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Evolution equations governed by families of weighted operators

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ABSTRACT. – In order to develop a Lebesgue approach for the fully non-linear non autonomous evolution problem, $CP_A^\alpha = \{\frac{du}{dt} + \alpha(t)A_\alpha(t)u \ni 0\}$ with $t \in I \subseteq [0,T]$, in an arbitrary Banach space X, we define an abstract L^1- comparison mode (called *coherence*) between multivalued time dependent families of operators $(A_\alpha(s))_{s\in I}$ and $(B_\beta(t))_{t\in J}$ defined on compact subintervals I and J of $[0,T]$ and weighted by functions α and β which belong to $L^\infty([0,T];\mathbb{R}^+)$. The solutions of these problems are limit of discrete schemes and the crucial point is to define these approximations in a Lebesgue sense. The results about this Cauchy problem consist in existence of an evolution operator, integral inequalities (extending Bénilan's inequalities for integral solutions), and continuous properties ; they extend the theory of evolution equations initiated at the beginning of the seventeenth by Crandall, Liggett, Bénilan, Kobayashi, Evans, ([10], [12], ...), and include more recent generalizations as in [18] and [6]. This general study motivated by the observation problem of a heat exchanger (see [16]) where a L^∞-control multiplies an unbounded operator, establishes in Theorem 3.4 a suitable continuity property with respect to the weak* topology on the weights (see applications in [3], [7], [20], ...). © Elsevier, Paris

Key words: Cauchy problem, infinite dimensional systems, coherence, mild solution, weak* convergence.

RÉSUMÉ. – Ce papier esquisse une approche de type Lebesgue des problèmes d'évolution pleinement non linéaires $CP_A^\alpha = \{\frac{du}{dt} + \alpha(t)A_\alpha(t)u \ni 0\}$ avec $t \in I \subseteq [0,T]$ dans un espace de Banach quelconque. Pour cela nous définissons un mode de comparaison (nommé *cohérence*) entre familles d'opérateurs multivoques $(A_\alpha(s))_{s\in I}$ et $(B_\beta(t))_{t\in J}$ définies sur des sous-intervalles compacts I et J de $[0,T]$ et pondérées par des fonctions α et β de $L^\infty([0,T];\mathbb{R}^+)$. Pour les problèmes d'évolution considérés les solutions sont des limites de schémas discrets : le point crucial est alors de définir ces approximations sur un ensemble dénombrable de nœuds (et donc de mesure nulle) en un sens compatible avec une infinité de classes de fonctions Lebesgue intégrables générées par notre approche. Nous mettons ainsi en évidence pour les problèmes de Cauchy CP_A^α un opérateur d'évolution, des inégalités intégrales (généralisant les inégalités des solutions intégrales de Bénilan) et des propriétés de continuité: ces résultats étendent des travaux de Crandall, Liggett, Bénilan, Kobayashi, Evans, ([10], [12], ...), et absorbent des généralisations plus récentes obtenues dans [18] et [6]. Cette étude motivée par un problème d'observabilité pour un échangeur thermique (voir [16]) où un contrôle L^∞ agit multiplicativement sur un opérateur non borné, contient en outre (Théorème 3.4) une propriété de continuité vis-à-vis de la topologie *-faible des poids dans L^∞ (cf. [3], [7], [20], ... pour les applications).

Mots clés : Problème de Cauchy, systèmes de dimension infinie, cohérence, bonnes solutions, convergence *-faible.

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1. INTRODUCTION

This paper deals with the abstract Cauchy problem $CP_A^\alpha(I, u^0)$ in a general Banach framework, for a class of nonlinear systems in which the control α acts on unbounded operators. These situations could be met, for instance, in the field of heat transfer applications, transport phenomena or biochemical processes, for which, so far as we know, the classical theorems of existence of discrete approximations, uniqueness and continuity with respect to the parameters of the solutions could not be applied directly (see [3], [14], [16]). This study unifies in a same approach different classes of systems as autonomous systems, quasi-autonomous systems, bilinear systems ...

We define a comparison mode of multivalued families of operators $(A_\alpha(\tau))_{\tau\in I}$ and $(B_\beta(\sigma))_{\sigma\in J}$ (for subintervals I and J of $[0,T]$), that is, for a.e. $s \in I$, for a.e. $t \in J$, $\forall(u,\widehat{u_\alpha}) \in A_\alpha(s)$, $\forall(v,\widehat{v_\beta}) \in B_\beta(t)$,

$$-[u-v,\widehat{u_\alpha}-\widehat{v_\beta}] \le \psi(s,t,\theta_{\alpha,\beta}(s,t),\|u-v\|)\,.$$

where the bracket $[u,w]$ denotes as usual $\lim\limits_{\lambda\searrow 0}((\|u+\lambda w\|-\|u\|)/\lambda)$ (see section 2.3 for the assumptions on the functions ψ, and $\theta_{\alpha,\beta}$).

In the case $(A_\alpha)_I = (B_\beta)_J$, and when, the null function is the unique positive continuous solution in $\mathcal{D}'(]0,a(T)[)$ of the inequation

$$SC(\psi,\theta_{\alpha,\alpha}) = \begin{cases} \dfrac{dx(\tau)}{d\tau} \le \psi(a^{-1}(\tau),a^{-1}(\tau),\theta_{\alpha,\alpha}\big(a^{-1}(\tau),a^{-1}(\tau)\big),x(\tau)) \\ x(0)=0 \end{cases}$$

with $a(t)=\int_0^t \alpha(\tau)\,d\tau$ we have a generalized L^1 time dependent accretivity condition, called "strong self-coherence". In view of these definitions, we see immediately that our generalization on CP_A^α relates on three directions: the time dependent framework, the accretivity conditions, the weight α. This framework contains the cases studied in [18], [5] or [6] and allows to study, without restriction on the weights (see [20] for instance).

As in the classical accretive case, the solutions of CP_A^α considered throughout this paper, called **mas** are continuous limits of discrete implicit schemes. More precisely, given a partition $\Lambda=(s_0,\dots,s_N)$, we approximate CP_A^α by the discrete system $u_i-u_{i-1}+(s_i-s_{i-1})\alpha(s_i)A_\alpha u_i \ni 0$. One of the main difficulties lies in the fact that the discrete schemes involve countable sets, and consequently neglectible subsets of I. According to our L^1 time dependence, suitable choices of partitions of I are needed to give a good approximation of $\alpha(s)$ and $A_\alpha(s)$. Since we take $\theta_{\alpha,\alpha}\in W$ (this space has been introduced by Crandall and Evans in [9]), there exist a sequence of C^1 functions $(\theta^k)_k$ and a sequence $(F_k)_k$ converging towards 0 in $L^1(I,\mathbb{R}^+)$ satisfying $\left|\theta_{\alpha,\alpha}(s,t)-\theta^k(s,t)\right| \le F_k(s)+F_k(t)$, a.e. $s\in I$, a.e. $t\in I$. Our choices of partitions Λ (called adapted partitions) are those which lead to Lebesgue sums for α and each F_k. We prove in Proposition 2.1 that such a partition exists. We say that the partition Λ is an ε-Lebesgue partition for the real function f if each s_i is a Lebesgue point for f and if the step function $\Lambda(f)$ built with the nodal values $f(s_i)$ satisfies $\int_I |f-\Lambda(f)|(s)ds \le \varepsilon$. Discrete schemes associated with a sequence of adapted partitions with step sizes decreasing towards zero are said adapted.

The theorem 3.1 states that, if the family $(A_\alpha)_I$ is strongly self-coherent and if there exists a bounded adapted discrete scheme, the Cauchy problem

CP_A^α has a unique solution ; moreover this solution does not depend upon the choice of any "free" parameters used in the construction of the **mas**. This theorem gives rise, with a range condition and a stability condition, to a continuous evolution operator (Theorem 3.3). This evolution operator is then endowed with a suitable continuity properties with respect to the weak* topology of L^∞ for the weights and the inferior limit of the families of operators (Theorem 3.4). In fact, all these results are deduced from an asymptotic maximum principle (Theorem 5.1) for discrete schemes which gives a fundamental upper bound for $\limsup_Q \|u_Q(s) - v_Q(t)\|$ where $(u_Q)_Q$ and $(v_Q)_Q$ are respectively discrete approximate solution sequences of $CP_A^\alpha(I, u^0)$ and $CP_B^\beta(J, v^0)$.

This paper is organized as follows. In Section 2 we introduce the basic notations and definitions and we give the time-dependence framework.

The main results are stated in Section 3.

In Section 4, we list in a long lemma (Lemma 4.1) the properties needed the solutions of our problem.

The proofs of our main results are given in Section 5.

Some technical proofs and considerations are postponed in an appendix 6.

2. NOTATIONS, CONVENTIONS AND BASIC DEFINITIONS

2.1. Definition of $CP_A\alpha(I, u^0)$

Let X be a real Banach space. The infinite intervals $I = [S_1, S_2]$, $J = [T_1, T_2]$ are compact subintervals of $[0, T]$. A weighted family on I, with weight $\alpha \in L^\infty([0,T]; \mathbb{R}^+)$, is a family $(A_\alpha(\tau))_{\tau \in I}$ of multivalued nonlinear operators (more precisely, a family of classes of operators) from X to X, and the notation $D_\tau^{A_\alpha}$ is used for the domain of $A_\alpha(\tau)$.

Let $CP_A^\alpha(I, u^0)$ be the following Cauchy problem

$$CP_A^\alpha = CP_A^\alpha(I, u^0) = \begin{cases} \dfrac{du}{dt} + \alpha(t) A_\alpha(t) u \ni 0 \\ u(S_1) = u^0 \in \overline{D_{S_1}^{A_\alpha}}, \quad t \in I . \end{cases}$$

DEFINITION 2.1. – *Given a weighted family $(A_\alpha)_I$, we write $\mathcal{S}(\alpha, I)$, the following stability condition: there exist a function $c \in L^1([0,T]; \mathbb{R}^+)$, and a set $\Xi_\alpha \subset I$ with Lebesgue measure equal to $S_2 - S_1$, such that for all $s_0 \in I$, and for all $w \in D_{s_0}^{A_\alpha}$, there exists $\widetilde{w_\alpha} \in X$ satisfying for all $s \in \Xi_\alpha$, and for all $(u, \widehat{u_\alpha}) \in A_\alpha(s)$,*

$$-[u - w, \widehat{u_\alpha} - \widetilde{w_\alpha}] \leq c(s)(\|u - w\| + 1)$$

Remark 2.1. – *Of course, the above stability condition holds in the classical case $A_\alpha(s) = A - g(s)$, where A is ω-accretive, and where g belongs to $L^1([0,T],X)$, as we see by setting $c(s) = \omega + \|g(s)\|$,and $\widetilde{w_\alpha} \in Aw$.*

2.2. NOTATION. – As in [9], W stands for the closure of the C^1 functions on $[0,T]^2$ in $L^1\Big([0,T]^2;\mathbb{R}\Big)$ under the norm $\|\cdot\|_*$, which is defined by :

$$\|h\|_* = \inf\{\|F\|_1 + \|G\|_1\,;\ |h(s,t)| \le F(s) + G(t) \text{ a.e. } s, \text{ a.e } t, \\ \text{with } F,G \in L^1\big([0,T];\mathbb{R}^+\big)\}.$$

where $\|\cdot\|_p$ denotes the $L^p([0,T];\mathbb{R})$ norm. By convention, the notations $\|\cdot\|_p$, $\|\cdot\|_*$, and W used for functions defined on I, or on $I \times J$, suppose that the functions are extended to $[0,T]$ or $[0,T]^2$. The sequence $\Lambda^I = \Lambda = (s_0, s_1, \cdots, s_N, s_{N+1})$ is called a partition of I. If we have $s_0 = S_1 < s_1 < \cdots < s_N \le S_2 = s_{N+1}$, we denote by $\pi(\Lambda) = \sup_{i=1,\cdots,N+1}(s_i - s_{i-1})$ the step size of the partition Λ^I. Let w be a function from I to E (for an arbitrary set E), we define the step function $\Lambda^I(w)$ from I to E by,

$$\begin{cases} \Lambda^I(w)(S_1) & = w(S_1) \\ \Lambda^I(w)(s) & = w(s_{i\wedge N} \quad \text{if } s \in\,]s_{i-1}, s_i] \text{ with } i = 1,\dots,N. \end{cases}$$

Now, let $\Lambda = \Lambda^I = (s_0, ..., s_{N+1})$ be a partition of I and $\Lambda' = \Lambda'^J = (t_0, ..., t_{P+1})$ be a partition of J. Then, for any function $w : I \times J \to E$ defined on $\Lambda \times \Lambda'$, we denote by $\Lambda \bigotimes \Lambda'(w)$ the step function from $I \times J$ to E satisfying

$$\Lambda \bigotimes \Lambda'(w)(s,t) = w(\Lambda(s), \Lambda(t)).$$

And the notation $\mathbf{I} \bigotimes \Lambda'(w)$ stands for the step function defined by :

$$\mathbf{I} \bigotimes \Lambda'(w) = w(s, \Lambda'(t)),\ (s,t) \in I \times J.$$

The functions $\alpha, \alpha_n, \beta, \beta_n$ belong to $L^\infty([0,T];\mathbb{R}^+)$, and the functions a, a_n, b, b_n are set for the respective integrals of the functions $\alpha, \alpha_n, \beta, \beta_n$ on $[0,T]$ (i.e. $\forall t \in [0,T]$, $a(t) = \int_0^t \alpha(\tau)\,d\tau$, $a_n(t) = \int_0^t \alpha_n(\tau)\,d\tau$, ...).

2.3. Definition of the coherence notion

First, we introduce two functions.

(1) We suppose that $\theta_{\alpha,\beta}$ belongs to W.

(2) The function $\psi : [0,T]^2 \times \mathbb{R}^2 \to \mathbb{R}^+$, is assumed to be (u.s.c. and) such that for each $K > 0$, there exists a decreasing sequence $(\psi_r^K)_r$ pointwise convergent on $\mathcal{K}_K = [0,T]^2 \times \mathbb{R} \times [-K,K]$ towards ψ. Moreover for each $r \in \mathbb{N}$, the function ψ_r^K is required to be C^1 and ω_r^K–Lipschitz on $\mathcal{K}_K$.

Remark 2.2. – *All the results of this article remain valid if we replace the condition* $\psi \geq 0$ *by the following : there exists* $\lambda_0 \leq 0$, *such that we have,* $\forall (s,t,\xi,x) \in [0,T]^2 \times \mathbb{R}^2$

$$\lambda_0(|\xi| + 1) \leq \psi(s,t,\xi,x) \tag{1}$$

Similarly, we can assume that $\theta_{\alpha,\beta}$ *takes its values in some* $\mathbb{R}^n$ *and then* $\xi \in \mathbb{R}^n$.

In the case $\{\psi(s,t,\xi,x) = \varphi(s,t,x) + \xi$, with $\varphi \geq 0\}$, the hypothesis 2 on ψ means simply that φ is u.s.c. (see [5]).

DEFINITION 2.2 (Cohence definition).

i) The weighted family $(A_\alpha)_I$ *is* $(\psi, \theta_{\alpha,\beta})$*-coherent (or* ψ *-coherent, or coherent) for* $(B_\beta)_J$, *if for a.e.* $s \in I$ *and for a.e.* $t \in J$ *we have :* $\forall (u, \widehat{u_\alpha}) \in A_\alpha(s), \forall (v, \widehat{v_\beta}) \in B_\beta(t)$,

$$-[u - v, \widehat{u_\alpha} - \widehat{v_\beta}] \leq \psi(s,t,\theta_{\alpha,\beta}(s,t), \|u - v\|). \tag{2}$$

ii) The weighted family $(A_\alpha)_I$ *is* $(\psi, \theta_{\alpha,\alpha})$*-self-coherent (or self-coherent, if it is* $(\psi, \theta_{\alpha,\alpha})$*-coherent for itself.*

iii) If $(A_\alpha)_I$ *is* ψ*–self-coherent, and if the null function is the unique positive continuous solution in* $\mathcal{D}'(]0, a(T)[)$ *of the inequation :*

$$SC(\psi, \theta_{\alpha,\alpha}) = \begin{cases} \dfrac{dx(\tau)}{d\tau} \leq \psi(a^{-1}(\tau), a^{-1}(\tau), \theta_{\alpha,\alpha}(a^{-1}(\tau), a^{-1}(\tau)), x(\tau)) \\ x(0) = 0 \end{cases} \tag{3}$$

then $(A_\alpha)_I$ *is said strongly* $(\psi, \theta_{\alpha,\alpha})$*-self-coherent.*

The set where a^{-1} is multivalued is at most countable.

Notation. – In the sequel, $I_{\alpha,\beta}$ and $J_{\alpha,\beta}$ are respectively neglectible subsets of I and J such that the relation (2) doesn't hold for all $(s,t) \notin I_{\alpha,\beta} \times J_{\alpha,\beta}$.

Example 2.1. – *Let A be accretive, then* $(A_\alpha)_I = A$ *and* $(A_\beta)_J = A$ *are strongly* 0*-coherent and* $(A_\alpha)_I$ *is strongly* 0*–self-coherent.*

Example 2.2. – *The classical ω-accretive quasi-autonomous case :* $\alpha = 1$, $\mathbf{A}_1(t) = A - f(t)$, *gives rise to a strong* (ψ, θ)*-self-coherence with,* $\psi(s, t, \xi, x) = \omega x + \xi$ *with* $\theta(s, t) = \|f(s) - f(t)\|$.

Example 2.3. – *If* $\mathbf{A}_\alpha(t) = \alpha(t)\, A$ *is everywhere defined, the strong self coherence of appears as a generalized Nagumo's condition for the ordinary differential equation* CP_0^α *(see [13]).*

Remark 2.3. – *By convention, when* $(A_\alpha)_I = A$ *and* $(B_\beta)_J = B$ *(the operators does not depend on time) a function* ψ *of the form* $(s, t, \xi, x) \mapsto \psi(x)$ *will be always required for the meaning of* ψ*-coherence between the families* $(A_\alpha)_I = A$ *and* $(B_\beta)_J = B$.

2.4. ASSUMPTIONS. – For all weighted family $(A_\alpha)_I$ we will suppose realized in the sequel the following conditions, denoted by $\mathcal{C}_\alpha$:

(a) $\forall t \in [0, T]$, $\forall \lambda > 0$, we have $R(I + \lambda A_\alpha(t)) = (I + \lambda A_\alpha(t))(X) = X$;

(b) the nonvoid values operator $t \mapsto \overline{D_t^{A_\alpha}}$ is closed ;

(c) the weighted family $(A_\alpha)_I$ satisfies the stability condition $\mathcal{S}(\alpha, I)$.

(d) the weighted family $(A_\alpha)_I$ is $(\psi, \theta_{\alpha,\alpha})$-self coherent.

2.5. DEFINITION OF MAS AND DAF. – We are ready to give the fundamental definitions of coherent dicretizations and coherent solutions of CP_0^α.

According to the hypothesis $\theta_{\alpha,\beta} \equiv \theta \in W$, let us introduce $\left(\theta_{\alpha,\beta}^k\right)_k \equiv \left(\theta^k\right)_k$ a sequence of C^1 functions on $[0, T]^2$ satisfying, for some $F \in L^1([0, T]; \mathbb{R}^+)$,

$$\begin{cases} \text{(a)} & \left|\theta(s,t) - \theta^k(s,t)\right| \le F^k(s) + F^k(t) \\ \text{(b)} & F^k \in L^1([0,T]; \mathbb{R}^+) \\ & \text{and for a.e. } s,\ \lim_k F^k(s) = 0, \text{ and } F^k(s) \le F(s), \end{cases} \tag{4}$$

DEFINITION 2.3. – *We say that a sequence* $\mathcal{F} = \mathcal{F}_{\alpha,\beta} = \left(\theta^k, F^k, F\right)_k$ *is* <u>*stemmed*</u> *from* $\theta \in W$, *if the above conditions (4) hold.*

We recall that it has been said in introduction that Λ is an ε-Lebesgue partition of I for the real valued function f on I, if $\pi(\Lambda) \le \varepsilon$, if $\int_I |f - \Lambda(f)| \le \varepsilon$ and if each point of Λ is a Lebesgue point for f.

DEFINITION 2.4. – *Let* $(A_\alpha)_I$ *be* ψ*-coherent with* $(B_\beta)_J$ *and* $\mathcal{F}_{\alpha,\beta} = \left(\theta^k, F^k, F\right)_k$ *be stemmed from* $\theta_{\alpha,\beta}$. *We will say that a partition* $\Lambda = (s_i)_{i=1,\cdots,R+1}$ *of* I *(resp.* J*) is* <u>*ε-adapted*</u> *with respect to* $\mathcal{F}_{\alpha,\beta}$, *if we have for all* $i = 1, \cdots, R$, $s_i \in \Xi_\alpha \cap I_{\alpha,\beta}$ *(resp.* $s_i \in \Xi_\alpha \cap J_{\alpha,\beta}$*) and if* Λ *is an* ε*-Lebesgue partition of the restrictions to* I *(resp.* J*) of* α *(resp.* β*),* c, *and every* F^k, $k \in \mathbb{N}$.

We will see in the next subsection that, for all $\varepsilon > 0$, ε-adapted partitions do exist.

DEFINITION 2.5. – *An* ε*-adapted solution with respect to* $\mathcal{F}_{\alpha,\beta}$ *of* $CP_A^\alpha(I, u^0)$ *is a step function* u *from* I *to* X*, such that there exist, an* ε*–adapted partition* $\Lambda = (s_0, ..., s_N, s_{N+1})$ *of* I*, and* $(u_i, \widehat{u_i}) \in A_\alpha(s_i)$ *for* $i = 1, ..., N$*, satisfying :*

$$\begin{cases} \Lambda(u) = u, \text{ and } \|u_0 - u^0\| \leq \varepsilon, \text{ and} \\ u_i - u_{i-1} + \delta_i \alpha(s_i)\widehat{u_i} = 0\,, \text{ where } \delta_i = s_i - s_{i-1}\,. \end{cases} \tag{5}$$

Remark 2.4.
(i) Given an ε*-adapted partition, with the strong range condition (assumption (a) of* C_α*), it is possible to find an* ε*–adapted solution of* CP_A^α *.*
(ii) We can replace 0 *in the second member of (5) by* ε_i *, with the condition:* $\sum_{i=1}^{N} \delta_i \|\varepsilon_i\| \leq \varepsilon\,.$

DEFINITION 2.6. – *Let* $(A_\alpha)_I$ *be* ψ*-coherent with* $(B_\beta)_J$ *and* $\mathcal{F}_{\alpha,\alpha}$ *(resp.*$\mathcal{F}_{\alpha,\beta}$*) be stemmed from* $\theta_{\alpha,\alpha}$ *(resp.* $\theta_{\alpha,\beta}$*). An* $(\mathcal{F}_{\alpha,\alpha}, \mathcal{F}_{\alpha,\beta}) - (\varepsilon_n)$ *-discrete adapted approximating family, denoted by* $(\mathcal{F}_{\alpha,\alpha}, \mathcal{F}_{\alpha,\beta}; \varepsilon_n)$*-* ***DAF*** *, of* $CP_A^\alpha(I, u^0)$*, is a sequence* (u_n) *of* (ε_n)*-adapted solutions with respect to both* $\mathcal{F}_{\alpha,\alpha}$*, and* $\mathcal{F}_{\alpha,\beta}$*, such that* $\lim_n \varepsilon_n = 0$*. If each* u_n *is associated with an* ε_n*-adapted partition* Λ_n *, we will say that* (u_n) *is a* (Λ_n)*-* ***DAF*** *. We will say more simply a* ***DAF*** *when no ambiguity is possible.*

Remark 2.5. – *Of course, when* $(A_\alpha)_I = (B_\beta)_J$ *and* $\mathcal{F}_{\alpha,\alpha} = \mathcal{F}_{\alpha,\beta}$ *in the previous definition we talk about* $(\mathcal{F}_{\alpha,\alpha}; \varepsilon_n)$*-* ***DAF*** *or more simply* $\mathcal{F}_{\alpha,\alpha}$*-**DAF**.*

Finally, we are in position to give the notion of solutions considered here.

DEFINITION 2.7. – *Then a mild adapted solution (* ***mas*** *)* u *of* $CP_A^\alpha(I, u^0)$ *is a uniform continuous limit on* I *of a* $\mathcal{F}_{\alpha,\alpha}$*-* ***DAF*** $(u_n)_n$ *of* $CP_A^\alpha(I, u^0)$*. We say that* $(u_n)_n$ *generates* u*.*

DEFINITION 2.8. – *Let us assume that* $(A_\alpha)_I$ *is coherent for* $(B_\beta)_J$*. Then, a* ***mas*** u *of* $CP_A^\alpha(I, u^0)$ *is coherent for a* ***mas*** v *of* $CP_B^\beta(J, v^0)$ *if* u *and* v *can be generated respectively by an* $(\mathcal{F}_{\alpha,\alpha}, \mathcal{F}_{\alpha,\beta})$*-* ***DAF*** *and an* $(\mathcal{F}_{\beta,\beta}, \mathcal{F}_{\alpha,\beta})$*-* ***DAF*** *.*

2.6. Existence of DAF

In view of the strong range condition we have just to prove that for each $\varepsilon > 0$, there exist ε-adapted partitions.

First, recall that a function f from $[0,T]$ to X is Bochner-integrable (written $f \in L^1([0,T];X)$) means $\int_0^T \|f(t)\|\, dt < \infty$ and that f is almost everywhere on $[0,T]$ limit of a sequence $(f_n)_n$ of simple functions (that is step functions on Borelians of $[0,T]$) (see [11]).

Second, let us mention without proof (see [2] or [17]) the following lemma.

LEMMA 2.1. – *Let $f \in L^1(I;X)$ and $\mathcal{N}$ be a subset of $I = [S_1, S_2]$ of measure zero. Then, for all $\varepsilon > 0$ there is a partition $\Lambda = (s_0, ..., s_N, s_{N+1})$ of I, satisfying,*
(i) for $i = 1, ..., N$, s_i is a Lebesgue point of f, and $s_i \notin \mathcal{N}$.
(ii) the step size of Λ verifies : $\pi(\Lambda) \leq \varepsilon$.
(iii) $\int_{S_1}^{S_2} \|f(\sigma) - \Lambda(f)(\sigma)\|\, d\sigma \leq \varepsilon$.

Third, according to this previous lemma, given $\varepsilon > 0$ the conditions of the definition 2.4 can be obviously satisfied if we prove the following proposition.

PROPOSITION 2.1. – *Let $l^\infty(\mathbb{R})$ be the space of the bounded sequences on $\mathbb{R}$, equipped with the supremum norm $|\cdot|_\infty$. Suppose that the integrable functions F and F^k for $k \in \mathbb{N}^*$ satisfy the relations (b) of (4). Then, the function $s \mapsto \left(F^k(s)\right)_{k\in\mathbb{N}^*}$, from $[0,T]$ to $l^\infty(\mathbb{R})$ is Bochner integrable. Therefore if $\mathcal{N}$ is a null subset of $[0,T]$, for all $\varepsilon > 0$ there exists a partition Λ of $[0,T]$ with points in $[0,T] \setminus \mathcal{N}$ satisfying,*

$$\forall k \in \mathbb{N}^*, \qquad \int_0^T \left|F^k(s) - \Lambda\big(F^k(s)\big)\right| ds \leq \varepsilon .$$

Proof. – In this proof, for $Y \subseteq [0,T]$, Y^c is written for $Y^c = [0,T] \setminus Y$.

a) Let F_∞ be the function $s \mapsto F_\infty(s) = (F^1(s), F^2(s), \cdots, F^k(s), \cdots)$. The relation $|F_\infty(s)|_\infty \leq F(s)$ a.e. on I, guarantees the inequality : $\int_0^T |F_\infty(\sigma)|_\infty\, d\sigma < \infty$. Therefore, we have to prove that F_∞ is limit almost everywhere of a sequence of simple functions.

b) By Egorov's theorem (see for instance [11], [4]) for each $n \in \mathbb{N}^*$, there exists $E_{n,1} \subseteq [0,T]$ with $\mu(E_{n,1}) \leq \frac{1}{2^{n+1}}$ and $N_n \in \mathbb{N}^*$ such that we have,

$$\left[\left(s \in (E_{n,1})^c, \text{ and } q > N_n\right) \Rightarrow \left(|F^q(s)| \leq \frac{1}{n}\right)\right] \tag{6}$$

c) For each $n \in \mathbb{N}^*$, once more by Egorov's theorem , and in view of the integrability of F^k for $k = 1, ..., N_n$, we can find $E_{n,2} \subseteq [0,T]$ with

$\mu(E_{n,2}) \leq \frac{1}{2^{n+1}}$ and simple functions F_n^k from $[0,T]$ to $\mathbb{R}$ for $k = 1, ..., N_n$ verifying,

$$(s \in (E_{n,2})^c, \text{ and } k \in \{1, ..., N_n\}) \Rightarrow \left(\left| F_n^k(s) - F^k(s) \right| \leq \frac{1}{n} \right) \quad (7)$$

d) Then, let us define for all $n \in \mathbb{N}^*$, the sets $E_n = E_{n,1} \cup E_{n,2}$, and the functions $g_n = (F_n^1, ..., F_n^{N_n}, 0, 0, ...)$. Then we deduce the following statements :

i) for each $n \in \mathbb{N}^*$, g_n is a simple function from $[0,T]$ to $l^\infty(\mathbb{R})$;

ii) Let $\mathcal{E}$ be the set $\mathcal{E} = \bigcap_{k_0} (\bigcup_{k \geq k_0} (E_k))$. Then for $s \notin \mathcal{E}$, there exists $k_0 \in \mathbb{N}^*$satisfying $s \in \bigcap_{k \geq k_0} E_k^c$. And therefore in view of relations (6) and (7), if $s \notin \mathcal{E}$, we have : $n \geq k_0 \Rightarrow |g_n(s) - F_\infty(s)|_\infty \leq \frac{1}{n}$. Thus, $s \notin \mathcal{E} \Rightarrow \lim_n g_n(s) = F_\infty(s)$.

iii) If μ stands for the Lebesgue measure on $[0,T]$, we have $\mu(\mathcal{E}) = 0$. Indeed, the inequality $\mu(E_k) \leq \mu(E_{k,1}) + \mu(E_{k,2}) = \frac{1}{2^k}$ implies, for all $k_0 \in \mathbb{N}^*$,
$\mu(\mathcal{E}) \leq \mu\left(\bigcup_{k \geq k_0} E_k\right) \leq \sum_{k \geq k_0} \mu(E_k) = \frac{1}{2^{k_0-1}}$.

e) Finally, the function F_∞ is therefore, limit almost everywhere of the sequence of simple functions $(g_n)_n$, and consequently F_∞ is Bochner-integrable on $[0,T]$.

2.7. Boundedness of DAF

Let us end this section by the following proposition which gives from the stability condition (assumption (c) of $\mathcal{C}_\alpha$), an a priori upper bound for the discrete schemes.

PROPOSITION 2.2. – *Fix for instance,* $w \in D_{S_1}^{A_\alpha}$. *Let us suppose that* $\varepsilon > 0$ *realizes,* .

$$\varepsilon \|\alpha\|_\infty \leq \frac{1}{4} \;\; \textit{and} \;\; \sup_{s \in [0, T-\varepsilon]} \left(\int_s^{s+\varepsilon} c(\xi)\, d\xi \right) \|\alpha\|_\infty \leq \frac{1}{4}.$$

Let u *be an* ε*-adapted solution of* $CP_A^\alpha(I, u^0)$. *Then we have, for all* $s \in I$,

$$\|u(s)\| \leq \|w\| + e^{\frac{3}{2}\|\alpha\|_\infty (\|c\|_1 + \varepsilon)} \left(\left\|u^0 - w\right\| + \varepsilon + \|\alpha\|_\infty (\|c\|_1 + \varepsilon) + \|\widetilde{w_\alpha}\| (a(T) + \varepsilon) \right).$$

Proof. – Let $\Lambda = (s_0, ..., s_{R+1})$ with $s_i \in \Xi_\alpha$ for $i = 1, ..., R$, $u = \Lambda(u)$ and $u(t_i) = u_i$. We have for $i = 1, ..., R$, if $\delta_i = s_i - s_{i-1}$,

$$-\frac{u_i - u_{i-1}}{\delta_i} + \alpha(s_i)\widehat{u_i} \ni 0 \text{ with } \widehat{u_i} \in A_\alpha(s_i) u_i$$

Then, the assumption 2.4-(c) of $\mathcal{C}_\alpha$ leads to, (setting $c(s_i) = c_i$),

$$(1 - \alpha(s_i)c_i\delta_i)\|u_i - w\| \leq \|u_{i-1} - w\| + \delta_i c_i + \alpha(s_i)\|\widetilde{w_\alpha}\|\,.$$

And, immediately by iterating, we have,

$$\|u_i - w\| \leq \prod_{k=1}^{i}\left(\frac{1}{1-\alpha(s_k)c_k\delta_k}\right)$$
$$\left[\|u_0 - w\| + \int_{S_1}^{s_i}(\Lambda(\alpha c)(\tau) + \Lambda(\alpha)(\tau)\|\widetilde{w_\alpha}\|)\,d\tau\right].$$

According to the following inequalities

$$0 \leq \alpha(s_k)c_k\delta_k \leq \|\alpha\|_\infty\left(\int_{s_{k-1}}^{s_k} c(\xi)\,d\xi + \varepsilon\right) \leq \frac{1}{2},$$
$$\text{and} \int_{S_1}^{s_i}\Lambda(\alpha c)(\tau)d\tau \leq \|\alpha\|_\infty\left(\int_0^{s_i} c(\xi)\,d\xi + \varepsilon\right),$$

and since we have $\ln(1-x) \geq -x - x^2$ for $x \in]0, \frac{1}{2}]$, we obtain,

$$\|u(s) - w\| \leq e^{\frac{3}{2}\|\alpha\|_\infty(\|c\|_1+\varepsilon)}\left(\|u^0 - w\| + \varepsilon + \|\alpha\|_\infty\left(\int_0^{s_i} c(\xi)\,d\xi + \varepsilon\right)\right.$$
$$\left. + \|\widetilde{w_\alpha}\|(a(s_i) + \varepsilon)\right).$$

Thus, the required inequality holds.

3. THE MAIN RESULTS

Let us point out that the operator $D_{\alpha,\beta} = \beta(t)(\partial/\partial s) + \alpha(s)(\partial/\partial t)$ considered in $\mathcal{D}'(\overset{\circ}{I} \times \overset{\circ}{I})$, plays the part of the operator $D = \partial/\partial s + \partial/\partial t\,(= D_{1,1})$ introduced at the first time in the classical case by Crandall - Evans [9].

THEOREM 3.1. – *Let $(A_\alpha)_I$ be strongly self-coherent. Then,*

(i) If $\mathcal{F}_{\alpha,\alpha}$ is stemmed from $\theta_{\alpha,\alpha}$, then, all (bounded) $\mathcal{F}_{\alpha,\alpha}$ - ***DAF*** *of the problem $CP_A^\alpha(I, u^0)$ is uniformly convergent on I towards its unique* ***mas*** *.*

(ii) The ***mas*** *of $CP_A^\alpha(I, u^0)$ does not depend upon any function α chosen in its class in L^∞, and upon any sequence $\mathcal{F}_{\alpha,\alpha}$ stemmed from $\theta_{\alpha,\alpha}$.*

Really, the part (i) of the theorem 3.1 can be applied in a wider framework than $\mathcal{C}_\alpha$ (which ensures boundedness and existence of a **DAF).** In particular, the tangential condition 2.4-$\mathcal{C}_\alpha$-a) is not needed.

THEOREM 3.2. – *Suppose that u is a* **mas** *of $CP_A^\alpha(I, u^0)$ coherent for a* **mas** *v of $CP_B^\beta(J, v^0)$. Let $\mathcal{B}$ be the continuous function on $a(I) - b(J)$ satisfying:*

$$\begin{cases} \mathcal{B}(\sigma - b(T_1)) = \|u(a^{-1}(d)) - v^0\| & \text{if } \sigma \in a(I) = [a(S_1), a(S_2)] \\ \mathcal{B}(a(S_1) - \tau) = \|u^0 - v(b^{-1}(-d))\| & \text{if } \tau \in b(J) = [b(T_1), b(T_2)]\,. \end{cases} \tag{8}$$

Then, $\mathcal{B}$ is a single valued continuous function, and we have $\forall (s,t) \in I \times J$

$$\|u(s) - v(t)\| \leq m_{\mathcal{B}}(a(s), b(t))\,. \tag{9}$$

More precisely, the function $e(s,t) = \|u(s) - v(t)\|$ satisfies in $\mathcal{D}'(\overset{\circ}{I} \times \overset{\circ}{I})$,

$$D_{\alpha,\beta}[e(s,t)] \leq \alpha(s)\beta(t)\psi(s,t,\theta_{\alpha,\beta}(s,t), e(s,t))\,. \tag{10}$$

And for $(\alpha A_\alpha)_I = (\beta B_\beta)_J$, we have in $\mathcal{D}'(\overset{\circ}{a(I)})$, if $x(\tau) = e(a^{-1}(\tau), a^{-1}(\tau))$

$$\frac{d}{d\tau} x(\tau) \leq \psi\big(a^{-1}(\tau), a^{-1}(\tau), \theta_{\alpha,\alpha}\big(a^{-1}(\tau), a^{-1}(\tau)\big), x(\tau)\big)\,.$$

The function $m_{\mathcal{B}}$ defined in the next paragraph (see definition 4.1) denotes (in some sense) a maximal solution in $\mathcal{D}'(\overset{\circ}{I} \times \overset{\circ}{I})$, of the inequation (10). A suitable choice of β, B_β and J, shows that the **mas** u is a Benilan's integral solution of $CP_A^\alpha(I, u^0)$ in a generalized but natural way (see [1] or [5]).

COROLLARY 3.1 (general variable change). – *Let $(\alpha(t)A)_{[0,T]}$ where A is strongly self-coherent on $[0, a(T)]$, then the unique* **mas** *u of $CP_A^\alpha([0,T], u^0)$ is given by the variable change $u(t) = v(a(t))$, where v is the unique mild solution of $CP_A^1([0, a(T)], u^0)$. If furthermore, the operator A is continuous the* **mas** *of $CP_A^\alpha([0,T], u^0)$ is a strong solution.*

Let $(A_\alpha)_{[0,T]}$ be a strongly ψ-self-coherent family on $[0,T]$. According to Theorem 3.1, for all $u^0 \in \overline{D_s^{A_\alpha}}$ the evolution problem $CP_A^\alpha([s,T], u^0)$ admits a unique **mas** u. We will set $S(t,s)u^0 = u(t)$ the value taken by u at t. Now we can state the two last results which concern strongly ψ-self-coherent families.

THEOREM 3.3. – *Let $(A_\alpha)_{[0,T]}$ be a strongly ψ-self-coherent family on $[0,T]$. Then,*

a) the family $S = (S(t,s))_{0 \leq s \leq t \leq T}$ is an evolution operator, that is ;

(i) For $0 \leq s \leq t \leq T$ *, the operator* $S(t,s)$ *maps* $\overline{D_s^{A_\alpha}}$ *into* $\overline{D_t^{A_\alpha}}$ *;*
(ii) For all $s \in [0,T]$ *, we have* $S(s,s) = \mathbb{I}$ *(*$\mathbb{I}$ *denotes the identity operator) ;*
(iii) For $0 \leq r \leq s \leq t \leq T$ *, we have* $S(t,s) \circ S(s,r) = S(t,r)$ *;*
b) the evolution operator S *is continuous on*

$$Y = \Big\{(s,t,w);\ 0 \leq s \leq t \leq T\,,\ w \in \overline{D_s^{A_\alpha}}\Big\}.$$

THEOREM 3.4. – *For all* $n \in \mathbb{N} \cup \{\infty\}$ *, let* $\mathcal{A}_n = \left(A^n_{\alpha_n}\right)_{[0,T]}$ *(satisfying* $\mathcal{C}_{\alpha_n}$*) be a strongly* ψ*-self-coherent family on* $[0,T]$*, let* S_n *be the evolution operator generated by* $\mathcal{A}_n$ *, and let* $u_n^0 \in \overline{D_0^{A^n_{\alpha_n}}}$ *. We suppose that* $(\alpha_n)_{n\in\mathbb{N}}$ *converges to* α_∞ *in the weak* topology in* $L^\infty([0,T],\mathbb{R})$ *and that we have for almost every* $s \in [0,T]$ *,* $A^\infty_{\alpha_\infty}(s) \subseteq \liminf_{n\in\mathbb{N}} A^n_{\alpha_n}(s)$ *. Finally, we suppose that* $(\theta_{\alpha_n,\alpha_n})_{n\in\mathbb{N}}$ *converges towards* $\theta_{\alpha_\infty,\alpha_\infty}$ *in* $(W, \|\cdot\|_*)$ *. Then, if* $(u_n^0)_{n\in\mathbb{N}}$ *converges towards* u_∞^0 *, and if the sequence of functions* $(S_n(.,0)u_n^0)_{n\in\mathbb{N}}$ *is bounded, then* $(S_n(.,0)u_n^0)_{n\in\mathbb{N}}$ *converges uniformly towards* $S_\infty(.,0)u_\infty^0$ *on* $[0,T]$ *.*

Remark 3.1. – *This last theorem appears like a Lebesgue dominated theorem for evolution equations (setting for instance* $\alpha_n = 1$ *and* $A^n_{\alpha_n}(.)u = f_n(.) \in L^1([0,T],X)$*).*

4. THE U.S.C. HULL LEMMA

Now, we will give a basic lemma called "the u.s.c. hull lemma" which is proved in the appendix. It summarizes in one proposition the most important results that we will need in this article on the inequation $D_{\alpha,\beta}[x] \leq \Phi(s,t,\theta,x)$.

The type of results and the methodology developed in this lemma 4.1 are similar to ones of [6]. But new problems appear here, because of the time dependence in L^1 , the multi-valued aspect and the lack of regularity of a^{-1} and b^{-1} .

We denote by $E_\Omega(\leq,\Phi,\theta,\mathcal{B},\alpha,\beta)$ the following inequation in $\mathcal{D}'(\Omega)$ with $\Omega = \Omega_{\alpha,\beta} =]a(S_1),a(S_2)[\times]b(T_1),b(T_2)[= a(\overset{\circ}{I}) \times b(\overset{\circ}{J})$,

$$E_\Omega(\leq,\Phi,\theta,\mathcal{B},\alpha,\beta) = \begin{cases} D[y(\sigma,\tau)] \leq \Phi(a^{-1}(\sigma),b^{-1}(\tau),\theta(a^{-1}(\sigma),b^{-1}(\tau)),y(\sigma,\tau)) \\ y(\sigma,b(T_1)) \leq \mathcal{B}(\sigma-b(T_1)) \text{ and}, \ y(a(S_1),\tau) \leq \mathcal{B}(a(S_1)-\tau)\,. \end{cases} \tag{11}$$

The function $\mathcal{B}$ is continuous on $a(I) - b(J) \subseteq [-b(T_2), a(S_2)]$ and θ belongs to W. We denote by $E_\Omega(=, \Phi, \theta, \mathcal{B}, \alpha, \beta)$ the equation in $\mathcal{D}'(\Omega)$ obtained by replacing the three symbols "$\leq$" in $E_\Omega(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$ by "=".

A similar notation is used for the characteristic equations. More precisely, we introduce for $d \in a(I) - b(J)$, the notation $\chi_{d,\Omega}(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta) = \chi_d(\leq, \Phi, \theta, \mathcal{B})$ for the following inequation in $\mathcal{D}'(\overset{\circ}{I}_d)$ with $I_d = [(a(S_1) - d) \vee b(T_1), b(T_2) \wedge (a(S_2) - d)]$,

$$\chi_d(\leq, \Phi, \theta, \mathcal{B}) = \begin{cases} \dfrac{d}{d\tau} z(\tau) \leq \Phi\big(a^{-1}(\tau + d), b^{-1}(\tau), \theta\big(a^{-1}(\tau + d), b^{-1}(\tau)\big), z(\tau)\big) \\ z((a(S_1) - d) \vee b(T_1)) \leq \mathcal{B}(d) \end{cases} \tag{12}$$

and $\chi_{d,\Omega}(=, \Phi, \theta, \mathcal{B}, \alpha, \beta)$ will denote the equation in $\mathcal{D}'(\overset{\circ}{I}_d)$ obtained by replacing the two symbols "$\leq$" in $\chi_{d,\Omega}(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$ by "=".

In the inequations or equations $E_\Omega(\cdot, \cdots)$ or $\chi_{d,\Omega}(\cdot, \cdots)$, we will sometimes forget the parameters when no ambiguity arises (for instance, $E(\leq, \Phi, \theta, \mathcal{B}) \equiv E_\Omega(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$). We emphasize that for all $d \in a(I) - b(J)$ there exists a unique $(\sigma_0, \tau_0) \in (a(I) \times \{b(T_1)\}) \cup (\{a(S_1)\} \times b(J))$ such that we have $\sigma_0 - \tau_0 = d$.

About inequation of $\chi_d(\leq, \Phi, \theta, \mathcal{B})$ type, we just recall that for $\varphi \in L^1([a, b]; \mathbb{R})$, a continuous function z on $[a, b]$ is solution in $\mathcal{D}'(]a, b[)$ of $\dfrac{d}{dt} z(t) \leq \varphi(t)$, is equivalent to, the integrated form $z(t) - z(s) \leq \int_s^t \varphi(\tau) d\tau$ with $a \leq s \leq t \leq b$.

LEMMA 4.1 (u.s.c. hull lemma). – *We suppose that there exists a decreasing sequence* $(\Phi_r)_{r \in \mathbb{N}}$ *of* C^1 *functions on* $[0, T]^2 \times \mathbb{R}^2$ *(pointwise) convergent towards* $\Phi \geq 0$. *We assume that for each* $r \in \mathbb{N}$, *the function* Φ_r *is* ω_r-*Lipschitz. Let* $\mathcal{B}$ *be a continuous function on* $a(I) - b(J) \subseteq [-b(T_2), a(S_2)]$ *and let* $\Omega_{\alpha,\beta} = \Omega =]a(S_1), a(S_2)[\times]b(T_1), b(T_2)[$.

i) The inequation $E_\Omega(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$ *in* $\mathcal{D}'(\Omega)$, *has a unique u.s.c. solution on* $\overline{\Omega}$, *denoted by* $m(\Phi, \theta, \mathcal{B}, \alpha, \beta)$ *or* $m(\Phi, \theta, \mathcal{B})$ *or* $m_{\mathcal{B}}$, *which bounds above each continuous solution of* $E_\Omega(\leq, \Phi, \theta, \mathcal{B})$ *and satisfies: for all* $d \in a(I) - b(J)$ *the function*

$$y_d = y_d(\Phi, \theta, \mathcal{B}) : \tau \mapsto m_{\mathcal{B}}(\tau + d, \tau)$$

is continuous on $I_d = [(a(S_1) - d) \vee b(T_1), b(T_2) \wedge (a(S_2) - d)]$, *and is the maximal continuous solution in* $\mathcal{D}'(\overset{\circ}{I}_d)$ *of the inequation* $\chi_{d,\Omega}(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$.

ii) *For* $s \in [0,T]$, *let* $\mathcal{B}_0$ *be defined on* $[-T,T]$ *by,* $\mathcal{B}_0(s) = \mathcal{B}(a(s) - b(T_1))$ *and* $\mathcal{B}_0(-s) = \mathcal{B}(a(S_1) - b(s))$. *Let* x *be a continuous (on* $\Omega_{1,1}$*) solution in* $\mathcal{D}'(]S_1,S_2[\times]T_1,T_2[)$ *of the inequation* $E_{\Omega_{1,1}}(\leq,\Phi,\theta,\mathcal{B}_0,1,1)$, *constant on each set value taken by* (a^{-1},b^{-1}). *Then we have, for* $(s,t) \in I \times J$,

$$x(s,t) \leq m(\Phi,\theta,\mathcal{B},\alpha,\beta)(a(s),b(t))$$

iii) *Let the functions* $\Phi_1,\Phi_2,\mathcal{B}_1,\mathcal{B}_2$, *be defined as at the beginning, then we have,*

$$(\Phi_1 \leq \Phi_2 \text{ and } \mathcal{B}_1 \leq \mathcal{B}_2) \Rightarrow m(\Phi_1,\theta,\mathcal{B}_1) \leq m(\Phi_2,\theta,\mathcal{B}_2)\,.$$

iv) *If* Φ *is* C^1 *on* $[0,T]^2 \times \mathbb{R}^2$ *, if*

$$\Phi(S_1,T_1,\theta(S_1,T_1),\mathcal{B}(a(S_1) - b(T_1))) = 0\,,$$

if θ *is* C^1 *on* $[0,T]^2$, *if* $\mathcal{B}$ *is* C^1 *on* $a(I) - b(J)$, *if* α *and* β *are strictly positive and continuous, then* $m_{\mathcal{B}}$ *is* C^1 *on* $\overline{\Omega}$ *and is the unique classical solution of* $E_\Omega(=,\Phi,\theta,\mathcal{B},\alpha,\beta)$.

v) *If* Φ *is* ω*–Lipschitz, then* $m_{\mathcal{B}}$ *is continuous on* $\overline{\Omega}$, *and verifies (in* $\mathcal{D}'(\Omega)$*) the equation* $E_\Omega(=,\Phi,\theta,\mathcal{B},\alpha,\beta)$.

vi) *Let* $(\alpha_0,\beta_0) \in \mathcal{Y} = (L^\infty([0,T],\mathbb{R}^+))^2$. *If* Φ *is* ω*–Lipschitz, then* $(\theta,\mathcal{B},\alpha,\beta) \mapsto m(\Phi,\theta,\mathcal{B},\alpha,\beta)$ *is sequentially continuous from* $W \times C^0(a_0(I) - b_0(J)) \times L_0$ *to* $C^0(\overline{\Omega})$ *where* L_0, *is the closed subset of* $\mathcal{Y}$ *equipped with its weak* topology, defined by* $L_0 = \{(\alpha,\beta) \in \mathcal{Y}\ ;\ \Omega_{\alpha_0,\beta_0} \subseteq \Omega_{\alpha,\beta}\}$.

vii) *The sequence of functions* $(y_d(\Phi_r,\theta,\mathcal{B}))_r$ *converges uniformly on* I_d *by decreasing towards* $y_d(\Phi,\theta,\mathcal{B})$ *(for all* $d \in [-b(T_2),a(S_2)]$*), and particularly* $(m(\Phi_r,\theta,\mathcal{B}))_r$ *converges (pointwise) by decreasing towards* $m(\Phi,\theta,\mathcal{B})$.

According to the following definition, all the results stated in section 3 remain valid whenever α or β are the null function on I or J.

DEFINITION 4.1. – *The maximal solution* $m_{\mathcal{B}} = m(\Phi,\theta,\mathcal{B},\Omega)$ *of* $E_\Omega(\leq,\Phi,\theta,\mathcal{B},\alpha,\beta)$ *given in the part i) of the lemma 4.1 is said* <u>*u.s.c. hull*</u> *of* $E_\Omega(\leq,\Phi,\theta,\mathcal{B},\alpha,\beta)$. *If* $\alpha \equiv 0$ *on* I *(resp.* $\beta \equiv 0$ *on* J*) we set for* $(\sigma,\tau) \in \overline{\Omega}$,

$$m_{\mathcal{B}}(\sigma,\tau) = \mathcal{B}(a(S_1) - \tau) \text{ (resp. } m_{\mathcal{B}}(\sigma,\tau) = \mathcal{B}(\sigma - b(T_1))\text{)}.$$

For $K > 0$, *we define the function* $\widehat{\psi_K}$ *on* $[0,T]^2 \times \mathbb{R}^2$ *by,* $\widehat{\psi_K}(s,t,\xi,x) = \psi(s,t,\xi,P^K(x))$, *where,* P^K *denotes the projection on* $[0,K]$.

We denote by $\mathcal{L}$ the assumption on Φ asked in the lemma 4.1, that is,

ASSUMPTION $\mathcal{L}$. – There exists a decreasing sequence $\left(\Phi_r^K\right)_{r\in\mathbb{N}}$ of C^1 functions on $[0,T]^2 \times \mathbb{R}^2$ convergent (pointwise) towards $\widehat{\psi_K}$, and, for each $r \in \mathbb{N}$, the function Φ_r^K is $\omega_{K,r}$-Lipschitz.

LEMMA 4.2. – *The function $\widehat{\psi_K}$ satisfies the assumption $\mathcal{L}$.*

The proof of this lemma is left to the reader.

LEMMA 4.3. – *Let us assume that the condition $SC(\psi, \theta_{\alpha,\alpha})$ of the definition 2.2 iii) holds. Then, for all $K > 0$, the property $SC\left(\widehat{\psi_K}, \theta_{\alpha,\alpha}\right)$ is also true.*

Proof. – Let $y \geq 0$ be a continuous solution in $[0, a(T)]$ of

$$\chi\left(\widehat{\psi_K}, [0, a(T)]\right) = \begin{cases} \dfrac{d}{d\tau} z(\tau) \leq \widehat{\psi_K}\left(a^{-1}(\tau), a^{-1}(\tau), \theta_{\alpha,\alpha}\left(a^{-1}(\tau), a^{-1}(\tau)\right), z(\tau)\right) \\ z(0) = 0\,, \quad \tau \in [0, a(T)]. \end{cases} \tag{13}$$

If we have $K \geq \|y\|_\infty$, clearly, we have $y \equiv 0$. If we have $K < \|y\|_\infty$, let

$$T_K = \sup\left\{t \in [0, a(T)];\ y(\tau) \leq K\,,\ \forall \tau \in [0, t]\right\}.$$

Then, the continuous function, y_1 defined by $y_1(t) = y(t)$ for $t \in [0, T_K]$ and $y_1(t) = y(T_K) = K$ for $t \in [T_K, a(T)]$, is solution of $\chi(\psi, [0, a(T)])$. Consequently, we have $y_1 \equiv 0$, and then $K = 0$, which is a contradiction.

Remark 4.1. – *Eventually by changing ψ into $\widehat{\psi_K}$ for a suitable constant $K > 0$ (derived from the stability condition),* ***in the sequel we will suppose always that the function ψ satisfies the assumption*** *$\mathcal{L}$ introduced before the lemma 4.2. With this convention, the existence of $m(\psi, \theta, \mathcal{B})$ is guaranteed. According to the remark 2.2, for the same reason, the results of this paper remain true (see [7]) with,*

$$\psi(s, t, \xi_1, \xi_2, x) = \xi_1 x + \xi_2\,.$$

5. PROOFS OF THE MAIN RESULTS

5.1. A discrete maximum principle

All the results stated in the section 3 are corollaries of the discrete maximum principle given at the end of this section.

Let us introduce the following notation. Given partitions $\Lambda = (s_0, s_1, ..., s_{N+1})$ and $\Lambda' = (t_0, ..., t_{P+1})$ respectively of I and J, and given a function $w : I \times J \to X$, the symbol $\Lambda \bigotimes \Lambda'[D_{\alpha,\beta}](w)$ is the step function v defined on $]S_1, S_2] \times]T_1, T_2]$ by the relations,

$$\begin{cases} v(s,t) = \alpha(t_{j\wedge P}) \dfrac{w(s_{i\wedge N}, t_{j\wedge P}) - w(s_{(i\wedge N)-1}, t_{j\wedge P})}{\delta_{i\wedge N}} \\ \qquad\qquad\qquad + \beta(s_{i\wedge N}) \dfrac{w(s_{i\wedge N}, t_{j\wedge P}) - w(s_{i\wedge N}, t_{(j\wedge P)-1})}{\gamma_{j\wedge P}} \\ \text{for } (s,t) \in]s_{i-1}, s_i] \times]t_{j-1}, t_j] \text{ and } i = 1, ..., N+1 \text{ and } j = 1, ..., P+1 . \end{cases} \tag{14}$$

LEMMA 5.1 (discrete lemma). – *We suppose* $0 \le \Gamma \le \Phi$ *on* $[0,T]^2 \times \mathbb{R}^2$ *and* Φ *ω–Lipschitz on* $[0,T]^2 \times \mathbb{R}^2$. *For all* $Q \in \mathbb{N}$, *let* $\Lambda_Q = \left(s_0^Q, ..., s_{N_{Q+1}}^Q\right)$ *(resp.* $\Lambda'_Q = \left(t_0^Q, ..., t_{N_{Q+1}}^Q\right)$*) be an* ε_Q*-Lebesgue partition of* I *(resp.* J*) for* α *(resp.* β*). Let* $(y_Q)_Q$ *and* $(\theta_Q)_Q$ *be sequences of functions on* $I \times J$ *verifying* $\forall Q \in \mathbb{N}$,

$$0 \le \Lambda_Q \bigotimes \Lambda'_Q \{-D_{\alpha,\beta}(y_Q)(s,t) + \alpha(s)\beta(t)\Gamma(s,t,\theta_Q(s,t), y_Q(s,t))\}$$

Let $\widetilde{a_Q} : [S_1, S_2] \to [S_1, S_2]$ *and* $\widetilde{b_Q} : [T_1, T_2] \to [T_1, T_2]$ *be integrable functions such that :*

$$\limsup_Q \int_{S_1}^{S_2} \alpha(s) |\Lambda_Q(\widetilde{a_Q}(s)) - s| \, ds = h$$

$$\textit{and} \quad \limsup_Q \int_{T_1}^{T_2} \beta(t) \left|\Lambda'_Q\left(\widetilde{b_Q}(t)\right) - t\right| dt = h' .$$

Moreover, we suppose that for all $Q \in \mathbb{N}$, *the functions* y^Q, $\widetilde{\theta_Q}$, H_Q, *satisfy on* $I \times J$, $\widetilde{\theta_Q} \in W$, $H_Q \in W$, *and,*

$$0 \ge \Lambda_Q \bigotimes \Lambda'_Q \{-D_{\alpha,\beta}(y^Q) \\ + \alpha(s)\beta(t)\Phi\left(\widetilde{a_Q}(s), \widetilde{b_Q}(t), \widetilde{\theta_Q}(s,t), y^Q(s,t)\right) + \alpha(s)\beta(t)H_Q(s,t)\right\}$$

$$\limsup_Q \sup_{s,t} \left\| \Lambda_Q \bigotimes \Lambda'_Q \left(H_Q^-(s,t)\right) \right\|_* = \kappa .$$

We assume realized the conditions, with $y_{ij}^Q = y_Q\left(s_i^Q, t_j^Q\right)$ *and* $Y_{ij}^Q = y^Q\left(s_i^Q, t_j^Q\right)$,

$$\begin{aligned} \alpha(s_i^Q) = 0 \Rightarrow \left(y_{ij}^Q \le y_{i-1j}^Q \quad \textit{and} \quad Y_{ij}^Q \ge Y_{i-1j}^Q\right) \\ \beta(t_i^Q) = 0 \Rightarrow \left(y_{ij}^Q \le y_{ij-1}^Q \quad \textit{and} \quad Y_{ij}^Q \ge Y_{ij-1}^Q\right), \end{aligned} \tag{15}$$

For some $b \in \mathbb{R}^+$ *we suppose,*

$$\limsup_Q \sup_{s,t} \Lambda_Q \bigotimes \Lambda'_Q \begin{pmatrix} y_Q(s,T_1) - y^Q(s,T_1) \\ y_Q(S_1,t) - y^Q(S_1,t) \end{pmatrix}^+ \leq b \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We assume finally that there exist sequences of nonnegative functions $(f_Q)_Q$ *and* $(g_Q)_Q$ *such that we have, for* $(s,t) \in I \times J$,

$$\Lambda_Q \bigotimes \Lambda'_Q \left| \theta_Q - \widetilde{\theta_Q} \right| (s,t) \leq \Lambda_Q f_Q(s) + \Lambda'_Q g_Q(t)$$

$$\textit{and let } \lambda = \limsup_Q \left(\int_{S_1}^{S_2} \Lambda_Q[\alpha(s) f_Q(s)]\, ds + \int_{T_1}^{T_2} \Lambda'_Q[\beta(t) g_Q(t)]\, dt \right).$$

Then, we conclude,

$$\limsup_Q \sup_{s,t} \Lambda_Q \bigotimes \Lambda'_Q \left(y_Q - y^Q\right)^+ (s,t)$$
$$\leq e^{\frac{3}{2}\omega(a(S_2)+b(T_2))} (\kappa(\|\alpha\|_\infty + \|\beta\|_\infty) + \omega\lambda + \omega(h+h') + b)$$

Proof. – Let $Y_{ij}^Q = y^Q\left(s_i^Q, t_j^Q\right)$ and $y_{ij}^Q = y_Q\left(s_i^Q, t_j^Q\right)$ and so on for the functions θ_Q, $\widetilde{\theta_Q}$, H_Q, ... According to the hypothesis $H_Q \in W$, pick up integrable nonnegative functions F_Q and G_Q realizing,

$$\Lambda_Q \bigotimes \Lambda'_Q \left| H_Q^- \right| (s,t) \leq F_Q(s) + G_Q(t) \text{ a.e. } s, \text{ a.e. } t.$$

Then, we have with the hypotheses on Γ and Φ,

$$\begin{aligned} \left(y_{ij}^Q - Y_{ij}^Q\right)^+ \leq & \frac{\sigma_{ij}^Q}{\delta_i^Q \alpha(s_i^Q)} \left(y_{i-1j}^Q - Y_{i-1j}^Q\right)^+ + \frac{\sigma_{ij}^Q}{\gamma_j^Q \beta(t_j^Q)} \left(y_{ij-1}^Q - Y_{ij-1}^Q\right)^+ \\ & + \sigma_{ij}^Q \left(\omega \left\{ \left| \theta_{ij}^Q - \widetilde{\theta_{ij}^Q} \right| + \left| y_{ij}^Q - Y_{ij}^Q \right| + \left| \left(\widetilde{a_Q}(s_i^Q) \right) - s_i^Q \right| \right.\right. \\ & \left.\left. + \left| \left(\widetilde{b_Q}(t_j^Q) \right) - t_j^Q \right| \right\} + \left(H_{ij}^Q \right)^- \right), \end{aligned} \tag{16}$$

with $\sigma_{ij}^Q = \frac{\alpha(s_i^Q)\delta_i^Q \beta(t_j^Q)\gamma_j^Q}{\alpha(s_i^Q)\delta_i^Q + \beta(t_j^Q)\gamma_j^Q}$, and $\delta_i^Q = s_i^Q - s_{i-1}^Q$, $\gamma_j^Q = t_j^Q - t_{j-1}^Q$. With conditions (16), we see that the relation (16) remains valid with the following conventions :

a) whenever, $\alpha(s_i^Q) = 0$ and $\beta(t_j^Q) \neq 0$, we take $\frac{\sigma_{ij}^Q}{\delta_i^Q \alpha(s_i^Q)} = 1$;

b) whenever, $\beta(t_j^Q) = 0$ and $\alpha(s_i^Q) \neq 0$, we take $\frac{\sigma_{ij}^Q}{\gamma_j^Q \beta(t_j^Q)} = 1$;

c) whenever, $\alpha(s_i^Q) = 0$ and $\beta(t_j^Q) = 0$, we take $\frac{\sigma_{ij}^Q}{\delta_i^Q \alpha(s_i^Q)} = \frac{\sigma_{ij}^Q}{\gamma_j^Q \beta(t_j^Q)} = \frac{1}{2}$, and $\sigma_{ij}^Q = 0$.

Therefore, it follows

$$\begin{aligned}
&\left(1 - \omega\sigma_{ij}^Q\right)\left(y_{ij}^Q - Y_{ij}^Q\right)^+ \\
&\leq \frac{\sigma_{ij}^Q}{\delta_i^Q \alpha(s_i^Q)}\left\{\left(y_{i-1j}^Q - Y_{i-1j}^Q\right)^+ + \omega\int_{s_{i-1}^Q}^{s_i^Q} \Lambda_Q[\alpha(s) f_Q(s)]\, ds \right. \\
&\quad \left. + \int_{s_{i-1}^Q}^{s_i^Q} \Lambda_Q[\alpha(s) F_Q(s)]\, ds + \omega\int_{s_{i-1}^Q}^{s_i^Q} \Lambda_Q[\alpha(s)|\widetilde{a_Q}(s) - s|]\, ds\right\} \\
&\quad + \frac{\sigma_{ij}^Q}{\gamma_j^Q \beta(t_j^Q)}\left\{\left(y_{ij-1}^Q - Y_{ij-1}^Q\right)^+ + \omega\int_{t_{j-1}^Q}^{t_j^Q} \Lambda'_Q[\beta(t) g_Q(t)]\, dt \right. \\
&\quad \left. + \int_{t_{j-1}^Q}^{t_j^Q} \Lambda'_Q[\beta(t) G_Q(t)]\, dt + \omega\int_{t_{j-1}^Q}^{t_j^Q} \Lambda'_Q\left[\beta(t)\left|\widetilde{b_Q}(t) - t\right|\right] dt\right\}.
\end{aligned}$$

Then by recurrence on the double suffix i, j, with for instance, Q sufficient to ensure $\omega\delta_i^Q \alpha(s_i^Q) \leq \frac{1}{2}$ and $\omega\gamma_j^Q \beta(t_j^Q) \leq \frac{1}{2}$, we obtain,

$$\begin{aligned}
\left(y_{ij}^Q - Y_{ij}^Q\right)^+ \leq &\left[\prod_{k=1}^{i} \frac{1}{1 - \delta_k^Q \alpha(s_k^Q)\omega} \prod_{k=1}^{j} \frac{1}{1 - \gamma_k^Q \beta(t_k^Q)\omega}\right] \\
&\left\{M_Q + \omega\left(\int_{S_1}^{s_i^Q} \Lambda_Q[\alpha(s)(f_Q(s) + (|\widetilde{a_Q}(s) - s|))]\, ds \right.\right. \\
&\left. + \int_{T_1}^{t_j^Q} \Lambda'_Q\left[\beta(t)\left(g_Q(t) + \left(\left|\widetilde{b_Q}(t) - t\right|\right)\right)\right] dt\right) \\
&\left. + \int_{S_1}^{s_i^Q} \Lambda_Q(\alpha(s) F_Q(s))\, ds + \int_{T_1}^{t_j^Q} \Lambda'_Q(\beta(t) G_Q(t))\, dt\right\},
\end{aligned}$$

where

$$\begin{aligned}
M_Q = \sup\Big(&\sup_s \Lambda_Q\big(y_Q(s, T_1) - y^Q(s, T_1)\big)^+, \\
&\sup_t \Lambda'_Q\big(y_Q(S_1, t) - y^Q(S_1, t)\big)\Big).
\end{aligned}$$

Therefore the upper bound announced follows easily.

LEMMA 5.2. – *Let us assume* $(A_\alpha)_I$ ψ*-coherent for* $(B_\beta)_J$. *Let* u *be an* ε*-adapted solution of* $CP_A^\alpha(I, u^0)$ *(with respect to* $\mathcal{F}_{\alpha,\beta}$*) bounded by a constant* L*, with associated partition* Λ *and let* v *be an* ε*-adapted solution of* $CP_B^\beta(J, v^0)$ *(with respect to* $\mathcal{F}_{\alpha,\beta}$*) bounded by a constant* L'*, with associated partition* Λ'. *Then , we have,*

$$0 \leq \Lambda \bigotimes \Lambda' \{-D_{\alpha,\beta}(\|u(s) - v(t)\|) + \alpha(t)\beta(t)\psi(s, t, \theta_{\alpha,\beta}(s,t), \|u(s) - v(t)\|)\} \; ; \tag{17}$$

and for all $w \in D_{S_1}^{A_\alpha}$, $w' \in D_{T_1}^{B_\beta}$, $s \in I$, $t \in J$,

$$\begin{cases} \Lambda(\|u(s) - v^0\|) \leq \|u^0 - w\| + \|v^0 - w\| + \varepsilon + (\Lambda(a(s) - a(S_1)) + \varepsilon)\|\widetilde{w_\alpha}\| \\ \qquad + (L + 1 + \|w\|)\left(\int_{S_1}^{\Lambda(s)} \Lambda[\alpha(\sigma)c(\sigma)]\, d\sigma\right) \\ \Lambda'(\|u^0 - v(t)\|) \leq \|u^0 - w'\| + \|v^0 - w'\| + \eta + (\Lambda'(b(t) - b(T_1)) + \varepsilon)\|\widetilde{w_\beta}\| \\ \qquad + (L' + 1 + \|w'\|)\left(\int_{T_1}^{\Lambda'(t)} \Lambda'[\beta(\tau)c(\tau)]\, d\tau\right). \end{cases} \tag{18}$$

Proof. – i) Let $\Lambda = (s_0, ..., s_{R+1})$ be the partition associated with the ε–adapted solution u , and let $\Lambda' = (t_0, ..., t_{P+1})$ be the partition associated with the η–adapted solution v , and let us write for the sake of simplicity $u_i = u(s_i)$, $v_j = v(t_j)$; then we have with $\delta_i = s_i - s_{i-1}$ and $i = 1, ..., N$, and $\gamma_j = t_j - t_{j-1}$ and $j = 1, ..., P$,

$$\begin{array}{ll} u_i - u_{i-1} + \delta_i \alpha(s_i)\widehat{u_i} \ni 0 \quad , & \widehat{u_i} \in A_\alpha u_i \text{ and,} \\ v_j - v_{j-1} + \gamma_j \beta(t_j)\widehat{v_j} \ni 0 \quad , & \widehat{v_j} \in B_\beta v_j \,. \end{array} \tag{19}$$

Let $d(s,t) = \|u(s) - v(t)\|$ and $d_{i,j} = d(s_i, t_j)$. Then we obtain from the coherence of (A_α) with (B_β),

$$\begin{aligned} &(\delta_i \alpha(s_i) + \gamma_j \beta(t_j)) d_{i,j} \\ &\leq \|(\delta_i \alpha(s_i) + \gamma_j \beta(t_j))(u_i - v_j) + (\delta_i \alpha(s_i)\gamma_j \beta(t_j))(\widehat{u_i} - \widehat{v_j})\| \\ &\quad + \delta_i \alpha(s_i)\gamma_j \beta(t_j)\psi(s_i, t_j, \theta(s_i, t_j), d_{i,j}) \,. \end{aligned}$$

With the equations (19) it follows easily from the above inequality, the required relation,

$$0 \leq \Lambda \bigotimes \Lambda' [-D_{\alpha,\beta} d(s,t) + \alpha(s)\beta(t)\psi(s, t, \theta(s,t), x)] \,. \tag{20}$$

According to the inequality lying in the stability condition $\mathcal{S}(A_\alpha, I)$, we can write,

$$\begin{aligned}\|u_i - w\| &\leq \delta_i\alpha(s_i)c(s_i)(\|u_i - w\| + 1) + \|u_i + \delta_i\alpha(s_i)\widehat{u_i} - w - \delta_i\alpha(s_i)\widetilde{w_\alpha}\| \\ &\leq \delta_i\alpha(s_i)c(s_i)(L + 1 + \|w\|) + \|u_{i-1} - w\| + \delta_i\alpha(s_i)\|\widetilde{w_\alpha}\| .\end{aligned}$$

Hence, by iteration, the claim of Lemma 5.2 is easily ended.

Remark 5.1. – *The inequality (20) could be rewritten in the following form,*

$$\begin{aligned} d_{ij} \leq \frac{\delta_i\alpha(s_i)}{\delta_i\alpha(s_i) + \gamma_j\beta(t_j)} d_{ij-1} + \frac{\gamma_j\beta(t_j)}{\delta_i\alpha(s_i) + \gamma_j\beta(t_j)} d_{i-1j} \\ + \frac{\delta_i\alpha(s_i)\gamma_j\beta(t_j)}{\delta_i\alpha(s_i) + \gamma_j\beta(t_j)} \psi(s_i, t_j, \theta(s_i, t_j), d_{i,j}) \text{ with,}\end{aligned}$$

a) $d_{ij} = d_{i-1j}$ if $\alpha(s_i) = 0$; b) $d_{ij} = d_{ij-1}$ if $\beta(t_j) = 0$; c) $d_{ij} = d_{i-1j} = d_{ij-1} = d_{i-1j-1}$ if $\alpha(s_i) = \beta(t_j) = 0$.

Let us state now the maximum principle which remains valid if $\alpha \equiv 0$ on I or $\beta \equiv 0$ on J.

THEOREM 5.1 (maximum principle). – *Assume $(A_\alpha)_I$ ψ–coherent for $(B_\beta)_J$. Let also the sequence $(u^Q)_{Q\in\mathbb{N}}$ be a $(\Lambda_Q)_Q$-* ***DAF*** *of $CP_A^\alpha(I, u^0)$ and let the sequence $(v^Q)_{Q\in\mathbb{N}}$, be a $(\Lambda'_Q)_Q$-* ***DAF*** *of $CP_B^\beta(J, v^0)$. Let $\mathcal{B}$ be continuous on $a(I) - b(J)$. We suppose,*

$$\begin{cases} \limsup\limits_Q \sup\limits_{s\in I} \left(\left\|u^Q(s) - v^0\right\| - \mathcal{B}(a(s) - b(T_1))\right) \leq 0 \\ \limsup\limits_Q \sup\limits_{t\in J} \left(\left\|u^0 - v^Q(t)\right\| - \mathcal{B}(a(S_1) - b(t))\right) \leq 0 \end{cases} \tag{21}$$

Then, for all $d \in a(I) - b(J) \subset [-b(T_2), a(S_2)]$, we have,

$$\limsup_Q \sup_{a(s)-b(t)=d} \Lambda \bigotimes \Lambda' \left[\left\|u^Q(s) - v^Q(t)\right\| - m_{\mathcal{B}}(a(s), b(t))\right] \leq 0 .$$

Proof of Theorem 5.1. – With the stability condition, it exists a constant $M > 0$ such that the sequences of functions $(\|u^Q\|)_Q$ and $(\|v^Q\|)_Q$ are bounded by M (resp. on I and J). Let $(\mathcal{B}_k)$ be a sequence of C^1 functions on the compact interval $a(I) - b(J)$, converging uniformly to $\mathcal{B}$; and let $\mathcal{F} = (\theta^k, F^k, F)_k$ be the considered sequence stemmed from $\theta_{\alpha,\beta}$. Let $(\alpha_n)_n$ and $(\beta_n)_n$ respectively, continuous strictly positive on I (resp. J) converging in $L^1([0,T], \mathbb{R}^+)$ respectively towards α and β. Let us recall that $a_n(s) = \int_0^s \alpha_n(\xi)\, d\xi$ and $b_n(t) = \int_0^t \beta_n(\xi)\, d\xi$. We can suppose, $a_n(S_1) = a(S_1)$, $a_n(S_2) > a(S_2)$, and $b_n(T_1) = b(T_1)$, $b_n(T_2) > b(T_2)$. Let $(\Phi_r)_r$ be a decreasing sequence of positive C^1 functions pointwise convergent towards ψ with Φ_r ω_r-Lipschitz on $[0,T]^2 \times \mathbb{R}^2$.We also introduce the following notations, (for $d \in a(I) - b(J)$, and $\tau \in I_d = [(a(S_1) - d) \vee b(T_1), b(T_2) \wedge (a(S_2) - d)])$

$$\begin{aligned} y_d^{k,r}(\tau) &= m\big(\Phi_r, \theta^k, \mathcal{B}_k, \alpha, \beta\big)(\tau + d, \tau) \\ y_d^{k,r,n}(\tau) &= m\big(\Phi_r, \theta^k, \mathcal{B}_k, \alpha_n, \beta_n\big)(\tau + d, \tau) \\ y_d(\tau) &= m(\psi, \theta_{\alpha,\beta}, \mathcal{B}, \alpha, \beta)(\tau + d, \tau) = m_{\mathcal{B}}(\tau + d, \tau), \\ e_Q(s,t) &= \big\|u^Q(s) - v^Q(t)\big\| \quad \forall (s,t) \in I \times J. \end{aligned}$$

We can suppose without lost of generality that $m\big(\Phi_r, \theta^k, \mathcal{B}_k, \alpha_n, \beta_n\big) = m^{k,r,n}$ is C^1 on $[0,T]^2$, because of the u.s.c. hull lemma 4.1-iv) and the proof part c) of this lemma 4.1.

Then, for all $d \in a(I) - b(J) \subset [-b(T_2), a(S_2)]$, and for $(s,t) \in I \times J$, we have, the following inequality,

$$\begin{aligned} &\sup_{a(s)-b(t)=d} \Big[\Lambda_Q \bigotimes \Lambda'_Q [e_Q(s,t)] - m_{\mathcal{B}}(a(s), b(t))\Big] \\ &= \sup_{a(s)-b(t)=d} \Big[\Lambda_Q \bigotimes \Lambda'_Q [e_Q(s,t)] - y_d(b(t))\Big] \\ &\leq \sup_{a(s)-b(t)=d} \Lambda_Q \bigotimes \Lambda'_Q \Big[e_Q(s,t) - y_d^{k,r,n}(b(t))\Big] \quad (1) \qquad (22) \\ &+ \sup_t \Big[\Lambda_Q \bigotimes \Lambda'_Q \Big\{y_d^{k,r,n}(b(t))\Big\} - y_d^{k,r,n}(b(t))\Big] \quad (2) \\ &+ \sup_t \Big[y_d^{k,r,n}(b(t)) - y_d(b(t))\Big] \quad (3) \end{aligned}$$

For all n, k, and $Q \geq Q'_n$ in order to have $a^Q(I) \times b^Q(J) \subseteq a_n(I) \times b_n(J)$ define, the following functions for $s, t \in [0,T]$

$$\begin{cases} a^Q(s) = a(S_1 \wedge s) + \displaystyle\int_{S_1}^{\Lambda_Q(s) \vee S_1} \Lambda_Q[\alpha(\xi)] d\xi \\ b^Q(t) = b(T_1 \wedge t) + \int_{T_1}^{\Lambda'_Q(t) \vee T_1} \Lambda'_Q[\beta(\xi)] d\xi, \\ \widetilde{a_n^Q} = a_n^{-1} \circ a^Q, \text{ and}, \widetilde{b_n^Q} = b_n^{-1} \circ b^Q, \\ \theta_{n,k}^Q(s,t) = \theta^k\Big(\widetilde{a_n^Q}(s), \widetilde{b_n^Q}(t)\Big). \end{cases}$$

If θ^k is ρ_k-Lipschitz (with $\rho_k \geq 0$), we have,

$$\begin{aligned} &\Big|\Lambda_Q \bigotimes \Lambda'_Q \Big[\theta(s,t) - \theta_{n,k}^Q(s,t)\Big]\Big| \\ &\leq \Lambda_Q \bigotimes \Lambda'_Q \big|\theta(s,t) - \theta^k(s,t)\big| + \Lambda_Q \bigotimes \Lambda'_Q \Big|\theta^k(s,t) - \theta_{n,k}^Q(s,t)\Big| \\ &\leq \Lambda_Q F^k(s) + \Lambda'_Q F^k(t) + \rho_k \Big[\Big|\Lambda_Q(s) - \widetilde{a_n^Q}(s)\Big| + \Big|\Lambda'_Q(t) - \widetilde{b_n^Q}(t)\Big|\Big] \end{aligned}$$

Let us define $\mu_Q^{k,n}$ on $I \times J$ by

$$\mu_Q^{k,n}(s,t) = \Lambda_Q \bigotimes \Lambda'_Q \big[-D_{\alpha,\beta}\big(m^{k,r,n}\big(a^Q(s), b^Q(t)\big)\big) \\ +\alpha(s)\beta(t)\Phi_r\bigg(\widetilde{a_n^Q}(s), \widetilde{b_n^Q}(t), \theta_k\bigg(\widetilde{a_n^Q}(s), \widetilde{b_n^Q}(t)\bigg), m^{k,r,n}\big(a^Q(s), b^Q(t)\big)\bigg)\bigg].$$

Since $m^{k,r,n}$ is C^1 and $\theta_k(a_n^{-1}(\sigma), b_n^{-1}(\tau))$ is continuous (even C^1) on $I \times J$, an immediate computation gives

$$\limsup_Q \sup_{(s,t)\in I\times J} \left|\mu_Q^{k,n}(s,t)\right| \leq \|\alpha\|_\infty \|\beta\|_\infty \sup_{s,t} \big| -D_{1,1} m^{r,k,n}(a(s), b(t)) \\ + \Phi_r\big(a_n^{-1} \circ a(s), b_n^{-1} \circ b(s), \theta_k\big(a_n^{-1} \circ a(s), b_n^{-1} \circ b(s)\big), \\ m^{r,k,n}(a(s), b(t))\big)\big|.$$

By definition of $m^{r,k,n}$ it follows :

$$\limsup_Q \sup_{(s,t)\in I\times J} \left|\mu_Q^{k,n}(s,t)\right| = 0, \text{ and then, } \lim_Q \left\|\mu_Q^{k,n}\right\|_* = 0.$$

Thus, setting,

$$h_n = \int_{S_1}^{S_2} \alpha(\xi)\big|\xi - a_n^{-1} \circ a(\xi)\big|\, d\xi \ \text{ and } h'_n = \int_{T_1}^{T_2} \beta(\xi)\big|\xi - b_n^{-1} \circ b(\xi)\big|\, d\xi,$$

and,

$$\lambda_k^n = \int_{S_1}^{S_2} \alpha(\xi) F^k(\xi)\, d\xi + \int_{T_1}^{T_2} \beta(\xi) F^k(\xi)\, d\xi + \rho_k(h_n + h'_n),$$

from the discrete lemma 5.1 we obtain,

$$\limsup_Q \sup_{s,t} \Lambda_Q \bigotimes \Lambda'_Q \big[e_Q(s,t) - m^{k,r,n}(a^Q(s), b^Q(t))\big]^+ \\ = \limsup_Q \sup_{s,t} \Lambda_Q \bigotimes \Lambda'_Q \big[e_Q(s,t) - m^{k,r,n}(a(s), b(t))\big]^+ \\ \leq e^{\frac{3}{2}\omega_r(a(S_2)+b(T_2))}((\|\alpha\|_\infty + \|\beta\|_\infty)\omega_r \lambda_k^n + \omega_r(h_1^n + h_2^n) + b_k),$$

with $b_k = \sup_{d\in[-b(T_2), a(S_2)]} \|\mathcal{B}_k(d) - \mathcal{B}(d)\|$. Since it is not difficult to show the relations,

$$h_1^n = \int_I \alpha(s)\big|a_n^{-1}(a(s)) - s\big| ds = \int_{a(I)} \big|a_n^{-1}(\sigma) - a^{-1}(\sigma)\big|\, d\sigma \underset{n\to\infty}{\longrightarrow} 0$$

$$h_2^n = \int_I \beta(t)\big|b_n^{-1}(b(t)) - t\big| dt = \int_{b(J)} \big|b_n^{-1}(\tau) - b^{-1}(\tau)\big|\, d\tau \underset{n\to\infty}{\longrightarrow} 0,$$

we obtain, if λ_k stands for $\lambda_k = \int_{S_1}^{S_2} \alpha(\xi)F^k(\xi)\, d\xi + \int_{T_1}^{T_2} \beta(\xi)F^k(\xi)\, d\xi$,

$$\limsup_n \limsup_Q \sup_{s,t} \Lambda_Q \bigotimes \Lambda'_Q \left[e_Q(s,t) - m^{k,r,n}(a(s), b(t))\right]^+ \\ \leq e^{\frac{3}{2}\omega_r(a(S_2)+b(T_2))}((\|\alpha\|_\infty + \|\beta\|_\infty)\omega_r \lambda_k + b_k),$$

Therefore clearly now, when $Q \to \infty$, then $n \to \infty$, and then $k \to \infty$, and then finally $r \to \infty$, using the u.s.c. hull lemma 4.1 (and according to the uniform continuity of $m^{k,r,n}$ on $a(I) \times b(J)$) all the terms (1) to (3) in the right side of the inequality (22) vanish and we get, for all $d \in a(I) - b(J)$,

$$\limsup_Q \sup_{a(s)-b(t)=d} (e_Q(s,t) - m(a(s), b(t))) \leq 0.$$

5.2. Proof of theorem 3.2

The function $\mathcal{B}$ satisfying (8) is single valued continuous since $u \circ a^{-1}$and $v \circ b^{-1}$ are single valued continuous and since for $(\sigma, \tau) \in a(I) \times b(J)$, the equality $\sigma - b(T_1) = a(S_1) - \tau$ is equivalent to $\sigma = a(S_1)$ and $\tau = b(T_1)$. Let (u_Q) be a suitable $(\Lambda_Q)_Q$-**DAF** associated with u, and a suitable $(\Lambda'_Q)_Q$-**DAF** (v_Q) converging towards v. Let us write $e_Q(s,t) = \|u_Q(s) - v_Q(t)\|$, and $e(s,t) = \|u(s) - v(t)\|$. The inequality (17) written with u_Q and v_Q leads (multiplying by $\Lambda_Q \bigotimes \Lambda'_Q[\varphi(s,t)]$ with $\varphi \in \mathcal{D}\Big((\overset{\circ}{I} \times \overset{\circ}{J})\Big)$ and $\varphi \geq 0$ and rearranging),

$$0 \leq \int_{I\times J} \Big\{ -e_Q(s,t) \Big(\mathbf{I} \bigotimes \Lambda'_Q \left[\beta(t) \frac{\partial \varphi}{\partial s}\right] + \Lambda_Q \bigotimes \mathbf{I} \left[\alpha(s) \frac{\partial \varphi}{\partial t}\right] \Big)(s,t) \\ + \Lambda_Q \bigotimes \Lambda'_Q [\alpha(s)\beta(t)\psi(s,t,\theta_{\alpha,\beta}(s,t), e_Q(s,t))\varphi(s,t)] \Big\} ds\, dt.$$

Thus, taking the limit when $Q \to +\infty$, the Lebesgue dominated theorem (since the **DAF** are adapted with respect to $\mathcal{F}_{\alpha,\beta}$) yields in $\mathcal{D}'\Big(\overset{\circ}{I} \times \overset{\circ}{J}\Big)$,

$$D_{\alpha,\beta}[e(s,t)] \leq \alpha(s)\beta(t)\psi(s,t,\theta(s,t), e(s,t))$$

Since $e(\cdot,\cdot)$ is continuous on $I \times J$, and of course constant on each set values taken by (a^{-1}, b^{-1}), thanks to the lemma 4.1 ii), the relation 9 in the theorem 3.2 is shown.

5.3. Proof of theorem 3.1

Let $(u_n)_n$ be a $(\mathcal{F}_{\alpha,\alpha}; (\varepsilon_n))$-**DAF**. Set $\mathcal{F}_{\alpha,\alpha} = \mathcal{F} = \left(\theta^k, F^k, F\right)_k$. Because of the stability condition $\mathcal{S}(\alpha, I)$, (u_n) is bounded by a constant

$M > 0$. Let $(\Phi_r)_r$ be a decreasing sequence of functions $(\Phi_r)_r$ on $\Delta = [0,T]^2 \times \mathbb{R}^2$, converging towards ψ on Δ, such that for all $r \in \mathbb{N}$, Φ_r is C^1 and ω_r–Lipschitz. Let $e_Q(s,t) = \|u_Q(s) - v_{k_Q}(t)\|$ where (k_Q) is a sequence of strictly increasing integers. Given $\varepsilon > 0$, let $w \in D_{S_1}^{A_\alpha}$ satisfying $\|u^0 - w\| \leq \varepsilon$ and let $\widetilde{w_{S_1}} \in X$ given by the stability condition $\mathcal{S}(\alpha, I)$; let $\mathcal{B}_\varepsilon$ be the continuous function on $a(I) - a(I) = [-(a(S_2) - a(S_1)),\, a(S_2) - a(S_1)]$ defined by,

$$\mathcal{B}_\varepsilon(x) = 2\varepsilon + \|\widetilde{w_{S_1}}\||x| + \left[M_1 + 1 + \|u^0\| + \varepsilon\right] \int_{a(S_1)}^{|x|+a(S_1)} c\big(a^{-1}(\sigma)\big)\, d\sigma\,.$$

According to the lemma 5.2 (relation (18)) and the definition of $\mathcal{B}_\varepsilon$ above, we have,

$$\begin{cases} \limsup_Q \sup_s \left(e_Q((s,S_1) - \mathcal{B}_\varepsilon(a(s) - a(S_1))\right) \leq 0\,, \text{ and,} \\ \limsup_Q \sup_t \left(e_Q((S_1,t) - \mathcal{B}_\varepsilon(a(S_1) - a(t))\right) \leq 0\,, \end{cases}$$

Let $\eta > 0$, then the maximum principle (with $a = b$, $d = 0$, see theorem 5.1) implies that there exists $Q \geq N(\varepsilon, \eta)$ such that ,

$$\sup_{s \in I} \left(e_Q(s,s) - y^\varepsilon(a(s))\right) \leq \eta\,. \tag{23}$$

where we have $y^\varepsilon(\sigma) = m(\psi, \theta, \mathcal{B}_\varepsilon)(\sigma, \sigma)$.

Moreover, the function y^ε satisfies in $\mathcal{D}'(]a(S_1), a(S_2)[)$ the inequation $\chi_0(\leq, \Phi, \theta, \varepsilon, \alpha)$. In view of the definition of the strong coherence (see definition 2.2) and Lemma 6.3 stated in Appendix, the family $(y^\varepsilon)_{\varepsilon > 0}$ converges uniformly on $[a(S_1), a(S_2)]$ towards the null function as $\varepsilon \downarrow 0$. Therefore, the relation (23) provides,

$$\lim_Q \sup_{s \in I} e_Q(s,s) = 0\,.$$

It follows that $(u_n)_n$ is a Cauchy sequence in the set of bounded functions on I endowed with the supremum norm. Let us denote by u the uniform limit of $(u_n)_n$, on I. Now, we have to prove the continuity of u (because the uniqueness of the $\mathcal{F}$-**mas**, is now obvious) to obtain that u is the unique $\mathcal{F}$-**mas** of $CP_A^\alpha(I, u^0)$.

For that, let $e(s,t) = \|u(s) - u(t)\|$, then the maximum principle (see theorem 5.1) allows to write, for all $\varepsilon > 0$ and for $s_0,\ s_0 + h \in I$,

$$0 \leq e(s_0 + h, s_0) \leq m(\psi, \theta_{\alpha,\alpha}, \mathcal{B}_\varepsilon)(a(s_0 + h), a(s_0))\,.$$

Therefore, according to the u.s.c. aspect of $m(\Phi, \theta, \mathcal{B}_\varepsilon)$, we have,

$$0 \leq \limsup_{h \to 0} e(s_0 + h, s_0) \leq m(\psi, \theta_{\alpha,\alpha}, \mathcal{B}_\varepsilon)(a(s_0), a(s_0)) = y^\varepsilon(a(s_0)).$$

Then, as $\varepsilon \downarrow 0$, we obtain

$$\lim_{h \to 0} e(s_0 + h, s_0) = 0,$$

So the part (i) of the theorem is proved. It remains to prove the second part.

b) For this purpose let $\alpha = \widehat{\alpha}$ in $L^\infty([0,T], \mathbb{R}^+)$ and let $\mathcal{F} = \mathcal{F}_{\alpha,\alpha} = (\theta_k, F_k, F)$ and $\widehat{\mathcal{F}} = \widehat{\mathcal{F}_{\widehat{\alpha},\widehat{\alpha}}} = \left(\widehat{\theta_k}, \widehat{F_k}, \widehat{F}\right)$ be stemmed from $\theta_{\alpha,\alpha} = \theta_{\widehat{\alpha},\widehat{\alpha}} = \theta$. Let $(u_Q)_Q$ be a **DAF** related to $(\alpha, \mathcal{F})$ and $(v_Q)_Q$ be a **DAF** related to $\left(\widehat{\alpha}, \widehat{\mathcal{F}}\right)$. Now , we consider a **DAF** $(w_Q)_Q$ both related to $(\alpha, \mathcal{F})$ and $\left(\widehat{\alpha}, \widehat{\mathcal{F}}\right)$. By the virtue of Lemma 2.1, such a choice is made possible. Then in view of the part (i) of Theorem 3.1 we claim that $(w_Q)_Q$ converges towards a **mas** w of $CP_A^\alpha(I, u^0)$. But we have $u = w$ since u and w are both $\mathcal{F}$-strongly coherent **mas** of $CP_A^\alpha(I, u^0)$. In the same way we deduce $v = w$ and therefore $u = v$. Hence the proof is complete.

5.4. Proof of corollary 3.1

Let $I_\alpha = [0, a(T)]$ and $I_1 = [0, T]$. According to the remark 2.3, we have,

$$\psi(s, t, \xi, x) = \psi(x).$$

By theorem 3.1 the **mas** u of $CP_A^\alpha(I_1, u^0)$ is coherent for the **mas** v of $CP_A^1(I_\alpha, u^0)$. Because of the estimate of Theorem 3.2, we obtain ($m_\mathcal{B}$ is defined on $(\mathbb{R}^+)^2$),

$$\|u(s) - v(t)\| \leq m_\mathcal{B}(a(s), t) \tag{24}$$

where $\mathcal{B}$ is the continuous function satisfying,

$$\mathcal{B}(d) = \left\|u\left(a^{-1}(d)\right) - u^0\right\| \text{ and } \mathcal{B}(-d) = \left\|u^0 - v(d)\right\|,$$

for $d \in [0, a(T)]$. Then $y_0(t) = m_\mathcal{B}(t, t)$ satisfies $\chi_0(\leq, \psi, \theta, 0, \alpha, 1)$ in $\mathcal{D}'(]0, a(T)[)$ .Thus, we obtain $y_0(t) = m_\mathcal{B}(t, t) = 0$ on $[0, a(T)]$. Then, the inequality (24) gives, $u(s) = v(a(s))$ for all $s \in [0, T]$.

The last conclusion of this corollary is evident, since with our hypotheses, v is C^1 on $[0, T]$ (see for instance [8]).

5.5. Proof of Theorem 3.3

Only the assertion (iii) in the part a) of theorem 3.3 is non trivial. In order to show this assertion, put $u(t) = S(t,r)u^0$ for $t \in [r,T]$ and some $u^0 \in \overline{D_r^{A_\alpha}}$ and $v(t) = S(t,s)u(s)$ for $t \in [s,T]$; then define the continuous function $\mathcal{B}$ on $[-(a(T)-a(s)), a(T)-a(s)]$ by,

$$\begin{cases} \mathcal{B}(\sigma - a(s)) = \|v(a^{-1}(\sigma)) - u(s)\| & \text{for } \sigma \in [a(s), a(T)]\,, \\ \mathcal{B}(a(s) - \tau) = \|u(s) - u(a^{-1}(\tau))\| & \text{for } \tau \in [a(s), a(T)]\,. \end{cases}$$

It follows from the strong coherence that we have $m_{\mathcal{B}}(\tau,\tau) = 0$ on $[-a(T), a(T)]$. Hence, theorem 3.2 applied with v and $u|_{[s,T]}$ yields

$$\|v(t) - u(t)\| \leq m_{\mathcal{B}}(a(t), a(t)) = 0 \text{ for all } t \in [s,T]\,.$$

Now, turn to the part b) of the theorem 3.3. Let $(s_n, t_n, w_n)_{n\in\mathbb{N}^*}$ be a sequence converging towards (s,t,w) in the metric space Y. Given $\varepsilon > 0$, let w^ε be such that we have, $w^\varepsilon \in D_s^{A_\alpha}$ and $\|w - w^\varepsilon\| \leq \varepsilon$. Then, let $\mathcal{B}^\varepsilon$ be the function defined on $[-T,T]$ by,

$$\mathcal{B}^\varepsilon(x) = 2\varepsilon + \left\|\widetilde{w^\varepsilon}\right\||x| + [M + \|w^\varepsilon\| + 2\varepsilon + 1] \int_{(a(s)-\varepsilon)\vee 0}^{|x|+a(s)+\varepsilon} c\big(a^{-1}(\sigma)\big)\, d\sigma\,,$$

where, $\widetilde{w^\varepsilon}$ is the element of X provided by the stability condition $\mathcal{S}(\alpha, I)$, and where $M < +\infty$ is an upper bound of $(\|S(.,s_n)w_n\|)_n$. For $N_\varepsilon \in \mathbb{N}^*$, satisfying,

$$n \geq N_\varepsilon \Rightarrow (\|w_n - w\| \leq \varepsilon\,, \text{ and } |a(s) - a(s_n)| \leq \varepsilon)\,,$$

the maximum principle theorem 5.3 yields ,

$$n \geq N_\varepsilon \Rightarrow [\|S(t_n, s_n)w_n - S(t,s)w\| \leq m(\theta_{\alpha,\alpha}, \mathcal{B}^\varepsilon)(a(t_n), a(t))]\,.$$

Consequently, in view of the u.s.c. aspect of $m(\theta, \mathcal{B}^\varepsilon)$, it follows, for all $\varepsilon > 0$,

$$\limsup_n \|S(t_n, s_n)w_n - S(t,s)w\| \leq m(\theta_{\alpha,\alpha}, \mathcal{B}^\varepsilon)(a(t), a(t)) = y^\varepsilon(a(t))\,.$$

Since $(y^\varepsilon)_\varepsilon$ decreases uniformly towards the null function on $a([s,T])$, when $\varepsilon \downarrow 0$ (see Lemma 6.3 in Appendix), it results,

$$\lim_n \|S(t_n, s_n)w_n - S(t,s)w\| = 0\,.$$

The proof is now complete.

5.6. Proof of Theorem 3.4

For each $n \in \mathbb{N} \cup \{\infty\}$, let u_n be the function $u_n = S_n(.,0)$, and let $\theta_n = \theta_{\alpha_n,\alpha_n}$.

a) The family $(u_n)_{n \in \mathbb{N}\cup\{\infty\}}$ is bounded by some constant $C > 0$ in $C^0([0,T],X)$, by hypothesis.

b) We give now a suitable bound for $\|u_n(s) - u_n^0\|$ for $n \in \mathbb{N}$. Given $\varepsilon > 0$, let $(w,\widehat{w}) \in A^{\infty}_{\alpha_\infty}(0)$, with $\|w - u^0_\infty\| \le \frac{\varepsilon}{2}$. According to the definition of the inferior limit, there exists for each $n \in \mathbb{N}$, $(w_n, \widehat{w_n}) \in A^n_{\alpha_n}(0)$, such as the sequence $((w_n,\widehat{w_n}))_n$ converges towards $(w,\widehat{w})$. Let us define an integer N_ϵ verifying,

$$n \ge N_\epsilon \Rightarrow \left(\left\|w_n - u^0_\infty\right\| + \left\|w - u^0_\infty\right\| + \left\|u^0_\infty - u^0_n\right\| \le \varepsilon\right).$$

The condition $\mathcal{S}(\alpha_n, c, [0,T])$ provides again, (see inequalities (18) in lemma 5.2)

$$\|u_n(s) - w_n\| \le \left\|w_n - u^0_n\right\| + a_n(s)\|\widehat{w_n}\| + (C+1+\|w_n\|)\int_0^{a_n(s)} c\big(a_n^{-1}(\sigma)\big)\,d\sigma.$$

Recall that we have $a_n(s) = \int_0^s \alpha_n(\sigma)\,d\sigma$ and, (from the change of variable lemma 6.1)

$$\int_0^{a_n(s)} c\big(a_n^{-1}(\sigma)\big)\,d\sigma = \int_0^s \alpha_n(\xi)c(\xi)\,d\xi.$$

Consequently, since $(\alpha_n)_n$ converges towards α_∞ in the weak* topology of $L^\infty([0,T],\mathbb{R})$, there exist a sequence of positive numbers $(\eta_n)_n$ converging towards zero and an integer $P_\varepsilon \ge N_\epsilon$ realizing, for all $s \in [0,T]$,

$$n \ge P_\epsilon \Rightarrow \left(\left\|u_n(s) - u^0_n\right\| \le 2\varepsilon + C_\varepsilon\left(a_\infty(s) + \int_0^{a_\infty(s)} c\big(a_\infty^{-1}(\sigma)\big)\,d\sigma + \eta_n\right)\right), \tag{25}$$

where, we put for instance, $C_\varepsilon = \sup_n\left(\|\widehat{w_n}\| + C + 1 + \|w_n\|\right)$.

c) Equicontinuity of $(u_n)_n$ in $C^0([0,T],X)$. For $x \in [-T,T]$, and $\eta > 0$, let $\mathcal{B}^\eta_\varepsilon$ be defined by ,

$$\mathcal{B}^\eta_\varepsilon(x) = 2\varepsilon + C_\varepsilon\left(|x| + \int_0^{|x|} c\big(a_\infty^{-1}(\sigma)\big)\,d\sigma + \eta\right).$$

Let $m_n^{\varepsilon,r} = m(\Phi_r, \theta_n, \mathcal{B}^{\eta_n}_\varepsilon, \alpha_n, \alpha_n)$ for $n \in \mathbb{N} \cup \{\infty\}$, and put $m_n^\varepsilon = m(\psi, \theta_n, \mathcal{B}^{\eta_n}_\varepsilon, \alpha_n, \alpha_n)$. The maximum principle (see theorem 5.1) gives, for $n \ge P_\varepsilon$,

$$\|u_n(s) - u_n(t)\| \le m_n^\varepsilon(a_n(s), a_n(t)) \le m_n^{\varepsilon,r}(a_n(s), a_n(t)). \tag{26}$$

The equicontinuity lemma 6.2 ensures that for all $\eta > 0$, there exists an integer $n(\varepsilon, r, \eta)$ such as we have,

$$n \geq n(\varepsilon, r, \eta) \Rightarrow (\|u_n(s) - u_n(t)\| \leq m_\infty^{\varepsilon,r}(a_\infty(s), a_\infty(t)) + \eta) . \quad (27)$$

Let y^ε be the maximal continuous solution in $\mathcal{D}'(]0, a(T)[)$ of $\chi_0(=, \psi, \theta, 2\varepsilon, \alpha)$. By lemma 4.1-vii), we get that the sequence $(m_\infty^{\varepsilon,r}(\tau,\tau))_r$ converges uniformly on $[0, a_\infty(T)]$ towards y^ε. Moreover, in view of the lemma 6.3 the generalized sequence $(y^\varepsilon)_{\varepsilon>0}$ converges uniformly on $[0, a_\infty(T)]$, towards zero as $\varepsilon \downarrow 0$. Then, from

$$\begin{aligned} m_\infty^{\varepsilon,r}(a_\infty(s), a_\infty(t)) \leq & |m_\infty^{\varepsilon,r}(a_\infty(s), a_\infty(t)) - m_\infty^{\varepsilon,r}(a_\infty(t), a_\infty(t))| \\ & + |m_\infty^{\varepsilon,r}(a_\infty(t), a_\infty(t))| , \end{aligned}$$

the relation (27) yields (choosing first a suitable ε_η and second a suitable r_η) to the existence of an integer $R_\eta = n(\varepsilon_\eta, r_\eta, \eta)$ such that,

$$\begin{aligned} n \geq R_\eta \Rightarrow & (\|u_n(s) - u_n(t)\| \\ & \leq |m_\infty^{\varepsilon_\eta,r_\eta}(a_\infty(s), a_\infty(t)) - m_\infty^{\varepsilon_\eta,r_\eta}(a_\infty(t), a_\infty(t))| + 2\eta) . \quad (28) \end{aligned}$$

Since $m_\infty^{\varepsilon_\eta,r_\eta}$ is continuous on $[0, a(T)]^2$, then the announced equicontinuity holds.

d) Convergence of $(u_n)_n$. Let $(u^Q)_Q$ be an adapted $(\Lambda_Q)_Q$-**DAF** of $CP_{A^\infty}^{\alpha_\infty}([0,T], u_\infty^0)$. Let $\mathcal{K}_0$ be the following compact subset of X,

$$\mathcal{K}_0 = \overline{u_\infty([0,T]) \bigcup \left(\bigcup_Q u^Q([0,T]) \right)} \; ;$$

put $g_n(t, v) = \|u_n(t) - v\|$ for $(t, v) \in [0,T] \times X$. According to the part b) of this proof, it is immediate to verify that $(g_n)_n$ is an equicontinuous sequence of continuous functions on $[0,T] \times X$. By the Ascoli-Arzelà theorem $(g_n)_n$ is relatively compact in $C^0([0,T] \times \mathcal{K}_0, \mathbb{R})$. Then, consider a cluster point $g = \lim g_{n_k}|_{[0,T]\times\mathcal{K}_0}$ of $(g_n)_n$. Let $\varepsilon > 0$ and Q be an integer. Set $\Lambda_Q = \left(t_0^Q, ..., t_{N_{Q+1}}^Q\right)$, with nodal points in the set of $s \in [0,T]$ such that $A_{\alpha_\infty}^\infty(s) \subset \liminf_n A_n^{\alpha_n}(s)$ and,

$$u^Q(t_i^Q) - u^Q(t_{i-1}^Q) + \delta_i^Q \alpha(t_i^Q)\widehat{u^Q}(t_i^Q) = 0 ,$$

with $\delta_i^Q = t_i^Q - t_{i-1}^Q$, and $\widehat{u^Q}(t_i^Q) \in A_{\alpha_\infty}^\infty(t_i^Q) u^Q(t_i^Q)$. For $i = 1, ..., N_Q$ choose,

$$\begin{cases} \lim_n \left(w_n^Q(t_i^Q), \widehat{w_n^Q}(t_i^Q) \right) = \left(u^Q(t_i^Q), \widehat{u^Q}(t_i^Q) \right) \\ \text{with, } \left(w_n^Q(t_i^Q), \widehat{w_n^Q}(t_i^Q) \right) \in A_{\alpha_n}^n(t_i^Q) . \end{cases}$$

We can find an integer $r(Q,\varepsilon)$ verifying, (with $M = \sup_{n\in\mathbb{N}\cup\{\infty\}} \|\alpha_n\|_\infty$),

$$n \geq r(Q,\varepsilon) \Rightarrow \left(\left\| w_n^Q(t_i^Q) - u^Q(t_i^Q) \right\| + M \left\| \widehat{w_n^Q}(t_i^Q) - \widehat{u^Q}(t_i^Q) \right\| \leq \varepsilon \inf_i \delta_i^Q \right),$$

If $\varphi \in \mathcal{D}\Big(]0,T[^2\Big)$, and $\varphi \geq 0$, for Q large enough and $n \geq r(Q,\varepsilon)$, it follows from a simple computation,

$$\begin{cases} 0 \leq \displaystyle\int_{[0,T]^2} g_n\big(s, w_n^Q(t)\big)[I \otimes \Lambda_Q]\left(\alpha_n(t)\frac{\partial\varphi}{\partial s}(s,t) + \alpha_n(s)\frac{\partial\varphi}{\partial t}(s,t) \right) \\ I \otimes \Lambda_Q\big\{ \big(\psi\big(s,t,\theta_n(s,t), g_n\big(s, w_n^Q(t)\big)\big) + 3\varepsilon\big)\alpha_n(s)\alpha_n(t)\varphi(s,t) \big\}\, dsdt. \end{cases} \tag{29}$$

Observe that the sequence of functions $\widetilde{\alpha_n}(s,t) = \alpha_n(s)\alpha_n(t)$ converges towards $\widetilde{\alpha}(s,t) = \alpha_\infty(s)\alpha_\infty(t)$ weakly* in $L^\infty\Big([0,T]^2;\mathbb{R}\Big)$. Thus, letting $n = n_k \to \infty$, and after $Q \to \infty$, in the relation (29), and since ε is arbitrary, we see that the continuous function $h : (s,t) \mapsto g(s, u_\infty(t)) = h(s,t)$ is solution in $\mathcal{D}'\Big(]0,T[^2\Big)$ of

$$\begin{cases} D_{\alpha_\infty,\alpha_\infty}[x(s,t)] \leq \alpha_\infty(s)\alpha_\infty(t)\psi(s,t,\theta_\infty(s,t), x(s,t)) \\ x(s,0) = \lim_k \|u_{n_k}(s) - u_\infty^0\| \text{ and } x(0,t) = \|u_\infty^0 - u_\infty(t)\|. \end{cases}$$

Moreover, the inequality (28) shows that $\limsup_n \|u_n(t) - u_n(s)\| = 0$ for $(s,t) \in [0,T]^2$ such as $a_\infty(s) = a_\infty(t)$. Therefore, h is constant on the set values taken by $(a_\infty^{-1}, a_\infty^{-1})$. Finally, using lemma 4.1 part ii), we obtain,

$$h(s,t) \leq m(\psi, \theta_{\alpha_\infty,\alpha_\infty}, \mathcal{B})(a_\infty(s), a_\infty(t)) \text{ for } (s,t) \in [0,T]^2,$$

where $\mathcal{B}$ is the continuous function on $[-a_\infty(T), a_\infty(T)]$ defined by,

$$\begin{cases} \mathcal{B}(\sigma) = \|u_{n_k}(a_\infty^{-1}(\sigma)) - u_\infty^0\| \text{ for } \sigma \in [0, a_\infty(T)], \\ \mathcal{B}(\sigma) = \|u_\infty^0 - u_\infty(a_\infty^{-1}(-\sigma))\| \text{ for } \sigma \in [-a_\infty(T), 0]. \end{cases}$$

Then, since we have $\mathcal{B}(0) = 0$, we claim $h(t,t) = 0$, for all $t \in [0,T]$, that is,

$$\forall t \in [0,T] \quad \lim_k u_{n_k}(t) = u_\infty(t).$$

Hence, u_∞ is the unique cluster value of $(u_n)_n$ in $C^0([0,T], X)$.

6. APPENDIX

Here, we state some useful results and we give summary indications about the proof of Lemma 4.1. More details can be found in [16] or [6].

6.1. Change of variable lemma

LEMMA 6.1. – *All selection* $\widetilde{a^{-1}}$ *of* a^{-1} *is measurable. Moreover, for* $f \in L^1(I,\mathbb{R})$, *we have,* $f \circ \widetilde{a^{-1}} \in L^1(a(I),\mathbb{R})$ *and,*

$$\int_I \alpha(s) f(s)\, ds = \int_{a(I)} f \circ \widetilde{a^{-1}}(\sigma)\, d\sigma . \tag{30}$$

The proof is clear and left to the reader.

Remark 6.1. – Since the quantity $\int_{a(I)} f \circ \widetilde{a^{-1}}(\sigma)\, d\sigma$ does not depend on the choice of (measurable) selections $\widetilde{a^{-1}}$of a^{-1} we agree on the notation $\int_{a(I)} f \circ a^{-1}(\sigma)\, d\sigma$.

6.2. Indications about the proof of the u.s.c. hull lemma

a) Suppose that x is a continuous solution in $\mathcal{D}'(]S_1,S_2[\times]T_1,T_2[)$ of $E_{\Omega_{1,1}}(\leq,\Phi,\mathcal{B}_0,\theta,1,1)$. Since x is bounded on $I \times J$, the operator $\varphi \mapsto R(\varphi)$ defined by

$$\begin{aligned} R(\varphi) = \int_{I\times J} & -x(s,t) D_{\alpha,\beta}[\varphi(s,t)] \\ & + \alpha(s)\beta(t)\Phi_r(s,t,\theta(s,t),x(s,t))\varphi(s,t)\, dsdt \end{aligned} \tag{31}$$

can be extended to the space of test functions $\widetilde{\mathcal{D}}(\mathring{I}\times\mathring{J})$ defined below,

$$\begin{aligned} \widetilde{\mathcal{D}}(\mathring{I}\times\mathring{J}) = \Big\{ & \varphi : I\times J \to \mathbb{R},\ \varphi \text{ continuous with compact support on } \mathring{I}\times\mathring{J}, \\ & \frac{\partial\varphi}{\partial s}(s,t) \in L^1(I\times J),\ \frac{\partial\varphi}{\partial t}(s,t) \in L^1(I\times J), \\ & \varphi(s,t) = \varphi(S_1,t) + \int_{S_1}^{s} \frac{\partial\varphi}{\partial s}(\sigma,t)\, d\sigma, \\ & \varphi(s,t) = \varphi(s,T_1) + \int_{T_1}^{t} \frac{\partial\varphi}{\partial t}(s,\sigma)\, d\tau \Big\}, \end{aligned}$$

by setting $R(\varphi) = \lim_k R(\varphi_k)$, for all sequence $(\varphi_k)_k$ in $\mathcal{D}(\mathring{I}\times\mathring{J})$ verifying

$$\lim_k \left[\|\varphi - \varphi_k\|_\infty + \left\| \frac{\partial\varphi}{\partial s} - \frac{\partial\varphi_k}{\partial s} \right\|_{L^1(I\times J)} + \left\| \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi_k}{\partial t} \right\|_{L^1(I\times J)} \right] = 0 .$$

It is clear therefore, that this definition does not depend upon the choice of $(\varphi_k)_k$ converging towards φ. And if we have $\varphi \geq 0$, we can take $\varphi_k \geq 0$ for all k (by taking for instance, classical regularized functions). Therefore, we have for $\varphi \in \widetilde{\mathcal{D}}(\overset{\circ}{I} \times \overset{\circ}{J})$, $(\varphi \geq 0 \Rightarrow R(\varphi) \geq 0)$.

Then let us consider $\varphi(s,t) = \xi(a(s), b(t))$ with $\xi \geq 0,\ \xi \in \mathcal{D}(\Omega)$. Then $\varphi \in \widetilde{\mathcal{D}}(\overset{\circ}{I} \times \overset{\circ}{J})$, and $\varphi \geq 0$. It follows $R(\varphi) \geq 0$; so using, in this last inequality (see relation (31)), the change of variable (Lemma 6.1) $\sigma = a(s)$ and $\tau = b(t)$, we see that $x(a^{-1}, b^{-1})$ is a continuous solution of $E_\Omega(\leq, \Phi, \theta, \mathcal{B}, \alpha, \beta)$. Therefore, if we prove the existence of the maximal solution $m_{\mathcal{B}}$, introduced in the part i) of the lemma, we will have for $(\sigma, \tau) \in a(I) \times b(J)$

$$x\big(a^{-1}(\sigma), b^{-1}(\tau)\big) \leq m(\Phi, \theta, \mathcal{B})(\sigma, \tau) = m_{\mathcal{B}}(\sigma, \tau)\,,$$

or in other words, with $(s,t) \in I \times J$, $x(s,t) \leq m_{\mathcal{B}}(a(s), b(t))$.

b) Consider y a continuous solution (if it exists) of the following inequation $E_\Omega(\leq, \Phi_r, \theta, \mathcal{B}, \alpha, \beta)$ in $\mathcal{D}'(\Omega)$. Let $\zeta_n(\sigma,\tau) = \zeta\big(\frac{\sigma+\tau-d}{2}\big)\rho\big(n\big(\frac{\sigma-\tau-d}{2}\big)\big)$, where $d \in a(I) - b(J)$, $\zeta \in \mathcal{D}\Big(\overset{\circ}{I}_d\Big)$, $\zeta \geq 0$, and $\rho \in \mathcal{D}(]-1,1[)$ with $\int_{-1}^{1} \rho(\tau)\, d\tau = 1$. The function ζ_n has a compact support in Ω for n large enough. Recall that I_d denotes the interval $I_d = [(a(S_1) - d) \vee b(T_1), b(T_2) \wedge (a(S_2) - d)]$. Then, with the change of variable,

$$\lambda = \frac{\sigma + \tau - d}{2} \quad , \quad \nu = n\left(\frac{\sigma - \tau - d}{2}\right),$$

we obtain easily,

$$\begin{aligned}0 \leq \int_{\mathbb{R}^2} &\Big\{ y\Big(\lambda + \frac{\nu}{n} + d, \lambda - \frac{\nu}{n}\Big)\zeta'(\lambda)\rho(\nu) \\ &+ \Phi_r\Big(a^{-1}\Big(\lambda + \frac{\nu}{n} + d\Big), b^{-1}\Big(\lambda - \frac{\nu}{n}\Big), \theta\Big(a^{-1}\Big(\lambda + \frac{\nu}{n} + d\Big), b^{-1}\Big(\lambda - \frac{\nu}{n}\Big)\Big), \\ &\quad y\Big(a^{-1}\Big(\lambda + \frac{\nu}{n} + d\Big), b^{-1}\Big(\lambda - \frac{\nu}{n}\Big)\Big)\Big)\zeta(\lambda)\rho(\nu)\Big\}\, d\sigma d\tau .\end{aligned}$$

No problem of integrability occurs thanks to the hypotheses on Φ_r and $\theta(\in W)$. Since y, Φ_r are continuous, a^{-1}, b^{-1} are a.e. continuous, and $\theta \in W$, letting $n \to \infty$, we get,

$$\begin{aligned}0 \leq \int_{I_d \times [-1,1]} &\{ y(\lambda + d, \lambda)\zeta'(\lambda)\rho(\nu) \\ &+ \Phi_r\big(a^{-1}(\lambda + d), b^{-1}(\lambda), \theta\big(a^{-1}(\lambda + d), b^{-1}(\lambda)\big), \\ &\quad y\big(a^{-1}(\lambda + d), b^{-1}(\lambda)\big)\big)\, \zeta(\lambda)\}\, \rho(\nu) d\nu .\end{aligned}$$

And $z_d(\tau) = y(\tau + d, \tau)$ is solution in $\mathcal{D}'(\overset{\circ}{I}_d)$ of $\chi_d^r = \chi_d(\leq, \Phi_r, \theta, \mathcal{B}, \alpha, \beta)$. Notice that we have for a.e. $\xi \in I_d$ (if we write $|\theta(s,t)| \leq F(s) + G(t)$, for a.e. s and a.e. t, with $F, G \in L^1([0,T], \mathbb{R}^+)$),

$$
\begin{aligned}
&\left|\Phi_r\big(a^{-1}(\xi+d), b^{-1}(\xi), \theta\big(a^{-1}(\xi+d), b^{-1}(\xi)\big), z_d(\xi)\big)\right| \\
&\leq \left|\Phi_r\big(a^{-1}(\xi+d), b^{-1}(\xi), 0, 0\big)\right| \\
&\quad + \omega_r\big(F\big(a^{-1}(\xi+d)\big) + G\big(b^{-1}(\xi)\big) + z_d(\xi)\big) \qquad (32)
\end{aligned}
$$

It follows from the Gronwall's lemma an a priori upper bound for z_d. Thus, the inequation χ_d^r has a maximal continuous solution y_d^r verifying the equation $\chi_{d,\Omega}(=, \Phi_r, \theta, \mathcal{B})$ in $\mathcal{D}'(\overset{\circ}{I}_d)$ (see [19], [6]).

c) We will examine now the C^1 case. We suppose $\mathcal{B}$ is C^1 on $[-b(T_2), a(S_2)]$, and Φ is C^1 on $[0,T]^2 \times \mathbb{R}^2$ and ω–Lipschitz. Let θ, be C^1 on $[0,T]^2$ and α and β strictly positive continuous in $[0,T]$. We suppose also that we have,

$$
\Phi(S_1, T_1, \theta(S_1, T_1), \mathcal{B}(a(S_1) - b(T_1))) = 0\,,
$$

then (see [6]) the solution $m(\Phi, \theta, \mathcal{B}, \alpha, \beta)$ is C^1 on $\overline{\Omega}$, and is a classical solution of $E(=, \theta, \mathcal{B}, \alpha, \beta)$.

d) We suppose here that Φ is C^1 ω–Lipschitz on $[0,T]^2 \times \mathbb{R}^2$. Let $(\alpha_k)_k$, (resp. $(\beta_k)_k$) be a sequence of strictly positive continuous functions on I (resp. J). We (can) suppose that we have for $q \in \{1,2\}$, $a_n(S_q) = a(S_q)$ and $b_n(S_q) = b(S_q)$. Let $(\mathcal{B}_k)_k$ be a sequence of functions C^1 on $a(I) \times b(J)$, converging in $C^0(a(I) \times b(J))$ towards $\mathcal{B}$. Let $(\theta_k)_k$ be a sequence of C^1 functions on $[0,T]^2$ converging towards θ in $(W, \|\cdot\|_*)$. Let us consider a sequence $(G_k)_k$ of functions from $[0, b(T_2)]$ to $\mathbb{R}$, such that G_k is C^1 with compact support included in $\left[0, \frac{1}{k}\right[$ and

$$
G_k(0) = -\Phi(S_1, T_1, \theta_k(S_1, T_1), \mathcal{B}_k(a(S_1) - b(T_1)))\,,
$$

with G_k bounded by $|G_k(0)|$ and $\|G_k\|_1\, k \to \infty \longrightarrow 0$. Let $m_k = m(\Phi + G_k, \theta_k, \mathcal{B}_k, \alpha_k, \beta_k)$, let $P^{\mathcal{I}}$ be the projection on the intervall $\mathcal{I}$, and m_k^* be the restriction to $a(I) \times b(J)$ of $m_k\big(P^{a_k(I)}, P^{b_k(J)}\big)$. Then, we claim that the family $(m_k^*)_k$ converges in $C^0(a(I) \times b(J))$ towards the unique continuous solution of $E_\Omega(=, \Phi, \theta, \mathcal{B}, \alpha, \beta)$. Indeed, this claim follows easily from the equicontinuity lemma 6.2 below.

e) In the general case, where Φ is the decreasing pointwise limit $\Phi = \lim \Phi_r$, $\mathcal{B}$ continuous, and $\theta \in W$. In this case, the reader can easily verify that the sequence $(m(\Phi_r, \theta, \mathcal{B}, \alpha, \beta))_r$ is pointwise convergent by decreasing towards the maximal (in the required meaning) solution of $E_\Omega(=,\Phi,\theta,\mathcal{B},\alpha,\beta)$. Indeed, $(y_d(\Phi_r,\theta,\mathcal{B},\alpha,\beta))_r$ converges uniformly by decreasing towards $y_d(\Phi,\theta,\mathcal{B},\alpha,\beta)$ on I_d.

The u.s.c. hull lemma results clearly from the previous steps a) to e).

LEMMA 6.2. – *We suppose that the sequences $(\alpha_k)_k$ and $(\beta_k)_k$ converge respectively towards α and β in the weak*-topology of $L^1([0,T],\mathbb{R})$. Let Φ be ω–Lipschitz on $[0,T]^2 \times \mathbb{R}^2$, and let $\left(\widetilde{\theta_q}\right)_q$ be a bounded sequence in $(W,\|\cdot\|_*)$. Put $\tilde{\Phi}_q = \Phi + \widetilde{\theta_q}$. Let $(\theta_k)_k$ a sequence of W functions, converging towards $\theta \in W$, and let $(\mathcal{B}_k)_k$ be a sequence of continuous functions on $a(I)-b(J)$, converging uniformly towards $\mathcal{B}$. Set (with the notations of the above step d)) $m^*_{k,q} = m(\Phi_q,\theta_k,\mathcal{B}_k)\circ\left(P^{a_k(I)},P^{b_k(J)}\right)$. Then the family $\left(m^*_{k,q}\right)_{k,q}$ is (bounded and) equicontinuous on $\overline{\Omega}$. More precisely, writing, $y_d^{k,q}(\tau) = m(\Phi_q,\theta_k,\mathcal{B}_k)(\tau+d,\tau)$, and $I_d^k = [(a_k(S_1)-d)\vee b(T_1), b_k(T_2)\vee(a_k(S_2)-d)]$ for $d\in a_k(I)-b_k(J)$ and $a_k(s)=\int_0^s \alpha_k(\sigma)\,d\sigma$ and $b_k(t)=\int_0^t \beta_k(\tau)\,d\tau$, we have ;*

1. For $\tau \in I_d^k \cap I_{d+\delta}^k$ and $(a_k(S_1)-d) < b_k(T_1)$,

$$\begin{aligned}\left|y_{d+\delta}^{k,q}(\tau)-y_d^{k,q}(\tau)\right| &\le |\mathcal{B}_k(d+\delta)-\mathcal{B}_k(d)|\\ &+\int_{b(T_1)}^{\tau}\Big(\Big|\Phi\Big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi),\theta_k\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big),y_{d+\delta}^{k,q}(\xi)\Big)\\ &-\Phi\Big(a_k^{-1}(\xi+d),b_k^{-1}(\xi),\theta_k\big(a_k^{-1}(\xi+d),b_k^{-1}(\xi)\big),y_d^{k,q}(\xi)\Big)\Big|\\ &+\Big|\widetilde{\theta}_q\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big)-\widetilde{\theta}_q\big(a_k^{-1}(\xi+d),b_k^{-1}(\xi)\big)\Big|\Big)d\xi. \quad (33)\end{aligned}$$

2. For $\tau \in I_d^k \cap I_{d+\delta}^k$ and $(a_k(S_1)-d) > b_k(T_1)$,

$$\begin{aligned}\left|y_{d+\delta}^{k,q}(\tau)-y_d^{k,q}(\tau)\right| &\le |\mathcal{B}_k(d+\delta)-\mathcal{B}_k(d)|\\ &+\int_{a(S_1)-d}^{\tau}\Big(\Big|\Phi\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi),\\ &\qquad\theta_k\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big),y_{d+\delta}^{k,q}(\xi)\Big)\\ &\qquad-\Phi\Big(a_k^{-1}(\xi+d),b_k^{-1}(\xi),\theta_k\big(a_k^{-1}(\xi+d),b_k^{-1}(\xi)\big),y_d^{k,q}(\xi)\Big)\Big|\\ &+\Big|\widetilde{\theta}_q\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big)-\widetilde{\theta}_q\big(a_k^{-1}(\xi+d),b_k^{-1}(\xi)\big)\Big|\Big)d\xi\end{aligned}$$

$$+\int_{a(S_1)-(d+\delta)}^{a(S_1)-d}\Big(\Big|\Phi\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi),$$
$$\theta_k\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big),y_{d+\delta}^{k,q}(\xi)\Big)\Big|$$
$$+\Big|\widetilde{\theta}_q\big(a_k^{-1}(\xi+d+\delta),b_k^{-1}(\xi)\big)\Big|\Big)d\xi\,. \qquad (34)$$

3. *For* $\tau \in I^k_{a_k(S_1)-b_k(T_1)} \cap I^k_{a_k(S_1)-b_k(T_1)+\delta}$, *(case* $(a_k(S_1)-d) = b_k(T_1)$*), the relation (33) holds if* $\delta > 0$, *and the relation (34) holds if* $\delta < 0$.

Proof. – Indications are given in [16].

6.3. A differential lemma

LEMMA 6.3. – *We assume that the function* Φ *is u.s.c. on* $[0,T] \times \mathbb{R}^2$. *Let* $(y_n^0)_n$ *be a sequence of real numbers converging towards* y_∞^0, *and* $(g_n)_n$ *be a sequence of functions converging towards* g_∞ *in* $L^1([0,T],\mathbb{R})$. *Furthermore, we suppose that the following inequality holds in* $[0,T]\times\mathbb{R}^2$ *for some positive constant* l, $|\Phi(s,\xi,x)| \le l(|\xi|+1)$. *For* $n \in \mathbb{N}\cup\{\infty\}$, *let us denote by* y_n *the maximal continuous solution in* $\mathcal{D}'(]0,T[)$ *of the inequation,*

$$\begin{cases} \dfrac{dz}{dt}(t) \le \Phi(s,g_n(t),z(t)) \\ z(0) \le y_n^0\,. \end{cases} \qquad (35)$$

Then, we have, $\limsup_n \sup_{t\in[0,T]}(y_n - y_\infty)(t) \le 0$.

Proof of lemma 6.3. – See [16].

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