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Existence and uniqueness results on Single-Peaked solutions of a semilinear problem

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ABSTRACT. – A one to one correspondence is established between the nondegenerate critical points of $Q(x)$ in Ω and single peaked solutions of the problem

$$-\epsilon^2\Delta u + u = Q(x)u^{p-1} \quad \text{in } \Omega$$
$$u > 0 \text{ in } \Omega \qquad \text{and } u = 0 \quad \text{on } \partial\Omega$$

where Ω is a bounded domain, $2 < p < (N+2)/(N-2)$, $\epsilon > 0$, and $Q(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$.

In particular, we establish the uniqueness of the least energy solution when $Q(x)$ attains its maximum in $\overline{\Omega}$ at only one nondegenerate critical point in Ω. © Elsevier, Paris

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RÉSUMÉ. – On établit une correspondance biunivoque entre les points critiques non-dégénérés de $Q(x)$ en Ω, et les solutions à un seul pic du problème

$$\begin{aligned} &-\epsilon^2 \Delta u + u = Q(x) u^{p-1} \quad \text{dans } \Omega \\ &u > 0 \text{ dans } \Omega \qquad \text{et } u = 0 \quad \text{sur } \partial\Omega \end{aligned}$$

où Ω est un domaine borné, $2 < p < (N+2)/(N-2)$, $\epsilon > 0$, et $Q(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$.

En particulier, nous démontrons l'unicité de la solution de moindre énergie lorsque $Q(x)$ achève son maximum dans $\overline{\Omega}$ en un seul point critique non-dégénéré.

1. INTRODUCTION

In this paper we consider the problem

$$\begin{aligned} &-\epsilon^2 \Delta u + u = Q(x) u^{p-1} \quad \text{in } \Omega \\ &u > 0 \text{ in } \Omega \qquad \text{and } u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$, with a smooth boundary $\partial\Omega, \epsilon > 0$ is a parameter, $2 < p < (N+2)/(N-2)$, and $Q(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$ has nondegenerate critical points at $a^1, \ldots, a^\ell \in \Omega$, *i.e.*, $D_j Q(a^i) = 0$ and $\det D^2 Q(a^i) \neq 0$, where $D_j = \dfrac{\partial}{\partial x_j}$ and $D^2 Q(x) = \left(\dfrac{\partial^2 Q(x)}{\partial x_k \partial x_j} \right)_{N \times N}$, $i = 1, \ldots, \ell, k = 1, \ldots, N;\ j = 1, \ldots, N$. The case of degenerate critical points is also considered.

Problem (1.1) arises in various applications, such as chemo taxis, population genetics, chemical reactor theory, etc. In applications, it is important to locate the maximum points of solutions in Ω, since these may correspond to locations of higher chemical concentrations, certain population, etc.

When $Q(x)$ is a positive constant, problem (1.1) has been considered by several authors. In these studies both the topology of Ω (see Benci and Cerami [3]), and the geometry of Ω, see [5], [6] play an important role in the existence and multiplicity of solutions of (1.1). Recently, Ni and Wei

[9] and Wei [12], constructed solutions with "single-peak", and the shape and peak location of "least energy" solutions were studied. Specifically, let

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega (\epsilon^2|\Delta u|^2 + u^2) - \frac{1}{p}\int_\Omega Qu_+^p$$

where $u_+ = \max\{u, 0\}$, for $u \in H_0^1(\Omega)$. The well known Mountain-Pass Lemma implies that

$$c_\epsilon = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} I_\epsilon(h(t))$$

is a positive critical value of I_ϵ, *i.e.*, $c_\epsilon = I_\epsilon(u_\epsilon)$ and u_ϵ is a solution of (1.1), where Γ is the set of all continuous paths joining the origin and a fixed $e \in H_0^1(\Omega)$ with $e \geq 0$ and $I_\epsilon(e) = 0$. It can be shown, see [9], that c_ϵ is independent of the choice of e. A critical point u_ϵ corresponding to c_ϵ is called a least energy solution (or a Mountain pass solution).

For Q a positive constant Ni and Wei [9] proved that u_ϵ has at most one local maximum and it is achieved at exactly one point $p_\epsilon \in \Omega, u_\epsilon(.+p_\epsilon) \to 0$ in $C^1_{\text{loc}}(\Omega - p_\epsilon\backslash\{0\})$, and $d(p_\epsilon, \partial\Omega) \longrightarrow \max_{p\in\Omega} d(p, \partial\Omega)$ as $\epsilon \longrightarrow 0$.

DEFINITION. – *We say that a function u defined on $\overline{\Omega}$ is single-peaked, if u has only one local maximum point in $\overline{\Omega}$.*

The aim of this paper is to show how the nondegenerate critical points of $Q(x)$ play a dominant role (compared to the geometry and topology of Ω) in the existence and the multiplicity of single peaked solutions. In particular, we establish a one-to-one correspondence between the nondegenerate critical points a^i of $Q(x)$ in Ω and single peaked solutions.

It will then follow that if $\max_{\overline{\Omega}} Q(x)$ is attained at only one nondegenerate critical point in Ω, then problem (1.1) has, for sufficiently small ϵ, a unique least energy solution, regardless of the shape or the topology of Ω.

The case of degenerate critical points is more delicate. We establish the existence of a single-peaked solution for each strict local maximum point a of $Q(x)$, and if $a \in \Omega$, we show that the peak point p_ϵ of such a solution converges, as $\epsilon \longrightarrow 0$, to a. However the question of uniqueness of such solutions is still open. That is, it is not known if there is one or more single-peaked solutions whose peak points converges, as $\epsilon \longrightarrow 0$, to a.

Our procedure is based on arguments similar to that used by Rey [11], by A. Bahri, Y. Li and O. Rey [2], and a degree argument similar to that used by L. Glangetas for a nonlinear elliptic problem involving the critical exponent [8].

In Section 2 we introduce our notations and establish a result on the profile of single-peaked solutions and the locations of their peaks. In

Section 3 we establish the existence and uniqueness of single-peaked solutions concentrating at any given nondegenerate critical point of Q.

In Section 4 we consider the case when Q has local maximum points in $\overline{\Omega}$. We are only able to establish the existence of single-peaked solutions and study their profile.

2. NOTATIONS AND PRELIMINARY RESULTS

Let V be the unique positive solution of

$$-\Delta V + V = V^{p-1} \quad \text{in } \mathbb{R}^N$$
$$V \in H^1(\mathbb{R}^N)$$

It is well known that V is radially symmetric about the origin, decreasing and

$$\lim_{|x|\to\infty} V(x)e^{|x|}\,|x|^{\frac{N-1}{2}} = \overline{c} > 0.$$

For a smooth bounded domain $D \subseteq \mathbb{R}^N, P_D V$ is the unique solution of

$$\begin{aligned} -\Delta u + u &= V^{p-1} \quad \text{in} D \\ u &= 0 \quad \text{on } \partial D \end{aligned} \tag{2.1}$$

It follows from the maximum principle that $P_D V(y) < V(y)$ for all $y \in D$. For $v \in H^1(\mathbb{R}^N), y \in \mathbb{R}^N$, and $\epsilon > 0$, let

$$v_{\epsilon,y}(.) = v((. - y)/\epsilon) \tag{2.2}$$

Let $P_{\Omega,\epsilon} v$ denote the unique solution of

$$\begin{aligned} &-\epsilon^2 \Delta u + u = |v|^{p-2} \quad \text{in } \Omega \\ &v \in H_0^1(\Omega) \end{aligned} \tag{2.3}$$

Notice that, in our notation, $P_{\Omega,1} \equiv P_\Omega$.

Let

$$\langle u, v\rangle_\epsilon = \epsilon^2 \int \nabla u . \nabla v + \int uv,$$
$$||u||_\epsilon^2 = \langle u, u\rangle_\epsilon,$$

for $u, v \in H_0^1(\Omega)$. All integrals are Lebesgue integrals over Ω unless otherwise stated.

For $y, z \in \mathbb{R}^N$ define

$$E_{\epsilon,y}(\Omega) = \{v \in H_0^1(\Omega) : \langle P_{\Omega,\epsilon}V_{\epsilon,y}, v\rangle_\epsilon = \langle \frac{\partial P_{\Omega,\epsilon}V_{\epsilon,y}}{\partial y_j}, v\rangle_\epsilon = 0, j = 1, \dots, N\}$$

$$\Omega_{\epsilon,y} = \{x \in \mathbb{R}^N : \epsilon x + y \in \Omega\}$$
$$B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$$

C will denote a positive constant.

PROPOSITION 2.1. – *u_ϵ is a single peaked solution of (1.1) which satisfies*

$$\|u_\epsilon\|_\epsilon = O(\epsilon^{N/2}) \tag{2.4}$$

if and only if

$$u_\epsilon = \alpha_\epsilon P_{\Omega,\epsilon}V_{\epsilon,x_\epsilon} + w_\epsilon \tag{2.5}$$

for some $\alpha_\epsilon \in \mathbb{R}, x_\epsilon \in \Omega$, and $w_\epsilon \in E_{\epsilon,x_\epsilon}$, satisfying

$$\epsilon^{-1}d(x_\epsilon, \partial\Omega) \longrightarrow \infty \tag{2.6}$$

$$\|w_\epsilon\|_\epsilon = o(\epsilon^{N/2}), \tag{2.7}$$

$$\alpha_\epsilon \longrightarrow (Q(x_0))^{-1/(p-2)}, \tag{2.8}$$

as $\epsilon \longrightarrow 0$, where $x_0 = \lim_{\epsilon\to 0} x_\epsilon = \lim_{\epsilon\to 0} p_\epsilon$, where p_ϵ is the peak of u_ϵ.

Proof. – Let u_ϵ be a single-peaked solution satisfying (2.4). Let p_ϵ be the point in Ω where u_ϵ achieves its maximum value on $\overline{\Omega}$. Following the same argument as in Ni and Wei [9], we have

$$\epsilon^{-1}d(p_\epsilon, \partial\Omega) \longrightarrow \infty, \text{ as } \epsilon \longrightarrow 0 \tag{2.9}$$

Suppose $p_0 = \lim_{\epsilon\to 0} p_\epsilon \in \overline{\Omega}$. Let

$$v_\epsilon(y) = u_\epsilon(\epsilon y + p_\epsilon), \quad y \in \Omega_{\epsilon,p_\epsilon}$$

Then v_ϵ satisfies

$$\begin{aligned} -\Delta v_\epsilon + v_\epsilon &= Q(\epsilon y + p_\epsilon)v_\epsilon^{p-1} \quad \text{in } \Omega_{\epsilon,p_\epsilon} \\ v_\epsilon &= 0 \text{ on } \partial\Omega_{\epsilon,p_\epsilon}, \\ \|v\|^2 &\le C \end{aligned} \tag{2.10}$$

for some positive constant C, where the last inequality follows from (2.4). Therefore

$$\begin{aligned} v_\epsilon &\rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^N), \\ v_\epsilon &\longrightarrow v \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N), \end{aligned} \tag{2.11}$$

from standard regularity results for solutions of (2.10). In the above we used v_ϵ to denote the extention of v_ϵ to $\mathbb{R}^N$ which is identically zero outside $\Omega_{\epsilon,p_\epsilon}$.

From (2.10) and (2.11) we have

$$\begin{aligned} -\Delta v + v &= Q(p_0)v^{p-1} \quad \text{in } \mathbb{R}^N \\ v &> 0 \\ v &\in H^1(\mathbb{R}^N) \end{aligned} \tag{2.12}$$

Since v_ϵ is single-peaked, the set $\{x \in \Omega_{\epsilon,p_\epsilon} : v_\epsilon(x) > \delta\}$ has only one connected component for any $\delta > 0$, the argument of Proposition 3.4 in [9] may be employed to show that (for any $\delta > 0$)

$$v_\epsilon(y) \le Ce^{-(1-\delta)|y|}, \ y \in \Omega_{\epsilon,p_\epsilon}$$

Hence

$$\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 + v_\epsilon^2 = \int_{\mathbb{R}^N} Q(\epsilon y + p_\epsilon)v_\epsilon^p,$$

and by taking the limit as $\epsilon \longrightarrow 0$, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 + v^2 = \int_{\mathbb{R}^N} Q(p_0)v^p,$$

which together with (2.11) yield

$$v_\epsilon \longrightarrow v \quad \text{strongly in } H^1(\mathbb{R}^N).$$

Since v satisfies (2.12), the uniqueness of solution of (2.12) and the definition of V implies that

$$v = (Q(p_0))^{-1/(p-2)}V$$

But it is easy to see from the definition of V that

$$||V - P_{\Omega_{\epsilon,p_\epsilon}}V|| \longrightarrow 0, \text{ as } \epsilon \longrightarrow 0.$$

Hence $v_\epsilon - (Q(p_0))^{-1/(p-2)} P_{\Omega,\epsilon} V \to 0$ strongly in $H_0^1(\mathbb{R}^N)$, and therefore

$$\epsilon^{-N} \| u_\epsilon - (Q(p_0))^{-1/(p-2)} P_{\Omega,\epsilon} V \|_\epsilon^2 \longrightarrow 0, \tag{2.13}$$

as $\epsilon \longrightarrow 0$. Using an argument similar to that used by A. Bahri and J.M. Coron [1], we then have that u_ϵ can be uniquely written in the form

$$u_\epsilon = \alpha_\epsilon P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon} + w_\epsilon$$

for some $\alpha_\epsilon \in R, x_\epsilon \in \Omega$, and $w_\epsilon \in E_{\epsilon,x_\epsilon}$, satisfying (2.7) and (2.8). It remains to show that (2.6) holds. This can be shown by the same argument as in Ni and Wei [9].

Now suppose that

$$u_\epsilon = \alpha_\epsilon P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon} + w_\epsilon$$

is a positive solution of (1.1), where $\alpha_\epsilon, x_\epsilon, w_\epsilon$ satisfy (2.6), (2.7) and (2.8). We show next that u_ϵ is a single-peaked solution of (1.1).

We proceed by contradiction. Suppose u_ϵ has two local maximum points $p_\epsilon^1, p_\epsilon^2$ in Ω. We notice first that if $x_0 = \lim_{\epsilon \to 0} x_\epsilon$, then for any fixed $\delta > 0$,

$$\begin{aligned} \int_{\Omega \setminus B_\delta(x_0)} \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 &\le \int_{\Omega \setminus B_{\frac{\delta}{2}}(x_\epsilon)} \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 \\ &= \epsilon^N \int_{\Omega_{\epsilon,x_\epsilon} \setminus B_{\frac{\delta}{2\epsilon}}(0)} |\nabla v_\epsilon|^2 + v_\epsilon^2 \\ &= \epsilon^N \int_{\mathbb{R}^N \setminus B_{\frac{\delta}{2\epsilon}}(0)} |\nabla v_\epsilon|^2 + v_\epsilon^2 \\ &= o(1)\epsilon^N, \quad \text{as} \quad \epsilon \longrightarrow 0. \end{aligned} \tag{2.14}$$

We consider now the following two cases:

$$\text{Case 1}: \qquad \epsilon^{-1} |p_\epsilon^1 - p_\epsilon^2| \longrightarrow \infty \quad \text{as} \quad \epsilon \longrightarrow 0.$$

In this case, we have

$$\int_{B_{R\epsilon}(p_\epsilon^1)} \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 = \epsilon^N \int_{B_R(0)} |\nabla v_\epsilon^1|^2 + |v_\epsilon^1|^2, \tag{2.15}$$

where

$$\begin{aligned} v_\epsilon^1(y) &= u_\epsilon(\epsilon y + p_\epsilon^1) \\ v_\epsilon^1 &\longrightarrow v \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^N), \end{aligned}$$

as in the first part of the proof, with v satisfying

$$\begin{aligned} -\Delta v + v &= Q(p^{(1)})v^{p-1} \quad \text{in } \mathbb{R}^N \\ v &> 0 \\ v &\in H^1(\mathbb{R}^N), \end{aligned}$$

where $p^1 = \lim_{\epsilon \longrightarrow 0} p^1_\epsilon$. Thus

$$\int_{B_{R\epsilon}(p^1_\epsilon)} \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 \geq C\epsilon^N \tag{2.16}$$

for some positive constant $C > 0$, and, similarly,

$$\int_{B_{R\epsilon}(p^2_\epsilon)} \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 \geq C\epsilon^N \tag{2.17}$$

From (2.14), (2.15) (2.16) and (2.17), we see that

$$\lim_{\epsilon \longrightarrow 0} p^1_\epsilon = \lim_{\epsilon \longrightarrow 0} p^2_\epsilon = x_0$$

But

$$\begin{aligned} I(u_\epsilon) &\equiv \frac{1}{2}\int_\Omega \epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 - \frac{1}{p}\int_\Omega Q(x) u_\epsilon^p \\ &= \epsilon^N \left\{ \int_{\mathbb{R}^N} [\frac{1}{2}(|\nabla v|^2 + v^2) - \frac{1}{p} Q(x_0) v^p] + o(1) \right\} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\epsilon^N \left(Q(x_0) \int_{\mathbb{R}^N} v^p + 0(1) \right), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} I(u_\epsilon) &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left\{ \int_{B_{R\epsilon}(p^1_\epsilon)} Q(x) u_\epsilon^p + \int_{B_{R\epsilon}(p^2_\epsilon)} Q(x) u_\epsilon^p \right\} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\left\{ \int_{B_R(0)} Q(\epsilon y + p^1_\epsilon) v_\epsilon^p(\epsilon y + p^1_\epsilon) \right. \\ &\quad \left. + \int_{B_R(0)} Q(\epsilon y + p^2_\epsilon) v_\epsilon^p(\epsilon y + p^2_\epsilon) \right\} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\epsilon^N \left\{ 2Q(x_0) \int_{B_R(0)} v^p + 0(1) \right\}, \end{aligned}$$

which contradicts (2.18), and hence case (1) is impossible

$$\text{Case 2}: \qquad \epsilon^{-1}|p^1_\epsilon - p^2_\epsilon| \longrightarrow \ell < \infty, \text{ as } \epsilon \longrightarrow 0$$

In this case we may establish a contradiction using a similar argument to that in Ni and Takagi [10].

The fact that $\lim_{\epsilon\to 0} p_\epsilon = \lim_{\epsilon\to 0} x_\epsilon$ follows by similar argument as in case (1).

PROPOSITION 2.2. – *Let u_ϵ be a single peaked solution of (1.1) of the form $u_\epsilon = \alpha_\epsilon P_{\epsilon,\Omega_\epsilon} V_{\epsilon,x_\epsilon} + w_\epsilon$, where $x_\epsilon, w_\epsilon, \alpha_\epsilon$ satisfy (2.6), (2.7), (2.8), and $x_0 = \lim_{\epsilon\to 0} x_\epsilon \in \Omega$. Then $\nabla Q(x_0) = 0$.*

Proof. – Since u_ϵ satisfies (1.1), multiplication of the equation (1.1) by $\dfrac{\partial u_\epsilon}{\partial y_j}$ and integration by parts yield

$$\frac{\epsilon^2}{2}\int_{\partial\Omega}\left(\frac{\partial u_\epsilon}{\partial n}\right)^2 n d\sigma = \frac{1}{p}\int_\Omega u_\epsilon^p \nabla Q(y)dy. \tag{2.19}$$

since on $\partial\Omega, \nabla u = (\nabla u.n)n = (\dfrac{\partial u}{\partial n})n$. Here n denotes the exterior unit normal to $\partial\Omega$. We estimate next the right hand side of (2.19):

$$\begin{aligned}&\int u_\epsilon^p(y)\frac{\partial Q}{\partial y_i}dy\\ &= \int_\Omega [\alpha_\epsilon P_{\epsilon,\Omega} V_{\epsilon,x_\epsilon}(y) + w_\epsilon(y)]^p \frac{\partial Q}{\partial y_i}dy\\ &= \epsilon^N \int_{\Omega_{\epsilon,x_\epsilon}} [\alpha_\epsilon P_{\Omega_{\epsilon,x_\epsilon}} V(x) + w_\epsilon(\epsilon x + x_\epsilon)]^p \frac{\partial}{\partial x_i} Q(\epsilon x + x_\epsilon)dx \end{aligned}\tag{2.20}$$

Since $w_\epsilon(\epsilon x + x_\epsilon) \longrightarrow 0$ strongly in $H_0^1(\mathbb{R}^N)$, $\dfrac{\partial Q}{\partial x_i}$ is bounded, and $\alpha_\epsilon \longrightarrow (Q(x_0))^{-1/(p-2)}$, we deduce form (2.20) that

$$\int u_\epsilon^p(y)\frac{\partial Q(y)}{\partial y_i}dy = \epsilon^N \frac{\partial Q}{\partial y_i}(x_0)\alpha_0^p + o(1), \tag{2.21}$$

where $\alpha_0 = \lim\limits_{\epsilon\longrightarrow 0}\alpha_\epsilon = (Q(x_0))^{-1/(p-2)}$. Now let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that

$$\varphi \equiv 1 \text{ on } \partial\Omega; \varphi \equiv 0 \text{ for } x \in \{y\in\Omega : \mathrm{d}\ (y,\partial\Omega) \geq \delta\}$$

Then φu_ϵ satisfies the equation

$$-\epsilon^2\Delta(\varphi u_\epsilon) = \varphi(Q(x)u_\epsilon^{p-1} - u_\epsilon) - \epsilon^2(2\nabla\varphi.\nabla u_\epsilon + u_\epsilon\Delta\varphi) \equiv f_\epsilon \tag{2.22}$$

Since the embedding $W^{1,2}(\Omega) \longrightarrow L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ is continuous, we deduce from (2.19) and (2.22) that

$$\int_{\partial\Omega} \left(\frac{\partial u_\epsilon}{\partial n}\right)^2 d\sigma = \int_{\partial\Omega} \left(\frac{\partial(\varphi u_\epsilon)}{\partial n}\right)^2 \le C\|\varphi u_\epsilon\|^2_{W^{2,2}(\Omega)} \le \frac{C}{\epsilon^2}\|f_\epsilon\|^2_{L^2(\Omega)} \tag{2.23}$$

The last inequality follows from the Schauder's inequality.

Since u_ϵ is single peaked, we may use the argument of Ni and Wei [9] to show that

$$u_\epsilon(x) \le Ce^{-(1-\alpha)|x-x_\epsilon|/\epsilon}$$

for any $\alpha > 0$, where $C = C(\alpha)$ is a positive constant.

But $x_\epsilon \longrightarrow x_0 \in \Omega, \epsilon \longrightarrow 0$, and therefore for sufficiently small $\delta > 0$ we have

$$u_\epsilon(x) \le Ce^{-\tau/\epsilon} \tag{2.24}$$

for some positive constants C, τ, and for all $x \in \{y \in \Omega : \mathrm{d}(y, \partial\Omega) \le \delta\}$.

Since $\varphi(x) = 0$ for $\mathrm{d}(x, \partial\Omega) > \delta$, we have

$$\frac{C}{\epsilon^A}\|\varphi(Qu_\epsilon^{p-1} - u_\epsilon) - \epsilon^2 u_\epsilon \Delta\varphi\|^2_{L^2(\Omega)} \longrightarrow 0 \tag{2.25}$$

for any $A > 0$, as $\epsilon \longrightarrow 0$ We estimate next the term $\int \nabla\varphi . \nabla u_\epsilon$. Multiply (1.1) by $\varphi^2 u_\epsilon$ and integrate by parts to obtain

$$\epsilon^2 \int \varphi^2 |\nabla u_\epsilon|^2 + 2\int \varphi u_\epsilon \nabla \varphi . \nabla u_\epsilon + \int \varphi^2 u_\epsilon^2 = \int Q(x)\varphi^2 u_\epsilon^p, \tag{2.26}$$

for any $\varphi \in C_0^\infty(R^N)$. Set $\varphi \equiv 0$ in $B_{\delta/2}(x_0); \varphi \equiv 1$ in $\Omega \backslash B_\delta(x_0)$. Then we have from (2.24) and (2.26) that

$$\epsilon^2 \int \varphi^2 |\nabla u_\epsilon|^2 \le Ce^{-\tau/\epsilon}$$

for some positive constants C, τ. Thus,

$$\epsilon^{-A} \int_{\Omega\backslash B_\delta(x_0)} |\nabla u_\epsilon|^2 \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$, for all $A > 0$. Hence

$$\epsilon^{-A}\|\nabla \varphi . \nabla u_\epsilon\|_{L^2(\Omega)} \longrightarrow 0, \tag{2.27}$$

as $\epsilon \longrightarrow 0$, for all $A > 0$. From (2.22), (2.23), (2.25) and (2.27) we obtain

$$\epsilon^{-A} \int_{\partial\Omega} \left| \frac{\partial u_\epsilon}{\partial n} \right|^2 \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$, for all $A > 0$, and hence by (2.19), (2.21), and the hypotheses on u_ϵ, we have

$$\alpha_0 \nabla Q(x_0) \int_{\mathbb{R}^N} V^p + o(1) = \epsilon^{-(N-2)} \int (\frac{\partial u_\epsilon}{\partial n})^2 n d\sigma \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. We conclude that

$$\nabla Q(x_0) = 0$$

This completes the proof of Proposition 2.2.

PROPOSITION 2.3. – *If u_ϵ is the least energy solution of (1.1) then*

(i) $\quad ||u_\epsilon||_\epsilon^2 = \epsilon^N \{(A/Q_M^{2/(p-1)}) + o(1)\}$

where $Q_M = \max_{x \in \overline{\Omega}} Q(x)$, *and*

$$A = \int_{\mathbb{R}^N} V^p$$

(ii) $\quad$ *u_ϵ is single peaked and the peak $p_\epsilon \longrightarrow x_0$, as $\epsilon \longrightarrow 0$, where,* $Q(x_0) = Q_M$

Proof. – Let $\tilde{x} \in \overline{\Omega}$ be a global maximum of $Q(x)$ on $\overline{\Omega}$. Choose $x_\epsilon \to \tilde{x}$ and $\epsilon^{-1} \mathrm{d}\,(x_\epsilon, \partial\Omega) \to \infty$ (if $\tilde{x} \in \Omega$, we may choose $x_\epsilon = \tilde{x}$). Then,

$$\begin{aligned} ||u_\epsilon||_\epsilon^{2-2/p} &= ||u_\epsilon||_\epsilon^2 / (\int Q(y) u_\epsilon^p)^{2/p} \\ &\leq \frac{||P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon}||_\epsilon^2}{(\int Q(y) |P_\epsilon V_{\epsilon,x_\epsilon}|^p)^{2/p}} \\ &= \epsilon^{N(1-2/p)} ||P_{\Omega_\epsilon,y} V||^2 / \left\{ \int_{\Omega_\epsilon,y} Q(\epsilon y + x_\epsilon) |P_{\Omega_\epsilon,y} V|^p \right\}^{2/p} \\ &= \epsilon^{N(1-2/p)} \frac{A + o(1)}{[Q(x_0) A + o(1)]^{2/p}}, \end{aligned}$$

as $\epsilon \longrightarrow 0$, and (i) follows. To show (ii) we proceed by contradiction. Assume u_ϵ has two local maximum points $p_\epsilon^1, p_\epsilon^2$. Then as in Proposition 2.1, we have two cases to consider:

Case 1.

$$\epsilon^{-1}|p_\epsilon^1 - p_\epsilon^2| \longrightarrow \infty, \text{ as } \epsilon \longrightarrow 0.$$

In this case, we have

$$\begin{aligned}\|u_\epsilon\|_\epsilon^2 &\geq \int_{B_{R\epsilon}(p_\epsilon^1)} (\epsilon^2|\nabla u_\epsilon|^2 + u_\epsilon) \\ &+ \int_{B_{R\epsilon}(p_\epsilon^2)} (\epsilon^2|\nabla u_\epsilon|^2 + u_\epsilon^2) \\ &\geq \epsilon^N \sum_{i=1}^{2} \int_{B_R(0)} |\nabla v_\epsilon^i|^2 + (v_\epsilon^i)^2,\end{aligned}$$

where $v_\epsilon^i(y) = u_\epsilon(\epsilon y + p_\epsilon^i)$, $i = 1, 2$, and $v_\epsilon^i \longrightarrow v^i$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where v^i solves the problem

$$\begin{aligned}-\Delta v + v &= Q(p^i)v^p \\ v > 0, v &\in H^1(\mathbb{R}^N),\end{aligned}$$

where $p^i = \lim_{\epsilon \longrightarrow 0} p_\epsilon^i$. Therefore, we have

$$\begin{aligned}\|u_\epsilon\|_\epsilon^2 &\geq \epsilon^N \left(\frac{A}{Q(p^1)^{\frac{2}{(p-2)}}} + \frac{A}{(Q(p^2))^{2/(p-2)}} + o(1) \right) \\ &\geq \epsilon^N \left(\frac{2A}{Q_M^{2/(p-2)}} + o(1) \right)\end{aligned}$$

This contradicts (i).

Case 2. $\epsilon^{-1}|p_\epsilon - p_\epsilon^2| \leq \ell$. We may argue as in Proposition 2.1 to show that this is impossible. Hence u_ϵ is single peaked. To show that the peak $p_\epsilon \longrightarrow x_0$, as $\epsilon \to 0$, we first notice that if $p_\epsilon \to \tilde{x} \neq x_0$, then

$$\|u_\epsilon\|_\epsilon^2 \geq \int_{B_{R\epsilon}(p_\epsilon)} (\epsilon^2|\nabla u_\epsilon|^2 + u_\epsilon^2) = \epsilon^N \int_{B_R(0)} |\nabla v_\epsilon|^2 + v_\epsilon^2,$$

where $v_\epsilon(y) = u_\epsilon(\epsilon y + p_\epsilon)$, and $v_\epsilon \longrightarrow v$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$. Arguing as above, we may choose R large enough to obtain

$$\|u_\epsilon\|_\epsilon^2 \geq \epsilon^N \left(\frac{A}{(Q(\tilde{x}))^{2/(p-2)}} + o(1) \right) > \epsilon^N \left(\frac{A}{Q_M^{2/(p-2)}} + o(1) \right)$$

Contradicting (i).

Remark 2.4. – The hypotesis that $x_0 = \lim_{\epsilon \longrightarrow 0} x_\epsilon \in \Omega$, in Proposition 2.2, is satisfied if we assume, for example, that Ω is convex and $\left.\frac{\partial Q}{\partial \nu}\right|_{\partial\Omega} < 0$. This can be shown by an argument similar to that used in Gidas, Ni, Nirenberg [7], using the moving planes method.

3. EXISTENCE AND UNIQUENESS IN THE NON-DEGENERATE CASE

In this section we assume $x_0 \in \Omega$ is a nondegenerate critical point of $Q(x)$. The main results of this section are:

THEOREM 3.1. – *If Q has k-nondegenerate critical points $a^1, \ldots, a^k$ in Ω, then the problem (1.1) has exactly k single peaked solutions of the form*

$$u_\epsilon^i = \alpha_\epsilon^i P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon^i} + w_\epsilon^i,$$

$i = 1, \ldots, k$, where $\alpha_\epsilon^i \in \mathbb{R}^+, x_\epsilon^i \in \Omega$ and $w_\epsilon^i \in E_{\epsilon,x_\epsilon^i}$, satisfy

$$\begin{aligned} \alpha_\epsilon^i &\longrightarrow (Q(a^i))^{-1/(p-2)} \\ x_\epsilon^i &\longrightarrow a^i \\ \|w_\epsilon^i\|_\epsilon &= o(\epsilon^{N/2}), \end{aligned}$$

as $\epsilon \longrightarrow 0$.

THEOREM 3.2. – *The problem (1.1) has, for small ϵ, a unique least energy solution of the form*

$$u_\epsilon = \alpha_\epsilon P_{\epsilon,\Omega} V_{\epsilon,x_\epsilon} + w_\epsilon,$$

$\alpha_\epsilon \in \mathbb{R}, x_\epsilon \in \Omega$ and $w_\epsilon \in E_{\epsilon,x_\epsilon}$, provided that $\mathrm{Max}_{\overline{\Omega}} Q(x)$ is uniquely attained at $x_0 \in \Omega$, and x_0 is a nondegenerate critical point of Q. Furthermore, $x_\epsilon \longrightarrow x_0$.

Let u_ϵ be a single-peaked solution of (1.1) of the form

$$\begin{aligned} u_\epsilon &= \alpha_\epsilon P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon} + w_\epsilon, \qquad (3.1) \\ x_\epsilon &\in \Omega, w_\epsilon \in E_{\epsilon,x_\epsilon}, \text{ and} \\ \alpha_\epsilon &\longrightarrow (Q(a))^{-1/(p-2)} \\ x_\epsilon &\longrightarrow a \\ \|w_\epsilon\|_\epsilon &= o(\epsilon^{N/2}), \qquad (3.2) \end{aligned}$$

as $\epsilon \longrightarrow 0$, where a is a nondegenerate critical point of Q in Ω. We assume for simplicity of notations that $a = 0$. By changing the variables $y = \frac{x}{\epsilon}$, we see that $\tilde{u}_\epsilon(y) = u_\epsilon(\epsilon y)$ is a solution of

$$\begin{aligned} &-\Delta u + u = Q(\epsilon y) u^{p-1} \quad \text{in } \Omega_\epsilon \\ &u > 0 \quad \text{in } \Omega_\epsilon \quad \text{and } u = 0 \quad \text{on } \partial\Omega_\epsilon, \qquad (3.3) \end{aligned}$$

where $\Omega_\epsilon = \{y : \epsilon y \in \Omega\}$, and Q has a nondegenerate critical point at $a = 0$.

Now

$$\tilde{u}_\epsilon(y) = \alpha_\epsilon P_{\Omega_\epsilon} V_{y_\epsilon}(y) + w_\epsilon(\epsilon y), \tag{3.4}$$

where $y_\epsilon = \dfrac{x_\epsilon}{\epsilon}$, and $x_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

We shall use the notations $\langle\ ,\ \rangle$ and $\|\ \|$ to denote the standard product and the norm in $H_0^1(\Omega_\epsilon)$. Define

$$K_\epsilon(u) = \left(\int_{\Omega_\epsilon} |\nabla u|^2 + u^2\right) \Big/ \left(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p\right)^{2/p}$$

for $u \in H_o^1(\Omega_\epsilon)$

$$F_{\epsilon,y} = \left\{ v \in H_0^1(\Omega_\epsilon) : \langle P_{\Omega_\epsilon} V_y, v\rangle = 0, \left\langle \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}, v \right\rangle = 0, j = 1, \dots, N \right\};$$

$$J_\epsilon(y, v) = K_\epsilon(P_{\Omega_\epsilon} V_y + v), \quad v \in F_{\epsilon,y}.$$

We notice that $\tilde{u}_\epsilon$ is a critical point of K_ϵ in $H_0^1(\Omega_\epsilon)$.

PROPOSITION 3.3. – *There exist $\epsilon_0 > 0, \delta_0 > 0$, such that for $y \in B_\delta(0), \epsilon \in (0, \epsilon_0], \delta \in (0, \delta_0]$, there exists a unique C^1–map : $y \longrightarrow v_y$, from $B_\delta(0)$ to $F_{\epsilon,y}$, such that*

$$\left\langle \frac{\partial J_\epsilon(y, v_y)}{\partial v}, w \right\rangle = 0$$

for all $w \in F_{\epsilon,y}$. Furthermore,

$$\|v_y\| = O(\epsilon^2)$$

as $\epsilon \longrightarrow 0$

Proof. – The argument is very similar to that used by A. Bahri and J. Coron [1], and O. Rey [11]. We will be sketchy.

Expand

$$J_\epsilon(y,v) = J_\epsilon(y,o) + f_{\epsilon,y}(v) + G_{\epsilon,y}(v) + R_{\epsilon,y}(v), \tag{3.5}$$

where

$$f_{\epsilon,y}(v) = -\frac{\|P_{\Omega_\epsilon} V_y\|^2}{(\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^p)^{\frac{2}{p}+1}} \int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^{p-1} v \tag{3.6}$$

$$\begin{aligned} G_{\epsilon,y}(v) = & \frac{\|P_{\Omega_\epsilon} V_y\|^2}{(\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^p)^{2/p}} \\ & \Bigg[\|v\|^2 - \frac{\|P_{\Omega_\epsilon} V_y\|^2}{\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^p}(p-1)\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^{p-2} v^2 \\ & + \frac{\|P_{\Omega_\epsilon} V_y\|^2}{(\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^p)^2}(2+p)\left\{\int Q(\epsilon x)(P_{\Omega_\epsilon} V_y)^{p-1} v\right\}^2\Bigg]. \end{aligned}$$

and $R_{\epsilon,y}(v)$ satisfies

$$\begin{aligned} R_{\epsilon,y}(v) &= 0(\|v\|^{\mathrm{Min}(3,p)}) \\ R'_{\epsilon,y}(v) &= 0(\|v\|^{\mathrm{Min}(2,p-1)}) \\ R''_{\epsilon,y}(v) &= 0(\|v\|^{\mathrm{Min}(1,p-2)}) \end{aligned}$$

$f_{\epsilon,y}$ is a continuous linear form over $F_{\epsilon,y}$ equipped with the scalar product $\langle , \rangle$ of $H_0^1(\Omega_\epsilon)$. Therefore $\exists 1 f_{\epsilon,y} \in F_{\epsilon,y}$ such that $f_{\epsilon,y}(v) = \langle f_{\epsilon,y}, v\rangle$ for all $v \in F_{\epsilon,y}$. Furthermore, $G_{\epsilon,y}$ is a continuous quadratic form over $F_{\epsilon,y}$. Moreover, there exists $\rho > 0$ such that for ϵ small enough

$$G_{\epsilon,y}(v) \geq \rho\|v\|^2, \; v \in F_{\epsilon,y} \tag{3.8}$$

A proof of the above inequality was given in [4]. This implies the existence of a unique symmetric and coercive operator $A_{\epsilon,y}$ from $F_{\epsilon,y}$ onto itself, such that

$$G_{\epsilon,y}(v) = \langle A_{\epsilon,y} v, v\rangle$$

for all $v \in F_{\epsilon,y}$.

Using these notations, we have

$$\left.\frac{\partial J_\epsilon}{\partial v}\right|_{F_{\epsilon,y}}(y,v) = f_{\epsilon,y} + 2A_{\epsilon,y} v + R'_{\epsilon,y}(v)$$

Using the implicit function theorem and arguing as in [11] we establish the existence of a unique C^1-map $y \longrightarrow v_y$, such that

$$\left\langle \frac{\partial J_\epsilon}{\partial v}(y, v_y), w \right\rangle = 0 \tag{3.9}$$

for all $w \in F_{\epsilon,y}$, and

$$||v_y|| \leq C||f_{\epsilon,y}|| \tag{3.10}$$

for some positive constant C. We estimate $||f_{\epsilon,y}||$ next:

$$\begin{aligned} \left| \int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y|^{p-1} v \right| &= \left| \int (Q(\epsilon x) - Q(0))|P_{\Omega_\epsilon} V_y|^{p-1} v \right| \\ &= O\left(\epsilon^2 \int |P_{\Omega_{\epsilon,x}} V|^{p-1} |v| \right) \\ &= O(\epsilon^2)||v||, \end{aligned} \tag{3.11}$$

where we have used the identity

$$0 = \langle P_{\Omega_\epsilon} V_y, v \rangle = \int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y|^{p-1} v, v \in F_{\epsilon,y},$$

and the hypotheses on Q. From (3.6) and (3.11) we deduce that

$$||f_{\epsilon,y}|| = O\,(\epsilon^2),$$

and the conclusion follows from (3.10).

From Proposition 3.3 we may define

$$L_\epsilon(y) \equiv J_\epsilon(y, v_y) = K_\epsilon(P_{\Omega_\epsilon} V_y + v_y), \tag{3.12}$$

for $y \in B_\delta$, where δ is small enough such that Proposition 3.3 holds.

Define

$$M_\epsilon = \{(y, v) : y \in B_\delta(0), v \in F_{\epsilon,y}\}$$

Remark 3.4. – $(y, v) \in M_\epsilon$ is a critical point of J_ϵ if and only if y is a critical point of L_ϵ in B_δ and $v = v_y$, where v_y is given by Proposition 3.3. Furthermore, for small $\epsilon, (y, v) \in M_\epsilon$ is a critical point of J_ϵ if and only if $u = P_{\Omega_\epsilon} V_y + v$ is a critical point of K_ϵ this may be proved as in [11]. We further notice that

$$\frac{\partial L_\epsilon(y)}{\partial y_i} = \langle K'_\epsilon(P_{\Omega_\epsilon} V_y + v_y), \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} + \frac{\partial v_y}{\partial y_i} \rangle \tag{3.13}$$

LEMMA 3.5. – *Let ϵ_0, δ_0 be as in Proposition 3.3 Then*

$$\deg(0, \nabla L_\epsilon, B_\delta(0)) = (-1)^n,$$

where n is the number of negative eigenvalues of the matrix $(D^2Q(0)$. In particular, $\nabla L_\epsilon(y) = 0$ has a solution in $B_\delta(0)$.

Proof. – We first approxiamate ∇L_ϵ. Let $v_y, y \in B_\delta(0)$ be as in Proposition 3.3. Let us write

$$\frac{\partial v_y}{\partial y_i} = w_i + \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}, \tag{3.14}$$

$i = 1, \ldots, N$, where w_i is the orthogonal projection of $\dfrac{\partial v_y}{\partial y_i}$ in $F_{\epsilon,y}$. The following estimates are established in Appendix A:

$$\begin{aligned} \alpha_i &= O(e^{-\ell/\epsilon}), \quad \text{for some} \quad \ell > 0 \\ \gamma_{ij} &= O(\epsilon^2), \\ \|w_i\| &= O(\epsilon^2), \end{aligned} \tag{3.15}$$

$i, j = 1, \ldots, N$. By (3.9) and (3.12) we have

$$\langle K'_\epsilon(P_{\Omega_\epsilon} V_y + v_y), w\rangle = 0 \tag{3.16}$$

for all $w \in F_{\epsilon,y}$. Hence by (3.13) and (3.14) we have

$$\begin{aligned} \frac{\partial L_\epsilon(y)}{\partial y_i} &= \left\langle K'_\epsilon(P_{\Omega_\epsilon} V_y + v_y), \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j} + \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} \right\rangle \\ &= \frac{2\langle P_{\Omega_\epsilon} V_y + v_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} + \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}\rangle}{(\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y + v_y|^p)^{2/p}} \\ &\quad - \left\{ \frac{2\|P_{\Omega_\epsilon} V_y + v_y\|^2}{\int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y + v_y|^p} \right\} \\ &\quad \times \int_{\Omega_\epsilon} Q(\epsilon x)|P_{\Omega_\epsilon} V_y + v_y|^{p-2}(P_{\Omega_\epsilon} V_y + v_y) \\ &\quad \times \left\{ \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} + \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j} \right\} \\ &\triangleq I_1 + I_2 \end{aligned}$$

From Appendix A, we have

$$\left\langle P_{\Omega_\epsilon} V_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} \right\rangle = O(e^{-\ell/\epsilon})$$

for some $\ell > 0$. Thus (noting that $\langle v_y, \dfrac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i}\rangle = 0$)

$$|I_1| = O(e^{-\ell/\epsilon}) \text{ for some } \ell > 0, \text{as } \epsilon \longrightarrow 0. \tag{3.18}$$

We estimate I_2 next: We first notice that for some $\ell > 0$,

$$|P_{\Omega_\epsilon} V_y(x) - V_y(x)| \leq C e^{-\ell/\epsilon}$$

for all $x \in \Omega_\epsilon$. This follows easily from the Maximum principle. We further notice that

$$\frac{\|P_{\Omega_\epsilon} V_y + v_y\|^2}{(\int Q(\epsilon x)|P_{\Omega_\epsilon} V_y + v_y|^p)^{\frac{2}{p}+1}} = \frac{A + o(1)}{[Q(0)A + o(1)]^{\frac{2}{p}+1}} \tag{3.19}$$

This follows easily from the estimate on v_y in proposition (3.3) and the hypothesis on $Q(x)$. Here $A = \|V_y\|^2$. We also have, by Proposition 3.3, that

$$\begin{aligned}
&\int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y + v_y|^{p-2}(P_{\Omega_\epsilon} V_y + v_y)\frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} \\
&= \int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y|^{p-2}\frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} + (p-1)\int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y|^{p-2}\frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} v_y \\
&+ O\left(\|v_y\|^{\min(2,p-1)}\right) = o(\epsilon^2),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_\epsilon} (Q(\epsilon x) - Q(\epsilon y))|V_y + v_y|^{p-2}(V_y + v_y)\frac{\partial V_y}{\partial y_i} + O(\epsilon^{\frac{-\ell}{\epsilon}}) \\
&= \int_{\Omega_\epsilon + y} (Q(\epsilon x + \epsilon y) - Q(\epsilon y)) \\
&|V + v_y(x+y)|^{p-1}(V + v_y(y+x))\frac{\partial V}{\partial y_i} + O(\epsilon^{-\ell/\epsilon}) \\
&= \epsilon \int_{\Omega_\epsilon + y} \sum_{j=1}^N D_j Q(\epsilon y) y_j V^{p-1}\frac{\partial V}{\partial y_i} \\
&+ \frac{\epsilon^2}{2}\int_{\Omega_\epsilon + y} \sum_{j,k=1}^N D_{jk}^2 Q(\epsilon y) y_j y_k V^{p-1}\frac{\partial V}{\partial y_i} + o(\epsilon^2) \\
&= \epsilon D_i Q(\epsilon y) \int_{\mathbf{R}^N} y_i V^{p-1}\frac{\partial V}{\partial x_i} + o(\epsilon^2) \\
&= \epsilon B D_i(\epsilon y) + o(\epsilon^2),
\end{aligned}$$

where $B = \int_{\mathbf{R}^N} y_i V^{p-1}\dfrac{\partial V}{\partial x_i}$ is independent of i, by symmetry.

Hence

$$I_2 = \frac{A + o(1)}{(Q(0)A + o(1))^{\frac{2}{p}+1}} B\epsilon D_i Q(\epsilon y) + o(\epsilon^2) \tag{3.20}$$

Thus

$$\nabla L_\epsilon = \frac{B}{(Q(0))^{\frac{2}{p}+1} A^{\frac{2}{p}}} \epsilon\, D_i\, Q(\epsilon y) + o(\epsilon^2) \tag{3.21}$$

Since det $D^2Q(0) \neq 0$, there is $\delta > 0$ such that

$$|\nabla Q(\epsilon x)| \geq C_0 \epsilon \tag{3.22}$$

for some $c_o > 0$ and for all $x \in \partial B_\delta(0)$.

We see from (3.21), (3.22) that

$$\begin{aligned}\deg(0, \nabla L_\epsilon, B_\delta(0)) &= \deg(0, \frac{B\epsilon}{(Q(0))^{\frac{2}{p}+1} A^{2/p}} \nabla Q(\epsilon y), B_\delta(0)) \\ &= \deg(0, \nabla Q(x), B_{\delta\epsilon}(0)) \\ &= \text{sign det } D^2Q(0) = (-1)^n\end{aligned}$$

This completes the proof of Lemma 3.5.

PROPOSITION 3.6. – *There exists* $\epsilon_0, \delta_0 > 0$ *such that for* $0 < \epsilon \leq \epsilon_0,\ 0 < \delta \leq \delta_0, L_\epsilon$ *has a unique critical point in* $B_\delta(0)$.

Proof. – We argue as in Glangetas [8]. We have the following uniform estimate for all x such that $\nabla L_\epsilon(x) = 0$:

$$\frac{\partial^2 L_\epsilon(x)}{\partial x_i \partial x_j} = \frac{2}{p} \frac{A^{1-2/p}}{(Q(0))^{2/p}} \epsilon^2 D_{ij} Q(0) + o(\epsilon^2) \tag{3.23}$$

The proof of (3.23) is given in Appendix A. Hence any critical point of L_ϵ is an isolated point, for ϵ sufficiently small.

Now choose ϵ_0, δ_0 such that Lemma 3.5 holds. Then L_ϵ has, for any $0 < \delta < \delta_0$, a finite number, say k_0, of critical points in $B_\delta(0)$ at $x_1, \ldots, x_{k_0}$. On the other hand, (3.23) implies that

$$\det\left(\frac{\partial^2 L_\epsilon(x)}{\partial x_i \partial x_j}\right) = \left(\frac{2}{p} \frac{A^{1-\frac{2}{p}}}{(Q(0))^{2/p}}\right)^N \epsilon^{2N} \det D^2 Q(0) + o(\epsilon^{2N}),$$

and hence

$$\text{sign}\left(\det\left(\frac{\partial^2 L_\epsilon(x)}{\partial x_i \partial x_j}\right)\right) = (-1)^n \tag{3.24}$$

for all critical points x of L_ϵ in $B_\delta(0)$. Using Proposition 3.3 and a classical property of the degree, we have

$$(-1)^n = \deg(0, \nabla L_\epsilon, B_\delta(0)) = \sum_{i=1}^{k_0} \deg(0, \nabla L_\epsilon, B_\delta(x_i)) = k_0(-1)^n,$$

and therefore $k_0 = 1$. This completes the proof of Proposition 3.6.

The proof of Theorem 3.1 will follow if we show that y_ϵ in (3.4) satisfies $y_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0$, since this implies that y_ϵ is a critical point of L_ϵ in B_δ for ϵ sufficiently small $0 < \delta \leq \delta_0$.

LEMMA 3.7. – *Let u_ϵ be a single-peaked solution of (1.1) of the form (3.1) with $\alpha_\epsilon, x_\epsilon, \omega_\epsilon$ satisfying (3.2), and $a = \lim_{\epsilon \longrightarrow 0} x_\epsilon$ is a nondegenerate critical point of Q in Ω. Then*

$$|x_\epsilon - a| = O(\epsilon^2)$$

Proof. – From (2.19) we have

$$\begin{aligned}\int_{\Omega_{\epsilon,x_\epsilon}} &\nabla Q(\epsilon y + x_\epsilon)|\alpha_\epsilon P_{\Omega_{\epsilon,x_\epsilon}} V(y) + \omega_\epsilon(\epsilon y + x_\epsilon)|^p \\ &= \epsilon^{-(N-2)} \int_{\partial\Omega_{\epsilon,x_\epsilon}} (\frac{\partial u_\epsilon}{\partial n})^2 n d\sigma = O(\epsilon^2)\end{aligned} \tag{3.25}$$

Expand the left side of (3.25) to obtain

$$\begin{aligned}\int_{\Omega_{\epsilon,x_\epsilon}} &\nabla Q(\epsilon y + x_\epsilon)\{\alpha_\epsilon^p |P_{\Omega_{\epsilon,x_\epsilon}} V|^p + \alpha_\epsilon^{p-1} |P_{\Omega_{\epsilon,x_\epsilon}} V|^{p-1} \omega_\epsilon(\epsilon y + x_\epsilon)\} \\ &+ O(\|\omega_\epsilon\|_\epsilon^2 \epsilon^{-N}) = O(\epsilon^2),\end{aligned}$$

and since $|P_{\Omega_{\epsilon,x_\epsilon}} V(y) - V(y)| \leq Ce^{-\tau/\epsilon}$ for all $y \in \Omega_{\epsilon,x_\epsilon}$ by the Maximum principle, where C, τ are positive constants, we have

$$\begin{aligned}\int_{\Omega_{\epsilon,x_\epsilon}} &\nabla Q(\epsilon y + x_\epsilon)\{\alpha_\epsilon^p V^p + \alpha_\epsilon^{p-1} V^{p-1} \omega_\epsilon(\epsilon y + x_\epsilon)\} \\ &+ O(\|\omega_\epsilon\|_\epsilon^2 \epsilon^{-N}) = O(\epsilon^2).\end{aligned} \tag{3.26}$$

Since $\omega_\epsilon \in E_{\epsilon,x_\epsilon}$, we have

$$\int_{\Omega_{\epsilon,x_\epsilon}} V^{p-1}(y)\omega_\epsilon(\epsilon y + x_\epsilon) = \langle P_{\Omega,\epsilon} V_{\epsilon,x_\epsilon}, \omega_\epsilon \rangle_\epsilon = 0$$

Therefore

$$\int_{\Omega_{\epsilon,x_\epsilon}} \nabla Q(\epsilon y + x_\epsilon) V^{p-1} \omega_\epsilon(\epsilon y + x_\epsilon)$$
$$= \int_{\Omega_{\epsilon,x_\epsilon}} (\nabla Q(\epsilon y + x_\epsilon) - \nabla Q(x_\epsilon)) V^{p-1} \omega_\epsilon(\epsilon y + x_\epsilon)$$
$$= O(\epsilon) \|\omega_\epsilon\|_\epsilon \epsilon^{-N/2} \tag{3.27}$$

We estimate $\|\omega_\epsilon\|_\epsilon^2$ next:

We notice first that $(x_\epsilon, \frac{1}{\alpha_\epsilon}\omega_\epsilon)$ is a critical point of $\tilde{J}_\epsilon$ defined by

$$\tilde{J}_\epsilon(x, \omega) = \tilde{K}(P_{\Omega,\epsilon} V_{\epsilon,x} + \omega),$$

in E_{ϵ,x_ϵ}, where $\tilde{K}_\epsilon$ is given by

$$\tilde{K}_\epsilon(u) = \left(\int_\Omega \epsilon^2 |\nabla u|^2 + u^2\right) \Big/ \left(\int_\Omega Q(y)|u|^p\right)^{2/p}$$

By following the argument in Proposition 3.3 we obtain

$$\|\omega_\epsilon\|_\epsilon \leq C \|\tilde{f}_{\epsilon,x_\epsilon}\|_\epsilon,$$

where $\tilde{f}_{\epsilon,x}$ is given by

$$\tilde{f}_{\epsilon,x}(\omega) = -\frac{\|P_{\Omega,\epsilon} V_{\epsilon,x}\|_\epsilon^2}{(\int_\Omega Q(x)|P_{\Omega,\epsilon} V_{\epsilon,x}|^p)^{\frac{2}{p}+1}} \int_\Omega Q(x)|P_{\Omega,\epsilon} V_{\epsilon,x}|^{p-1}\omega$$

By estimating $\tilde{f}_{\epsilon,x_\epsilon}$ as in Proposition 3.3, we obtain

$$\|\omega_\epsilon\|_\epsilon^2 = O(\epsilon^{N+2}) \tag{3.28}$$

We also have

$$\int_{\Omega_{\epsilon,x_\epsilon}} \nabla Q(\epsilon y + x_\epsilon) V^p$$
$$= \int_{\Omega_{\epsilon,x_\epsilon}} \nabla Q(x_\epsilon) V^p + \int_{\Omega_{\epsilon,x_\epsilon}} [\nabla Q(\epsilon y + x_\epsilon) - \nabla Q(x_\epsilon)] V^p$$
$$= \nabla Q(x_\epsilon) \int_{\mathbf{R}^N} V^p + \epsilon \int_{\mathbf{R}^N} D^2 Q(x_\epsilon) y V^p + O(\epsilon^2)$$
$$= \nabla Q(x_\epsilon) \int_{\mathbf{R}^N} V^p + O(\epsilon^2), \tag{3.29}$$

where we used the fact that

$$\int_{\mathbb{R}^N} D^2Q(x_\epsilon)yV^p(y) = 0,$$

by the radial symmetry of V.

Combining (3.25)-(3.29) we obtain

$$\nabla Q(x_\epsilon)\int_{\mathbb{R}^N} V^p = O(\epsilon^2)$$

But $\det D^2Q(x_0) \neq 0$, and therefore we conlude that

$$|x_\epsilon - x_0| = O(\epsilon^2)$$

Proof of Theorem 3.1. – Let a be a non-degenerate critical point of Q. We may assume that $a = 0$. Now u_ϵ is a single-peaked solution of (1.1) of the form

$$u_\epsilon = \alpha_\epsilon P_{\Omega,\epsilon}V_{\epsilon,x_\epsilon} + w_\epsilon \tag{3.30}$$

with

$$\begin{aligned} \alpha_\epsilon &\longrightarrow (Q(0))^{-1/(p-2)} \\ x_\epsilon &\longrightarrow 0 \\ \|w_\epsilon\|_\epsilon &= o(\epsilon^{N/2}), \end{aligned}$$

as $\epsilon \longrightarrow 0, w_\epsilon \in E_{\epsilon,x_\epsilon}$, if and only if $y_\epsilon = \dfrac{x_\epsilon}{\epsilon}$ is a critical point of L_ϵ, and $v_\epsilon(y) = \dfrac{\omega_\epsilon(\epsilon y)}{\alpha_\epsilon} \in F_{\epsilon,y_\epsilon}$.

By Lemma 3.7, $y_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0$, and therefore $y_\epsilon \in B_\delta(0)$ for any small δ, provided ϵ is sufficiently small. Since L_ϵ has a unique critical point in $B_\delta(0)$, for small δ,(1.1) has a unique single-peaked solution of the form (3.40), and Theorem (3.1) follows.

Proof of Theorem 3.2. – By Propositions 2.1, 2.3, u_ϵ is a least energy solution, and is a single-peaked solution of the form

$$u_\epsilon = \alpha_\epsilon P_{\epsilon,\Omega}V_{\epsilon,x_\epsilon} + w_\epsilon,$$

where $x_\epsilon \longrightarrow x_0$, $\alpha_\epsilon \longrightarrow (Q(x_0))^{-1/(p-2)}$, $\|w_\epsilon\|_\epsilon \longrightarrow 0$, and $w_\epsilon \in E_{\epsilon,x_\epsilon}$, as $\epsilon \longrightarrow 0$.

Since $\mathrm{Max}_{\overline{\Omega}}Q(x)$ is uniquely attained at x_0, the conclusion follows from Theorem 3.1.

4. EXISTENCE IN THE DEGENERATE CASE

In this section we establish the existence of single-peaked solutions when $Q(x)$ has strict local maximum points in $\overline{\Omega}$, which are not neccessarily nondegenerate critical points, as required in sections 2 and 3. In fact, we will only require that Q is Lipschitz continuous on $\overline{\Omega}$, and so it may have no critical points in Ω. The main result of this section is the following.

THEOREM 4.1. – *Assume Q is Lipschitz continuous in $\overline{\Omega}$. Let $x_0 \in \overline{\Omega}$ be strict local maximum point of $Q(x)$, that is, $Q(x_0) > Q(x)$ for $x \in B_\delta(x_0) \cap \Omega \backslash \{x_0\}$, for some $\delta > 0$. Then (1.1) has a single peaked solution of the form*

$$u_\epsilon = \alpha_\epsilon P_{\epsilon,\Omega} V_{\epsilon,x_\epsilon} + \omega_\epsilon, \tag{4.1}$$

where

$$\begin{aligned} &\alpha_\epsilon \longrightarrow (Q(x_0))^{-1/(p-2)}, \\ &x_\epsilon \longrightarrow x_0, \\ &\epsilon^{-1} \mathrm{d}(x_\epsilon, \partial\Omega) \longrightarrow \infty, \\ &\|\omega_\epsilon\|_\epsilon^2 = o(\epsilon^N) \\ &\omega_\epsilon \in E_{\epsilon,x_\epsilon} \end{aligned} \tag{4.2}$$

Furthermore,

$$\|u_\epsilon\|_\epsilon^2 \leq \epsilon^N \Big(\frac{A}{(Q(x_0))^{2/(p-2)}} + o(1) \Big).$$

We will prove Theorem 4.1 when $x_0 \in \partial\Omega$. The case $x_0 \in \Omega$ can be discussed in a similar way.

An example will be given to show that, contrary to the case of nondegenerate critical points in Ω, a nondegenerate critical point on $\partial\Omega$ doesn't correspond to a single peaked solution of (1.1) with its peak tending to x_0 as $\epsilon \longrightarrow 0$.

Let $x_0 \in \partial\Omega$ denote a point where Q has a strict local maximum.

Define

$$A_\epsilon = \Big\{ x \in \Omega \cup B_\delta ; d(x, \partial\Omega) > \frac{1}{H} \epsilon \ln \epsilon \Big\} \tag{4.3}$$

where H is a large positive constant to be determined, and δ is a fixed small positive constant such that $Q(x_0) > Q(x)$ for all $x \in B_\delta(x_0) \cup \Omega \backslash \{x_0\}$. Define

$$\tilde{K}_\epsilon(u) = \Big(\int_\Omega \epsilon^2 |\nabla u|^2 + u^2 \Big) \Big/ \Big(\int_\Omega Q(x)|u|^p \Big)^{2/p}$$

for $u \in H_0^1(\Omega)$; let

$$\tilde{J}_\epsilon(x,\omega) = \tilde{K}_\epsilon(P_{\Omega,\epsilon}V_{\epsilon,x} + \omega),$$

$\omega \in E_{\epsilon,x}$.

Consider the following minimization problem:

$$\inf\{\tilde{J}_\epsilon(x,\omega); x \in \overline{A}_\epsilon, \|\omega\|_\epsilon^2 \le \delta\epsilon^N, \omega \in E_{\epsilon,x}\} \tag{4.4}$$

It is easy to show that the infimum in (4.4) is achieved, since $2 < p < 2N/(N-2)$. We now state a proposition which is crucial in the proof of Theorem 4.1.

PROPOSITION 4.2. – *Let*

$$M = \{(x,\omega) : x \in A_\epsilon, \omega \in E_{\epsilon,x}, \text{ and } \|\omega\|_\epsilon^2 \le \delta\epsilon^2\}$$

Then for sufficiently small $\epsilon > 0$,

$$u = P_{\epsilon,\Omega}V_{\epsilon,x} + \omega$$

is a critical point of K_ϵ *if* (x,ω) *is a critical point of* $\tilde{J}_\epsilon$ *in* M.

The proof is very similar to the proof given in [4], [11]. We omit it here.

To prove Theorem 4.1 it is enough to establish the existence of critical points in M. We need the following estimates. First we introduce the functions $\varphi_{\epsilon,p}, \psi_{\epsilon,p}$; as in [9]: For $p \in \Omega, y \in \Omega_{\epsilon,p}$, set $x = \epsilon y + p$,

$$\begin{aligned} \varphi_{\epsilon,p}(y) &= V(y) - P_{\Omega_{\epsilon,p}}V(y) \\ \psi_{\epsilon,p} &= -\epsilon \ln \varphi_{\epsilon,p}(y) \end{aligned} \tag{4.5}$$

LEMMA 4.3. – *Assume* $\partial\Omega$ *is of class* C^1. *Let* $p_\epsilon \in \Omega$ *satisfies* $\epsilon/d(p_\epsilon,\partial\Omega) \longrightarrow 0$ *as* $\epsilon \longrightarrow 0$. *Then for any* $c_0 > 0$ *there is* ϵ_0 *such that for* $\epsilon < \epsilon_0$

$$\frac{1}{2}d(p_\epsilon,\partial\Omega) \le \psi_{\epsilon,p_\epsilon}(p_\epsilon) \le c_0 d(p_\epsilon,\partial\Omega) \tag{4.6}$$

Proof. – The inequality $\psi_{\epsilon,p_\epsilon}(p_\epsilon) \le c_0 d(p_\epsilon,\partial\Omega)$ is proved in Lemma 4.6 of [9]. We prove the other inequality.

Let $\overline{p}_\epsilon \in \partial\Omega$ be such that $d(p_\epsilon,\partial\Omega) = |p_\epsilon - \overline{p}_\epsilon|$. Let y_ϵ be a point on the ray $\overline{p_\epsilon\overline{p}_\epsilon}$ such that $|y_\epsilon - p_\epsilon| = (1+\eta)|p_\epsilon - \overline{p}_\epsilon|$, where $\eta > 0$ is small enough so that $\overline{B}_{\eta/(p_\epsilon - \overline{p}_\epsilon)} \cap \overline{\Omega} = \{\overline{p}_\epsilon\}$.

Set

$$v_\epsilon(x) = (1-\eta)(|p_\epsilon - \overline{p}_\epsilon| - \eta|y_\epsilon - x|)$$

We now use Lemma 4.5 of [9] to obtain

$$\begin{aligned}\psi_{\epsilon,p_\epsilon} &\geq (1-\eta)|x-p_\epsilon| \geq |1-\eta||p_\epsilon - \overline{p}_\epsilon| \\ &> (1-\eta)(|p_\epsilon - \overline{p}_\epsilon| - \eta|y_\epsilon - x|) = v_\epsilon(x)\end{aligned} \tag{4.7}$$

for sufficiently small ϵ.

But simple calculations show that

$$\begin{aligned}\epsilon\Delta v_\epsilon - |\nabla v_\epsilon|^2 + 1 &\geq \frac{-C\epsilon}{\eta|y_\epsilon - \overline{p}_\epsilon|} - \eta^2(1-\eta)^2 + 1 \\ &= \frac{-C\epsilon}{\eta(1+\eta)|p_\epsilon - \overline{p}_\epsilon|} - \eta^2(1-\eta)^2 + 1 > 0\end{aligned}$$

since $\epsilon/|p_\epsilon - \overline{p}_\epsilon| \longrightarrow 0$, as $\epsilon \longrightarrow 0$, by hypothesis. Hence by the Maximum principal, we conclude that

$$\begin{aligned}\psi_{\epsilon,p_\epsilon}(p_\epsilon) \geq v_\epsilon(p_\epsilon) &= (1-\eta)(|p_\epsilon - \overline{p}_\epsilon| - \eta|y_\epsilon - p_\epsilon|) \\ &= (1-\eta)[1-\eta(1+\eta)]|p_\epsilon - \overline{p}_\epsilon| \\ &> \frac{1}{2}\, d(p_\epsilon, \partial\Omega)\end{aligned}$$

This completes the proof of Lemma 4.3.

LEMMA 4.4. – *Let $x_0 \in \partial\Omega$ denote a point where Q has a strict local maximum. Let $z_\epsilon \in \{x_0 + t\nu : t < 0\}$ be a point such that*

$$|z_\epsilon - x_0| = \epsilon \ln \frac{1}{\epsilon},$$

where ν is the unit outward normal to $\partial\Omega$ at x_0. Then

$$\tilde{J}_\epsilon(z_\epsilon, 0) \leq \frac{\epsilon^{N(1-2/p)}}{(Q(x_0)A)^{2/p}}\{A + 2\gamma\epsilon^{1/2} + o(\epsilon^{1/2})\}$$

for some positive constant γ.

Proof. – The following estimates were established by Ni and Wei [9]:

$$\begin{aligned}||P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}||_\epsilon^2 &= \epsilon^N ||P_{\Omega_\epsilon,z_\epsilon}V||^2 \\ &= \epsilon^N(A - 2\gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)}))\end{aligned} \tag{4.8}$$

$$\int |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p = \epsilon^N \int_{\Omega_{\epsilon,z_\epsilon}} |P_{\Omega_{\epsilon,z_\epsilon}}V|^p$$
$$= \epsilon^N (A - 2\gamma p e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)})) \quad (4.9)$$

But

$$\begin{aligned}
&\left| \int (Q(y) - Q(x_0)) |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \right| \\
&\leq C \int |y - x_0| |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \\
&= C \int_{B_{\epsilon \ln \frac{1}{\epsilon}}(z_\epsilon)} |y - x_0| |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \\
&+ C \int_{\Omega \setminus B_{\epsilon \ln \frac{1}{\epsilon}}(z_\epsilon)} |y - x_0| |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \\
&\leq C\left(\epsilon \ln\left(\frac{1}{\epsilon}\right) + |z_\epsilon - x_0|\right) \int_{B_{\epsilon \ln \frac{1}{\epsilon}}(z_\epsilon)} |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \\
&+ C \int_{\Omega \setminus B_{\epsilon \ln(\frac{1}{\epsilon})}(z_\epsilon)} |P_{\epsilon,\Omega}V_{\epsilon,z_\epsilon}|^p \\
&\leq C_1 \epsilon \ln(\frac{1}{\epsilon}) A \epsilon^N + C\epsilon^N \int_{R^N \setminus B_{\ln \frac{1}{\epsilon}}(0)} V^p \\
&\leq C\epsilon^N \left(A\epsilon \ln\left(\frac{1}{\epsilon}\right) + \epsilon^p\right) \\
&= C\epsilon^N o(\epsilon^{3/2})
\end{aligned} \quad (4.10)$$

Using (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned}
\tilde{J}_\epsilon(z_\epsilon, 0) &= \epsilon^{N(1-\frac{2}{p})} \frac{\{A - 2\gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)})\}}{\left\{ \begin{array}{c} \{Q(x_0)(A - 2\gamma p e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)}) \\ + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + \epsilon^{3/2})\}^{2/p} \end{array} \right\}} \\
&= \frac{\epsilon^{N(1-2/p)} A^{1-2/p}}{Q(x_0)^{2/p}} \\
&\quad \times \{1 + 2\gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} + \epsilon^{3/2})\} \quad (4.11)
\end{aligned}$$

By Lemma 4.3, we have

$$e^{-\frac{1}{\epsilon}\psi_{\epsilon,z_\epsilon}(z_\epsilon)} \leq e^{-\frac{1}{2\epsilon}d(z_\epsilon,\partial\Omega)} = e^{-\frac{1}{2}\ln(\frac{1}{\epsilon})} = \epsilon^{1/2}$$

Therefore,

$$\tilde{J}_\epsilon(z_\epsilon, 0) \leq \frac{\epsilon^{N(1-2/p)}}{Q(x_0)^{2/p}} A^{1-2/p}(1 + 2\gamma\epsilon^{1/2} + o(\epsilon^{1/2})),$$

which completes the proof of Lemma 4.4.

Proof of Theorem 4.1. – We first derive a lower bound for $\tilde{J}_\epsilon(x_\epsilon, \omega_\epsilon)$, where

$$\tilde{J}_\epsilon(x_\epsilon, \omega_\epsilon) = \inf\{J_\epsilon(x, \omega) : x \in \overline{A}_\epsilon, \|\omega\|_\epsilon^2 \leq \delta\epsilon^N, \omega \in E_{\epsilon,x}\}$$

we have

$$\begin{aligned} \|P_{\epsilon,\Omega} V_{\epsilon,x_\epsilon} + \omega_\epsilon\|_\epsilon^2 &= \epsilon^N \|P_{\Omega_{\epsilon,x_\epsilon}} V + \omega_\epsilon(\epsilon y + x_\epsilon)\|^2 \\ &= \epsilon^N(\|P_{\Omega_{\epsilon,x_\epsilon}} V\|^2 + \|\omega_\epsilon(\epsilon y + x_\epsilon)\|^2) \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\int Q(y)|P_{\epsilon,\Omega} V_{\epsilon,x_\epsilon} + \omega_\epsilon|^p \\ &= \epsilon^N \int_{\Omega_{\epsilon,x_\epsilon}} Q(\epsilon y + x_\epsilon)|P_{\Omega_{\epsilon,x_\epsilon}} V + \omega_\epsilon(\epsilon y + x_\epsilon)|^p \\ &= \epsilon^N\{\int_{\Omega_{\epsilon,x_\epsilon}} Q(\epsilon y + x_\epsilon)(|P_{\Omega_{\epsilon,x_\epsilon}} V|^p + p|P_{\Omega_{\epsilon,x_\epsilon}} V|^{p-1}\omega_\epsilon(\epsilon y + x_\epsilon) \\ &+ \frac{p(p-1)}{2}(P_{\Omega_{\epsilon,x_\epsilon}} V)^{p-2}\omega_\epsilon^2(\epsilon y + x_\epsilon)) \\ &+ \|\omega_\epsilon(\epsilon y + x_\epsilon)\|^{\min(p,3)}\} \\ &= \epsilon^N Q(x_\epsilon)\{\int (|P_{\Omega_{\epsilon,x_\epsilon}} V|^p + p|P_{\Omega_{\epsilon,x_\epsilon}} V|^{p-1}\omega_\epsilon(\epsilon y + x_\epsilon) \\ &+ \frac{p(p-1)}{2}(P_{\Omega_{\epsilon,x_\epsilon}} V)^{p-2}\omega_\epsilon^2(\epsilon y + x_\epsilon)) \\ &+ \|\omega_\epsilon(\epsilon y + x_\epsilon)\|^{\min(p,3)} + O(\epsilon)\} \\ &= \epsilon^N Q(x_\epsilon) \int_{\Omega_{\epsilon,x_\epsilon}} (P_{\Omega_{\epsilon,x_\epsilon}} V)^p\{1 + \frac{p(p-1)}{2} \frac{\int (P_{\Omega_{\epsilon,x_\epsilon}} V)^{p-2}\omega_\epsilon^2(\epsilon y + x_\epsilon)}{\int_{\Omega_{\epsilon,x_\epsilon}} (P_{\Omega_{\epsilon,x_\epsilon}} V)^p} \\ &+ O(e^{-(\frac{1}{2}+\sigma)\frac{\psi_{\epsilon,x_\epsilon}(x_\epsilon)}{\epsilon}})\|\omega_\epsilon(\epsilon y + x_\epsilon)\| \\ &+ O(\epsilon) + \|\omega_\epsilon(\epsilon y + x_\epsilon)\|^{\min(p,3)}\} \end{aligned} \tag{4.13}$$

where σ is some positive constant.

Combining (4.7) and (4.8) we obtain

$$
\begin{aligned}
\tilde{J}_\epsilon(x_\epsilon,\omega_\epsilon) &= \epsilon^{(1-\frac{2}{p})N}(Q(x_\epsilon))^{-2/p}\frac{\|P_{\Omega_{\epsilon,x_\epsilon}}V\|^2}{\int_{\Omega_{\epsilon,x_\epsilon}}|P_{\Omega_{\epsilon,x_\epsilon}}V|^p}\{1+\frac{\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2}{\|P_{\Omega_{\epsilon,x_\epsilon}}V\|^2}\\
&-(p-1)\frac{\int_{\Omega_{\epsilon,x_\epsilon}}(P_{\Omega_{\epsilon,x_\epsilon}}V)^{p-2}\omega_\epsilon(\epsilon y+x_\epsilon)}{\int_{\Omega_{\epsilon,x_\epsilon}}|P_{\Omega_{\epsilon,x_\epsilon}}V|^p}\\
&+O(\epsilon)+O(e^{-(\frac{1}{2}+\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\|\omega_\epsilon(\epsilon y+x_\epsilon)\|\\
&+O(\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^{\min(p,3)}\}\\
&\geq \epsilon^{(1-\frac{2}{p})N}(Q(x_\epsilon))^{-2/p}\frac{\|P_{\Omega_{\epsilon,x_\epsilon}}V\|^2}{(\int_{\Omega_{\epsilon,x_\epsilon}}|P_{\Omega_{\epsilon,x_\epsilon}}V|^p)^{2/p}}\{1+\rho'\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2\\
&+O(\epsilon)+O(e^{-(\frac{1}{2}+\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\|\omega_\epsilon(\epsilon y+x_\epsilon)\|\\
&+O(\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^{\min(p,3)})\}
\end{aligned}
$$

From the above inequatity, (4.12), and (4.13), we obtain

$$
\begin{aligned}
&\tilde{J}_\epsilon(x_\epsilon,\omega_\epsilon)\\
&\geq \epsilon^{N(1-2/p)}(Q(x_\epsilon))^{-2/p}\{A^{1-2/p}+2\gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)}+o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)}\}\\
&\times\{1+\rho'\|w_\epsilon(\epsilon y+x_\epsilon)\|^2+O(\epsilon)+O(e^{-(\frac{1}{2}+\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\|\omega_\epsilon(\epsilon y+x_\epsilon)\|\\
&+O(\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^{\min(p,3)})\}
\end{aligned}
\tag{4.14}
$$

We are now ready to prove that $x_\epsilon \in A_\epsilon$, and $\|\omega_\epsilon(\epsilon y+x_\epsilon)\| \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

Claim 1. $\|\omega_\epsilon(\epsilon y+x_\epsilon)\| \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

In fact, since $\|\omega_\epsilon\|_\epsilon^2 \leq \delta\epsilon^N$, we have $\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2 \leq \delta$. Therefore, for small δ, we have

$$
\begin{aligned}
&1+\rho'\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2+O(\epsilon)+O(e^{-(\frac{1}{2}+\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\|\omega_\epsilon(\epsilon y+x_\epsilon)\|\\
&+O(\|\omega_\epsilon(\epsilon y+x_\epsilon)\|)^{\min(p,3)}\\
&\geq 1+\rho''\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2+O(\epsilon+e^{-(1+2\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})
\end{aligned}
$$

Thus, from Lemma 4.4, (4.14) and the fact that $(x_\epsilon,\omega_\epsilon)$ is a minimizer of $\tilde{J}_\epsilon$, we get

$$
\begin{aligned}
&\epsilon^{N(1-2/p)}(Q(x_0))^{-2/p}A^{1-2/p}(1+\rho''\|\omega_\epsilon(\epsilon y+x_\epsilon)\|^2+o(1))\\
&\leq \tilde{J}_\epsilon(x_\epsilon,\omega_\epsilon)\leq \tilde{J}_\epsilon(z_\epsilon,0)\\
&\leq \epsilon^{N(1-2/p)}(Q(x_0))^{-2/p}A^{1-2/p}(1+o(1))
\end{aligned}
$$

Therefore, $\|\omega_\epsilon(\epsilon y + x_\epsilon)\| \longrightarrow 0$, as $\epsilon \longrightarrow 0$. In particular, if $\epsilon > 0$ is small enough,

$$\|\omega_\epsilon\|_\epsilon^2 = \epsilon^N \|\omega_\epsilon(\epsilon y + x_\epsilon)\|^2 < \frac{\epsilon^N}{2}\delta.$$

Claim 2. $x_\epsilon \in A_\epsilon$ for small $\epsilon > 0$.

We procedd by contradiction. Suppose $x_\epsilon \in \partial A_\epsilon$ for all small ϵ. There are two clases to consider:

(i) $x_\epsilon \in \partial B_\delta(x_0) \cup \Omega$. Then

$$Q(x_0) - \tau > Q(x_\epsilon)$$

for some positive τ. From Lemma 4.4 and inequality (4.14), we have

$$\begin{aligned}
&\epsilon^{N(1-2/p)}(Q(x_0) - \tau)^{-2/p} A^{1-2/p}(1 + o(1)) \\
&\leq \tilde{J}_\epsilon(x_\epsilon, \omega_\epsilon) \leq \tilde{J}_\epsilon(z_\epsilon, 0) \\
&\leq \epsilon^{N(1-2/p)}(Q(x_0))^{-2/p} A^{1-2/p}(1 + o(1))
\end{aligned}$$

where $o(1) \longrightarrow 0$ as $\epsilon \longrightarrow 0$. This is a contradiction.

(ii) Suppose for any $\epsilon_0, H_0 > 0$, there is $0 < \epsilon < \epsilon_0, H > H_0$, such that x_ϵ satisfies

$$d(x_\epsilon, \partial\Omega) = \frac{1}{H}\epsilon \ln(1/\epsilon)$$

But from claim 1 and (4.14) we have

$$\begin{aligned}
&\tilde{J}_\epsilon(x_\epsilon, \omega_\epsilon) \\
&\quad \geq \epsilon^{N(1-2/p)}(Q(x_0))^{-2/p}\{A^{1-2/p} + 2\gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\} \\
&\quad \times \{1 + \rho''\|\omega_\epsilon(\epsilon y + x_\epsilon)\|^2 + O(\epsilon + e^{-(1+2\sigma)\frac{1}{\epsilon}\psi_{\epsilon,\dot{x}_\epsilon}(x_\epsilon)})\} \\
&\quad \geq \epsilon^{N(1-2/p)}(Q(x_0))^{\frac{-2}{p}}\{A^{1-2/p} + \gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)}\} \\
&\quad \times \{1 + O(\epsilon + e^{-(1+2\sigma)\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)})\},
\end{aligned}$$

and from $\tilde{J}_\epsilon(x_\epsilon, \omega_\epsilon) \leq J_\epsilon(z_\epsilon, 0)$, and Lemma 4.4, we then have

$$A^{1-2/p} + \gamma e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)} + o(e^{-\frac{1}{\epsilon}\psi_{\epsilon,x_\epsilon}(x_\epsilon)}) + O(\epsilon) \leq A^{1-2/p} + 2\gamma\epsilon^{1/2} + o(\epsilon^{1/2})$$

But

$$\psi_{\epsilon,x_\epsilon} \leq c_0 d(x_\epsilon, \partial\Omega) = \frac{C_0}{H}\epsilon \ln\frac{1}{\epsilon}$$

by (4.6). Hence

$$\epsilon^{c_0/H} + o(\epsilon^{c_0/H}) \leq C\epsilon^{1/2}$$

for some constant $C > 0$. This is a contradiction if we choose $H > H_0 = 2c_0$

From claims 1 and 2 we have that $(x_\epsilon, \omega_\epsilon)$ is an interior point of M for small ϵ, and therefore a critical point of $\tilde{J}$ in M. By Proposition 4.2 we then have that $u_\epsilon = P_{\epsilon,\Omega}V_{\epsilon,x_\epsilon} + \omega_\epsilon$ is a critical point of K_ϵ, and Theorem 4. follows.

The following example shows that for a local minimum point of Q, a single peaked solution, with its peak approaching the minimum point, may not exist.

Example. – Let Ω be the unit ball $B_1(0)$ in $\mathbb{R}^N$. Let $\varphi(x) \in C_0^2[0,\infty)$ with $\varphi'(0) = 0, \varphi'(r) < 0, r > 0$, and $\varphi''(0) < 0$. Thus φ attains its global maximum at $r = 0$. Define $Q(x)$ by

$$Q(x) = \begin{cases} C - \varphi(\sqrt{(x_1-1)^2 + x_2^2 + \ldots + x_N^2}) & x_1 > 0 \\ Q(-x_1 x_2, \ldots, x_N) & x_1 \leq 0 \end{cases}$$

where C is a positive constant large enough so that $Q(x) \geq 1$. Then $Q(x)$ is decreasing in the x_1–direction.

Using the moving plane method in the x_1–direction, as in [7], we see that every positive solution u of

$$\begin{aligned} -\epsilon^2 \Delta u + u = Q(x)u^{p-1}, \quad x \in \Omega \\ u(x) = 0 \quad \text{on } \partial\Omega \end{aligned}$$

attains its maximum in the set $\{x_1 = 0\} \cap B_1(0)$. This shows that there is no positive solution of the above problem with single peak near the minimum point $(1, 0, 0, \ldots, 0)$ of Q.

APPENDIX A

In this appendix, we provide some of the estimates used in sections 2, 3, and 4. We first state the following result, which is a direct consequence of the maximum principle.

LEMMA A.1. – *There is a constant $\ell > 0$ such that*

$$|V_x - P_{\Omega_\epsilon} V_x| \leq e^{-\ell/\epsilon},$$
$$\left|\frac{\partial}{\partial x_i}(V_x - P_{\Omega_\epsilon} V_x)\right| \leq e^{-\ell/\epsilon},$$
$$\left|\frac{\partial^2}{\partial x_i \partial x_j}(V_x - P_{\Omega_\epsilon} V_x)\right| \leq e^{-\ell/\epsilon},$$

$i, j = 1, \ldots, N.$

From Lemma A1 we obtain

LEMMA A2. – *There exists a constant $\ell > 0$ such that the following estimates hold:*

$$\langle P_{\Omega_\epsilon} V_x, P_{\Omega_\epsilon} V_x \rangle = \int_{\Omega_\epsilon} V_x^{p-1} P_{\Omega_\epsilon} V_x = A + o(e^{-\frac{\ell}{\epsilon}}) \qquad (A.1)$$

where

$$A = \int_{\mathbb{R}^N} V^p$$

$$\left\langle P_{\Omega_\epsilon} V_x, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \right\rangle = \int_{\Omega_\epsilon} V_x^{p-1} \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \qquad (A.2)$$
$$= \int_{\Omega_\epsilon} V_x^{p-1} \frac{\partial V_x}{\partial x_i} + o(e^{-\frac{\ell}{\epsilon}})$$
$$= \int_{\mathbb{R}^N} V_x^{p-1} \frac{\partial V_x}{\partial x_i} + o(e^{-\frac{\ell}{\epsilon}})$$
$$= O(e^{-\frac{\ell}{\epsilon}})$$

$$\left\langle \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i}, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} \right\rangle = \int_{\Omega_\epsilon} \frac{\partial V_x^{p-1}}{\partial x_i} \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} \qquad (A.3)$$
$$= (p-1) \int_{\Omega_\epsilon} V^{p-2} \frac{\partial V_x}{\partial x_i} \frac{\partial V_x}{\partial x_j} + o(e^{-\ell/\epsilon})$$
$$= (p-1) \int_{\Omega_\epsilon} V^{p-2} \frac{\partial V_x}{\partial x_i} \frac{\partial V_x}{\partial x_j} + o(e^{-\ell/\epsilon})$$
$$= (p-1) \int_{\mathbb{R}^N} V^{p-1} \frac{\partial V_x}{\partial x_i} \frac{\partial V}{\partial x_j} + O(e^{-\ell/\epsilon})$$
$$= \begin{cases} O(e^{-\ell/\epsilon}), & i \neq j \\ A + O(e^{-\ell/\epsilon}) & i = j \end{cases}$$

LEMMA A.3. – *Let $\epsilon_0, \delta_0, v_y, y \in B_\delta(0), 0 < \delta < \delta_0$, be as in Proposition 3.3. For $0 < \epsilon < \epsilon_0$ set*

$$\frac{\partial v_y}{\partial y_i} = \omega_i + \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j},$$

$i = 1, \ldots, N$, where ω_i is the orthogonal projection of $\frac{\partial v_y}{\partial y_i}$ on $F_{\epsilon,y}$. Then the following estimates hold:

$$\begin{aligned} \alpha_i &= O(e^{-\ell/\epsilon}), \quad \text{for some} \quad \ell > 0, \\ \gamma_{ij} &= O(\epsilon^2), \\ \|\omega_i\| &= O(\epsilon^2), \end{aligned}$$

$i, j = 1, \ldots, N$, and $0 < \epsilon < \epsilon_0$.

Proof. – We first consider the scalar product in $H_0^1(\Omega_\epsilon)$ of $\dfrac{\partial v_y}{\partial y_i}$ with $P_{\Omega_\epsilon} V_y, \dfrac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}$ for $j, \ell = 1, \ldots, N$:

$$\begin{aligned} &\alpha_i \langle P_{\Omega_\epsilon} V_y, P_{\Omega_\epsilon} V_y \rangle + \sum_{j=1}^{N} \gamma_{ij} \left\langle \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}, P_{\Omega_\epsilon} V_y \right\rangle \\ &= \left\langle \frac{\partial v_y}{\partial y_i}, P_{\Omega_\epsilon} V_y \right\rangle = -\left\langle v_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j} \right\rangle = 0 \\ &\alpha_i \left\langle P_{\Omega_\epsilon} V_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_\ell} \right\rangle + \sum_{j=1}^{N} \gamma_{ij} \left\langle \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_j}, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_\ell} \right\rangle \\ &= -\left\langle v_y, \frac{\partial^2 P_{\Omega_\epsilon} V_y}{\partial y_i \partial y_\ell} \right\rangle = O\left(\|v_y\|\right) \end{aligned}$$

From (A1),(A2), (A3), and the estimate on $\|v_y\|$ in Proposition 3.3, we can solve the above equations for α_i and γ_{ij} and show that

$$\begin{aligned} \alpha_i &= O(e^{-\ell\epsilon}), \quad \text{for some} \quad \ell > 0, \\ \gamma_{ij} &= O(\|v_y\|) = O(\epsilon^2) \end{aligned}$$

To estimate $\|\omega_i\|$ we follow the argument in [8, Proposition 3.2]:

$$\begin{aligned} &D^2 K_\epsilon(P_{\Omega_\epsilon} V_y + v_y)\langle \omega_i, \omega_i \rangle + D^2 K_\epsilon(P_{\Omega_\epsilon} V_y + v_y) \\ &\times \left(\frac{\partial P_{\Omega_\epsilon}}{\partial y_i} + \alpha_i P_{\Omega_\epsilon} V_y + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i}, \omega_i \right\rangle = 0 \qquad (A.4) \end{aligned}$$

Direct calculations show

$$D^2K_\epsilon(u)(\varphi,\psi)$$
$$= \frac{2\langle\varphi,\psi\rangle}{(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p)^{2/p}} - \frac{4\langle u,\varphi\rangle}{\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p)^{\frac{2}{p}+1}} \int_{\Omega_\epsilon} Q(\epsilon y)|u|^{p-2}u\psi \tag{A.5}$$
$$- \frac{4\langle u,\psi\rangle}{(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p)^{\frac{2}{p}+2}} \left(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^{p-2}u\psi\right)\left(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^{p-2}u\varphi\right)$$
$$- \frac{2\|u\|^2(p-1)}{(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p)^{\frac{2}{p}+1}} \int_{\Omega_\epsilon} Q(\epsilon y)|u|^{p-2}\varphi\psi$$

We have

Claim (1):

$$D^2K_\epsilon(P_{\Omega_\epsilon}V_y + v_y)\langle\omega_i,\omega_i\rangle \geq \rho'\|\omega_i\|^2$$

for some $\rho' > 0$, uniformly for $0 < \epsilon < \epsilon_0$, $y \in B_\delta(0)$, $0 < \delta < \delta_0$. In fact, from $\omega_i \in F_{\epsilon,y}$, and the estimate of $\|v_y\|$, we have

$$\frac{2}{(\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y + v_y|^p)^{2/p}} \left\{\|\omega_i\|^2 - \left((p-1)\frac{\|P_{\Omega_\epsilon}V_y + v_y\|^2}{\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y + v_y|^p}\right)\right.$$
$$\left.\times \left(\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y + v_y|^{p-2}\omega_i^2\right)\right\} \geq \rho\|\omega_i\|^2$$

But

$$\langle P_{\Omega_\epsilon}V_y + v_y, \omega_i\rangle = \langle v_y,\omega_i\rangle = o(1)\|\omega_i\|$$
$$\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y + v_y|^{p-2}(P_{\Omega_\epsilon}V_y + v_y)\omega_i$$
$$= \int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y|^{p-1}\omega_i + o(1)\|\omega_i\|$$
$$= \int_{\Omega-\epsilon} (Q(\epsilon y) - Q(x))|P_{\Omega_\epsilon}V_y|^{p-1}\omega_i + o(1)\|\omega_i\|$$
$$= o(1)\|\omega_i\|$$

Claim (1) follows by putting the above estimates into (A5)

Claim (2):

$$D^2K_\epsilon(P_{\Omega_\epsilon}V_y + v_y)\left\langle\frac{\partial P_{\Omega_\epsilon}V_y}{\partial y_i}, \omega_i\right\rangle = O(\epsilon^2)\|\omega_i\|$$

In fact,

$$\left\langle \frac{\partial P_{\Omega_\epsilon} V_y}{\partial x_i}, \omega_i \right\rangle = 0$$

$$\begin{aligned}\langle P_{\Omega_\epsilon} V_y + v_y, \omega_i \rangle &= \langle v_y, \omega_i \rangle \\ &= O(\|v_y\|)\|\omega_i\| \\ &= O(\epsilon^2)\|\omega_i\|\end{aligned}$$

$$\left\langle P_{\Omega_\epsilon} V_y + v_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} \right\rangle = \left\langle P_{\Omega_\epsilon} V_y, \frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} \right\rangle = O(e^{-\ell/\epsilon}) \tag{A.6}$$

$$\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_y + v_y|^{p-2}(P_{\Omega_\epsilon} V_y + v_y)\frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i} = O(\epsilon^2) \tag{A.7}$$

$$\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_y + v_y|^{p-2}(P_{\Omega_\epsilon} V_y + v_y)\omega_i \tag{A.8}$$

$$\begin{aligned}&= O(\epsilon^2)\|\omega_i\| + Q(0)\int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y + v_y|^{p-2}(P_{\Omega_\epsilon} V_y + v_y)\omega_i \\ &= O(\epsilon^2)\|\omega_i\| + Q(0)\int_{\Omega_\epsilon} |P_{\Omega_\epsilon} V_y|^{p-1}\omega_i + O(\|v_y\|)\|\omega_i\| \\ &= O(\epsilon^2)\|\omega_i\|\end{aligned}$$

$$\begin{aligned}&\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_y + v_y|^{p-2}\frac{\partial P_{\Omega_\epsilon} V_y}{\partial y_i}\omega_i \\ &= \int_{\Omega_\epsilon} Q(\epsilon y)|V_y + v_y|^{p-2}\frac{\partial V_y}{\partial y_i}\omega_i + o(e^{-\ell/\epsilon})\|\omega_i\| \\ &= Q(0)\int_{\Omega_\epsilon} |V_y + v_y|^{p-2}\frac{\partial V_y}{\partial y_i}\omega_i + O(\epsilon^2)\|\omega_i\| \\ &= \frac{1}{p-1}Q(0)\int_{\Omega_\epsilon} \frac{\partial V_y^{p-1}}{\partial y_i}\omega_i + R + O(\epsilon^2)\|\omega_i\|,\end{aligned}$$

where

$$R = Q(0)\int_{\Omega_\epsilon} (|V_y + v_y)^{p-2} - V_y^{p-2})\frac{\partial V_y}{\partial y_i}\omega_i$$

$$|R| \le C\begin{cases} \displaystyle\int_{\Omega_\epsilon} V_y^{p-3}|\frac{\partial V_y}{\partial y_i}||v_y||\omega_i| & \text{if } p \le 3 \\ \displaystyle\int_{\Omega_\epsilon} V_y^{p-3}|\frac{\partial V_y}{\partial y_i}|(|v_y| + |v_y|^{\min(p-2,2)})|\omega_i| & \text{if } p > 3 \end{cases}$$

Since $\left|V_y^{p-3}\frac{\partial V_y}{\partial y_i}\right| \le C$ for $2 < p < 3$,

$$|R| \le C\|v_y\|\|\omega_i\| = O(\epsilon^2)\|\omega_i\|,$$

$$\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon}V_y + v_y|^{p-2}\left(\frac{\partial P_{\Omega_\epsilon}V_y}{\partial y_i}\right)\omega_i \\ = O(\epsilon^2)\|\omega_i\| \tag{A.9}$$

Claim (2) follows from (A.5), (A.6), (A.7), (A.8) and (A.9).

From (A.4), claims (1) and (2), and the estimate on α_i, we obtain

$$\rho'\|\omega_i\|^2 = O(\epsilon^2)\|\omega_i\|$$

Thus

$$\|\omega_i\| = O(\epsilon^2)$$

This completes the proof of Lemma A3.

LEMMA A4. – *There is* $\epsilon_0, \delta_0 > 0$ *such that*

$$\frac{\partial^2 L_\epsilon(x)}{\partial x_i \partial x_j} = \frac{2}{p}\frac{A^{1-2/p}}{(Q(0))^{2/p}}\epsilon^2 D_{ij}Q(0) + o(\epsilon^2)$$

for $0 < \epsilon \le \epsilon_0$, $0 < \delta \le \delta_0$.

Proof. – Arguing as in Glagetas [7], we get

$$\begin{aligned}\frac{\partial^2 L_\epsilon(x)}{\partial x_i \partial x_j} &= D^2K_\epsilon(P_\Omega V_x + v_x)\Big\langle \frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_j} + \alpha_j P_{\Omega_\epsilon}V_x \\ &\quad + \sum_{\ell=1}^N \gamma_{j\ell}\frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_\ell}, \frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_i} + \sum_{\ell=1}^N \gamma_{i\ell}\frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_\ell}\Big\rangle \\ &\quad + D^2K_\epsilon(P_{\Omega_\epsilon}V_x + v_x)\Big\langle \omega_j, \frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_i} + \sum_{k=1}^N \gamma_{ik}\frac{\partial P_{\Omega_\epsilon}V_x}{\partial x_i}\Big\rangle \\ &\triangleq I_3 + I_4 \end{aligned} \tag{A.10}$$

From Claim (2) in Lemma A3, we have

$$I_4 = O(\epsilon^2)\|\omega_i\| = O(\epsilon^4)$$

Now we estimate I_3:

I_3 consists of terms of the form

$$I(\epsilon) = D^2 K_\epsilon(P_{\Omega_\epsilon} V_x + v_x)\left\langle \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j}, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \right\rangle$$

Set $l_\epsilon(u) = \|u\|^2/(\int_{\Omega_\epsilon} Q(\epsilon y)|u|^p)$

Since $P_{\Omega_\epsilon} V_x + v_x$ is a critical point of K_ϵ, we have

$$\left\langle P_{\Omega_\epsilon} V_x + v_x, \frac{\partial^2}{\partial x_i \partial x_j} P_{\Omega_\epsilon} V_x \right\rangle - l_\epsilon(P_{\Omega_\epsilon} V_x + v_x) \int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^{p-2}(P_{\Omega_\epsilon} V_x + v_x) \frac{\partial^2 P_{\Omega_\epsilon} V_x}{\partial x_i \partial x_j} = 0 \quad (A.11)$$

From the estimates in (A2), (A6), (A7), and from (A5), we see that

$$\begin{aligned} I(\epsilon) = {} & \frac{2}{[\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^p]^{2/p}} \left\{ \left\langle \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j}, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \right\rangle \right. \\ & - (p-1)\, l_\epsilon(P_{\Omega_\epsilon} V_x + v_x) \\ & \left. \times \int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^{p-2} \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} \right\} \\ & + O(\epsilon^4) \triangleq \frac{2}{[\int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^p]^{2/p}} J(\epsilon) + O(\epsilon^4) \quad (A.12) \end{aligned}$$

Form (A.11), we have

$$\begin{aligned} J(\epsilon) = {} & \left\langle \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i}, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} \right\rangle \\ & + \left\langle P_{\Omega_\epsilon} V_x + v_x, \frac{\partial^2 P_{\Omega_\epsilon} V_x}{\partial x_i \partial x_j} \right\rangle \\ & - l_\epsilon(P_{\Omega_\epsilon} V_x + v_x) \left\{ (p-1) \int_{\Omega-\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^{p-2} \right. \\ & \left. \times \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i} \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} + \int_{\Omega_\epsilon} Q(\epsilon y)|P_{\Omega_\epsilon} V_x + v_x|^{p-1} \frac{\partial^2 P_{\Omega_\epsilon} V_x}{\partial x_i \partial x_j} \right\} \end{aligned} \quad (A.13)$$

We also have

$$\begin{aligned} & \left\langle \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_i}, \frac{\partial P_{\Omega_\epsilon} V_x}{\partial x_j} \right\rangle + \left\langle P_{\Omega_\epsilon} V_x, \frac{\partial^2 P_{\Omega_\epsilon} V_x}{\partial x_j \partial x_j} \right\rangle \\ & = \left\langle \frac{\partial V_x}{\partial x_i}, \frac{\partial V_x}{\partial x_j} \right\rangle + \left\langle V_x, \frac{\partial^2 V_x}{\partial x_i \partial x_j} \right\rangle + O(\epsilon^{-\ell/\epsilon}) \\ & = \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \langle V_x, V_x \rangle + O(e^{-\ell/\epsilon}) = O(e^{-\ell/\epsilon}), \\ & \left\langle V_x, \frac{\partial^2 P_{\Omega_\epsilon} V_x}{\partial x_i \partial x_j} \right\rangle = \left\langle V_x, \frac{\partial^2 V_x}{\partial x_i \partial x_j} \right\rangle + O(e^{-\ell/\epsilon}) \end{aligned}$$

Combining the above estimates and Lemma A1, we easily obtain

$$J(\epsilon) = \left\langle v_x, \frac{\partial^2 V_x}{\partial x_i \partial x_j} \right\rangle - l_\epsilon(V_x + v_x)\left\{(p-1)\int_{\mathbb{R}^N} Q(\epsilon y)|V_x + v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j} + \int_{\mathbb{R}^N} Q(\epsilon y)|V_x + v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i \partial x_j}\right\} + O(e^{-\ell/\epsilon}) \tag{A.14}$$

But

$$\int_{\mathbb{R}^N} |V_x + v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j} = \int_{\mathbb{R}^N} V_x^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j} + (p-2)\int_{\mathbb{R}^N} V_x^{p-3}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j}v_x + O(\|v_x\|^2) \tag{A.15}$$

$$\int_{\mathbb{R}^N} |V_x + v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i \partial x_j} = \int_{\mathbb{R}^N} V_x^{p-1}\frac{\partial^2 V_x}{\partial x_i \partial x_j} + (p-1)\int_{\mathbb{R}^N} V_x^{p-2}\frac{\partial^2 V_x}{\partial x_i \partial x_j}v_x + O(\|v_x\|^2) \tag{A.16}$$

Thus

$$\begin{aligned}(p-1)&\int_{\mathbb{R}^N} |V_x + v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j} + \int_{\mathbb{R}^N} |V_x + v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i \partial x_j}\\ &= \frac{1}{p}\int_{\mathbb{R}^N} \frac{\partial^2 V_x^p}{\partial x_i \partial x_j} + \int_{\mathbb{R}^N} \frac{\partial^2 V^{p-1}}{\partial x_i \partial x_j}v_x + O(\epsilon^4)\\ &= \int_{\mathbb{R}^N} \frac{\partial^2 V_x^{p-1}}{\partial x_i \partial x_j}v_x + O(\epsilon^4)\end{aligned} \tag{A.17}$$

Therefore

$$\begin{aligned}&\left\langle v_x, \frac{\partial^2 V_x}{\partial x_i \partial x_j} \right\rangle - l_\epsilon(V_x + v_x)Q(0)\left\{(p-1)\int_{\mathbb{R}^N} |V_x + v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j} + \int_{\mathbb{R}^N} |V_x + v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i \partial x_j}\right\}\\ &= \int_{\mathbb{R}^N} v_x\frac{\partial^2 V_x^{p-1}}{\partial x_i \partial x_j} - \frac{A + O(\epsilon^4)}{Q(0)A + O(\epsilon^2)}Q(0)\int_{\mathbb{R}^N} \frac{\partial^2 V_x^{p-1}}{\partial x_i \partial x_j}v_x + O(\epsilon^4)\\ &= O(\epsilon^2)\int \frac{\partial^2 V_x^{p-1}}{\partial x_i \partial x_j}v_x + O(\epsilon^4) = O(\epsilon^4)\end{aligned} \tag{A.18}$$

But

$$(p-1)\int_{\mathbb{R}^N}(Q(\epsilon y)-Q(0))|V_x+v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j}$$
$$+\int_{\mathbb{R}^N}(Q(\epsilon y)-Q(0))|V_x+v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i\partial x_j}$$
$$=\frac{p-1}{2}\epsilon^2\int_{\mathbb{R}^N}\sum_{\ell,k=0}^{N}D^2_{\ell k}Q(0)y_\ell y_k|V_x+v_x|^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j}$$
$$+\frac{1}{2}\epsilon^2\int_{\mathbb{R}^N}\sum_{\ell,k=1}^{N}D^2_{\ell k}Q(0)y_\ell y_k|V_x+v_x|^{p-1}\frac{\partial^2 V_x}{\partial x_i\partial x_j}+o(\epsilon^2)$$
$$=\frac{p-1}{2}\epsilon^2\int_{\mathbb{R}^N}\sum_{\ell,k=0}^{N}D^2_{\ell k}Q(0)y_\ell y_k V_x^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j}$$
$$+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{\ell,k=1}^{N}D^2_{\ell k}Q(0)y_\ell y_k V_x^{p-1}\frac{\partial^2 V_x}{\partial x_i\partial x_j}+o(\epsilon^2)$$
$$=\epsilon^2 D^2_{ij}Q(0)\int_{\mathbb{R}^N}y_iy_j\left\{(p-1)V_x^{p-2}\frac{\partial V_x}{\partial x_i}\frac{\partial V_x}{\partial x_j}+V_x^{p-1}\frac{\partial^2 V_x}{\partial x_i\partial x_j}\right\}+o(\epsilon^2)$$
$$=\frac{1}{p}\epsilon^2 D^2_{ij}Q(0)\int_{\mathbb{R}^N}y_iy_j\frac{\partial^2 V_x^p}{\partial x_i\partial x_j}+o(\epsilon^2)$$
$$=\frac{1}{p}\epsilon^2 D^2_{ij}Q(0)\frac{\partial^2}{\partial x_i\partial x_j}\int_{\mathbb{R}^N}y_iy_jV_x^p+o(\epsilon^2)$$
$$=\frac{1}{p}\epsilon^2 D^2_{ij}Q(0)\frac{\partial^2}{\partial x_i\partial x_j}\int_{\mathbb{R}^N}(y_i+x_i)(y_j+x_j)V^p+o(\epsilon^2)$$
$$=\frac{1}{p}\epsilon^2 D^2_{ij}Q(0)\int_{\mathbb{R}^N}V^p+o(\epsilon^2) \tag{A.19}$$

Combining (A.14), (A.18), and (A.19), we obtain

$$J(\epsilon)=\frac{1}{p}\epsilon^2 D^2_{ij}Q(0)\int_{\mathbb{R}^N}V^p+o(\epsilon^2) \tag{A.20}$$

Hence

$$I(\epsilon)=\frac{2}{[Q(0)A]^{2/p}+o(1)}\frac{A}{p}\epsilon^2D^2_{ij}Q(0)+o(\epsilon^2)$$

Therefore,

$$\frac{\partial^2 L_\epsilon(x)}{\partial x_i\partial x_j}=\frac{2}{p}\frac{A^{1-2/p}}{|Q(0)|^{2/p}}\epsilon^2D^2_{ij}Q(0)+o(\epsilon^2),$$
$$\det\left(\frac{\partial^2 L_\epsilon(x)}{\partial x_i\partial x_j}\right)=\left(\frac{2}{p}\frac{A^{1-2/p}}{|Q(0)|^{2/p}}\right)^N\epsilon^{2N}\det\left(D^2Q(0)\right)+o(\epsilon^{2N})$$

This completes the proof of Lemma A4.

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