ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 14, nº 3 (1997), p. 339-352
<http://www.numdam.org/item?id=AIHPC_1997__14_3_339_0>

Ann. Inst. Henri Poincaré,

Vol. 14, n° 3, 1997, p. 339-352 *Analyse non linéaire*

Minimizing a functional depending on ∇u and on u

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ABSTRACT. – We prove existence of solutions for a class of minimum problems of the Calculus of Variations, where the integrand depends both on ∇u and on u.

Key words: Minimization, gradient.

RÉSUMÉ. – On donne un théorème d'existence de solutions pour un problème du calcul des variations ou la fonctionnelle dépend de u et de ∇u.

INTRODUCTION

The purpose of the present paper is to contribute to the theory of existence of solutions to minimum problems of the Calculus of Variations when there is no assumptions of convexity with respect to the variable gradient. When convexity is not assumed, to prove existence of a solution one cannot rely on passing to the limit along minimizing sequences, but in most cases one has to actually provide a construction yielding the solution. Several examples of this approach exist: [1], [3], [4], [5], [8], [13]. The constructions appearing in these papers are, so to say, local, in the sense that the problem is solved locally and then the construction is extended to the full region by means of covering arguments. These constructions are used

A.M.S. Classification: 49 A 21, 49 B 22.

to solve non-convex problems both in the scalar and in the vector cases of the Calculus of Variations. However, when the functional to be minimized depends, besides on the gradient of the function u, on the function u itself, a local constructions in general cannot yield the solution. The purpose of this paper is to provide one non-local construction for a class of problems depending both on ∇u and on u.

In a paper by B. Kawohl, J. Stara and G. Wittum [11], a non convex functional, arising in a problem of shape optimization, is investigated, partially by numerical methods. The functional to be minimized is of the form

$$\int_{\Omega} (h(\|\nabla u(x)\|) + u(x))\, dx$$

under the condition $u = 0$ at $\partial\Omega$, and Ω is a two dimensional square. The function h is non convex, the infimum of two parabolas (the investigation of this functional has actually a longer history, *see* [10] and [14]). Numerical evidence suggests that (for the parameters considered in the numerical evaluations) solutions to the minimum problem do not exist. Notice that, in case Ω was a disk, solutions to the above problem would exist and be unique, essentially without any assumptions on h, except some lower semicontinuity and growth condition [2], [15], [16], conditions satisfied by the function h in [11]. In the case of a disk, in fact, one can show the existence of a radially symmetric solution to the convexified problem and modify this solution to obtain a (radially symmetric) solution to the original non convex problem.

The problem we consider is the minimization problem stated above, where h belongs to a class of (not necessarily convex) functions, and Ω is any bounded, open, convex subset of $\Re^2$ with a piecewise smooth boundary (so as to include a square as a special case). We provide a result that states that when the width of the set Ω is small (depending on a property of the function h), a solution to the problem exists. Moreover, this estimate on the width cannot be improved, in a sense to be made precise, as it is shown by an example. Another condition, connecting more accurately the properties of h and of Ω, is also presented. Again, this condition cannot be improved.

Our results, as such, give no informations for the map h considered in [11]. However, were the lower parabola, appearing in the definition of h, be replaced by a half line (a degenerate parabola; we obtain the functional considered by Kohn and Strang in [12]), our result would guarantee existence of solutions on squares whose side is of length not larger than twice the angular coefficient of the convexification of h. It is conceivable that, under similar conditions on the length of the side of the square, a solution to the true problem should exist.

NOTATIONS AND BASIC ASSUMPTIONS

By $\|x\|$ we mean the euclidean norm of x. When B is a matrix, $|B|$ denotes the determinant of B. The complement of a set A is denoted by $C(A)$ and the interior of A by $\text{int}(A)$. By $\Pi(x)$ we mean the set of nearest projections on $\partial\Omega$ from x in Ω, *i.e.* $\Pi(x) = \{y \in \partial\Omega : \|x-y\| = d(x, \partial\Omega)\}$. Throughout this paper we make the following assumptions on Ω.

The set Ω is convex, open and bounded. Moreover there exist: M points $O_i, i = 1, ..., M$, in $\partial\Omega$, such that Ω admits a unique (inward) normal **n** at O_i; M functions ϕ_i, each defined on a closed interval I_i containing the origin in its interior, such that:

a) with respect to the pair of coordinate axis with origin in O_i and directions defined by the tangent and the (inward) normal at O_i, the points $(\xi, \phi_i(\xi))$ belong to $\partial\Omega$ for every ξ in I_i; for ξ in $\text{int}(I_i)$, points (ζ, ξ) belong to Ω for $\zeta > \phi_i(\xi)$ sufficiently small.

b) the functions ϕ_i are of class C^2 on an open set containing I_i; as a consequence of a) and of the regularity of ϕ_i, we have that $\phi_i(0) = 0$ and $\phi_i'(0) = 0$.

c) Every point of $\partial\Omega$ is represented in the form above; moreover the representation is unique except for the finitely many points of $\partial\Omega$, images of the endpoints of the intervals I_i.

We recall that the radius of curvature R at a point $(\xi, \phi_i(\xi))$ of the boundary of Ω, ξ in $\text{int}(I_i)$, is $R = +\infty$ when $\phi_i''(\xi) = 0$ and

$$R = \frac{(1 + (\phi_i'(\xi)))^2)^{3/2}}{\phi_i''(\xi)}$$

when $\phi_i''(\xi) \neq 0$.

The width W_Ω of Ω is defined to be $W_\Omega = \sup\{d(x, C(\Omega)) : x \in \Omega\}$. Notice that the word width has been used in convex anaysis with a somewhat different meaning.

We will consider a class of maps h satisfying the following assumption:

Assumption A

The map $h : [0, \infty) \to [0, \infty]$ is a non-negative lower semicontinuous extended valued function with minimum value 0. Moreover, $\sup\{r \geq 0 : h(r) = 0\}$ is finite.

Definitions of ρ and Λ

Set $\rho = \sup\{r \geq 0 : h(r) = 0\}$, and call A the set of supporting linear functions at ρ: $A = \{a : h(s) \geq a(s - \rho), \text{ for every } s \geq 0\}$. We have that $0 \in A$. Set Λ in $[0, \infty]$ to be : $\Lambda = \sup\ A$.

Whenever $\Lambda < \infty, \Lambda$ is in A. Whenever h is smooth, $\Lambda = 0$. In the case the map h is convex, we have that

$$\Lambda = \lim_{r \to 0^+} \frac{h(\rho + r)}{r}.$$

MAIN RESULTS

It is our purpose to consider the following problem

Problem P): Minimize

$$\int_\Omega (h(\|\nabla u(x)\|) + u(x))\, dx$$

on $u \in W_0^{1,1}(\Omega)$.

LEMMA 1. – *Let $y \in \Pi(x), x \in \Omega$. Then $\partial\Omega$ is differentiable at y.*

Proof. – The open disk centered at x having radius $\rho = \|x - y\|$ contains no points of $\partial\Omega$. Since x is interior to $\overline{\Omega}$, there is a ball about x contained in $\overline{\Omega}$. Each half line issuing from points in this ball in the direction of $y - x$ meets the boundary of Ω in exactly one point. By moving the origin of the coordinates to y and the axis oriented as the normal to $x - y$ and as $x - y$, there exists an open interval I containing 0 such that the boundary of Ω is represented by $\zeta = \phi(\xi)$ where ϕ is a convex function defined on I. The convex function ϕ has both a left derivative ϕ'_- and a right derivative ϕ'_+ at 0. Since there are no points of $\partial\Omega$ in the disk centered at $(0, \rho)$ and radius ρ we must have $\phi'_- \geq 0$ and $\phi'_+ \leq 0$ while, being ϕ a convex function, $\phi'_- \leq \phi'_+$, so that $\phi'(0)$ exists and equals 0.

DEFINITION. – Let $x \in \partial\Omega$ be a point of differentiability of $\partial\Omega$. Set $\ell(x)$ to be

$$\ell(x) = \sup\{\lambda \geq 0 : \text{for } x' \in C(\Omega), x' \neq x, \lambda < d(x + \lambda\mathbf{n}(x), x')\}.$$

The set inside parenthesis always contains at least $\lambda = 0$. About the properties of ℓ, we have:

LEMMA 2. – *a*) $\ell(x) \leq W_\Omega$;

b) *For y in Ω and x in* $\Pi(y)$, $d(y, C(\Omega)) \leq \ell(x)$;

c) *Let $\partial\Omega$ be of class C^2 in a neighborhood of x. Then*; $\ell(x) \leq R(x)$; *moreover*;

d) *when $\ell(x) < R(x)$, there exists $z \in \partial\Omega, z \neq x$, such that* $d(x + \lambda\mathbf{n}(x), z) = \ell(x)$.

Proof. – *a*) Obvious when $\ell = 0$. For $\lambda < \ell(x), \lambda = d(x+\lambda\mathbf{n}(x), C(\Omega))$, so that $\ell(x) = d(x + \ell(x)\mathbf{n}(x), C(\Omega)) \leq W_\Omega$;

b) For every $\lambda < d(y, C(\Omega))$, x is the unique point in $\partial\Omega$ nearest to $x + \lambda\ell(x)$; hence the supremum of such λ is $\geq d(y, C(\Omega))$;

c) There is nothing to prove when $R(x) = \infty$. Assume $R(x) < \infty$ and $\ell(x) = R(x) + \eta, \eta > 0$, and choose $\lambda = R(x) + \frac{\eta}{2}$. The open disk centered at $x + \lambda\mathbf{n}(x)$ of radius λ does not contain points of $C(\Omega)$. Move the origin of the coordinates to x and the axis in the directions of the tangent and normal in x, so that $\partial\Omega$ is represented, in a neighborhood N of $(0,0)$, as $\zeta = \phi(\xi)$, with $\phi(0) = 0$ and $\phi'(0) = 0$. Hence, for $|\xi|$ sufficiently small, we have $\phi(\xi) = \frac{1}{2}\phi''(\overline{\xi})\xi^2$ with$|\overline{\xi}| \leq |\xi|$. In a neighborhood of 0, $\phi''(\overline{\xi}) \geq \frac{1}{R+\eta/4}$, while

$$\begin{aligned} d((\xi,\phi(\xi)),(0,R+\eta)) &= \sqrt{\xi^2 + (\phi(\xi) - (R+\eta))^2} \\ &= \sqrt{\xi^2 + (\phi(\xi))^2 + (R+\eta)^2 - 2\phi(\xi)(R+\eta)} \\ &= \sqrt{(R+\eta)^2 + \xi^2(1 - (R+\eta)\phi''(\overline{\xi}) + o(\xi))} \end{aligned}$$

and, for $|\xi|$ small, $1-(R+\eta)\phi''(\overline{\xi})+o(\xi)) < 0$, so that $d((\xi,\phi(\xi)),(0,R+\eta)) < R+\eta$, a contradiction to the definition of $\ell(x)$ for $x = 0$.

d) Let $\ell(x) = R(x) - \eta$ and again consider the origin of the coordinates to be in x. From the definition of ℓ, there exist: $\varepsilon_n \downarrow 0, z_n \in \partial\Omega, z_n \neq 0$, such that $d(z_n, 0 + (\ell + \varepsilon_n)\mathbf{n}(0)) \leq \ell + \varepsilon_n$. For n large, all the disks centered at $0 + (\ell + \varepsilon_n)\mathbf{n}(0)$ with radius $\ell + \varepsilon_n$ are contained in the disk with center $0 + (\ell + \frac{\eta}{2})\mathbf{n}(0)$ and radius $\ell + \frac{\eta}{2}$. In a neighborhood of 0, the equation of this circle is

$$\zeta = C(\xi) = \frac{1}{2}[C''(0) + o_C(\xi)]\xi^2 = \frac{1}{2}\left[\frac{1}{R+(\eta/2)} + o_C(\xi)\right]\xi^2$$

In a neighborhood of $(0,0)$ the points of $\partial\Omega$ are represented by

$$\zeta = \phi(\xi) = \frac{1}{2}[\phi''(0) + o_\phi(\xi)]\xi^2 = \frac{1}{2}\left[\frac{1}{R} + o_\phi(\xi)\right]\xi^2$$

Hence there exists a neighborhood N of $(0,0)$ such that points $z_n = (\xi_n, \zeta_n)$ cannot be in N, since otherwise we would have

$$\frac{1}{2}\left[\frac{1}{R+(\eta/2)} + o_C(\xi_n)\right]\xi_n^2 \geq \zeta_n = \frac{1}{2}\left[\frac{1}{R} + o_\phi(\xi_n)\right]\xi_n^2$$

Then, a subsequence of the z_n converges to a point $z \in \partial\Omega \cap C(N)$, and we have

$$d(z, 0 + \ell\mathbf{n}) \leq \ell.$$

LEMMA 3. – *For each i, the function $\ell(\xi,\phi_i(\xi))$ is continuous on I_i.*

Proof. – (i) Continuity on $\mathrm{int}\,(I_i)$.

a) It cannot happen that there exists $(x_n)_n$ with x_n in $\partial\Omega$ and $x_n \to x*$ such that $\ell(x_n) \to \overline{\ell} = \ell(x*) - \eta$. If this is the case, in fact, three situations can happen, in view of Lemma 2: 1) on a subsequence, $\ell(x_n) = R(x_n)$; 2) there are points z_n in $\partial\Omega$, $z_n \neq x_n$ but $d(x_n, z_n) \to 0$ and $d(z_n, x_n + \ell(x_n)\mathbf{n}(x_n)) = \ell(x_n)$, and 3) the points z_n are such that $d(x_n, z_n)$ are bounded away from zero. In case 1), since $R(x_n) \to R(x*)$, then $R(x*) < \ell(x*)$ and this contradicts Lemma 2 *c*).

Consider case 2). In a neighborhood N of $x*$, $R(x) > R(x*) - \frac{\eta}{2}$ when $R(x*)$ is finite, or larger than the diameter of Ω when $R(x*) = \infty$. Since the open disk with center $x_n + \ell(x_n)\mathbf{n}(x_n)$ and radius $\ell(x_n)$ has empty intersection with $\partial\Omega$ and has the two points z_n and x_n on its boundary, at some point on the boundary intermediate between z_n and x_n, the radius is not larger than $\ell(x_n)$. When both z_n and x_n are in N, we have $\ell(x*) - \frac{\eta}{2} \leq R(x*) - \frac{\eta}{2} \leq R(x_n) \leq \ell(x_n)$, a contradiction.

Case 3) cannot happen: the sequence $(z_n)_n$ would converge to $z*$ in $\partial\Omega$, $z* \neq x*$ having distance $\overline{\ell}$ from $x* + \overline{\ell}\mathbf{n}(x*)$. The point $z*$ would have distance less than $\ell(x*)$ from $x* + \ell(x*)\mathbf{n}(x_n)$.

b) It cannot happen either that there exists $(x_n)_n$ with x_n in $\partial\Omega$ and $x_n \to x*$ such that $\ell(x_n) \to \overline{\ell} = \ell(x*) + \eta$. Let the origin be in $x*$ and the axis oriented as the tangent and normal. The distance $d_{x'}$ from $x' = (\xi', \zeta')$ in $\partial\Omega$ to $0 + \mathbf{n}(0)\overline{\ell}$ is $\sqrt{(\xi')^2 + (\overline{\ell} - \zeta')^2}$. When $\ell(0) = R(0)$, locally $\partial\Omega$ is represented by

$$\phi(\xi) = \left[\frac{1}{2}\left(\frac{1}{\ell(0)}\right) + o(\xi)\right]\xi^2$$

so that

$$d_{x'} = \sqrt{(\xi')^2 + \left[\overline{\ell} - \left(\frac{1}{2}\left(\frac{1}{\ell(0)}\right) + o(\xi)\right)\xi^2\right]^2} < \overline{\ell}$$

for ξ' small. For n large, $d(x_n + \ell(x_n)\mathbf{n}(x_n), x') < \overline{\ell}$ so that for large n one would have $d(x_n + \ell(x_n)\mathbf{n}(x_n), x') < \ell(x_n)$. Finally, in the case there exists $x' \neq x$ such that $\ell(0) = d(0 + \mathbf{n}(0)\ell(0), x')$, again one would have $d_{x'} < \eta + \ell(0) = \overline{\ell}$ and the same conclusion would follow.

(ii) Continuity at the boundary points of I_i.

When ξ is a boundary point of I_i, either the normal at $\partial\Omega$ exists at $(\xi, \phi_i(\xi))$ or it does not. In the first case ℓ is defined and the considerations above apply as one-sided considerations. In the second case it is easy to see that for ξ close to the boundary of I_i, $\ell(\xi)$ tends to zero. In this case, by defining ℓ to be zero at these boundary points, one achieves the proof of the continuity on I_i.

DEFINITION AND PROPERTIES OF THE MAPS g_i AND f_i. – With respect to the system of coordinates centered at O_i, consider the transformation g_i that associates to the pair (ξ, l) : ξ in $\operatorname{int}(I_i)$ and $0 \leq l \leq \ell((\xi, \phi_i(\xi)))$, the vector of components (ξ_1, ξ_2) given by

$$\xi_1 = \xi + \frac{-\phi_i'(\xi)}{\sqrt{1 + (\phi_i')^2}}\, l$$

$$\xi_2 = \phi_i(\xi) + \frac{1}{\sqrt{1 + (\phi_i')^2}}\, l.$$

Here $\frac{-\phi_i'(\xi)}{\sqrt{1+\phi_i'^2}}$ and $\frac{1}{\sqrt{1+\phi_i'^2}}$ are the components of the normal $\mathbf{n}$ at $(\xi, \phi_i(\xi))$. We will set $\ell(\xi)$ to be $\ell((\xi, \phi_i(\xi))$. Set S_i to be the image of the map g_i on its domain and S_i^0 to be the image of $\{(\xi, l) : \xi \in \operatorname{int}(I_i), 0 \leq l < \ell(\xi)\}$. On S_i^0 the map g_i is invertible, since $(\xi, \phi_i(\xi))$ is the unique point in $\partial\Omega$, nearest to $g_i(\xi, l)$. Call f_i the map from S_i^0 to $\Re$ defined by the first component of g^{-1}, so that, for ξ in $\operatorname{int}(I_i)$,

$$\xi = f_i(g_i(\xi, l)).$$

For the derivatives of f_i, setting (g_i^1) and (g_i^2) to be the two components of g_i, we have the system

$$1 = (f_i)_{\xi_1}(g_i^1)_\xi + (f_i)_{\xi_2}(g_i^2)_\xi$$

$$0 = (f_i)_{\xi_1}(g_i^1)_l + (f_i)_{\xi_2}(g_i^2)_l$$

so that

$$(f_i)_{\xi_1} = \frac{1}{|\nabla g_i|}(g_i^2)_l$$

$$(f_i)_{\xi_2} = \frac{-1}{|\nabla g_i|}(g_i^1)_l$$

Hence the norm of the gradient of f_i is

$$\|\nabla f_i\| = \frac{1}{|\nabla g_i|}\sqrt{((g_i^2)_l)^2 + ((g_i^1)_l)^2} = \frac{1}{|\nabla g_i|}$$

computed at $g_i^{-1}(\xi_1, \xi_2)$. Computing $|\nabla g_i|$ we obtain

$$|\nabla g_i| = \sqrt{1+(\phi_1')^2}\left(1 - \frac{\phi_i''(\xi)}{(1+(\phi_i'(\xi))^2)^{3/2}}\right) = \sqrt{1+(\phi_1')^2}\left(1 - \frac{l}{R(\xi)}\right)$$

so that, since $\ell(\xi) \leq R(\xi)$, $|\nabla g_i|$ is $\neq 0$ on $\text{int}(I_i) \times \{0 \leq l < \ell(\xi)\}$. Since our assumptions imply that ϕ_i'' and $\sqrt{1+(\phi_i')^2}$ are uniformly bounded on I_i, the map g_i is lipschitzean. Moreover, by setting S_i^ε to be the image of $\text{int}(I_i) \times \{0 \leq l < \ell(\xi) - \varepsilon\}$, one has that the map f_i is Lipschitzean on S_i^ε. We will need both properties in what follows.

About the sets S_i^0 and S_i^ε we have the following result.

LEMMA 4. – *We have:* $\lim_{\varepsilon\to 0} \mu(\Omega\backslash(\cup S_i^\varepsilon)) = 0$. *In particular,* $\mu(\Omega\backslash(\cup S_i^0)) = 0$.

Proof. – Fix $y \in \Omega$ and let x be in $\Pi(y)$. At x, by Lemma 1, the normal $\mathbf{n}(x)$ exists, and y can be written as $y = x + \mathbf{n}(x)d(y, C\Omega)$ and by b) of Lemma 2, $d(y, C(\Omega)) \leq \ell(x)$. Hence y may fail to be in $\cup S_i^\varepsilon$ when either the points x in $\Pi(y)$ are represented as $(\xi, \phi_i(\xi))$ with ξ in ∂I_i or when there is some x in $\Pi(y)$ represented by ξ in $\text{int}(I_i)$ but $d(y, C(\Omega)) < \ell(x) - \varepsilon$. Points of the first type are contained in the union of finitely many segments (on the normal lines through x), a set of measure zero. About the other points, notice that, in the space $I_i \times \Re$, the set $\{(\xi, l) : \ell(\xi) - \varepsilon \leq l \leq \phi_i(\xi)\}$ has measure $\varepsilon\ \mu(I_i)$. Its image by the lipschitzean map g_i is of measure that can be made arbitrarily small in Ω by decreasing ε. The union of these images contains all points of the second type.

The following is our existence Theorem. A more precise condition is expressed in Theorem 2.

THEOREM 1. – *Let Ω be an open, bounded, convex subset of $\Re^2$ with piecewise smooth boundary, having width W_Ω. Let h satisfy Assumption A) and let ρ and Λ be defined as above. When $W_\Omega \leq \Lambda$, the function $u(x) = -\rho\ d(x, \partial\Omega)$ is a solution to the minimization problem P.*

Proof. – *a*) The map $x \to d(x, C(\Omega))$ is differentiable a.e. and its gradient is (a.e.) $-\mathbf{n}(\Pi(x))$ (*see* [9], p. 354) . In particular, $\Pi(x)$ is single valued for a.e. x in Ω. The map u is $-\rho d(x, C(\Omega))$ so that a.e., $\nabla u(x) = -\rho\mathbf{n}(y)$, y the unique point in $\Pi(x)$. In the case $\rho > 0$, $\frac{\nabla u(x)}{\|\nabla u(x)\|} = -\mathbf{n}(y)$, while for $\rho = 0$ we set $\frac{\nabla u(x)}{\|\nabla u(x)\|}$ to be $-\mathbf{n}(y)$ by definition.

Let α be a function in $L^\infty(\Omega)$ and such that: $0 \leq a(x) \leq \Lambda$ for a.e. x in Ω when $\Lambda < \infty$, and $0 \leq \alpha$ when $\Lambda = \infty$. Then, for any vector v, when $\rho > 0$

$$\begin{aligned} h(\|\nabla u(x) + v\|) &= h(\|\nabla u(x)\| + \|\nabla u(x) + v\| - \|\nabla u(x)\|) \\ &\geq h(\|\nabla u(x)\|) + \alpha(x)(\|\nabla u(x) + v\| - \|\nabla u(x)\|) \\ &\geq h(\|\nabla u(x)\|) + \alpha(x)\left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, v \right\rangle \end{aligned}$$

and, for $\rho = 0$

$$h(\|v\|) \geq \alpha(x)\|v\| \geq \alpha(x)\langle -\mathbf{n}(y), v\rangle.$$

Hence, for every ρ and for every function η in $W_0^{1,1}(\Omega)$, we have

$$\begin{aligned} &\int_\Omega (h(\|\nabla u + \nabla \eta\|) + (u + \eta))\, dx \\ &\quad \geq \int_\Omega (h(\|\nabla u\|) + u)\, dx + \int_\Omega \left(\alpha(x)\left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla \eta \right\rangle + \eta \right) dx \end{aligned}$$

We are planning to show that there exists a function α in $L^\infty(\Omega)$, with the properties stated above, and such that for every function η in $C_0^\infty(\Omega)$,

$$\int_\Omega \left(\alpha(x)\left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla \eta(x) \right\rangle + \eta(x) \right) dx = 0$$

Finding this function α, then, amounts to proving that the function u is a solution to the minimization problem P. In fact, by approximating a function η in $W_0^{1,1}(\Omega)$ by standard mollifiers, one sees that the above equation must actually be true for every function η in $W_0^{1,1}(\Omega)$, so that u solves P.

b) The function $|\nabla g_i(\xi, s)|$ is uniformly bounded on $I_i \times \{0 \leq l \leq \ell(\xi)\}$. Consider the function $G_i(\xi, l)$ defined by

$$G_i(\xi, l) = \int_l^{\ell(\xi)} |\nabla g_i(\xi, s)| ds$$

so that $G_i \geq 0$ and $G_i(\xi, \ell(\xi)) = 0$. Since ℓ is a continuous function of $\xi \in I_i$, G_i is a continuous function of its variables (ξ, l). Set the function β_i to be

$$\beta_i(\xi, l) = \frac{G_i(\xi, l)}{|\nabla g_i(\xi, l)|}$$

so that:

(i) $\beta_i(\xi, l)$ is continuous for ξ in $\mathrm{int}(I_i)$ and $0 \le l \le \ell(\xi)$, $\beta_i(\xi, \ell(\xi)) = 0$ and $\beta_i(\xi, 0) \le \ell(\xi)$. These last assertions follow by actually computing the map $\beta_i(\xi, l)$ using the expression found for $|\nabla g_i|$: one obtains

$$\beta_i(\xi, l) = \begin{cases} \ell(\xi) - l, & \text{when } \phi''(\xi) = 0 \\ \dfrac{1}{2}(\ell(\xi) - l)\dfrac{(R(\xi) - l) + (R(\xi) - \ell(\xi))}{R(\xi) - l}, & \text{when } \phi''(\xi) \neq 0. \end{cases}$$

From the above expression one can see that the derivative with respect to l of $\beta_i(\xi, l)$ exists and a small computation shows that it is negative: β_i achieves its maximum at $l = 0$. We have: $\beta_i(\xi, \ell(\xi)) = 0$; $\beta_i(\xi, l) \le \ell(\xi) - l$ and $\beta_i(\xi, 0) = \ell(\xi)$ when $\phi''(\xi) = 0$ and $\beta_i(\xi, 0) = \ell(\xi) - (\ell(\xi))^2/2R(\xi)$ when $\phi''(\xi) \neq 0$. In either case, $\beta_i(\xi, 0) \le \ell(\xi)$.

(ii) For every ξ and l, we have $\beta_i(\xi, l)|\nabla g_i(\xi, l)| - G_i(\xi, l) = 0$.

c) Having defined $\beta_i(\xi, l)$, define α_i on S_i^0 by setting

$$\alpha_i(x) = \beta_i(g_i^{-1}(x))$$

The map α_i is continuous on the open (relative to $\overline{\Omega}$) set O_i^0. By our previous claim, the set $\Omega \backslash \cup O_i^0$ has measure zero. The map α, defined to be α_i on O_i^0 and 0 elsewhere, is measurable, non negative and uniformly bounded.

d) Let η be any function in $C_0^\infty(\Omega)$ and let us compute

$$I = \int_\Omega \left(\alpha(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta(x) \right\rangle + \eta(x) \right) dx$$

Since the integrand is in $L^\infty(\Omega)$, by our claim on O_i^ε we have also

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \sum_i \int_{S_i^\varepsilon} \left(\alpha_i(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta \right\rangle + \eta \right) dx \\ &= \lim_{\varepsilon \to 0} \sum_i \int_{S_i^\varepsilon} \left(\alpha_i(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta \right\rangle + \eta \right) \frac{1}{\|\nabla f_i\|} \|\nabla f_i\| \, dx \end{aligned}$$

On O_i^ε, f_i is a Lipschitzean map with values in $\Re$; by the coarea formula ([7], p. 117) we have

$$\begin{aligned} &\int_{S_i^\varepsilon} \left(\alpha_i(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta \right\rangle + \eta \right) \frac{1}{\|\nabla f_i\|} \|\nabla f_i\| \, dx \\ &= \int_{I_i} \left(\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \left(\alpha_i(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta \right\rangle + \eta \right) \right) \frac{1}{\|\nabla f_i\|} \, dH \right) d\xi \end{aligned}$$

where H is the one-dimensional Hausdorff measure.

The set $f_i^{-1}(\xi)$ is the segment described by

$$\xi_1 = \xi + \frac{-\phi_i'(\xi)}{\sqrt{1+(\phi_i')^2}}\, l$$

$$\xi_2 = \phi_i(\xi) + \frac{1}{\sqrt{1+(\phi_i')^2}}\, l.$$

for $0 \leq l < \ell(\xi)$. On it the Hausdorff measure coincides with the Lebesgue measure. We have:

$$\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \frac{1}{\|\nabla f_i\|}\, dH = \int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \eta |\nabla g_i|\, dH$$

$$= \int_0^{\ell(\xi)-\varepsilon} \eta((\xi, \phi_i(\xi)) + l\mathbf{n}(\xi)) |\nabla g_i(\xi, l)|\, dl.$$

By integrating by parts, since $|\nabla g_i(\xi, l)| = -\frac{d}{dl} G_i(\xi, l)$ and $\frac{d}{dl}\eta((\xi, \phi_i(\xi)) + l\mathbf{n}(\xi)) = \langle \mathbf{n}, \nabla \eta \rangle$, we have

$$\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \frac{1}{\|\nabla f_i\|}\, dH = -\eta G_i|_0^{\ell(\xi)-\varepsilon} + \int_0^{\ell(\xi)-\varepsilon} \langle \mathbf{n}, \nabla \eta \rangle G_i\, dl.$$

Since $\eta|_{\partial\Omega} = 0$,

$$\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \frac{1}{\|\nabla f_i\|}\, dH = -\, \eta((\xi, \phi_i(\xi)) + (\ell(\xi) - \varepsilon)\mathbf{n}(\xi)) G_i(\xi, \ell(\xi) - \varepsilon)$$
$$+ \int_0^{\ell(\xi)-\varepsilon} \langle \mathbf{n}, \nabla \eta \rangle G_i\, dl.$$

Then

$$\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \left(\alpha_i(x) \left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla \eta \right\rangle + \eta \right) \frac{1}{\|\nabla f_i\|}\, dH$$
$$= \int_0^{\ell(\xi)-\varepsilon} \langle \mathbf{n}, \nabla \eta \rangle \{-\alpha_i |\nabla g_i| + G_i\}\, dl - \eta((\xi, \phi_i(\xi))$$
$$+ (\ell(\xi) - \varepsilon)\mathbf{n}(\xi) G_i(\xi, \ell(\xi) - \varepsilon)$$

where $\mathbf{n} = \mathbf{n}(\xi)$, $|\nabla g_i| = |\nabla g_i(\xi, l)|$ and the functions $\nabla \eta$ and α_i appearing inside the integral are computed along $\{((\xi, \phi_i(\xi)) + l\mathbf{n}(\xi)) : 0 \leq l \leq \ell(\xi) - \varepsilon\}$. At these points the function α_i equals $\beta_i(\xi, l)$.

By point i) of b) above, $\alpha_i \leq \ell$ hence, by Lemma 2, a), $\alpha_i \leq W_\Omega$ and, by the assumption of the Theorem, $\alpha_i \leq \Lambda$. Moreover, since by point ii) of b) we have that $\{-\beta_i|\nabla g_i| + G_i\} \equiv 0$, we obtain

$$\int_{S_i^\varepsilon \cap f_i^{-1}(\xi)} \left(\alpha_i(x)\left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta\right\rangle + \eta\right) \frac{1}{\|\nabla f_i\|}\, dH$$
$$= \eta((\xi, \phi_i(\xi)) + (\ell(\xi) - \varepsilon)\mathbf{n}(\xi))\; G_i(\xi, \ell(\xi) - \varepsilon)$$

Hence

$$I = \lim_{\varepsilon \to 0} \sum_i \int_{I_i} \eta((\xi, \phi_i(\xi)) + (\ell(\xi) - \varepsilon)\mathbf{n}(\xi))\; G_i(\xi, \ell(\xi) - \varepsilon)\, d\xi$$

Each integrand is a continuous function uniformly converging to 0 so that

$$\int_\Omega \left(\alpha(x)\left\langle \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta\right\rangle + \eta\right) dx = 0$$

The map α satisfies all the requirements of a). This completes the proof.

EXAMPLE 1. – Let h be the indicator function of any closed and bounded set. Then $\Lambda = \infty$ and u is a solution on any bounded convex set Ω, with a piecewise smooth boundary.

Next example shows that the condition expressed in terms of W_Ω cannot be improved.

EXAMPLE 2. – Consider the function h defined by $h(r) = r$, for $0 \leq r \leq 1$ and $h(r) = \infty$ for $r > 1$. Then $\rho = 0$ and $\Lambda = 1$. Hence problem P consists in minimizing a convex coercitive functional on $W_0^{1,1}$ and admits a solution v. By our previous theorem, $u \equiv 0$ is a solution whenever $W_\Omega \leq 1$. Let us show that, given any positive ε, there are convex sets Ω with $W_\Omega = 1 + \varepsilon$ such that a solution must have its gradient different from zero on a set of positive measure.

Consider the rectangle $R_{\varepsilon,\Lambda}$ with sides of length $2(1+\varepsilon)$ and $2(1+\varepsilon)+\Lambda$ and a second concentric rectangle $R_{0,\Lambda}$ with sides 2 and $2 + \Lambda$. For $\Omega = R_{\varepsilon,\Lambda}$, $W_\Omega = 1 + \varepsilon$, independent of Λ. The value of the functional computed along the map $u_0 \equiv 0$ is zero. Consider u_1, where u_1 is negative with gradient in norm $= 1$, and orthogonal to the sides, on the strip difference of the rectangle $R_{\varepsilon,\Lambda}$ and the rectangle $R_{0,\Lambda}$, and gradient 0 on $R_{0,\Lambda}$. Computing the functional along u_1 we have the value $(8+2\Lambda)\varepsilon + (4\varepsilon^2) - [(4+2\Lambda)\varepsilon + (4+\Lambda)\varepsilon^2 + (4/3)\varepsilon^3] = 4\varepsilon - \Lambda\varepsilon^2 - (4/3)\varepsilon^3$ and, for Λ large, this value is negative. Hence u_0 is not a solution.

Remark. – The condition appearing in the preceding Theorem is expressed in terms of W_Ω, a quantity easily computed. For the validity of the result, however, the following condition is actually sufficient, as one can see from property (ii) of b) in the Proof of Theorem 1:

THEOREM 2. – *Under the same conditions on h and Ω assume that, for a.e. x in $\partial\Omega$, $\ell(x) \leq \Lambda$ when $R(x) = \infty$, and $\ell(x) - (\ell(x))^2/2R(x) \leq \Lambda$ when $R(x) < \infty$. Then problem P admits the solution $u(x) = -\rho\ d(x, \partial\Omega)$.*

Next Example shows that this second condition cannot be improved.

EXAMPLE 3. – Consider the case where Ω is a disk of radius R. Then, for every x in $\partial\Omega$ we have $\ell(x) = R$, and $\ell(x) - (\ell(x))^2/2R(x) = R/2$. So the condition becomes: $R/2 \leq \Lambda$. Let h be as in Example 2 and set Ω to be $B_{2+2\varepsilon}$, a disk of radius $2 + 2\varepsilon$. Since $R/2 = 1 + \varepsilon$ and $\Lambda = 1$, the above condition is violated. Again for $u_0 \equiv 0$ the value of the functional is 0. Consider a concentric disk B_2 of radius 2 and let u_1 be such that the gradient is in norm 1 on the annulus from $r = 2 + \varepsilon$ to $r = 2$ and 0 otherwise. Computing the value of the functional one obtains $-\frac{1}{3}\pi 8[3\varepsilon + 3\varepsilon^2 + \varepsilon^3] + 4\pi[2\varepsilon + \varepsilon^3]$, a negative number. Hence u_0 is not a solution.

It is obvious that the examples above refer to our function u not being a solution. That other solutions might exist is not a problem easily solvable.

REFERENCES

[1] A. CELLINA, On minima of a functional of the gradient: sufficient conditions, *Nonlinear Analysis, TMA*, Vol. **20**, 1993, pp. 343-347.

[2] A. CELLINA and S. PERROTTA, On minima of radially symmetric functionals of the gradient, *Nonlinear Analysis, TMA*, Vol. **23**, 1994, pp. 239-249.

[3] A. CELLINA and S. PERROTTA, On a problem of Potential Wells, *Journal of Convex Anal.z*, Vol. **2**, 1995, pp. 103-116.

[4] A. CELLINA and S. ZAGATTI, A version of Olech's Lemma in a Problem of the Calculus of Variations, *SIAM J. Control and Optimization*, Vol. **32**, 1994, pp. 1114-1127.

[5] A. CELLINA and S. ZAGATTI, An Existence Result in a Problem of the Vectorial Case of the Calculus of Variations, *SIAM J. Control and Optimization*, Vol. **333**, 1995.

[6] B. DACOROGNA and P. MARCELLINI, Existence of Minimizers for non quasiconvex integrals, *Preprint Ecole Polytechnique Federale Lausanne*, June 1994, to appear in: *Arch. Rat. Mech. Anal.*

[7] L. C. EVANS and R. F. GARIEPY, Measure Theory and Fine Properties of Functions, *CRC Press, Boca Raton*, 1992.

[8] G. FRIESECKE, A necessary and sufficient condition for nonattainement and formation of microstructure almost everywhere in scalar variational problems, *Proc. Royal Soc. Edinburgh*, Vol. **124**, 1984, pp. 437-471.

[9] D. GILBARG and N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1977.

[10] J. GOODMAN, V. R. KOHN and L. REYNA, Numerical study of a relaxed variational problem from optimal design, *Computer Methods in Appl. Math. and Eng.*, Vol. **57**, 1986, pp. 107-127.

[11] B. KAWOHL, J. STARA and G. WITTUM, Analysis and Numerical Studies of a Problem of Shape Design, *Arch. Rat. Mech. Anal.*, Vol. **114**, 1991, pp. 349-363.

[12] R. V. KOHN and G. STRANG, Optimal Design and Relaxation of Variational Problems I, *Comm. Pure Appl. Math.*, Vol. **39**, 1986, pp. 113-137.

[13] E. MASCOLO and R. SCHIANCHI, Existence Theorems for Nonconvex Problems, *J. Math. Pures Appl.*, Vol. **62**, 1983, pp. 349-359.

[14] F. MURAT and L. TARTAR, Calcul des variations et Homegenization, in: Les Méthodes de l'homogenization, Eds: D. Bergman *et al.*, Collection de la Direction des Études et recherche de l'Électricité de France, Vol. **57**, 1985, pp. 319-369.

[15] R. TAHRAOUI, Sur une classe de fonctionnelles non convexes et applications, *SIAM J. Math. Anal.*, Vol. **21**, 1990, pp. 37-52.

[16] R. TAHRAOUI, Théoremes d'existence en calcul des variations et applications a l'élasticité non lineaire, *Proc. Royal Soc. Edinburgh*, Vol. **109 A**, 1988, pp. 51-78.

(Manuscript received May 15, 1995;
Revised October 6, 1995.)