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Elastic knots in Euclidean 3–space

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ABSTRACT. – This paper deals with the problem of minimizing the curvature functional $\int \kappa^2 \, ds$ on isotopy classes of closed knotted curves in $\mathbb{R}^3$. We show existence of minimizers under a given topological knot type and develop a regularity theory by analyzing different touching situations. © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous minimisons la fonctionnelle de courbure $\int \kappa^2 \, ds$ dans des classes d'isotopie des courbes fermées et nouées. L'existence des courbes minimales étant donné un type de nœud topologique est démontrée et une théorie sur la régularité est développée par l'analyse de situations de touchage différentes. © Elsevier, Paris

1. THE PROBLEM

Knotted loops of elastic wire spring into stable configurations as soon as they are released. Due to the physical fact that it is impossible for a wire to pass through itself the knot type is preserved in the experiment. To model this behavior we consider the well-known curvature functional

$$\int \kappa^2 \, ds \tag{1}$$

as elastic energy to be minimized on isotopy classes of closed curves in $\mathbb{R}^3$. In addition, we define an obstacle condition that prevents selfintersections

AMS Classification: 49J40, 49N60, 49Q10, 53A04 of the curves under consideration in order to preserve the given isotopy class, i.e., knot type in the minimization process. We show existence of minimizers and develop the regularity theory for a variety of different touching situations.

The curvature functional (1) was suggested as early as 1738 by D. Bernoulli as a model for the elastic energy of springy wires. In 1743 L. Euler considered the corresponding variational equations and classified the solutions called *elastica* in the plane ([3]). In the first half of this century J. Radon and R. Irrgang examined more general curvature functionals also in the case of curves in space ([15], [7]). Until very recently there have been numerous publications regarding existence and form of solutions of related variational problems such as the investigations by J. Langer and D.A. Singer as well as R. Bryant and P. Griffiths concerning critical points of the functional in different space forms ([9]–[11], [1]) and the corresponding gradient flow ([12]), see also [14] for the evolution problem. Different knot energies suitable for describing nonelastic but electrically charged wires were considered by J. O'Hara ([13]), M. Freedman, Z. He, Z. Wang ([4]), R. Kusner and J. Sullivan ([8]). But the treatment as an isoperimetric obstacle problem excluding selfintersections is new. A special feature of our setting is that the solution itself determines the shape of the obstacle, which is therefore not known a priori.

We model the centerline of knotted wires as regular, closed space curves in the Sobolev class

$$H^{2,2}(S^1,\mathbb{R}^3) := \{\, \mathbf{x}\in H^{2,2}((0,2\pi),\mathbb{R}^3) \mid \mathbf{x}(0)=\mathbf{x}(2\pi)\,,\ \dot{\mathbf{x}}(0)=\dot{\mathbf{x}}(2\pi),\ \dot{\mathbf{x}}(s)\neq 0 \text{ for all } s\in S^1 \,\}$$

Note that the first derivatives of $\mathbf{x}\in H^{2,2}((0,2\pi),\mathbb{R}^3)$ are defined everywhere on $[0,2\pi]$ because of the embedding $H^{2,2}((0,2\pi)) \hookrightarrow C^{1,1/2}([0,2\pi])$.

In order to exclude selfintersections we assign to such curves a small "thickness" $0 < \delta \ll 1$ in the form of an obstacle condition, by which different curve points $\mathbf{x}(s)$ and $\mathbf{x}(s')$ cannot have euclidean distance less than δ unless the parameters s and s' are close to each other:

$$C_\delta := \left\{ \mathbf{x}\in H^{2,2}(S^1,\mathbb{R}^3) \mid |\mathbf{x}(s)-\mathbf{x}(s')| \geq \min\left\{ \delta,\ \frac{1}{2}L_{[s,s']}(\mathbf{x}),\ \frac{1}{2}L_{[s',s]}(\mathbf{x}) \right\} \right\},$$

where $L_{[s,s']}(\mathbf{x})$ denotes the length of the arc connecting the points $\mathbf{x}(s)$ and $\mathbf{x}(s')$.

Given a curve in C_δ we are able to determine its knot or *isotopy* type by deforming it continuously and without selfintersections into a standard knot in $\mathbb{R}^3$. To be more precise, let $\omega_0, \omega_1, \omega_2 \dots$ be equivalence classes of such standard knots in $\mathbb{R}^3$. Then the *isotopy class* C_δ^n is defined as

$$C_\delta^n := \{\, \mathbf{x} \in C_\delta \;\big|\; \exists \text{ a parametrization } \mathbf{y} \in \omega_n \text{ isotopic to } \mathbf{x} \,\}.$$

A curve $\mathbf{x}$ is *isotopic* to $\mathbf{y}$, if and only if there is a continuous deformation $\Phi : S^1 \times [0,1] \longrightarrow \mathbb{R}^3$ with the following properties: $\Phi(.,0) = \mathbf{x}(.)$, $\Phi(.,1) = \mathbf{y}(.)$, and $\Phi(.,\tau)$ closed and 1–1 for all $\tau \in [0,1]$. Isotopy is an equivalence relation, see the simple argument in [18, p. 28].

Restricting our attention to curves of prescribed length l we finally obtain the class of admissible knots $C_{\delta,l}^n$ as

$$C_{\delta,l}^n := C_\delta^n \cap \{\, \mathbf{x} \in C^1(S^1, \mathbb{R}^3) \;\big|\; L_{S^1}(\mathbf{x}) := \int_0^{2\pi} |\dot{\mathbf{x}}(\sigma)|\, d\sigma = l \,\}, \quad 0 < \delta < l.$$

Neglecting the effects of physical torsion or twisting and gravity we look at the following variational problem:

$$\textit{Minimize the functional } F(\mathbf{x}) := \int_{\mathbf{x}} \kappa^2\, ds \textit{ in } C_{\delta,l}^n.$$

Remark. – Without any normalization like the isoperimetric side condition one cannot expect to find a minimizer, since the scaling $\mathbf{x} \mapsto R\mathbf{x}$ yields $F(R\mathbf{x}) = F(\mathbf{x})/R \longrightarrow 0$ as $R \nearrow \infty$. The *total curvature* $T(\mathbf{x}) := \int_{\mathbf{x}} |\kappa|\, ds$, on the other hand, provides a lower bound for the functional F for closed regular curves $\mathbf{x}$ of fixed length l by Hölder's inequality:

$$(T(\mathbf{x}))^2/l \le F(\mathbf{x}).$$

By the classical Fáry–Milnor theorem we have the estimate $T(\mathbf{x}) \ge 4\pi$ for knotted curves $\mathbf{x}$, i.e. for $\mathbf{x} \in C_\delta^n, n \ge 1$; see [18, Chapter 2] for an alternative proof in the $H^{2,2}$–context.

Using a direct method and drawing extensively from the fact that $H^{2,2}$ embeds into $C^{1,1/2}$ we are able to show the following existence result (Section 2):

THEOREM 1.1. – *Let $\delta < l/8$. If $C_{\delta,l}^n$ is nonempty, then there is a curve $\mathbf{x}_\delta \in C_{\delta,l}^n$ with $|\dot{\mathbf{x}}_\delta(s)| = l/2\pi$ for all $s \in S^1$ and*

$$F(\mathbf{x}_\delta) \quad = \quad \inf\{\, F(\mathbf{y}) \,|\, \mathbf{y} \in C_{\delta,l}^n \,\}.$$

The physical experiments show that we have to take points of selfcontact into consideration when investigating the regularity of the minimizers. These are points, where one has equality in the obstacle condition – points, in fact, with euclidean distance equal δ, as will be shown in Section 3.1. Our regularity results for different touching situations are based on a lemma by S. Hildebrandt and H.C. Wente ([6]) that guarantees the existence of a Lagrange multiplier for obstacle problems with side conditions. Using a measure theoretic argument we show

THEOREM 1.2. – *A minimizer* $\mathbf{x}=\mathbf{x}_\delta \in C^n_{\delta,l}$ *has bounded curvature near isolated simple touching points.*

In fact, we derive $H^{3,1}$–regularity for such points, which – according to the experiment – seem to constitute the only type of touching that occurs in nature. For certain "unhooked", so-called *convex* touching situations, we are able to improve the result up to $H^{3,2}$–regularity employing Nirenberg's difference quotient method (§3.4). Finally, we treat *two-sided*, i.e., "clamped" contact points in Section 3.5, where we use inverse Hölder inequalities and Gehring's lemma to show $H^{2,p}$–regularity for a $p>2$ near such a point.

We conclude this introduction by mentioning some interesting open problems:

1. Is the $H^{3,1}$–regularity optimal for general isolated simple touching points ? Due to the fact that the obstacle is not fixed but determined by the solution itself one might conjecture higher regularity.

2. Are there minimizers that have any other than isolated touching points? For instance, one could think of a curve that possesses two touching arcs winding around each other like a part of a circular double helix.

3. The application of Hildebrandt's and Wente's Lagrange multiplier lemma is based on the assumption that the minimizer is not extremal for the length functional L in the class C^n_δ. In the case of isolated touching points this assumption is not necessary, but is it conceivable that there are minimizing knots that are L–extremal ? Geometrically this means that such a minimizing knot would not have any freely variable arc; in other words, every point on that curve would be a touching point. In [18, Chapter 4] we have considerably reduced the class of curves where this problem could occur.

Let us remark that this paper is self–contained, although at some places we refer to the author's thesis [18], where the straightforward but somewhat tedious admissibility proofs for certain comparison curves are carried out in detail.

The appendix contains a slight generalization of the result by Hildebrandt and Wente and some technical material.

2. EXISTENCE OF MINIMIZING KNOTS

Proof of Theorem 1.1. – Observing that the functional F is translation invariant we may assume that there is a minimal sequence $\{\mathbf{y}_p\}_{p=1}^{\infty} \subset C^n_{\delta,l}$ with

$$(2) \qquad \lim_{p\nearrow\infty} F(\mathbf{y}_p) = \inf\{ F(\mathbf{y}) \,|\, \mathbf{y} \in C^n_{\delta,l} \} \quad \text{and } \mathbf{y}_p(0)=\mathbf{y}_p(2\pi)=0.$$

Using the embedding $H^{2,2}((0,2\pi)) \hookrightarrow C^{1,1/2}([0,2\pi])$ and the fact that $\dot{\mathbf{y}}_p(t)\neq 0$ for all $t\in S^1$ we find constants $c_p>0$, s.th. $|\dot{\mathbf{y}}_p(t)| \geq c_p > 0$ for all $t \in S^1$.

As usual one considers the function $\sigma_p : [0,2\pi] \longrightarrow [0,l]$

$$\sigma_p(t) := \int_0^t |\dot{\mathbf{y}}_p(s)|\, ds,$$

which is in $C^1([0,2\pi])$ and invertible, since $\dot{\sigma}_p(t) = |\dot{\mathbf{y}}_p(t)| \geq c_p > 0$.

For the derivatives of the inverse function $\tau_p : [0,l] \longrightarrow [0,2\pi] \in C^1([0,l])$ one finds

$$\frac{d}{ds}\tau_p(s) = \frac{1}{|\dot{\mathbf{y}}_p(t)|}$$

and

$$\frac{d^2}{ds^2}\tau_p(s) = -\frac{\langle \dot{\mathbf{y}}_p(t), \ddot{\mathbf{y}}_p(t)\rangle}{|\dot{\mathbf{y}}_p(t)|^4}$$

for $t=\tau(s)$ for almost all $s\in[0,l]$.

In particular, τ_p is a C^1–diffeomorphism with

$$\int_0^l |\ddot{\tau}_p(s)|^2\, ds \leq C_p < \infty.$$

Then one can show (Lemma A.1 in the appendix) that the composition $\mathbf{y}_p \circ \tau_p : [0,l] \longrightarrow \mathbb{R}^3$ is in $H^{2,2}((0,l),\mathbb{R}^3)$. Composing this with the linear transformation $\Lambda(t) := l \cdot t/2\pi$ one obtains a regularized minimal sequence $\mathbf{x}_p := \mathbf{y}_p \circ \tau_p \circ \Lambda : [0,2\pi] \longrightarrow \mathbb{R}^3 \in H^{2,2}((0,2\pi),\mathbb{R}^3)$ with

$$(3) \qquad |\dot{\mathbf{x}}_p(s)| = l/2\pi \quad \text{for all } s\in S^1.$$

To get compactness we note (recalling the parametric invariance of F and using (3)) that there is a positive constant M, such that for all $p \in \mathbb{N}$

$$\begin{aligned} M \geq F(\mathbf{y}_p) = F(\mathbf{x}_p) &= \int_0^{2\pi} \frac{|\dot{\mathbf{x}}_p(\sigma) \wedge \ddot{\mathbf{x}}_p(\sigma)|^2}{|\dot{\mathbf{x}}_p(\sigma)|^6} |\dot{\mathbf{x}}_p(\sigma)| \, d\sigma \\ &= (2\pi/l)^3 \int_0^{2\pi} |\ddot{\mathbf{x}}_p(\sigma)|^2 \, d\sigma, \\ &\Rightarrow \|\ddot{\mathbf{x}}_p\|_{L^2} \leq \sqrt{M(\frac{l}{2\pi})^3}. \end{aligned}$$

In addition, we have $\|\dot{\mathbf{x}}_p\|_{L^2} = l/\sqrt{2\pi}$. Together with (2), which implies $\mathbf{x}_p(0) = \mathbf{x}_p(2\pi)$ by the definition of τ_p and Λ, we find a constant C independent of p, such that

$$\|\mathbf{x}_p\|_{H^{2,2}} \leq C < \infty. \tag{4}$$

Hence, there is a weakly convergent subsequence $\mathbf{x}_{p'} \rightharpoonup \mathbf{x} \in H^{2,2}((0, 2\pi), \mathbb{R}^3)$. The embedding $H^{2,2} \hookrightarrow C^{1,1/2}$, inequality (4) and the theorem by Arzela–Ascoli imply also the strong convergence $\mathbf{x}_p \longrightarrow \mathbf{x}$ in $C^1([0, 2\pi], \mathbb{R}^3)$ for a subsequence $\{\mathbf{x}_p\}_{p=1}^{\infty} \subset \{\mathbf{x}_{p'}\}_{p'=1}^{\infty}$.

CLAIM. – $\mathbf{x} \in C^n_{\delta,l}$.

Proof. – 1. The strong convergence in C^1 implies the conditions

$$\mathbf{x}(0) = \mathbf{x}(2\pi), \quad \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}(2\pi), \tag{5}$$

$$|\dot{\mathbf{x}}(t)| = 2\pi/l \quad \text{for all } t \in S^1, \tag{6}$$

since this is true for all $\mathbf{x}_p$, $p \in \mathbb{N}$.

2. The parametric invariance of the length functional implies that the obstacle condition for the original minimal sequence $\{\mathbf{y}_p\}_{p=1}^{\infty} \subset C^n_{\delta,l}$ carries over to the regularized minimal sequence $\{\mathbf{x}_p\}_{p=1}^{\infty}$:

$$\begin{aligned} |\mathbf{x}_p(s) - \mathbf{x}_p(s')| &= |\mathbf{y}_p \circ \tau_p \circ \Lambda(s) - \mathbf{y}_p \circ \tau_p \circ \Lambda(s')| \\ &\geq \min\left\{ \delta, \frac{1}{2} L_{[\tau_p(\Lambda(s)), \tau_p(\Lambda(s'))]}(\mathbf{y}_p), \right. \\ &\qquad \left. \frac{1}{2} L_{[\tau_p(\Lambda(s')), \tau_p(\Lambda(s))]}(\mathbf{y}_p) \right\} \\ &= \min\left\{ \delta, \frac{1}{2} L_{[s,s']}(\mathbf{x}_p), \frac{1}{2} L_{[s',s]}(\mathbf{x}_p) \right\}. \end{aligned}$$

This fact together with the strong C^1–convergence $\mathbf{x}_p \longrightarrow \mathbf{x}$ guarantees that $\mathbf{x}$ satisfies the obstacle condition as well, which is shown in Lemma A.2 in the appendix.

3. The reparametrization of the original minimal sequence does not change the isotopy type of the curve, i.e., $\mathbf{x}_p \in C_\delta^n$ for all $p \in \mathbb{N}$. The following lemma together with the C^1–convergence yields $\mathbf{x} \in C_\delta^n$, which concludes the proof of the claim.

LEMMA 2.1. – *Let* $\boldsymbol{\eta} \in H^{2,2}(S^1, \mathbb{R}^3)$ *satisfy*

$$(7) \qquad |\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')| \geq \min\{\, d, \theta l |s - s'|_{S^1} \,\} \quad \textit{for all } s, s' \in S^1,$$

where $d>0$, $\theta \in (0,1)$. *Then there exists a constant* $\epsilon>0$, *such that all* $\boldsymbol{\xi} \in H^{2,2}(S^1, \mathbb{R}^3)$, *with* $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \leq \epsilon$ *are isotopic to* $\boldsymbol{\eta}$.

Proof. – The homotopy $\Phi : S^1 \times [0,1] \longrightarrow \mathbb{R}^3$ defined by

$$\Phi(s,t) := (1-t)\boldsymbol{\eta}(s) + t\boldsymbol{\xi}(s)$$

satisfies $\Phi(s,0) = \boldsymbol{\eta}(s)$, $\Phi(s,1) = \boldsymbol{\xi}(s)$ and the curves $\Phi(.,t)$ are closed for all $t \in [0,1]$.

In addition, $\Phi(.,t)$ is injective for $\epsilon>0$ sufficiently small, since by (7)

$$\begin{aligned} |\Phi(s,t) - \Phi(s',t)| &= \Big|\, (1-t)(\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')) + t \int_{[s,s']_{S^1}} \dot{\boldsymbol{\eta}}(\sigma)\, d\sigma \\ &\qquad - t \int_{[s,s']_{S^1}} [\, \dot{\boldsymbol{\eta}}(\sigma) - \dot{\boldsymbol{\xi}}(\sigma) \,]\, d\sigma \,\Big| \\ &\geq |\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')| - t \|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \cdot |s - s'|_{S^1} \\ &\geq \min\{\, d, \theta l |s - s'|_{S^1} \,\} - \|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \cdot |s - s'|_{S^1} \\ &\geq \frac{1}{2} \min\{\, d, \theta l |s - s'|_{S^1} \,\} \quad \text{for all } t \in [0,1], \end{aligned}$$

if $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \leq \epsilon := \min\{d/2\pi, \theta l/2\}$. For convenience, we have denoted the minimal distance between two parameters s, s' on $S^1 \cong [0, 2\pi)$ by $|s - s'|_{S^1} := \min\{\, |s - s'|, 2\pi - |s - s'| \,\}$ and the corresponding arc on S^1 by $[s, s']_{S^1}$. (Obviously, this proof also works for vector functions $\boldsymbol{\eta}, \boldsymbol{\xi}$ that are Lipschitz continuous.) □

It remains to show that $\mathbf{x}_\delta := \mathbf{x}$ actually minimizes the curvature energy F. Since $|\dot{\mathbf{x}}_p(t)| = |\dot{\mathbf{x}}(t)| = l/2\pi$ for all $t \in S^1$, the functional F is a bounded nonnegative quadratic form on the regularized minimal sequence. A standard reasoning then shows that F is lower semicontinuous with respect to the weak convergence $\mathbf{x}_p \rightharpoonup \mathbf{x}$ in $H^{2,2}$. Consequently, since $\mathbf{x} \in C^n_{\delta,l}$,

$$\inf\{\, F(\mathbf{y}) \,|\, \mathbf{y} \in C^n_{\delta,l} \,\} \le F(\mathbf{x}) \le \liminf_{p\nearrow\infty} F(\mathbf{x}_p) = \inf\{\, F(\mathbf{y}) \,|\, \mathbf{y} \in C^n_{\delta,l} \,\}.$$

□

3. REGULARITY

3.1. Preliminaries

We first observe that the minimizer $\mathbf{x}_\delta \in C^n_{\delta,l}$ in Theorem 1.1 also minimizes the functional $D(\mathbf{y}) := \int_{S^1} \frac{|\ddot{\mathbf{y}}(\sigma)|^2}{|\dot{\mathbf{y}}(\sigma)|^3}\, d\sigma$ in $C^n_{\delta,l}$, since by (6)

$$\begin{aligned} D(\mathbf{x}_\delta) &= \int_{S^1} \frac{|\dot{\mathbf{x}}_\delta(\sigma) \wedge \ddot{\mathbf{x}}_\delta(\sigma)|^2}{|\dot{\mathbf{x}}_\delta(\sigma)|^5}\, d\sigma = F(\mathbf{x}_\delta) \\ &\le F(\mathbf{y}) \le \int_{S^1} \frac{|\ddot{\mathbf{y}}(\sigma)|^2}{|\dot{\mathbf{y}}(\sigma)|^5} \cdot |\dot{\mathbf{y}}(\sigma)|^2\, d\sigma = D(\mathbf{y}). \end{aligned}$$

The following lemma simplifies matters for touching points, i.e., for points, where there is equality in the obstacle condition:

LEMMA 3.1. – *Assuming there exists a $\delta_1 < l/8$, with $C^n_{\delta_1,l} \neq \emptyset$ we find a $\delta_0 \le \delta_1$, such that for all $0<\delta\le\delta_0$ and the corresponding minimizers $\mathbf{x}_\delta \in C^n_{\delta,l}$ the following holds:*

If $|\mathbf{x}_\delta(s) - \mathbf{x}_\delta(s')| = \min\{\, \delta,\ \frac{1}{2}L_{[s,s']}(\mathbf{x}_\delta),\ \frac{1}{2}L_{[s',s]}(\mathbf{x}_\delta) \,\}$ for $s \neq s'$, then $|\mathbf{x}_\delta(s) - \mathbf{x}_\delta(s')| = \delta$.

Proof. – 1. For $0<\delta\le\delta_1$ we have by definition $C^n_{\delta_1,l} \subset C^n_{\delta,l}$ and therefore

$$\inf\{F(\mathbf{y}) \,|\, \mathbf{y} \in C^n_{\delta,l}\} \le \inf\{F(\mathbf{y}) \,|\, \mathbf{y} \in C^n_{\delta_1,l}\} =: i_1.$$

Equation (6) then implies

$$\text{(8)} \qquad \|\ddot{\mathbf{x}}_\delta\|^2_{L^2} \le C_1 := l^3 \cdot i_1/(2\pi)^3 < \infty \quad (C_1 \neq 0,\ \text{since } C^n_{\delta,l} \neq \emptyset\,).$$

2.

$$
\begin{aligned}
&|\mathbf{x}_\delta(s)-\mathbf{x}_\delta(s')| \\
&\geq \Big| \int\limits_{[s,s']_{S^1}} \dot{\mathbf{x}}_\delta(s')\,d\sigma \Big| - \Big| \int\limits_{[s,s']_{S^1}} (\dot{\mathbf{x}}_\delta(\sigma)-\dot{\mathbf{x}}_\delta(s'))d\sigma \Big| \\
&\geq \frac{l}{2\pi}\cdot|s-s'|_{S^1} - \Big| \int\limits_{[s,s']_{S^1}} \Big|\int_{s'}^{\sigma} |\ddot{\mathbf{x}}_\delta(\tau)|^2\,d\tau\Big|^{1/2} \Big|\int_{s'}^{\sigma} d\tau\Big|^{1/2} d\sigma \Big| \qquad (9)\\
&\geq \frac{l}{2\pi}\cdot|s-s'|_{S^1} - \|\ddot{\mathbf{x}}_\delta\|_{L^2}\cdot|s-s'|_{S^1}^{3/2} \\
&\geq \frac{3l}{8\pi}\cdot|s-s'|_{S^1} \quad \text{for } |s-s'|_{S^1}^{1/2}\leq \frac{l}{8\pi C_1^{1/2}}.
\end{aligned}
$$

We define $\delta_0 := \min\{\,\delta_1,\ \frac{l}{4\pi}\cdot\frac{l^2}{64\pi^2 C_1}\,\}$ and

$$
L_{[s,s']_{S^1}}(\mathbf{x}) := \begin{cases} L_{[s,s']}(\mathbf{x}) & \text{if } [s,s']=[s,s']_{S^1}, \\ L_{[s',s]}(\mathbf{x}) & \text{else.} \end{cases}
$$

(If $s\geq s'$ the length $L_{[s,s']}(\mathbf{x})$ is given by $\int_s^{2\pi}|\dot{\mathbf{x}}|\,dt+\int_0^{s'}|\dot{\mathbf{x}}|\,dt$.)

Since $\delta_1<l/8$ (compare (45) in the appendix) we have

$$
\min\Big\{\,\frac{1}{2}L_{[s,s']}(\mathbf{x}_\delta),\ \frac{1}{2}L_{[s',s]}(\mathbf{x}_\delta)\,\Big\}=\frac{1}{2}L_{[s,s']_{S^1}}(\mathbf{x}_\delta)=\frac{l}{4\pi}\cdot|s-s'|_{S^1}.
$$

Consequently, we obtain for $\frac{1}{2}L_{[s,s']_{S^1}}(\mathbf{x}_\delta)\leq\delta\leq\delta_0$ the inequality

$$
|s-s'|_{S^1}^{1/2}\leq\frac{l}{8\pi C^{1/2}}.
$$

The estimate (9) then implies

$$
|\mathbf{x}_\delta(s)-\mathbf{x}_\delta(s')|\geq\frac{3}{4}L_{[s,s']_{S^1}}(\mathbf{x}_\delta)>\min\{\,\delta,\ \frac{1}{2}L_{[s,s']}(\mathbf{x}_\delta),\ \frac{1}{2}L_{[s',s]}(\mathbf{x}_\delta)\,\}.
$$

That is, for $\delta\leq\delta_0$ equality in the obstacle condition can only occur when $\delta<\min\Big\{\,\frac{1}{2}L_{[s,s']}(\mathbf{x}_\delta),\ \frac{1}{2}L_{[s',s]}(\mathbf{x}_\delta)\,\Big\}$. □

From now on we make the

General Assumption (G): Let $\delta\leq\delta_0<\delta_1<l/8$ be fixed, and $\mathbf{x}:=\mathbf{x}_\delta$ a minimizer of the functional F in the class $C^n_{\delta,l}$.

3.2. Regularity of free arcs of the minimizer

THEOREM 3.2. – *If there is a parameter* $s \in S^1$ *with*

$$|\mathbf{x}(s) - \mathbf{x}(s')| > \min\{\, \delta\,,\, \frac{1}{2} L_{[s,s']_{S^1}}(\mathbf{x})\,\} \quad \textit{for all } s' \neq s,$$

then there is an arc $B_r(s) \subset S^1$ *centered in* s*, such that* $\mathbf{x} \in C^\infty(B_r(s), \mathbb{R}^3)$.

The proof is more or less standard in the calculus of variations and will only be sketched briefly: First one shows the admissibility of comparison functions $\mathbf{z}_{\epsilon,t} := \mathbf{x} + \epsilon\phi + t\psi$ for all $|\epsilon|, |t| < \epsilon_0$ with ϵ_0 sufficiently small, i.e., $\mathbf{z}_{\epsilon,t} \in C^n_\delta$, using continuity arguments to establish the validity of the obstacle and isotopy condition for $\mathbf{z}_{\epsilon,t}$, (Lemma 3.5 in [18]). Then one can derive a differential equation involving a Lagrange multiplier, and standard regularity theory including a "bootstrap" argument gives the desired smoothness of $\mathbf{x}$, ([18], pp 50–52).

3.3. $H^{3,1}$–Regularity

We start out with a simple case of a contact situation, namely with touching points that are isolated and simple:

DEFINITION. – *We call* $s \in S^1$ *an isolated simple touching parameter (with respect to* $\mathbf{x}$*)* $:\Longleftrightarrow$

(i) $|\mathbf{x}(s) - \mathbf{x}(s')| = \min\{\, \delta,\, \frac{1}{2} L_{[s,s']_{S^1}}(\mathbf{x})\,\}$ for one and only one $s' \neq s$ and

(ii) there is a radius $R = R(s)$, such that for all $\sigma \in B_R(s) \backslash \{s\}$

$$|\mathbf{x}(\sigma) - \mathbf{x}(\sigma')| > \min\{\, \delta,\, \frac{1}{2} L_{[\sigma,\sigma']_{S^1}}(\mathbf{x})\,\} \quad \text{for all } \sigma' \in S^1, \sigma' \neq \sigma.$$

The corresponding image point $\mathbf{x}(s)$ is called an *isolated simple touching point.*

Theorem 1.2 is a consequence of the Sobolev embedding $H^{3,1} \hookrightarrow H^{2,\infty}$ and the following theorem:

THEOREM 3.3. – *Let* $s \in S^1$ *be an isolated simple touching parameter of a minimizing curve* $\mathbf{x}$ *satisfying the assumption (G). Then there is a radius* $\tilde{R} \leq R$*, such that* $\mathbf{x} \in H^{3,1}(B_{\tilde{R}}(s), \mathbb{R}^3)$*, where* $R = R(s)$ *is the radius in the definition above.*

Proof of Theorem 3.3.

Step 1. – If $\mathbf{x}(s)$ is contained in a straight part of the curve $\mathbf{x}$, then we have C^∞–regularity near s, and nothing has to be proved. Excluding this case we show that we can "correct" the length infinitesimally in the following way:

LEMMA 3.4. – *There exist arcs* $I_1, I_2 \subset B_R(s)\backslash\{s\}, \overline{I_1}\cap\overline{I_2}=\emptyset$ *and vector valued functions* $\zeta_i\in C_0^\infty(I_i,\mathbb{R}^3)$ *for* $i=1,2$ *and* $\tau_0>0$ *with*

a) $\mathbf{x}+\tau\zeta_i \in C_\delta^n$ *for all* $|\tau|<\tau_0$,
b) $\delta L(\mathbf{x},\zeta_i) = 1$ *for* $i=1,2$.

Proof. – By choice of a coordinate system and a shift of parameters if necessary, we can assume that $\dot{\mathbf{x}}(s) = (l/2\pi)\mathbf{e}_1$ and $s \neq 0$. (Here and throughout this paper $\mathbf{e}_i$ denotes the i–th standard basis vector in $\mathbb{R}^3$.) The assumption that $\mathbf{x}(s)$ lies on a curved part of $\mathbf{x}$ and the continuity of $\dot{\mathbf{x}}$ imply that there are parameters $s_1,s_2,s_3,s_4\in B_R(s)$, $s_1<s_2<s<s_3<s_4$ with $\dot{\mathbf{x}}(s_1) \neq \dot{\mathbf{x}}(s_2)$, $\dot{\mathbf{x}}(s_3) \neq \dot{\mathbf{x}}(s_4)$.

That means, $\mathbf{x}$ restricted to $I_1 := (s_1,s_2)$ or $I_2 := (s_3,s_4)$ is not a straight line, hence there are vector functions $\xi_i\in C_0^\infty(I_i,\mathbb{R}^3)$ with $\delta L(\mathbf{x},\xi_i) \neq 0$. The normalization $\zeta_i := \xi_i/\delta L(\mathbf{x},\xi_i)$ gives the desired result, if one proves that $\mathbf{x}+\tau\zeta_i\in C_\delta^n$ for $|\tau|<\tau_0 \ll 1$, which can be shown as in [18, Lemma 3.5]. □

Remark. – It is straightforward to extend this lemma to an arbitrary finite number $N\in\mathbb{N}$ of disjoint arcs $I_1,I_2,\ldots,I_N \subset S^1$ and corresponding vectorfunctions $\zeta_i, i=1,\ldots,N$ with the properties a) and b).

Step 2. – The following result due to S. Hildebrandt and H.C. Wente is a valuable tool in deriving a differential inequality for obstacle problems with a side condition. We will prove a slight generalization in the appendix (Lemma A.4) in order to treat more general contact situations, see Section 3.5.

LEMMA 3.5. – *Suppose, there are functions* $\phi_1,v_1\in C^1([0,\epsilon_0))$*;* $\phi_2,v_2\in C^1((-t_0,t_0))$ *for* $\epsilon_0,t_0>0$ *and constants* $\phi_0,c \in \mathbb{R}$ *with the properties*

(i) $0 = \phi_i(0) = v_i(0)$ *for* $i=1,2$,
(ii) $v_2'(0) = 1$,
(iii) *the function* $\phi(\epsilon,t) := \phi_0+\phi_1(\epsilon)+\phi_2(t)$ *satisfies*

$$\phi(\epsilon,t)\geq\phi(0,0) \quad \textit{for all } (\epsilon,t)\in[0,\epsilon_0)\times(-t_0,t_0) \textit{ with } v(\epsilon,t)=c,$$

where $v(\epsilon,t) := c+v_1(\epsilon)+v_2(t)$.

Then we obtain the inequality

$$\phi_1'(0)-\phi_2'(0)v_1'(0)\geq 0. \tag{10}$$

The number $\lambda := -\phi_2'(0)$ is the Lagrange multiplier in this situation.

Step 3. – By a further rotation and translation of the coordinate system we may assume that $\mathbf{e} := \frac{\mathbf{x}(s)-\mathbf{x}(s')}{|\mathbf{x}(s)-\mathbf{x}(s')|} = \mathbf{e}_3$ and $\mathbf{x}(s)=0$, where s' is the (unique) touching parameter corresponding to s.

CLAIM. – There exists a radius $R_1<R(s)$, such that $\langle \mathbf{x}, \mathbf{e}\rangle = x^3 \in H^{3,1}(B_{R_1}(s))$.

Proof. – One has to check that there are constants $K_0>0$, $0<t_0 \ll 1$ and $\epsilon_0=\epsilon_0(\eta, K_0, t_0)>0$, such that for $0\leq K<K_0$ the comparison curves $\mathbf{z}^t_{\epsilon,K} := \mathbf{x} + \epsilon\eta[K\mathbf{e}_2 + \mathbf{e}_3] + t\boldsymbol{\zeta}^*$ are admissible, i.e., in the class C^n_δ for all $(\epsilon, t)\in[0, \epsilon_0) \times (-t_0, t_0)$. Here η is an arbitrary nonnegative function in $C^\infty_0(B_r(s), \mathbb{R}^+)$, $r< \min\{\,|s-s_2|_{S^1}, |s-s_3|_{S^1}\,\}$ and $\boldsymbol{\zeta}^*\in C^\infty_0(I^*, \mathbb{R}^3)$ is one of the vector valued functions $\boldsymbol{\zeta}_i$ in Lemma 3.4. By the choice of r we have $\overline{B_r(s)} \cap \overline{I^*} = \emptyset$.

In order to apply Lemma 3.5 we define

$$\begin{aligned}
\phi_0 &:= D_{S^1}(\mathbf{x}), & c &:= l = L_{S^1}(\mathbf{x}),\\
\phi_1(\epsilon) &:= D_{B_r(s)}(\mathbf{z}^t_{\epsilon,K}) - D_{B_r(s)}(\mathbf{x}), & v_1(\epsilon) &:= L_{B_r(s)}(\mathbf{z}^t_{\epsilon,K}) - L_{B_r(s)}(\mathbf{x}),\\
\phi_2(t) &:= D_{I^*}(\mathbf{z}^t_{\epsilon,K}) - D_{I^*}(\mathbf{x}), & v_2(t) &:= L_{I^*}(\mathbf{z}^t_{\epsilon,K}) - L_{I^*}(\mathbf{x}).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\phi(\epsilon, t) := \phi_0 + \phi_1(\epsilon) + \phi_2(t) = D_{S^1}(\mathbf{z}^t_{\epsilon,K}),\\
&v(\epsilon, t) := c + v_1(\epsilon) + v_2(t) = L_{S^1}(\mathbf{z}^t_{\epsilon,K}),\\
\Rightarrow\quad &\phi(0,0) = D_{S^1}(\mathbf{x}) \leq D_{S^1}(\mathbf{z}^t_{\epsilon,K}) = \phi(\epsilon, t)
\end{aligned}$$

for all $\mathbf{z}^t_{\epsilon,K}\in C^n_\delta$ with $v(\epsilon, t) = L_{S^1}(\mathbf{z}^t_{\epsilon,K}) = l = c$.

One observes that Lemma 3.4 implies condition (ii) of Lemma 3.5, hence inequality (10) holds, which in turn gives us a differential inequality in the coordinates x^2 and x^3 for all $0\leq K<K_0$:

$$\begin{aligned}
(11)\qquad \int_{B_r(s)} &[(K\ddot{x}^2(\sigma) + \ddot{x}^3(\sigma))\ddot{\eta}(\sigma) - \{c_1|\ddot{\mathbf{x}}(\sigma)|^2 + c_2\}\\
&\cdot (K\dot{x}^2(\sigma) + \dot{x}^3(\sigma)) \cdot \dot{\eta}(\sigma)\,]\, d\sigma \geq 0
\end{aligned}$$

with $c_1 := 6\pi^2/l^2$, $c_2 := -\lambda \cdot l^2/(16\pi^3)$, $\lambda := -\phi_2'(0) = -\delta D(\mathbf{x}, \boldsymbol{\zeta}^*)$.

Setting $K = 0$ and integrating by parts one obtains

$$\begin{aligned}
(12)\qquad &\textstyle\int_{B_r(s)}[\,\ddot{x}^3(\sigma) + \int_s^\sigma \{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\} \cdot \dot{x}^3(\tau)\, d\tau\,] \cdot \ddot{\eta}(\sigma)\,]\, d\sigma \geq 0\\
&\text{for all } \eta\in C^\infty_0(B_r(s), \mathbb{R}^+).
\end{aligned}$$

We can interpret the left-hand side in (12) as a positive distribution T on $C_0^\infty(B_r(s))$ setting

$$T(\psi) := \int_{B_r(s)}[\,\ddot{x}^3(\sigma) + \int_s^\sigma \{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\} \cdot \dot{x}^3(\tau)\,d\tau\,] \cdot \ddot{\psi}(\sigma)\,d\sigma \quad \text{for } \psi \in C_0^\infty(B_r(s)).$$

Employing an argument of L. Schwartz ([16], p. 29) we find a Radon measure μ on $B_r(s)$, such that

$$T(\psi) = \int_{B_r(s)} \psi\,d\mu \quad \text{for all } \psi \in C_0^\infty(B_r(s)).$$

In the appendix (Lemma A.3) it is shown that for $\theta \in (0,1)$ one can find a nondecreasing bounded function $g = g_\theta$, such that

$$T(\psi) = -\int_{B_{\theta r}(s)} \dot{\psi}(\sigma) g(\sigma)\,d\sigma \quad \text{for all } \psi \in C_0^\infty(B_{\theta r}(s)).$$

An integration by parts yields

$$\int_{B_{\theta r}(s)} \left[\ddot{x}^3(\sigma) + \int_s^\sigma [\,\{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\}\dot{x}^3(\tau) - g(\tau)\,]\,d\tau\right] \cdot \ddot{\psi}(\sigma)\,d\sigma = 0 \quad \text{for } \psi \in C_0^\infty(B_{\theta r}(s)),$$

which implies

$$\ddot{x}^3(\sigma) + \int_s^\sigma [\,\{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\}\dot{x}^3(\tau) - g(\tau)\,]\,d\tau = a\sigma + b \tag{13}$$

for some numbers $a, b \in \mathbb{R}$ and for almost all $\sigma \in B_{\theta r}(s)$ by a generalized version of the fundamental lemma in the calculus of variations. Since $\mathbf{x} \in H^{2,2}(S^1, \mathbb{R}^3) \hookrightarrow C^{1,1/2}([0, 2\pi], \mathbb{R}^3)$ and g is bounded, the integrand in (13) is in L^1, hence $\ddot{x}^3 \in H^{1,1}(B_{\theta r}(s))$, i.e. $\langle \mathbf{x}, \mathbf{e}\rangle = x^3 \in H^{3,1}(B_{\theta r}(s))$.

Step 4. – For $K := K_0/2$ in (11) we apply the same method to find a nondecreasing bounded function $g = g_{\theta,K} \in L^1(B_{\theta r}(s))$, such that

$$K\ddot{x}^2(\sigma) + \ddot{x}^3(\sigma) + \int_s^\sigma (\{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\}[\,K\dot{x}^2(\tau) + \dot{x}^3(\tau) - g(\tau)\,]\,d\tau = \overline{a}\sigma + \overline{b} \tag{14}$$

for some numbers $\overline{a}, \overline{b} \in \mathbb{R}$ for almost all $\sigma \in B_{\theta r}(s)$. Since $\ddot{x}^3 \in H^{1,1}$ and $g_{\theta,K} \in L^1$, we obtain

$$\ddot{x}^3(\sigma) + \int_s^\sigma (\{c_1|\ddot{\mathbf{x}}(\tau)|^2 + c_2\}[\,K\dot{x}^2(\tau) + \dot{x}^3(\tau) - g(\tau)]\,d\tau \in H^{1,1}(B_{\theta r}(s)) \Rightarrow \ddot{x}^2 \in H^{1,1}(B_{\theta r}(s)).$$

Step 5. – Differentiating the equation $|\dot{\mathbf{x}}(\sigma)|^2=l^2/4\pi^2$ and recalling that $\dot{x}^1(s)=l/2\pi$ one obtains $\ddot{x}^1\in H^{1,1}(B_{R_1}(s))$ with $\ddot{x}^1(\sigma) = -[\dot{x}^2(\sigma)\ddot{x}^2(\sigma) + \dot{x}^3(\sigma)\ddot{x}^3(\sigma)]/\dot{x}^1(\sigma)$ for all $\sigma\in B_{R_1}(s)$, where $R_1 := \min\{ l^2/(4\pi\|\ddot{\mathbf{x}}\|^2_{L^2}),\, \theta r \}$.

In fact, one estimates

$$\begin{aligned} |\dot{x}^1(\sigma)| &\geq |\dot{x}^1(s)| - |\dot{x}^1(\sigma) - \dot{x}^1(s)| \\ &\geq l/2\pi - \|\ddot{\mathbf{x}}\|_{L^2}\cdot|\sigma - s|^{1/2} \\ &\geq \frac{l}{4\pi} \quad \text{for all } \sigma\in B_{R_1}(s). \end{aligned}$$ □

Remark. – With a suitable modification of the admissibility proof in step 2 one can extend this regularity result to more general *one–sided contact points*, where the constant K_0 now depends on the touching parameter $s\in S^1$, see [18, pp. 65–72]:

DEFINITION. – A parameter $s\in S^1$ with

$$|\mathbf{x}(s) - \mathbf{x}(s')| = \min\{ \delta,\, \frac{1}{2}L_{[s,s']_{S^1}}(\mathbf{x}) \} \tag{15}$$

for at least one $s'\in S^1\setminus\{s\}$ is called a *parameter with one–sided contact* if and only if there is a vector $\boldsymbol{\nu}=\boldsymbol{\nu}(s)\in S^2$, such that $\langle \mathbf{x}(s') - \mathbf{x}(s),\, \boldsymbol{\nu}\rangle < 0$ for all $s'\in S^1\setminus\{s\}$, for which (15) holds. Geometrically this means that all touching points $\mathbf{x}(s')$ corresponding to $\mathbf{x}(s)$ lie in an open halfspace H_ν with $\boldsymbol{\nu} \perp \partial H_\nu$ and $\mathbf{x}(s)\in\partial H_\nu$.

3.4. Higher Regularity

It is an open question, if the $H^{3,1}$–regularity is optimal for touching points of the minimizer $\mathbf{x}=\mathbf{x}_\delta$; there are, however, contact situations, where one can prove higher regularity.

DEFINITION. – Let $V_s := \{ s' \in S^1\setminus\{ s \}\,|\, |\mathbf{x}(s) - \mathbf{x}(s')| = \min\{\delta, \frac{1}{2}L_{[s,s']_{S^1}}(\mathbf{x})\}\}$ be the set of all touching parameters corresponding to s. Then we call $\mathbf{x}(s)$ a *convex touching point*, if and only if there are radii $R>0$ and $R'>0$, such that

$$\operatorname{dist}\left(\mathbf{x}(B_{R'}(V_s)),\, \overline{\operatorname{conv}(\mathbf{x}(B_R(s)))}\right) \geq \delta. \tag{16}$$

Remark. – This means that we can vary the curve $\mathbf{x}$ locally near the point $\mathbf{x}(s)$, as long as we stay in the convex hull of a short arc containing $\mathbf{x}(s)$. Not every isolated touching point is convex – consider for instance two arcs that are "hooked" in the sense that the normals at s and s' point at each other. On the other hand, there are convex contact situations that cannot be treated with the previous method. For simplicity, however, we will concentrate on convex touching points that are also isolated and simple. With some technical modifications one can prove the following results for more general convex touching points, see [18, Chapter 3.4.1].

THEOREM 3.6. – *Let $s \in S^1$ be an isolated simple touching point that is convex with respect to the minimizer $\mathbf{x}=\mathbf{x}_\delta$ satisfying assumption* (G). *Then there is a radius $r<\tilde{R}$, such that $\mathbf{x} \in H^{3,2}(B_r(s), \mathbb{R}^3)$, where $\tilde{R}=\tilde{R}(s)$ is the radius of $H^{3,1}$–regularity (Theorem* 3.3*).*

The embedding $H^{3,2} \hookrightarrow C^{2,1/2}$ implies that near such contact points the curvature of $\mathbf{x}$ is Hölder continuous with exponent $1/2$.

Proof. – As before, one shows first that there exists a radius $R_1=R_1(s)$, such that for all functions $\eta \in C_0^\infty(B_{2R_1}(s))$, there are constants $\epsilon_0, t_0 \in (0,1)$, such that for all $|h|<R_1$, $0<\epsilon<h^2\epsilon_0$, $|t|<t_0$ the comparison curves $\mathbf{z}_{\epsilon,h}^t := \mathbf{x} + \epsilon \triangle_{-h}(\eta^4 \triangle_h \mathbf{x}) + t\boldsymbol{\zeta}^*$ are admissible, i.e., in C_δ^n. The vector valued function $\boldsymbol{\zeta}^* \in C_0^\infty(I^*, \mathbb{R}^3)$ is chosen from the finitely many $\boldsymbol{\zeta}_i$ in the remark following Lemma 3.4, such that $s \notin \overline{I_i}$ and $s' \notin \overline{I_i}$. Furthermore we have used the notation $\triangle_h f(\sigma) := (f(\sigma+h) - f(\sigma))/h$ for difference quotients. (For the details of the admissibility proof see [18, Lemma 3.24].)

We take $\eta \in C_0^\infty(B_{2r}(s), [0,1])$, $r < \min\{\tilde{R}/4, R_1\}$ with $\eta \equiv 1$ on $B_r(s)$, $|\dot{\eta}| \le C/r$, $|\ddot{\eta}| \le C/r^2$.

As in the proof of Theorem 3.3 we apply Lemma 3.5 to obtain a differential inequality in terms of $\mathbf{x}$ and $\boldsymbol{\phi} := \triangle_{-h}(\eta^4 \triangle_h \mathbf{x}) \in H_0^{2,2}(B_{3r}(s), \mathbb{R}^3)$:

$$\int_{B_{3r}(s)} [\langle \ddot{\mathbf{x}}(\sigma), \ddot{\boldsymbol{\phi}}(\sigma)\rangle - \{c_1|\ddot{\mathbf{x}}(\sigma)|^2 + c_2\} \cdot \langle \dot{\mathbf{x}}(\sigma), \dot{\boldsymbol{\phi}}(\sigma)\rangle]\, d\sigma \ge 0$$

for $c_1 := 6\pi^2/l^2, c_2 := -\lambda \cdot l^2/(16\pi^3)$, $\lambda := -\delta D(\mathbf{x}, \boldsymbol{\zeta}^*)$.

Applying the well-known calculus for difference quotients (see e.g. in [2], vol. II, p. 84) we arrive at

$$\begin{aligned} &\int_{B_{3r}(s)} \langle \triangle_h \ddot{\mathbf{x}}, \triangle_h \ddot{\mathbf{x}}\rangle \cdot \eta^4\, d\sigma \\ &\le -\int_{B_{3r}(s)} \langle \triangle_h \ddot{\mathbf{x}}, (12\eta^2\dot{\eta}^2 \triangle_h \mathbf{x} + 4\eta^3 \ddot{\eta} \triangle_h \mathbf{x} + 8\eta^3 \dot{\eta} \triangle_h \dot{\mathbf{x}})\rangle d\sigma \\ &\quad - \int_{B_{3r}(s)} \langle \triangle_h(\{c_1|\ddot{\mathbf{x}}|^2 + c_2\}\dot{\mathbf{x}}), 4\eta^3\dot{\eta}\triangle_h \mathbf{x} + \eta^4 \triangle_h \dot{\mathbf{x}}\rangle\, d\sigma. \end{aligned} \tag{17}$$

For vector valued functions $\mathbf{f}=\mathbf{f}(\mathbf{p},\mathbf{r})$ we have

$$\begin{aligned}
&\triangle_h f(\mathbf{p}(t),\mathbf{r}(t)) \\
&= \frac{1}{h}\int_0^1 f_{\mathbf{p}}(\mathbf{p}(t)+\tau h\triangle_h\mathbf{p}(t),\, \mathbf{r}(t)+\tau h\triangle_h\mathbf{r}(t))\cdot h\triangle_h\mathbf{p}(t)\,d\tau \\
&+ \frac{1}{h}\int_0^1 f_{\mathbf{r}}(\mathbf{p}(t)+\tau h\triangle_h\mathbf{p}(t),\, \mathbf{r}(t)+\tau h\triangle_h\mathbf{r}(t))\cdot h\triangle_h\mathbf{r}(t)\,d\tau.
\end{aligned}$$

Hence, from (17) we obtain

$$\begin{aligned}
&\int_{B_{3r}(s)}\langle \triangle_h\ddot{\mathbf{x}},\, \triangle_h\ddot{\mathbf{x}}\rangle\cdot\eta^4\,d\sigma \\
&\leq -\int_{B_{3r}(s)}\langle \triangle_h\ddot{\mathbf{x}},\, (12\eta^2\dot\eta^2\triangle_h\mathbf{x}+4\eta^3\ddot\eta\triangle_h\mathbf{x}+8\eta^3\dot\eta\triangle_h\dot{\mathbf{x}})\rangle\,d\sigma \\
&-\int_{B_{3r}(s)}\int_0^1\{c_1|\dot{\mathbf{x}}+\tau h\triangle_h\dot{\mathbf{x}}|^2+c_2\}\,d\tau \\
&\qquad\cdot\langle \triangle_h\dot{\mathbf{x}},\, 4\eta^3\dot\eta\triangle_h\mathbf{x}+\eta^4\triangle_h\dot{\mathbf{x}}\rangle\,d\sigma \qquad (18)\\
&-2c_1\int_{B_{3r}(s)}\int_0^1[\ddot{\mathbf{x}}+\tau h\triangle_h\ddot{\mathbf{x}}] \\
&\qquad\otimes[\dot{\mathbf{x}}+\tau h\triangle_h\dot{\mathbf{x}}]\,d\tau\langle \triangle_h\ddot{\mathbf{x}},\, 4\eta^3\dot\eta\triangle_h\mathbf{x}+\eta^4\triangle_h\dot{\mathbf{x}}\rangle\,d\sigma \\
&=: I_1+I_2+I_3.
\end{aligned}$$

Using a generic notation for constants we estimate

$$|I_1|\leq\epsilon\int_{B_{3r}(s)}\eta^4|\triangle_h\ddot{\mathbf{x}}|^2\,d\sigma+C(\epsilon,r),$$

since

$$\int(|\triangle_h\mathbf{x}|^2+|\triangle_h\dot{\mathbf{x}}|^2)\,d\sigma\leq c\|\mathbf{x}\|_{H^{2,2}}.$$

$$\begin{aligned}
|I_2| &\leq C\int_{B_{3r}(s)}\int_0^1\underbrace{[|\dot{\mathbf{x}}|^2+|\dot{\mathbf{x}}(.+h)|^2+|c_2|]}_{\text{indep. of }\tau}\,d\tau \\
&\qquad\cdot|\triangle_h\dot{\mathbf{x}}|\{\eta^4|\triangle_h\dot{\mathbf{x}}|+\frac{C}{r}\eta^3|\triangle_h\mathbf{x}|\}\,d\sigma \\
&\leq C\int_{B_{4r}(s)}[|\dot{\mathbf{x}}|^2+|c_2|]\cdot\{|\eta^2\triangle_h\dot{\mathbf{x}}|^2+\frac{C}{r^2}|\triangle_h\mathbf{x}|^2\}\,d\sigma \\
&\leq C(\|\mathbf{x}\|_{H^{2,\infty}},r)
\end{aligned}$$

Similarly,

$$|I_3| \le \epsilon \int_{B_{3r}(s)} \eta^4 |\triangle_h \ddot{\mathbf{x}}|^2 \, d\sigma + \frac{C}{\epsilon} \int_{B_{4r}(s)} |\ddot{\mathbf{x}}|^2 \cdot (\, \eta^2 \dot{\eta}^2 (\triangle_h \mathbf{x})^2 + \eta^4 (\triangle_h \dot{\mathbf{x}})^2 \,) \, d\sigma$$

$$\le \epsilon \int_{B_{3r}(s)} \eta^4 |\triangle_h \ddot{\mathbf{x}}|^2 \, d\sigma + C(\|\mathbf{x}\|_{H^{2,\infty}}, \epsilon, r)$$

Summarizing the estimates for $I_1 - I_4$ we can choose $\epsilon > 0$ sufficiently small to absorb the leading term on the right-hand side to get

$$\int_{B_r(s)} |\triangle_h \ddot{\mathbf{x}}|^2 \, d\sigma \le \int_{B_{3r}(s)} |\eta^2 \triangle_h \ddot{\mathbf{x}}|^2 \, d\sigma \le C < \infty,$$

which implies $\int_{B_r(s)} |\,\dddot{\mathbf{x}}\,|^2 \, d\sigma \le C < \infty$. □

3.5. Two-sided contact points

Definition. – We call $s \in S^1$ an *isolated double touching parameter* $:\Longleftrightarrow$

(i) $|\mathbf{x}(s) - \mathbf{x}(s')| = \min\{\, \delta, \, \frac{1}{2} L_{[s,s']_{S^1}}(\mathbf{x}) \,\}$ for exactly two different parameters $s_1', s_2' \in S^1 \backslash \{s\}$ and at least one of the s_i' is isolated simple;

(ii) there is a radius $R = R(s)$, such that for all $\sigma \in B_R(s) \backslash \{s\}$

$$|\mathbf{x}(\sigma) - \mathbf{x}(\sigma')| > \min\{\, \delta, \, \frac{1}{2} L_{[s,s']_{S^1}}(\mathbf{x}) \,\} \quad \text{for all } \sigma' \in S^1 \backslash \{\sigma\}.$$

The corresponding image point $\mathbf{x}(s)$ is called an *isolated double touching point.*

Remark. – Such a point is a one-sided contact point (see the definition at the end of Section 4.3), unless $\mathbf{x}(s), \mathbf{x}(s_1')$ and $\mathbf{x}(s_2')$ lie on a straight line. Hence, we can concentrate on that special situation and fix the coordinate system in the following way:

$$\mathbf{x}(s) = 0 \in \mathbb{R}^3, \quad \mathbf{x}(s_1') = \delta \mathbf{e}_3, \quad \mathbf{x}(s_2') = -\delta \mathbf{e}_3,$$

$$\dot{\mathbf{x}}(s) = (l/2\pi)\mathbf{e}_1 \quad \text{and} \quad (\mathbf{x}(s) - \mathbf{x}(s_2'))/\delta = (\mathbf{x}(s_1') - \mathbf{x}(s))/\delta = \mathbf{e}_3.$$

The idea is to vary simultaneously near the point of interest $\mathbf{x}(s)$ and near the contact point that is isolated and simple, say $\mathbf{x}(s_1')$. One has to show that there is a constant K_0, such that the comparison curves

$$\mathbf{z}_\epsilon^t := \mathbf{x} + \epsilon\varphi[\, K\mathbf{e}_2 + \mathbf{e}_3 \,] + 2\epsilon \|\varphi\|_{L^\infty} \eta^2 \mathbf{e}_3 + t\boldsymbol{\zeta}^*$$

are admissible, i.e., in C_δ^n for all $0 \le K < K_0$, $|t| < t_0$, $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 = \epsilon_0(\varphi, \eta, K_0, t_0) \ll 1$, where $\eta \in C_0^\infty(B_{R_1}(s_1'), [0,1])$, $\eta \equiv 1$ on $B_{R_1}(s_1')$ with $|\dot\eta| \le C/R_1$, $|\ddot\eta| \le C/R_1^2$ for some $R_1 \ll 1$. Here $\varphi \in H_0^{2,2}(B_\rho(s), \mathbb{R}^+)$ is an arbitrary nonnegative testfunction for $\rho \le R$ sufficiently small. (See [18, pp. 87–90] for the detailed computations.)

We define

$$\begin{aligned}
\phi_0 &:= D_{S^1}(\mathbf{x}),\\
c &:= l = L_{S^1}(\mathbf{x}),\\
\phi_1(\epsilon) &:= D_{B_\rho(s)}(\mathbf{z}_\epsilon^t) - D_{B_\rho(s)}(\mathbf{x}),\\
v_1(\epsilon) &:= L_{B_\rho(s)}(\mathbf{z}_\epsilon^t) - L_{B_\rho(s)}(\mathbf{x}),\\
\phi_2(t) &:= D_{I^*}(\mathbf{z}_\epsilon^t) - D_{I^*}(\mathbf{x}),\\
v_2(t) &:= L_{I^*}(\mathbf{z}_\epsilon^t) - L_{I^*}(\mathbf{x}),\\
\phi_3(s(\epsilon)) &:= D_{B_{R_1}(s_1')}(\mathbf{z}_\epsilon^t) - D_{B_{R_1}(s_1')}(\mathbf{x}),\\
v_3(s(\epsilon)) &:= L_{B_{R_1}(s_1')}(\mathbf{z}_\epsilon^t) - L_{B_{R_1}(s_1')}(\mathbf{x}),
\end{aligned}$$

where $s(\epsilon) := 2\epsilon\|\varphi\|_\infty$. Then we obtain

$$\begin{aligned}
\phi(\epsilon,t) &:= \phi_0 + \phi_1(\epsilon) + \phi_2(t) + \phi(s(\epsilon)) = D_{S^1}(\mathbf{z}_\epsilon^t),\\
v(\epsilon,t) &:= c + v_1(\epsilon) + v_2(t) + v_3(s(\epsilon)) = L_{S^1}(\mathbf{z}_\epsilon^t).
\end{aligned}$$

Since $\mathbf{x} \in C_{\delta,l}^n$ is minimal,

$$\phi(0,0) = D_{S^1}(\mathbf{x}) \le D_{S^1}(\mathbf{z}_\epsilon^t) = \phi(\epsilon,t)$$

for all $\mathbf{z}_\epsilon^t \in C_\delta^n$ with $v(\epsilon,t) = L_{S^1}(\mathbf{z}_\epsilon^t) = l = c$.

Consequently, the generalized Lagrange multiplier lemma (Lemma A.4 in the appendix) is applicable. From the inequality (48) one finds after a short calculation

$$(19)\quad \int_{B_\rho(s)} [(K\ddot x^2 + \ddot x^3)\ddot\varphi - \{c_1|\ddot{\mathbf{x}}|^2 + c_2\}\cdot(K\dot x^2 + \dot x^3)\dot\varphi]\, d\sigma + 2\lambda_1\|\varphi\|_\infty \ge 0$$

for all $0 \le K < K_0$ with $c_1 := 6\pi^2/l^2$, $c_2 := -\lambda\cdot l^2/(16\pi^3)$, $\lambda := -\delta D(\mathbf{x}, \boldsymbol{\zeta}^*)$ and $\lambda_1 = \delta D_{B_{R_1}(s_1')}(\mathbf{x}, \eta^2\mathbf{e}_3) - \delta D_{I^*}(\mathbf{x}, \boldsymbol{\zeta}^*) \cdot \delta L_{B_{R_1}(s_1')}(\mathbf{x}, \eta^2\mathbf{e}_3)$.

CLAIM. – There is a radius $r_0 = r_0(s)$ and $p > 2$, such that $x^3 \in H^{2,p}(B_{r_0}(s))$.

Proof.

Step 1. – We test (19) for $K{=}0$ with $\varphi(\sigma) := \zeta^2(\sigma)\cdot(x_*^3 - x^3)(\sigma) \in H_0^{2,2}(\Omega(s,r),\mathbb{R}^+)$, where $r{\le}\rho_0 := 5\rho/6$, $\Omega(s,r) := B_{6r/5}(s)$,

$$x_*^3 := \max\{\, x^3(t)\,|\, t\in\overline{\Omega(s,r)}\,\} \quad (\ge 0,\ \text{since } x^3(s){=}0) \tag{20}$$

and $\zeta{\in}C_0^\infty(\,\Omega(s,r),[0,1]\,)$, $\zeta\equiv 1$ on $B_r(s)$, $|\dot\zeta|{\le}C/r$, $|\ddot\zeta|{\le}C/r^2$.

Inserting this into (19) we arrive at

$$\begin{aligned}\int_{\Omega(s,r)}\zeta^2(\ddot x^3)^2d\sigma \le & \int_{\Omega(s,r)}\ddot x^3\cdot[((2\dot\zeta^2{+}2\zeta\ddot\zeta)(x_*^3 - x^3) - 4\zeta\dot\zeta\dot x^3]d\sigma\\ & -\int_{\Omega(s,r)}\{c_1|\dot{\mathbf{x}}|^2{+}c_2\}\dot x^3\cdot(2\zeta\dot\zeta(x_*^3{-}x^3){-}\zeta^2\dot x^3)d\sigma\\ & +2\lambda_1\|\zeta^2\cdot(x_*^3-x^3)\|_\infty\\ := & I_1{+}I_2{+}I_3{+}I_4{+}I_5.\end{aligned} \tag{21}$$

Now we estimate (using a generic notation for constants)

$$|I_1| = |\int_{\Omega(s,r)}\ddot x^3{\cdot}[((2\dot\zeta^2{+}2\zeta\ddot\zeta)(x_*^3{-}x^3)]\,d\sigma| \le \frac{C}{r^2}\int_{\Omega(s,r)}|\ddot x^3|{\cdot}|x_*^3{-}x^3|\,d\sigma.$$

By continuity of x^3 there is $s_*{\in}\overline{\Omega(s,r)}$, such that $x_*^3{=}x^3(s_*)$, hence

$$\begin{aligned}|x_*^3 - x^3(\sigma)| = |x^3(s_*) - x^3(\sigma)| &= |\int_\sigma^{s_*}(\dot x^3(\tau) - \underbrace{\dot x^3(s)}_{=0})\,d\tau\,|\\ &\le |\int_\sigma^{s_*}\int_{\Omega(s,r)}|\ddot x^3(t)|\,dt\,d\tau\,| \le Cr\int_{\Omega(s,r)}|\ddot x^3(t)|\,dt,\end{aligned} \tag{22}$$

$$\Rightarrow\quad |I_1| \le \frac{C}{r}\Big[\int_{\Omega(s,r)}|\ddot x^3(t)|\,dt\Big]^2 \le Cr\Big[\fint_{\Omega(s,r)}|\ddot x^3(t)|\,dt\Big]^2. \tag{23}$$

$$\begin{aligned}|I_2| &= |\int_{\Omega(s,r)}4\zeta\dot\zeta\dot x^3\ddot x^3\,d\sigma\,| \le \frac{C}{r}\int_{\Omega(s,r)}|\dot x^3|\cdot|\ddot x^3|\,d\sigma\\ &= \frac{C}{r}\int_{\Omega(s,r)}|\ddot x^3(\sigma)|\cdot|\dot x^3(\sigma) - \dot x^3(s)|\,d\sigma\\ &= \frac{C}{r}\int_{\Omega(s,r)}|\ddot x^3|\cdot|\int_s^\sigma\ddot x^3(t)\,dt\,|\,d\sigma\\ &\le \frac{C}{r}\cdot\Big[\int_{\Omega(s,r)}|\ddot x^3(\sigma)|\,d\sigma\Big]^2 = Cr\Big[\fint_{\Omega(s,r)}|\ddot x^3(\sigma)|\,d\sigma\Big]^2.\end{aligned} \tag{24}$$

$$
\begin{aligned}
|I_3| &= \Big| \int_{\Omega(s,r)} \{c_1|\ddot{\mathbf{x}}|^2 + c_2\}\dot{x}^3(2\zeta\dot{\zeta}(x_*^3 - x^3))\,d\sigma \Big| \\
&\leq \frac{C}{r}\int_{\Omega(s,r)} \{\,|c_1||\ddot{\mathbf{x}}|^2 + |c_2|\,\}|\dot{x}^3|\cdot|x_*^3 - x^3|\,d\sigma \\
&\underset{(22)}{\leq} \frac{C}{r}\cdot r\int_{\Omega(s,r)} \{\,|c_1||\ddot{\mathbf{x}}|^2 + |c_2|\,\}\,d\sigma\cdot\left[\int_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt\right]^2 \\
&\leq Cr^2\left[⨍_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt\right]^2,
\end{aligned}
\tag{25}
$$

where we have also used

$$
|\dot{x}^3(\sigma)| = |\dot{x}^3(\sigma) - \underbrace{\dot{x}^3(s)}_{=0}| \leq \int_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt. \tag{26}
$$

For the remaining terms one obtains

$$
|I_4| = \Big| -\int_{\Omega(s,r)} \{\,c_1|\ddot{\mathbf{x}}|^2 + c_2\,\}(\dot{x}^3)^2\zeta^2\,d\sigma \Big| \underset{(26)}{\leq} Cr^2\left[⨍_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt\right]^2, \tag{27}
$$

$$
|I_5| = 2\lambda_1\|\zeta^2(x_*^3 - x^3)\|_\infty \underset{(22)}{\leq} Cr\int_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt \leq Cr^{3/2}. \tag{28}
$$

Inserting the inequalities (23)–(28) into (21) we arrive at

$$
⨍_{B_r(s)} |\ddot{x}^3(t)|^2\,dt \leq C\left[⨍_{\Omega(s,r)} |\ddot{x}^3(t)|\,dt\right]^2 + Cr^{1/2} \quad \text{for all } r\leq\rho_0. \tag{29}
$$

Step 2. – For $r<\rho_1<(5/6)\rho_0$ and $\frac{6}{5}r < |\sigma - s| < \rho_0 - \frac{6}{5}\rho_1$ we can vary freely around σ, since s is isolated by assumption, and we have the differential equation

$$
\int_{B_r(\sigma)} [\langle\ddot{\mathbf{x}}(\sigma), \ddot{\boldsymbol{\varphi}}(\sigma)\rangle - \{c_1|\ddot{\mathbf{x}}(\sigma)|^2 + c_2\}\cdot\langle\dot{\mathbf{x}}(\sigma), \dot{\boldsymbol{\varphi}}(\sigma)\rangle]\,d\sigma = 0 \tag{30}
$$

for $c_1 := 6\pi^2/l^2$ and $c_2 := -\lambda_0\cdot l^2/(16\pi^3)$ for all $\varphi\in H_0^{2,2}(\Omega(\sigma,r),\mathbb{R}^3)$.

We test this equation with $\varphi(t) := \eta^2(x^3(t) - l_3(t))\cdot\mathbf{e}_3 \in H_0^{2,2}(\Omega(\sigma,r),\mathbb{R}^3)$, where $\eta\in C_0^\infty(\Omega(\sigma,r))$, satisfies $0\leq\eta\leq1$; $\eta\equiv1$ on $B_r(\sigma)$; $|\dot{\eta}|\leq C/r$ and $|\ddot{\eta}|\leq C/r^2$. The function l_3 is the linear interpolating function to x^3 with values

$$
l_3(\sigma - (6/5)r) = x^3(\sigma - (6/5)r), \quad l_3(\sigma + (6/5)r) = x^3(\sigma + (6/5)r).
$$

By Rolle's theorem there is $\tau_* \in \Omega(\sigma, r)$ with $\dot{x}^3(\tau_*) = \dot{l}_3(\tau_*)$. Inserting this into (30) we find

$$\begin{aligned}
&\int_{\Omega(\sigma,r)} \eta^2 |\ddot{x}^3|^2 \, dt \\
&\quad = -2 \int_{\Omega(\sigma,r)} \ddot{x}^3 \cdot [\, (\dot{\eta}^2 + \eta\ddot{\eta})(x^3 - l_3) \,] \, dt \\
&\quad\quad - 4 \int_{\Omega(\sigma,r)} \ddot{x}^3 \cdot (\dot{x}^3 - \dot{l}_3) \eta\dot{\eta} \, dt \\
&\quad\quad + \int_{\Omega(\sigma,r)} \{c_1 |\ddot{\mathbf{x}}|^2 + c_2\} \dot{x}^3 [\, 2\eta\dot{\eta}(x^3 - l_3) + \eta^2(\dot{x}^3 - \dot{l}_3) \,] \, dt \\
&\quad \leq \frac{C}{r^2} \int_{\Omega(\sigma,r)} |\ddot{x}^3| |x^3 - l_3| \, dt + \frac{C}{r} \int_{\Omega(\sigma,r)} |\ddot{x}^3| |\dot{x}^3 - \dot{l}_3| \, dt \\
&\quad\quad + \frac{C}{r} \int_{\Omega(\sigma,r)} \{\, |c_1| |\ddot{\mathbf{x}}|^2 + |c_2| \,\} |\dot{x}^3| |x^3 - l_3| \, dt \\
&\quad\quad + C \int_{\Omega(\sigma,r)} \{\, |c_1| |\ddot{\mathbf{x}}|^2 + |c_2| \,\} |\dot{x}^3| |\dot{x}^3 - \dot{l}_3| \, dt \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{31}$$

For $\tau \in \Omega(\sigma, r)$ we have, since $\ddot{l}_3 \equiv 0$

$$\begin{aligned}
|x^3(\tau) - l_3(\tau)| &= |x^3(\tau) - l_3(\tau) - [\underbrace{x^3(\sigma - (6/5)r) - l_3(\sigma - (6/5)r)}_{=0}]| \\
&= |\int_{\sigma-(6/5)r}^{\tau} (\dot{x}^3(t) - \dot{l}_3(t)) \, dt \,| \\
&= |\int_{\sigma-(6/5)r}^{\tau} [\, \dot{x}^3(t) - \dot{l}_3(t) - (\dot{x}^3(\tau_*) - \dot{l}_3(\tau_*)) \,] \, dt \,| \\
&= |\int_{\sigma-(6/5)r}^{\tau} \int_{\tau_*}^{t} \ddot{x}^3(\bar{t}) \, d\bar{t} \, dt \,| \leq |\int_{\sigma-(6/5)r}^{\tau} \int_{\Omega(\sigma,r)} |\ddot{x}^3| \, dt \,| \\
&\leq Cr \int_{\Omega(\sigma,r)} |\ddot{x}^3| \, dt
\end{aligned}$$

and similarly

$$|\dot{x}^3 - \dot{l}_3| \leq C \int_{\Omega(\sigma,r)} |\ddot{x}^3| \, dt.$$

Now we estimate the terms $I_1 - I_4$:

$$\begin{aligned}|I_1| + |I_2| &\le \frac{C}{r}\left[\int_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\right]^2 = Cr\left[⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\right]^2,\\ |I_3| &\le C\int_{\Omega(\sigma,r)} \{|c_1||\ddot{\mathbf{x}}|^2 + |c_2|\}|\ddot{x}^3|\, dt \cdot \int_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\\ &\le Cr ⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\\ &\le Cr + r\left[⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\right]^2,\\ |I_4| &\le Cr + r\left[⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\right]^2.\end{aligned}$$

We insert these inequalities into (31) and obtain

$$(32)\qquad ⨍_{B_r(\sigma)} |\ddot{x}^3|^2\, dt \le C\left[\left(⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\, dt\right)^2 + 1\right] \quad \text{for all } r<\rho_1.$$

Step 3. – We claim that there are constants $C, R_0, \tilde{R}>0$, such that for all $\sigma\in B_{\tilde{R}}(s)$

$$(33)\qquad ⨍_{B_r(\sigma)} |\ddot{x}^3|^2\, dt \le C\left[⨍_{B_{4r}(\sigma)} |\ddot{x}^3|\, dt\right]^2 + C \quad \text{for all } r\le R_0.$$

In fact, for $\sigma\in B_{\tilde{R}}(s)$, where $\tilde{R} := \rho_0 - (6/5)\rho_1$ and $R_0 := \min\{\rho_1, (5/11)\rho_0\}$, we distinguish between two cases for $r\le R_0$:

$$\text{I.}\quad |\sigma - s|<\frac{6}{5}r \quad \text{and} \quad \text{II.}\quad \frac{6}{5}r\le|\sigma - s|<\tilde{R}.$$

In case I we have $|\sigma' - s|\le|\sigma' - \sigma| + |\sigma - s|<\frac{11}{5}r$ for all $\sigma'\in B_r(\sigma)$, and therefore

$$\begin{aligned}⨍_{B_r(\sigma)} |\ddot{x}^3|^2\, dt &\le C ⨍_{B_{\frac{11}{5}r}(s)} |\ddot{x}^3|^2\, dt \underset{(29)}{\le} C\left[⨍_{B_{\frac{66}{25}r}(s)} |\ddot{x}^3|\, dt\right]^2 + C\left(\frac{11}{5}r\right)^{1/2}\\ &\le C\left[⨍_{B_{4r}(\sigma)} |\ddot{x}^3|\, dt\right]^2 + C,\end{aligned}$$

where we used $|\sigma' - \sigma|\le|\sigma' - s| + |\sigma - s|<\frac{66}{25}r + \frac{6}{5}r<4r$ for all $\sigma'\in B_{\frac{66}{25}r}(s)$. In case II one immediately obtains the desired inequality from (32).

Step 4. – The estimate (33) implies in terms of the local maximal function (see e.g. [17]) $M_R[f](t) := \sup\{⨍_{B_r(t)} f \mid 0 < r < R\}$

$$M_{R_0}[\,|\ddot{x}^3|^2\,](\sigma) \le C \cdot M^2_{4R_0}[\,|\ddot{x}^3| + 1\,](\sigma) \quad \text{for all } \sigma \in B_{\tilde{R}}(s).$$

Redefining $\ddot{x}^3$ and the constant function $f \equiv 1$ in $\mathbb{R}\backslash B_{\tilde{R}+4R_0}(s)$ by 0 we arrive at

$$M_{R_0}[\,|\ddot{x}^3|^2\,](\sigma) \le C \cdot M^2_{\infty}[\,|\ddot{x}^3| + 1\,](\sigma) \quad \text{for all } \sigma \in B_{\tilde{R}+4R_0}(s).$$

Thus we can apply Gehring's lemma ([17]) to get the desired higher integrability on $B_{r_0}(s)$, $r_0 := \tilde{R}/2$:

$$(34)\quad \left[⨍_{B_{r_0}(s)} |\ddot{x}^3|^p \, dt \right]^{1/p} \le K_1 \cdot \left[\left(\frac{1}{|B_{r_0}(s)|} \int_{B_{\tilde{R}+4R_0}(s)} |\ddot{x}^3|^2 \, dt \right)^{1/2} + \left(\frac{|B_{\tilde{R}+4R_0}(s)|}{|B_{r_0}(s)|} \right)^{1/p} \right]$$

for some $p>2$ and a constant K_1 depending on p and the ratio $\tilde{R}/R_0$. □

Now we are in the position to prove higher integrability for the full vector $\ddot{\mathbf{x}}$:

THEOREM 3.7. – *Let $s \in S^1$ be an isolated double touching parameter with corresponding touching parameters $s'_1, s'_2 \in S^1\backslash\{s\}$, such that the image points $\mathbf{x}(s), \mathbf{x}(s'_1), \mathbf{x}(s'_2)$ lie on a straight line, where $\mathbf{x}{=}\mathbf{x}_\delta \in C^n_{\delta,l}$ is the minimizer satisfying the assumption* (G).

Then there is a radius $r_1{=}r_1(s)$ and $p>2$, such that $\mathbf{x} \in H^{2,p}(B_{r_1}(s))$.

Proof. – Setting $K := K_0/2$ we test the differential inequality (19) with $\varphi(\sigma) := \zeta^2(\sigma)(x_*^2 - x^2(\sigma)) \in H_0^{2,2}(\Omega(s,r), \mathbb{R}^+)$ for radii r chosen as in the first step in the proof of the last claim.

The only new terms (up to constant factors depending on K_0) are

$$J_1 := \int_{\Omega(s,r)} \ddot{x}^3 \cdot [\,(2\dot{\zeta}^2 + 2\zeta\ddot{\zeta})(x_*^2 - x^2)\, dt,$$

$$J_2 := -\int_{\Omega(s,r)} 4\zeta\dot{\zeta}\dot{x}^2\ddot{x}^3 \, dt,$$

$$J_3 := -\int_{\Omega(s,r)} \{\, c_1|\ddot{\mathbf{x}}|^2 + c_2 \,\}\dot{x}^3 \cdot 2\zeta\dot{\zeta}(x_*^2 - x^2) \, dt,$$

$$J_4 := \int_{\Omega(s,r)} \{\, c_1|\ddot{\mathbf{x}}|^2 + c_2 \,\}\dot{x}^3 \cdot \zeta^2\dot{x}^2 \, dt,$$

$$J_5 := \int_{\Omega(s,r)} \zeta^2\ddot{x}^2\ddot{x}^3 \, dt.$$

Using the same techniques as before (in particular inequalities (22), (26) with x^3 replaced by x^2) we estimate

$$
\begin{aligned}
|J_1| &\leq \frac{C}{r^2}\int_{\Omega(s,r)} |\ddot{x}^3|\cdot|x_*^2 - x^2|\,dt \leq Cr ⨍_{\Omega(s,r)} |\ddot{x}^3|\,dt ⨍_{\Omega(s,r)} |\ddot{x}^2|\,dt,\\
|J_2| &\leq \frac{C}{r}\int_{\Omega(s,r)} |\ddot{x}^3|\cdot|\dot{x}^2|\,dt \leq Cr ⨍_{\Omega(s,r)} |\ddot{x}^3|\,dt ⨍_{\Omega(s,r)} |\ddot{x}^2|\,dt,\\
|J_3| &\leq \frac{C}{r}\int_{\Omega(s,r)} \{\,|c_1||\dot{\mathbf{x}}|^2 + |c_2|\,\}|\ddot{x}^3|\cdot|x_*^2 - x^2|\,dt\\
&\leq C\int_{\Omega(s,r)} \{\,|c_1||\dot{\mathbf{x}}|^2 + |c_2|\,\}\,dt \int_{\Omega(s,r)} |\ddot{x}^3|\,dt \int_{\Omega(s,r)} |\ddot{x}^2|\,dt\\
&\leq Cr^2 ⨍_{\Omega(s,r)} |\ddot{x}^3|\,dt ⨍_{\Omega(s,r)} |\ddot{x}^2|\,dt
\end{aligned}
$$

and analogously,

$$|J_4| \leq Cr^2 ⨍_{\Omega(s,r)} |\ddot{x}^3|\,dt ⨍_{\Omega(s,r)} |\ddot{x}^2|\,dt.$$

Applying Young's inequality we finally obtain

$$|J_5| \leq \epsilon \int_{\Omega(s,r)} \zeta^2|\ddot{x}^2|^2\,dt + C(\epsilon)\int_{\Omega(s,r)} \zeta^2|\ddot{x}^3|^2\,dt. \tag{35}$$

Summarizing these estimates and absorbing the leading term of (35) for sufficiently small $\epsilon>0$ we arrive at

$$
\begin{aligned}
⨍_{B_r(s)} |\ddot{x}^2|^2\,dt \leq C\Bigg[&\bigg(⨍_{\Omega(s,r)} |\ddot{x}^2|\,dt\bigg)^2\\
&+\bigg(⨍_{\Omega(s,r)} |\ddot{x}^3|\,dt\bigg)^2 + ⨍_{\Omega(s,r)} |\ddot{x}^3|^2\,dt\Bigg] + Cr^{1/2}.
\end{aligned}
$$

Similarly, one calculates on $\Omega(\sigma, r)$ with $\frac{6}{5}r < |\sigma - s| < \rho_0 - \frac{6}{5}\rho_1$ testing the corresponding differential equation for $K := K_0/2$ with $\varphi(t) := \zeta^2(t)(x^2(t) - l_2(t))$, where l_2 is the linear interpolator to x^2 with the same boundary values as x^2 on $\partial\Omega(\sigma, r)$.

One obtains the inequality

$$
\begin{aligned}
⨍_{B_r(\sigma)} |\ddot{x}^2|^2\,dt \leq C\Bigg[&\bigg(⨍_{\Omega(\sigma,r)} |\ddot{x}^2|\,dt\bigg)^2\\
&+\bigg(⨍_{\Omega(\sigma,r)} |\ddot{x}^3|\,dt\bigg)^2 + ⨍_{\Omega(\sigma,r)} |\ddot{x}^3|^2\,dt + 1\Bigg].
\end{aligned}
$$

Treating two different cases as before we finally find constants $C, R_0, \tilde{R}>0$, such that for all $\sigma \in B_{\tilde{R}}(s)$

$$\fint_{B_r(\sigma)} |\ddot{x}^2|^2\, dt \leq C\left[\left(\fint_{B_{4r}(\sigma)} |\ddot{x}^2|\, dt\right)^2 + \left(\fint_{B_{4r}(\sigma)} |\ddot{x}^3|\, dt\right)^2 + \fint_{B_{4r}(\sigma)} |\ddot{x}^3|^2\, dt + 1\right] \quad \text{for all } r \leq R_0$$

or in terms of maximal functions after zero–extension of all involved functions on $\mathbb{R}\backslash B_{\tilde{R}+4R_0}(s)$:

$$M_{R_0}[\,|\ddot{x}^2|^2\,](\sigma) \leq CM_\infty^2[\,|\ddot{x}^2| + |\ddot{x}^3| + 1\,](\sigma) + CM_\infty[\,|\ddot{x}^3|^2\,]$$

$$\text{for all } \sigma \in B_{\tilde{R}+4R_0}(s).$$

A more general version of the Gehring lemma (see e.g. [5, p. 122] gives us higher L^p–integrability of $\ddot{x}^2$ for a $p>2$ on $B_{r_1}(s)$, $r_1 := \tilde{R}/4 = r_0/2$:

$$\left[\fint_{B_{r_1}(s)} |\ddot{x}^2|^p\, dt\right]^{1/p} \leq K \cdot \left[\left(\frac{1}{|B_{r_1}(s)|}\int_{B_{2r_1}(s)} |\ddot{x}^2|^2\, dt\right)^{1/2} + \left(\frac{1}{|B_{r_1}(s)|}\int_{B_{2r_1}(s)} |\ddot{x}^3|^p\, dt\right)^{1/p} + \left(\frac{|B_{2r_1}(s)|}{|B_{r_1}(s)|}\right)^{1/p}\right].$$

As in step 5, Section 4.3 we get an analogous estimate for x^1, which completes the proof on account of (34). ☐

A. APPENDIX

LEMMA A.1. – *Let $I, \tilde{I}$ be open intervals in $\mathbb{R}$, $f \in H^{2,2}(I)$ and $\tau : \tilde{I} \longrightarrow I$ a C^1–diffeomorphism. Assume in addition that $\ddot{\tau}$ exists almost everywhere on $\tilde{I}$ and satisfies*

$$\int_{\tilde{I}} |\ddot{\tau}(s)|^2\, ds \leq C < \infty. \tag{36}$$

Then $f \circ \tau \in H^{2,2}(\tilde{I})$ *with*

$$(f \circ \tau)^{\cdot\cdot} = (\ddot{f} \circ \tau) \cdot \dot{\tau}^2 + (\dot{f} \circ \tau) \cdot \ddot{\tau}$$

in the sense of distributions.

Proof. – Taking an approximating sequence $f_k \in C^\infty(I) \cap H^{2,2}(I)$ with $\|f - f_k\|_{H^{2,2}} \longrightarrow 0$ for $k \nearrow \infty$ we obtain

$$(37) \qquad (f_k \circ \tau)^{\cdot}(s) = (\dot{f}_k \circ \tau) \cdot \dot{\tau}(s) \quad \text{for all } s \in \tilde{I},$$

$$(38) \qquad (f_k \circ \tau)^{\cdot\cdot}(s) = (\ddot{f}_k \circ \tau) \cdot \dot{\tau}^2(s) + (\dot{f}_k \circ \tau) \cdot \ddot{\tau}(s) \quad \text{for almost all } s \in \tilde{I}.$$

Since $|\dot{\tau}| \geq c$ for some constant $c>0$, the functions $\ddot{f}_k \circ \tau$ constitute a Cauchy sequence in $L^2(\tilde{I})$:

$$\begin{aligned}\int_{\tilde{I}} |\ddot{f}_k \circ \tau(s) - \ddot{f}_l \circ \tau(s)|^2 \, ds &\leq \frac{1}{c} \int_{\tilde{I}} |\ddot{f}_k \circ \tau(s) - \ddot{f}_l \circ \tau(s)|^2 \cdot |\dot{\tau}(s)| \, ds \\ &= \int_I |\ddot{f}_k(t) - \ddot{f}_l(t)|^2 \, dt \underset{k,l \nearrow \infty}{\longrightarrow} 0.\end{aligned}$$

For a subsequence we have $\ddot{f}_k \longrightarrow \ddot{f}$ almost everywhere. The inverse $\tau^{-1} \in C^1$ maps sets $N \subset I$ of measure zero into sets $\tilde{N} \subset \tilde{I}$ of measure zero. Hence, $\ddot{f}_k \circ \tau \longrightarrow \ddot{f} \circ \tau$ almost everywhere on $\tilde{I}$ and consequently $\ddot{f}_k \circ \tau \longrightarrow \ddot{f} \circ \tau$ in $L^2(\tilde{I})$. Similarly we get

$$(39) \qquad \dot{f}_k \circ \tau \longrightarrow \dot{f} \circ \tau \quad \text{and}$$

$$(40) \qquad f_k \circ \tau \longrightarrow f \circ \tau \quad \text{in } L^2(\tilde{I}).$$

Multiplying (38) by $\varphi \in C_0^\infty(\tilde{I})$ and integrating over $\tilde{I}$ one obtains after two integrations by parts

$$(41) \quad \int_{\tilde{I}} \varphi \cdot [\ddot{f}_k \circ \tau \cdot \dot{\tau}^2 + \dot{f}_k \circ \tau \cdot \ddot{\tau}] \, dt = \int_{\tilde{I}} \varphi \cdot (f_k \circ \tau)^{\cdot\cdot} \, dt = \int_{\tilde{I}} \ddot{\varphi} \cdot (f_k \circ \tau) \, dt.$$

It is easy to show that the left–hand side of (41) converges to $\int_{\tilde{I}} \varphi \cdot [\ddot{f} \circ \tau \cdot \dot{\tau}^2 + \dot{f} \circ \tau \cdot \ddot{\tau}] \, dt$, which together with (40) proves the lemma. □

LEMMA A.2. – *Let* $d>0$, $\theta \in (0,1)$ *and* $c>0$ *satisfy*

$$(42) \qquad d < \frac{\pi \theta c}{2}$$

and $\eta\in C^1(S^1,\mathbb{R}^3)$, *such that* $|\dot{\eta}(t)|\geq c>0$ *for all* $t\in S^1$ *and*

$$|\eta(s)-\eta(s')|\geq c(s,s')\cdot\min\{\,d,\ \theta L_{[s,s']}(\eta),\ \theta L_{[s',s]}(\eta)\,\} \tag{43}$$

for all $s,s'\in S^1$, *where* $c(s,s')$ *is a positive, uniformly bounded function:* $0<c(s,s')\leq C<\infty$.

CLAIM. – *For all* $\epsilon>0$ *there is an* $\bar{\epsilon}=\bar{\epsilon}(\epsilon)$, *such that for all* $\boldsymbol{\xi}\in C^1(S^1,\mathbb{R}^3)$ *with* $\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\leq\bar{\epsilon}$ *we have*

$$|\boldsymbol{\xi}(s)-\boldsymbol{\xi}(s')|\geq(c(s,s')-\epsilon)\cdot\min\{\,d,\ \theta L_{[s,s']}(\boldsymbol{\xi}),\ \theta L_{[s',s]}(\boldsymbol{\xi})\,\} \tag{44}$$

for all $s,s'\in S^1$.

Proof. – a) For $\boldsymbol{\xi}\in C^1(S^1,\mathbb{R}^3)$ with $\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\leq\epsilon_1:=c/2$ we have $|\dot{\boldsymbol{\xi}}(t)|\geq c/2$ for all $t\in S^1$. The technical condition (42) secures that

$$\min\{\,d,\ \theta L_{[s,s']}(\boldsymbol{\xi}),\ \theta L_{[s',s]}(\boldsymbol{\xi})\,\}=\min\{\,d,\ \theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\,\} \tag{45}$$

for all $\boldsymbol{\xi}$ with $\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\leq\epsilon_1$, i.e., the correspondence of the shorter arc between s and s' on S^1 with the shorter arc on the curve between the image points $\mathbf{x}(s)$ and $\mathbf{x}(s')$.

b) For $s,s'\in S^1$ one estimates using (43)

$$\begin{aligned}
|\boldsymbol{\xi}(s)-\boldsymbol{\xi}(s')| &= \Big|\int_{[s,s']_{S^1}}\dot{\boldsymbol{\xi}}(t)\,dt\Big|\\
&\geq|\eta(s)-\eta(s')|-\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\cdot|s-s'|_{S^1}\\
&\geq c(s,s')\cdot\min\{\,d,\theta L_{[s,s']_{S^1}}(\eta)\,\}-\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\cdot|s-s'|_{S^1}\\
&\geq c(s,s')\cdot\min\{\,d,\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\,\}\\
&\quad-[\,1+c(s,s')\theta\,]\cdot\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\cdot|s-s'|_{S^1}.
\end{aligned}$$

c) If $\min\{d,\ \theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\}=d$, then we have for $\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\leq\epsilon_2:=\epsilon d/((1+C)\pi)$ by the previous estimate

$$|\boldsymbol{\xi}(s)-\boldsymbol{\xi}(s')|\geq(c(s,s')-\epsilon)d=(c(s,s')-\epsilon)\min\{d,\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\}. \tag{46}$$

If $\min\{\,d,\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\,\}=\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})$, one obtains for $\|\dot{\boldsymbol{\xi}}-\dot{\eta}\|_{C^0}\leq\epsilon_3:=\epsilon\theta c/(2(1+C))$:

$$\begin{aligned}
|\boldsymbol{\xi}(s)-\boldsymbol{\xi}(s')| &\geq (c(s,s')-\epsilon)\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\\
&=(c(s,s')-\epsilon)\min\{\,d,\theta L_{[s,s']_{S^1}}(\boldsymbol{\xi})\}.
\end{aligned} \tag{47}$$

The inequalities (46), (47) together with (45) prove the claim, if one takes $\bar{\epsilon}:=\min\{\epsilon_1,\epsilon_2,\epsilon_3\}$. □

LEMMA A.3. – *Let μ be a Radon measure on the open interval $J \subset \mathbb{R}$. Then for all open subintervals $I \subset\subset J$ we find a nondecreasing bounded function $g=g_I : I \longrightarrow \mathbb{R}$, such that $\mu = g'$ on I in the sense of distributions.*

Proof. – Let $I := (a,b) \subset\subset J$; define $g = g_I : I \longrightarrow \mathbb{R}$ by $g(s) := \mu(\,(a,s]\,)$ for $a<s<b$. Since μ is a Radon measure on J, g is bounded on I : $g(s) \le \mu(\,[a,b]\,) < \infty$ for all $s \in I$. By construction g is nondecreasing, because

$$\begin{aligned} g(s) &= \mu((a,b]) = \mu((a,b] \cap (a,\sigma]) + \mu((a,b] \setminus (a,\sigma]) \\ &= g(\sigma) + \mu((\sigma,s]) \quad \text{for } \sigma<s. \end{aligned}$$

Furthermore $\mu([\sigma,c]) = \mu((\sigma,c]) = g(c) - g(\sigma)$ for almost all $a<\sigma<c<b$. For $\psi \in C_0^\infty(I)$ with $\operatorname{supp}\psi \subset\subset (\alpha,\beta) \subset (a,b)$ we write using Fubini's theorem

$$\begin{aligned} \int_I \psi(\sigma)d\mu(\sigma) &= \int_\alpha^\beta \left(\int_\alpha^\sigma \psi'(t)\,dt\right)d\mu(\sigma) = \int_\alpha^\beta \left(\int_t^\beta \psi'(t)\,d\mu(\sigma)\right)dt \\ &= \int_\alpha^\beta \mu((t,\beta]) \cdot \psi'(t)dt = \int_\alpha^\beta (g(\beta) - g(t)) \cdot \psi'(t)dt \\ &= -\int_I g(t)\psi'(t)\,dt. \end{aligned}$$

□

LEMMA A.4. – *Suppose, there are functions ϕ_1, $v_1 \in C^1([0,\epsilon_0))$; ϕ_2, $v_2 \in C^1((-t_0,t_0))$ and $\phi_3, v_3 \in C^1(\,[0,s_0))$ with $\epsilon_0, t_0, s_0 > 0$ and constants $\phi_0, c \in \mathbb{R}$ with the properties*

(i) $0 = \phi_i(0) = v_i(0)$ *for* $i=1,2,3$,

(ii) $v_2'(0) = 1$,

(iii) $\phi(\epsilon,t) := \phi_0 + \phi_1(\epsilon) + \phi_2(t) + \phi_3(s(\epsilon))$ *satisfies*

$$\phi(\epsilon,t) \ge \phi(0,0) \quad \textit{for all } (\epsilon,t) \in [0,\epsilon_0)\times(-t_0,t_0) \quad \textit{with } v(\epsilon,t) = c,$$

where $v(\epsilon,t) := c + v_1(\epsilon) + v_2(t) + v_3(s(\epsilon))$ *and* $s : [0,\epsilon_0) \longrightarrow \mathbb{R} \in C^1(\,[0,\epsilon_0),[0,s_0)\,), s(0) = 0$.

CLAIM. – *Then we have the inequality*

$$\phi_1'(0) + \lambda_0 v_1'(0) + \lambda_1 s'(0) \ge 0, \tag{48}$$

where $\lambda_0 := -\phi_2'(0)$, $\lambda_1 := \phi_3'(0) - \phi_2'(0) \cdot v_3'(0)$.

Remark. – Lemma 3.5 in Section 4.3 follows if one sets $\phi_3 \equiv v_3 \equiv 0$.

Proof. – Applying the mean value theorem to v_2 one obtains $v_2(t) = [1 + \eta(t)] \cdot t$ with a function η satisfying $|\eta(t)| \to 0$ as $|t| \searrow 0$. Then by the intermediate-value theorem we find a constant $d_0 \in (0, t_0/2)$, such that $[-d, d] \subset v_2([-2d, 2d])$ for all $0 \le d \le d_0$.

On the other hand, there is an $\epsilon_1 \in (0, \epsilon_0]$, such that $|v_1(\epsilon) + v_3(s(\epsilon))| < \delta_0 (< t_0/2)$ for all $\epsilon \in [0, \epsilon_1)$, since v_1, v_2 and s are continuous and $v_1(0) = v_3(0) = s(0) = 0$. Consequently, for every $\epsilon \in [0, \epsilon_1)$ there exists $t = t(\epsilon)$ with

$$|t(\epsilon)| \le 2|v_1(\epsilon) + v_3(s(\epsilon))| < 2\delta_0 \quad (< t_0), \text{ such that}$$

$$[1 + \eta(t(\epsilon))] \cdot t(\epsilon) = v_2(t(\epsilon)) = -v_1(\epsilon) - v_3(s(\epsilon)) \tag{49}$$

$$\Rightarrow \quad \lim_{\epsilon \searrow 0} t(\epsilon) = 0 \quad \text{and} \quad v(\epsilon, t(\epsilon)) = c. \tag{50}$$

The identities (49) and (50) imply

$$\lim_{\epsilon \searrow 0} \left[\frac{t(\epsilon) - t(0)}{\epsilon} \right] = \lim_{\epsilon \searrow 0} \left[\frac{1}{1 + \eta(t(\epsilon))} \cdot \frac{-v_1(\epsilon) - v_3(s(\epsilon))}{\epsilon} \right] = -v_1'(0) - v_3'(0) \cdot s'(0). \tag{51}$$

Using (iii), (50) and (51) we arrive at

$$\begin{aligned} 0 &\le \frac{\phi(\epsilon, t(\epsilon)) - \phi(0,0)}{\epsilon} \\ &\xrightarrow[\epsilon \searrow 0]{} \phi_1'(0) + \phi_2'(0) \cdot t'(0) + \phi_3'(0) \cdot s'(0) \\ &= \phi_1'(0) \underbrace{-\phi_2'(0)}_{=\lambda_0} \cdot v_1'(0) + [\underbrace{\phi_3'(0) - \phi_2'(0) \cdot v_3'(0)}_{=\lambda_1}] \cdot s'(0). \end{aligned}$$

□

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