscous compressible fluid, to appear in *J. Math. Pure Appl.*, Vol. **77**, 1998, pp. 585-627.

(Manuscript received April 8, 1997.)