SERGIO PLAZA

JAIME VERA

Bifurcation and stability of families of hyperbolic vector fields in dimension three

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of hyperbolic vector fields in dimension three

by

Sergio PLAZA *

Depto. de Matemática, Universidad de Santiago de Chile.
Casilla 307, Correo 2, Santiago, Chile.
E-mail: splaza@fermat.usach.cl

and

Jaime VERA **

Depto. de Matemáticas, Universidad Católica del Norte,
Avda. Angamos 0610, Antofagasta, Chile.
E-mail: jvera@socompa.cecun.ucn.cl

ABSTRACT. – In this article we study the global stability of one-parameter families of hyperbolic vector fields with simple bifurcations in three-dimensional manifolds at least in all known cases (*see* introduction).

RÉSUMÉ. – L'objet de ce travail est d'étudier la stabilité globale des familles à un paramètre de champs de vecteurs hyperboliques avec bifurcations simples dans les variétés de dimension trois. Cette étude est faite au moins dans les cas connus (voir le préliminaire).

1. INTRODUCTION

We first recall that the global stability (*see* definition in Section 2) of one-parameter families of vector fields $\{X_\mu\}$, $\mu \in [0, 1]$ with simple

recurrences (that is, for each parameter value, μ, the nonwandering set of X_μ is constituted solely by a finite number of critical elements, *i.e.*, singularities and periodic orbits) was studied by Palis and Takens in [P-T] for gradient vector fields and by Labarca in [La] for many cases of quasi-transversal intersection orbits and the remaining cases were completed by himself and Plaza in [La-Pℓ]. The study of the stability or not of the bifurcation due to a saddle–node periodic orbit was done by Plaza in [Pℓ] and in all remaining cases the global stability or not follows from the articles of Palis and Takens [P-T] and Newhouse, Palis and Takens [N-P-T]. For families of vector fields on three–dimensional manifolds which are *hyperbolic*, say for $\mu < \overline{\mu}$, and for $\mu = \overline{\mu}$ there exists an orbit of quasi-transversal intersection between an unstable and a stable manifolds, the global stability was studied by Vera in [Ve]. It is to be noted that for higher number of parameters, the question is quite open, except for the case of two-parameter families of gradient vector fields which was solved by Carneiro and Palis in [C-P].

In this article we study the global stability of one-parameter families of vector fields $\{X_\mu\}$ that are *hyperbolic*, say for $\mu < \overline{\mu}$, and at $\mu = \overline{\mu}$ the vector field $X_{\overline{\mu}}$ has a simple bifurcation that unfolds generically, which will be (i) a saddle-node that is, singularity or periodic orbit, (ii) a flip periodic orbit, and (iii) a Hopf singularity. Adding this result to those mentioned above we have a complete study of the global stability of one-parameter families of vector fields in dimension three, modulus a Conjecture of Palis and Newhouse (*see* [N-P]).

In order to obtain the result we must impose some mild nondegenerated conditions which we explain in the following section.

2. BASIC CONCEPTS

In what follows, M will denote a C^∞ compact boundaryless n-dimensional manifold. In the following sections we will impose that $n = 3$.

Let $\chi^\infty(M)$ and $\chi_1^\infty(M)$ denote, respectively, the spaces of C^∞ vector fields and C^∞ arcs $\xi : I = [-1, 1] \to \chi^\infty(M)$, both endowed with the C^∞ Whitney topology. If $\xi \in \chi_1^\infty(M)$, we let $\xi = \{X_\mu\}$ where $X_\mu = \xi(\mu)$, for each $\mu \in I$.

First we recall some concepts and results on hyperbolic vector fields. Let $X \in \chi^\infty(M)$, we denote by X_t its flow. The nonwandering set $\Omega(X)$ of X is the set of $x \in M$ for which each neighborhood U of x and

each $T > 0$, satisfies $U \cap \left(\cup_{|t|>T} X_t(U) \right) \neq \emptyset$. Note that the periodic orbits and singularities, which in the sequel will be called critical elements of X, belong to $\Omega(X)$ and that $\Omega(X)$ is a closed X-invariant set, *i.e.*, $X_t(\Omega(X)) = \Omega(X)$ for all $t \in \mathbb{R}$.

A closed X-invariant set Γ is a *hyperbolic set* for X if the tangent bundle of M restricted to Γ, $T_\Gamma M$, can be written as a Whitney sum of three DX_t-invariant subbundles, $T_\Gamma M = E^s \oplus E^c \oplus E^u$, and there are constants $c > 0$, $\lambda > 0$ such that for each $x \in \Gamma$, we have:

a) E^c is the one-dimensional bundle tangent to the flow of X;

b) $\| DX_t(x)v \| \leq c \cdot e^{-\lambda t} \| v \|$, $v \in E^s_x$ and $t \geq 0$;

c) $\| DX_{-t}(x)w \| \leq c \cdot e^{-\lambda t} \| w \|$, $w \in E^u_x$ and $t \geq 0$,

where, respectively, E^s_x and E^u_x is the fiber of E^s and E^u over x.

Let F be the set of singulaties of X (*i.e.*, $x \in F$ if and only if $X(x) = 0$) and let Λ be the closure of the set of periodic orbits. We say X satisfies the *Axiom A* if, (i) $\Omega(X) = F \cup \Lambda$, (ii) F is finite and hyperbolic; in this case hyperbolicity yields $E^c = \{0\}$ and it is equivalent to that for each $x \in F$, $DX(x)$ has no eigenvalues with null real part, (iii) Λ is hyperbolic, as well (iv) $F \cap \Lambda = \emptyset$.

It is well known that if X is Axiom A, then there exists a spectral decomposition of $\Omega = \Omega(X)$, *i.e.*, it can be writen as a disjoint union, $\Omega = \Omega_1 \cup \cdots \cup \Omega_\ell$, where each Ω_i is a closed X-invariant set which contains a dense orbit of X (*i.e.*, the flow $X_t|_{\Omega_i}$ is topologically transitive) (*see* [Sm]). The sets Ω_i used to be called *basic sets* of X.

In what follows we will say that Ω_i is *nontrivial* if it is not a singularity or an isolated periodic orbit. Let $x \in M$, we define, respectively, the stable and the unstable set of x as $\mathcal{W}^s(x) = \{y \in M \ : \ d(X_t(x), X_t(y)) \to 0,\ t \to \infty\}$ and $\mathcal{W}^u(x) = \{y \in M \ : \ d(X_t(x), X_t(y)) \to 0, \ t \to -\infty\}$, where d is a distance function on M induced by a riemannian metric. Now let $\gamma = \cup_{t \in \mathbb{R}} X_t(x)$ be the orbit of x. We define, respectively, the stable and the unstable set of γ as $\mathcal{W}^s(\gamma) = \cup_{y \in \gamma} \mathcal{W}^s(y)$ and $\mathcal{W}^u(\gamma) = \cup_{y \in \gamma} \mathcal{W}^u(y)$. Note that $\mathcal{W}^s(\gamma)$ and $\mathcal{W}^u(\gamma)$ are invariant by the flow of X. It is well known that if γ is a hyperbolic critical element of X or if X satifies the Axiom A and γ is the orbit of a point $x \in \Omega(X)$, then $\mathcal{W}^s(x)$, $\mathcal{W}^u(x)$, $\mathcal{W}^s(\gamma)$ and $\mathcal{W}^u(\gamma)$ are C^∞ injectively immersed submanifolds of M. Let $B_\varepsilon(x)$ denote the disc of radius ε centered at x in M then the connected component of $\mathcal{W}^s(x) \cap B_\varepsilon(x)$ that contains x, $\mathcal{W}^s_\varepsilon(x)$, and the connected component of $\mathcal{W}^u(x) \cap B_\varepsilon(x)$ that contains x, $\mathcal{W}^u_\varepsilon(x)$, depend differentiably on x. Furthermore, $T_x \mathcal{W}^s_\varepsilon(x) = E^s_x$ and $T_x \mathcal{W}^u_\varepsilon(x) = E^u_x$.

Let X be an Axiom A vector field, we say that X satisfies the *strong transversality condition* if for each orbit $\gamma \subset M$, $\mathcal{W}^s(\gamma)$ transversally intersects $\mathcal{W}^u(\gamma)$.

We say that $X \in \chi^\infty(M)$ is *hyperbolic* if it satifies the Axiom A and the strong transversality condition. It is well known that if X is hyperbolic then it is structurally stable (*see* [Ro], [Rob]), that is, there is a neighborhood $\mathcal{U}$ of X in $\chi^\infty(M)$ such that each $Y \in \mathcal{U}$ is topologically equivalent to X, this last assertion means that there is a homeomorphism, called topological equivalence, $h : M \to M$ which sends orbits of X into orbits of Y preserving their orientation time. Respect to this last assertion we have the following remark: let X be a hyperbolic vector field and let Ω_i be a basic set of X, then for each $Y \in \chi^\infty(M)$, C^r–close to X, which may be chosen to be hyperbolic (the set of hyperbolic vector fields is an open set in $\chi^\infty(M)$), there is a homeomorphism (over its images) $h : \Omega_i \to M$, C^r–close to the inclusion map such that $h(\Omega_i) = \Omega_i(Y)$ is a hyperbolic basic set for Y and h is unique up to a composition of h with a homeomorphism $\eta : \Omega_i \to \Omega_i$ close to the inclusion map which leaves invariant the orbits of X restricted to Ω_i.

Now let $X \in \chi^\infty(M)$. We say that X has a *weakest contracting (resp. expanding) eigenvalue*, A, at a singularity p if: (i) A is a simple eigenvalue of $DX(p)$, (ii) $\Re(A) < 0$ (resp. $\Re(A) > 0$), and (iii) for all eigenvalue B of $DX(p)$ with $\Re(B) < 0$ (resp. $\Re(B) > 0$), $B \neq A, \overline{A}$, $\Re(B) < \Re(A)$ (resp. $\Re(B) > \Re(A)$). (Here $\Re(z)$ denotes the real part of the complex number z). Similarly, assume that X has a periodic orbit σ. Let Σ be a transversal section to X at $q \in \sigma$ and let P be the corresponding Poincaré map. The vector field X is said to have a weakest contracting (resp. expanding) eigenvalue, A, at σ if: (i) A is a simple eigenvalue of $DP(q)$, (ii) $|A| < 1$ (resp. $|A| > 1$), and (iii) for all eigenvalue B of $DP(q)$, with $|B| < 1$ (resp. $|B| > 1$), $B \neq A, \overline{A}$, $|B| < |A|$ (resp. $|B| > |A|$).

We now recall some concepts related to one-parameter families of vector fields. Let $\{X_\mu\}, \{Y_\mu\} \in \chi_1^\infty(M)$, $\overline{\mu}, \tilde{\mu} \in I$ and $\overline{\gamma}, \tilde{\gamma}$ be orbits of $X_{\overline{\mu}}$ and $Y_{\tilde{\mu}}$, respectively. We say that $\{X_\mu\}$ at $(\overline{\gamma}, \overline{\mu})$ is *locally equivalent* to $\{Y_\mu\}$ at $(\tilde{\gamma}, \tilde{\mu})$ if there are intervals $\overline{I}, \tilde{I} \subset I$, with $\overline{\mu} \in \overline{I}$, $\tilde{\mu} \in \tilde{I}$ and a neighborhood U of the closure of $\overline{\gamma}$ in M and a homeomorphism (over its images) $H : U \times \overline{I} \to M \times \tilde{I}$, $H(x, \mu) = (h_\mu(x), \rho(\mu))$, where $\rho : (\overline{I}, \overline{\mu}) \to (\tilde{I}, \tilde{\mu})$ is a reparametrization and $h_\mu : U \to h_\mu(U)$ is a topological equivalence between $X_\mu|_U$ and $Y_{\rho(\mu)}|_{h_\mu(U)}$ and $h_{\overline{\mu}}(\overline{\gamma}) = \tilde{\gamma}$. We say that $\{X_\mu\} \in \chi_1^\infty(M)$ is *locally stable* at $(\overline{\gamma}, \overline{\mu})$ or that the pair $(\overline{\gamma}, \overline{\mu})$ is *stable* if there is a neighborhood $\mathcal{U}$ of $\{X_\mu\}$ in $\chi_1^\infty(M)$ such that for each $\{Y_\mu\} \in \mathcal{U}$, there is a parameter value $\tilde{\mu}$ near $\overline{\mu}$ and an orbit $\tilde{\gamma}$ of $Y_{\tilde{\mu}}$ near $\overline{\gamma}$

such that $\{X_\mu\}$ at $(\overline{\gamma}, \overline{\mu})$ is equivalent to $\{Y_\mu\}$ at $(\tilde{\gamma}, \tilde{\mu})$. Furthermore, we say that $\{X_\mu\} \in \chi_1^\infty(M)$ is stable at $\overline{\mu} \in I$, if there exists a neighborhood $\mathcal{U}$ of $\{X_\mu\}$ in $\chi_1^\infty(M)$ such that for each $\{Y_\mu\} \in \mathcal{U}$, there is a parameter value $\tilde{\mu} \in I$ near $\overline{\mu}$ and a homeomorphism $H : M \times \overline{I} \to M \times \tilde{I}$ where $\overline{I}$, respectively $\tilde{I}$, is a neighborhood of $\overline{\mu}$, respectively of $\tilde{\mu}$, in I and $H(x, \mu) = (h_\mu(x), \rho(\mu))$, with $\rho : (\overline{I}, \overline{\mu}) \to (\tilde{I}, \tilde{\mu})$ a reparametrization and $h_\mu : M \to M$ is a topological equivalence between X_μ and $Y_{\rho(\mu)}$, and the map $I \to Homeo(M, M)$, $\mu \to h_\mu$ is continuous.

Let now Ω_i and Ω_j be basic sets of the vector field X we put $\Omega_i \preceq \Omega_j$ if $(\mathcal{W}^u(\Omega_i) - \Omega_i) \cap (\mathcal{W}^s(\Omega_j) - \Omega_j) \neq \emptyset$ and we say that X has no cycles among its basic sets if there are no sequences $\Omega_{i_1}, \cdots, \Omega_{i_k}$, $k \geq 1$ of basic sets such that $\Omega_{i_k} = \Omega_{i_1}$ and $\Omega_{i_j} \preceq \Omega_{i_{j+1}}$, $j = 1, \cdots, k-1$. It is well known that if X is Axiom A and there are no cycles among its basic sets, then the partial order "$\preceq$" can be extended to a total order, "$\preceq$", so that $\Omega_i \preceq \Omega_j$ if and only if $i \leq j$.

In what follows we will consider one-parameter families of vector fields, $\{X_\mu\} \in \chi_1^\infty(M)$, for which there exists a first bifurcation value $\overline{\mu} \in I$, that is, $X_{\overline{\mu}}$ is non structurally stable and for each $\mu < \overline{\mu}$, X_μ is hyperbolic, and for each $\mu \in I$, X_μ has no cycles among its basic sets. Furthermore, we suppose that for each $\mu < \overline{\mu}$, the basic sets of X_μ are ordered as follows:

$$\Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i,\mu} \preceq \Omega_{i+1,\mu} \preceq \cdots \preceq \Omega_{n,\mu} ,$$

and for $\mu = \overline{\mu}$ the vector field $X_{\overline{\mu}}$ has one and only one orbit $\overline{\gamma}$ along which it is non locally stable. In this article we will only consider the cases in which $\overline{\gamma}$ is an orbit of the following type:

(1) an isolated saddle-node singularity,

(2) an isolated Hopf singularity,

(3) an isolated flip periodic orbit,

(4) an isolated saddle-node periodic orbit,

(5) a flip periodic orbit arising from two hyperbolic periodic orbits inside a basic set (which may be an attractor, a repellor or of saddle type).

Remark. – If the vector field $X_{\overline{\mu}}$ has a saddle-node periodic orbit inside a basic set, then $\{X_\mu\}$ is non stable at $\overline{\mu}$; this follows from [Ma-P] or [Pℓ].

3. RESULTS

In what follows we will suppose $\dim M = 3$. In this paragraph we will specify in a more precise way the set of one-parameter families of vector

fields that we will study. For this we specify some generic conditions (the nondegenerated mild conditions mentioned in the Introduction).

Let $\Gamma_1 \subset \chi_1^\infty(M)$ be the set characterized as follows: $\{X_\mu\} \in \Gamma_1$ if and only if

1) there exists a parameter value $\overline{\mu} \in I$, so that for $\mu < \overline{\mu}$, X_μ is a hyperbolic vector field, and for each $\mu \in I$, X_μ has no cycles among its basic sets and the order above holds.

2) for $\mu = \overline{\mu}$, the vector field $X_{\overline{\mu}}$ has one and only one basic set, say $\Omega_{i,\overline{\mu}}$, which is nonhyperbolic and may be of the following type:

(i) if $\Omega_{i,\overline{\mu}}$ is a singularity, then it is an isolated saddle-node or a Hopf singularity, which unfolds generically at $\overline{\mu}$ (for definition *see* [P-T] or [La-Pℓ]);

(ii) if $\Omega_{i,\overline{\mu}}$ is a periodic orbit, then it is an isolated saddle-node or an isolated flip periodic orbit, which unfolds generically at $\overline{\mu}$;

(iii) if $\Omega_{i,\overline{\mu}}$ is a nontrivial basic set, then there exists a unique non hyperbolic periodic orbit γ contained in $\Omega_{i,\overline{\mu}}$ which is a flip periodic orbit unfolding generically at $\overline{\mu}$.

Let now $\Gamma_2 \subset \Gamma_1$ be the set characterized as follows: $\{X_\mu\} \in \Gamma_2$ if and only if at the first bifurcation value $\mu = \overline{\mu}$ of $\{X_\mu\}$ the vector field $X = X_{\overline{\mu}}$, depending on the case, satisfies the following generic conditions:

1) **saddle-node singularity.** Let $\Omega_{i,\overline{\mu}} = \sigma$ be the saddle-node singularity of X. In this case, it is well known that $\mathcal{W}^s(\sigma) = \{y \in M : d(X_t(y), X_t(x)) \to 0, \ t \to \infty\}$ is a C^∞ injectively immersed submanifold with boundary, $\partial \mathcal{W}^s(\sigma) = \mathcal{W}^{ss}(\sigma)$, called the strong stable manifold, and characterized as follows $\mathcal{W}^{ss}(\sigma) = \{y \in M : d(X_t(x), X_t(y)) \leq c \cdot e^{-\lambda t} \cdot d(x, y), \ t \geq 0\}$, where c and λ are positive constants. Analogously, for $\mathcal{W}^u(\sigma) = \{y \in M : d(X_{-t}(x), X_{-t}(y)) \to 0, \ t \to \infty\}$, its boundary $\partial \mathcal{W}^u(\sigma) = \mathcal{W}^{uu}(\sigma) = \{y \in M : d(X_{-t}(x), X_{-t}(y)) \leq c \cdot e^{-\lambda t} \cdot d(x, y), \ t \geq 0\}$ is called the strong unstable manifold. In this case, we impose the following generic condition: for each basic set, $\Omega_{j,\overline{\mu}}$, $j \neq i$, $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$ is transversal to $\mathcal{W}^{ss}(\sigma)$ and to $\mathcal{W}^{cs}(\sigma)$, and $\mathcal{W}^s(\Omega_{j,\overline{\mu}})$ is transversal to $\mathcal{W}^{uu}(\sigma)$ and to $\mathcal{W}^{cu}(\sigma)$, where $\mathcal{W}^{cs}(\sigma)$, respectively $\mathcal{W}^{cu}(\sigma)$, denote the classical center–stable, respectively center-unstable, manifold of σ.

2) **Hopf singularity.** Let $\Omega_{i,\overline{\mu}} = \sigma$, be the Hopf singularity of X. We suppose that for $\mu < \overline{\mu}$, $\dim \mathcal{W}^s(\sigma_\mu) = 2$. At the parameter value $\mu = \overline{\mu}$, we impose the following generic condition: for each basic set $\Omega_{j,\overline{\mu}}$, $j \neq i$, $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$, and $\mathcal{W}^s(\Omega_{j,\overline{\mu}})$ are transversal, respectively, to $\mathcal{W}^s(\sigma)$ and to $\mathcal{W}^u(\sigma)$. Analogously, if $\dim \mathcal{W}^u(\sigma_\mu) = 2$ for $\mu < \overline{\mu}$.

3) **isolated flip periodic orbit.** Let $\Omega_{i,\overline{\mu}} = \sigma$ be the isolated flip periodic orbit of X. It is well known that in this case, there exist the strong stable and the strong unstable manifold for the Poincaré map, $P_{\overline{\mu}}$, associated to the flip periodic orbit σ, denoted respectively by $\mathcal{W}^{ss}(q)$ and by $\mathcal{W}^{uu}(q)$, $q \in \sigma$, these manifolds are $P_{\overline{\mu}}$-invariant. In this case we impose the following generic condition: for each basic set $\Omega_{j,\overline{\mu}}$, $j \neq i$, $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$ and $\mathcal{W}^s(\Omega_{j,\overline{\mu}})$ are transversal, respectively, to $\mathcal{W}^{ss}(q)$ and to $\mathcal{W}^{uu}(q)$.

4) **isolated saddle-node periodic orbit.** Let $\Omega_{i,\overline{\mu}} = \sigma$ be the isolated saddle-node periodic orbit of the vector field X. We let $\Sigma_q \subset M$ be a transversal section to the flow of X at a point $q \in \sigma$, so that $\Sigma = \cup_{\mu \in I} \Sigma_q \times \{\mu\}$ is a transversal section to the vector field $X(x,\mu) = (X_\mu(x), 0)$ defined on $M \times I_1$, where $I_1 \subset I$ is a small neighborhood of $\overline{\mu}$. Then the Poincaré map $P = \{P_\mu\}$ of X is an arc of saddle-node diffeomorphisms and in this case at the bifurcation value, $\mu = \overline{\mu}$, there are the following rigidity conditions: (i) in a center manifold of q, $\mathcal{W}^c(q)$, there exists a unique C^∞ vector field, Z, such that $P_\mu|_{(\mathcal{W}^c(q) \cap \Sigma_q \times \{\overline{\mu}\})} = Z_{t=1}$, and if we let $h_\mu : \mathcal{W}^c(q) \to \mathcal{W}^c(\tilde{q})$ be a conjugacy between the corresponding Poincaré maps of two nearby arcs having a bifurcation due to a saddle-node periodic orbit, then $h_{\overline{\mu}}$ is a conjugacy between the corresponding C^∞ vector fields Z and $\tilde{Z}$; (ii) a conjugacy between the Poincaré maps associated to two nearby one-parameter families $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$ of vector fields bifurcating through saddle-node periodic orbits, say σ and $\tilde{\sigma}$, respectively, must send leaves of the strong stable (resp. of the strong unstable) foliation of $X_{\overline{\mu}}$ into the corresponding ones of $\tilde{X}_{\tilde{\mu}}$. We say that a basic set $\Omega_{j,\overline{\mu}}$ of $X_{\overline{\mu}}$ is s-critical (resp. u-critical) if there exists a tangency orbit between $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$ and the strong stable foliation, $\mathcal{F}^{ss}$ (resp. between $\mathcal{W}^s(\Omega_{j,\overline{\mu}})$ and the strong unstable foliation $\mathcal{F}^{uu}$). From [Pℓ] it follows that if $\Omega_{j,\overline{\mu}}$ is a non trivial basic set or a periodic orbit or a singularity with complex weakest contraction (resp. expansion) which is s-critical (resp. u-critical), then $\{X_\mu\}$ is non stable at $\overline{\mu}$. Taking the latter into account, we impose the following generic condition: *If there exists an s-critical (resp. u-critical) basic set, $\Omega_{j,\overline{\mu}}$, then it is a singularity with real weakest contraction (resp. real weakest expansion)* . On the other hand it is easy to see that if the s-criticality (resp. u-criticality) is non generic, *i.e.*, the contact between $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$ and $\mathcal{F}^{ss}$ (resp. between $\mathcal{W}^s(\Omega_{j,\overline{\mu}})$ and $\mathcal{F}^{uu}$) along the orbit of s-criticality (resp. u-criticality) is non quadratic, then $\{X_\mu\}$ is non stable at $\overline{\mu}$ (*see* [Pℓ] or [Ma-P]), analogously if there are two or more s-criticalities (resp. u-criticalities), then $\{X_\mu\}$ is non stable at $\overline{\mu}$ (*see* [Pℓ] or [Ma-P]). Therefore we impose the following generic condition: *If there exists an orbit of s-criticality (resp. u-criticality), then it*

is the unique orbit of tangency between $\mathcal{W}^u(\Omega_{j,\bar{\mu}})$ *and* $\mathcal{F}^{ss}$ *(resp. between* $\mathcal{W}^s(\Omega_{j,\bar{\mu}})$ *and* $\mathcal{F}^{uu}$*), it is generic, and* $\Omega_{j,\bar{\mu}}$ *is a singularity with real weakest contraction (resp. real weakest expansion).* This condition is not sufficient for the stability of our family, thus we furthermore impose the following condition: *Let* x *the unique point of tangency between* $\mathcal{W}^u(\Omega_{j,\bar{\mu}})$ *(resp. between* $\mathcal{W}^s(\Omega_{j,\bar{\mu}})$*) and* $\mathcal{F}^{ss}$ *(resp. and* $\mathcal{F}^{uu}$*).* We suppose that a center unstable (resp. center stable) manifold, $\mathcal{W}^{cu}(\Omega_{j,\bar{\mu}})$, (resp. $\mathcal{W}^{cs}(\Omega_{j,\bar{\mu}})$) of $\Omega_{j,\bar{\mu}}$ is transversal to $\mathcal{F}^{ss}$ (resp. to $\mathcal{F}^{uu}$) in a neighborhood of x in Σ_q. (*See* Figure 1.)
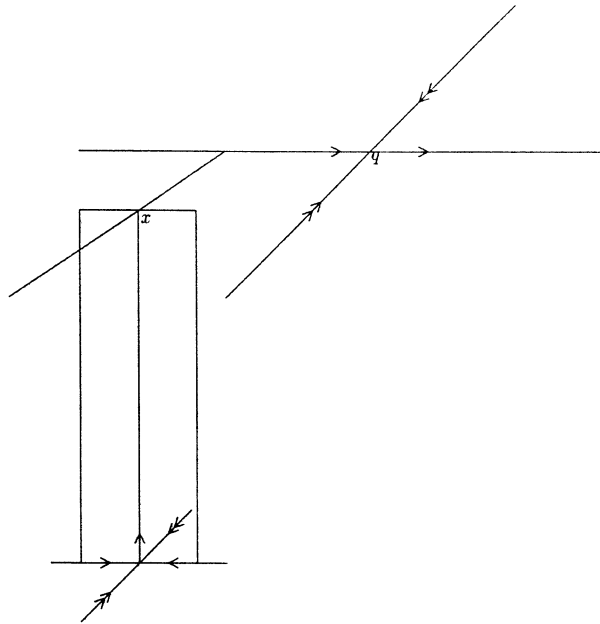

Fig. 1.

Let $\{X_\mu\} \in \Gamma_2$ and let $\bar{\mu}$ its first bifurcation value. Since, for $\mu < \bar{\mu}$, X_μ is hyperbolic, we have an order:

$$\Omega_{1,\mu} \preceq \Omega_{2,\mu} \preceq \cdots \preceq \Omega_{i,\mu} \preceq \Omega_{i+1,\mu} \preceq \cdots \preceq \Omega_{n,\mu}$$

between the basic sets of X_μ. At the parameter value $\mu = \bar{\mu}$, we have a simple bifurcation which we will study in the following order:

(I) Isolated saddle-node singularity;

(II) Isolated Hopf singularity;

(III) Isolated flip periodic orbit;

(IV) Isolated saddle-node periodic orbit;

(V) Flip periodic orbit arising from a hyperbolic periodic orbit inside a basic set.

Note that in the cases (I), (II), (III) and (IV) the bifurcation occurs outside the nontrivial basic sets of $X_{\bar{\mu}}$. In each case we will assume that we have the following order among their basic sets:

$$(\text{I}') \quad \begin{cases} \mu < \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i-1,\mu} \preceq \Omega_{i,\mu} \preceq \Omega_{i+1,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \\ \mu = \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \Omega_{i-1,\bar{\mu}} \preceq \overline{\Omega}_{i,\bar{\mu}} \preceq \Omega_{i+2,\bar{\mu}} \preceq \cdots \preceq \Omega_{n,\bar{\mu}} \\ \mu > \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i-1,\mu} \preceq \Omega_{i+2,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \end{cases}$$

i.e., for $\mu < \bar{\mu}$ there are two hyperbolic singularities $\Omega_{i,\mu} = s_{i,\mu}$ and $\Omega_{i+1,\mu} = s_{i+1,\mu}$ collapsing at $\mu = \bar{\mu}$, creating the saddle–node singularity $\overline{\Omega}_{i,\mu} = s$ that disappears for $\mu > \bar{\mu}$.

$$(\text{II}') \quad \begin{cases} \mu < \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \\ \mu = \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \overline{\Omega}_{i,\bar{\mu}} \preceq \cdots \preceq \Omega_{n,\bar{\mu}} \\ \mu > \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \overline{\Omega}_{i,\mu} \preceq \overline{\overline{\Omega}}_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \end{cases}$$

i.e., for $\mu < \overline{\mu}$, the singularity $\Omega_{i,\mu} = s_\mu$ is hyperbolic and for $\mu = \overline{\mu}$, $\overline{\Omega}_{i,\bar{\mu}}$ is non hyperbolic and $DX_{\bar{\mu}}(s_{\bar{\mu}})|_{\mathcal{W}^s(s_{\bar{\mu}})}$ has eigenvalues $\lambda = b\mu$, $\overline{\lambda} = -b\mu$, $b \neq 0$ and for $\mu > \bar{\mu}$ appear a periodic orbit $\overline{\overline{\Omega}}_{i,\mu}$ and the singularity $\overline{\Omega}_{i,\mu}$ became hyperbolic, but changes its index of stability.

$$(\text{III}') \quad \begin{cases} \mu < \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \\ \mu = \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \overline{\Omega}_{i,\bar{\mu}} \preceq \cdots \preceq \Omega_{n,\bar{\mu}} \\ \mu > \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \overline{\Omega}_{i,\mu} \preceq \overline{\overline{\Omega}}_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \end{cases}$$

i.e., for $\mu < \bar{\mu}$, $\Omega_{i,\mu}$ is a hyperbolic periodic orbit and for $\mu = \bar{\mu}$, $\overline{\Omega}_{i,\bar{\mu}}$ is a generic flip periodic orbit and for $\mu > \bar{\mu}$, $\overline{\overline{\Omega}}_{i,\mu}$ is a hyperbolic periodic orbit arising from the flip bifurcation.

$$(\text{IV}') \quad \begin{cases} \mu < \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i-1,\mu} \preceq \Omega_{i,\mu} \preceq \Omega_{i+1,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \\ \mu = \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \Omega_{i-1,\bar{\mu}} \preceq \overline{\Omega}_{i,\bar{\mu}} \preceq \Omega_{i+2,\bar{\mu}} \preceq \cdots \preceq \Omega_{n,\bar{\mu}} \\ \mu > \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \overline{\Omega}_{i-1,\mu} \preceq \Omega_{i+2,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \end{cases}$$

i.e., for $\mu < \bar{\mu}$ there are two hyperbolic periodic orbits, $\Omega_{i,\mu}$ and $\Omega_{i+1,\mu}$ collapsing at $\mu = \bar{\mu}$, creating the saddle-node periodic orbit $\overline{\Omega}_{i,\mu}$ that disappears for $\mu > \bar{\mu}$.

$$(\text{V}') \quad \begin{cases} \mu < \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \Omega_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \\ \mu = \bar{\mu}: & \Omega_{1,\bar{\mu}} \preceq \cdots \preceq \overline{\Omega}_{i,\bar{\mu}} \preceq \cdots \preceq \Omega_{n,\bar{\mu}} \\ \mu > \bar{\mu}: & \Omega_{1,\mu} \preceq \cdots \preceq \overline{\Omega}_{i,\mu} \preceq \overline{\overline{\Omega}}_{i,\mu} \preceq \cdots \preceq \Omega_{n,\mu} \end{cases}$$

i.e., for $\mu < \bar{\mu}$ in $\Omega_{i,\mu}$ there is a hyperbolic periodic orbit such that for $\mu = \bar{\mu}$, it became a generic flip periodic orbit and for $\mu > \bar{\mu}$, $\overline{\overline{\Omega}}_{i,\mu}$ is a hyperbolic basic set and $\overline{\Omega}_{i,\mu}$ is a periodic orbit arising from the flip bifurcation.

Now, let $B(\{X_\mu\}) = \{\bar{\mu} \in I : \bar{\mu}$ is a bifurcation value of $\{X_\mu\}\}$. The set $B(\{X_\mu\})$ is called the *bifurcation set* of the family $\{X_\mu\}$.

With the above notations, we have

THEOREM. – *Let $\{X_\mu\} \in \Gamma_2$ and let $\bar{\mu} \in B(\{X_\mu\})$ be its first bifurcation value. Then $\{X_\mu\}$ is stable at $\bar{\mu}$.*

Proof. – We give the proof in each case as was itemized above.

I. Isolated saddle-node singularity

Let $\{X_\mu\}$, $\{\tilde{X}_\mu\} \in \Gamma_2$ be close families, and $\bar{\mu} \in B(\{X_\mu\})$, $\tilde{\mu} \in B(\{\tilde{X}_\mu\})$ be the corresponding first bifurcation values. Each time we make a construction for the family $\{X_\mu\}$, we assume that a similar construction is made for each close family $\{Y_\mu\} \in \Gamma_2$.

Without loss of generality we suppose that $\dim \mathcal{W}^s(\Omega_{i,\bar{\mu}}) = 2$. For simplicity we use the notation $s = \Omega_{i,\bar{\mu}}$. We also assume that for $\mu < \bar{\mu}$, X_μ has two hyperbolic singularities $s_{1,\mu}$ and $s_{2,\mu}$ near s collapsing for $\mu = \bar{\mu}$ in the saddle-node singularity s and dissapearing for $\mu > \bar{\mu}$ (*see* Figure 2).

$$\mu < \bar{\mu} \qquad\qquad \mu = \bar{\mu} \qquad\qquad \mu > \bar{\mu}$$
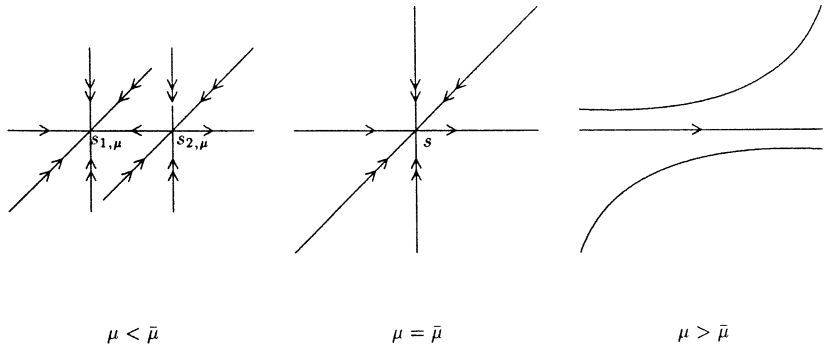
Fig. 2.

As in [Ve] we define compatible systems of stable and unstable foliations $\{\mathcal{F}_k^s\}_k$ and $\{\mathcal{F}_k^u\}_k$, respectively, and a compatible selection of leaves, $\mathcal{R}$, between the respective foliations. The construction of the compatible system of foliations is the same as in [Ve] but we have

to take some care in the construction of $\mathcal{F}_i^s$, $\mathcal{F}_i^u$ and $\mathcal{R}$ over these two foliations: Let U be a small neighborhood of s. If $\overline{I}$ is a small neighborhood of $\overline{\mu}$, then in the neighborhood $U \times \overline{I}$ of $(s, \overline{\mu})$ we have a center stable and a center unstable manifolds, and a center manifold for the vector field $X(x, \mu) = (X_\mu(x), 0)$ denoted, respectively, by $\mathcal{W}^{cs}(s) = \cup_{\mu \in \overline{I}} \mathcal{W}_\mu^{cs}(s) \times \{\mu\}$, $\mathcal{W}^{cu}(s) = \cup_{\mu \in \overline{I}} \mathcal{W}_\mu^{cu}(s) \times \{\mu\}$ and $\mathcal{W}^c(s) = \cup_{\mu \in \overline{I}} \mathcal{W}_\mu^c(s) \times \{\mu\}$. Furthermore, in $U \times \overline{I}$ we have an X-invariant foliation of codimension two, $\mathcal{F}^{su}$, called the stable-unstable foliation, with leaves transversal to $\mathcal{W}^c(s)$. The intersections $\mathcal{F}^{su} \cap \mathcal{W}^{cs}(s)$ and $\mathcal{F}^{su} \cap \mathcal{W}^{cu}(s)$ give us, respectively, the strong stable foliation, $\mathcal{F}^{ss}$, and the strong unstable foliation, $\mathcal{F}^{uu}$, of s. These foliations having as distinguished leaves, the strong stable manifold, $\mathcal{W}^{ss}(s)$, and the strong unstable manifold, $\mathcal{W}^{uu}(s)$, of s, respectively. Note that for each $\mu \in \overline{I}$, $\mathcal{W}_\mu^{cs}(s) = \mathcal{W}^{cs}(s) \cap (M \times \{\mu\})$ (resp. $\mathcal{W}_\mu^{cu}(s) = \mathcal{W}^{cu}(s) \cap (M \times \{\mu\})$) is the union of leaves of $\mathcal{F}^{ss}$ (resp. $\mathcal{F}^{uu}$).

Fundamental domain in $\mathcal{W}^{cs}(s)$ ($\mathcal{W}^{cu}(s)$) and fences

In order to construct a fundamental domain for the vector field X in $\mathcal{W}^{cs}(s)$ we proceed as in [P-T]: first, in $\mathcal{W}^{cs}(s)$ we take a cylinder C transversal to the strong stable foliation $\mathcal{F}^{ss}$ and disjoint from all connected components of $\mathcal{W}^u(\Omega_{j,\overline{\mu}})$ that not intersects $\mathcal{W}^{ss}(s)$, $j \le i - 1$, i.e., C is taken so that $\mathcal{W}^u(\Omega_{j,\overline{\mu}}) \cap C$ is transversal to $\mathcal{F}^{ss}$ in C. Take a disc D in a leaf of $\mathcal{F}^{ss}$ so that the boundary of D intersects the boundary of C, we have that the union $C \cup D$ is a fundamental domain for $X_{\overline{\mu}}|_{\mathcal{W}^s(s)}$. Now we define a fence $\Sigma_{i,\overline{\mu}}^u$ over this fundamental domain in such a way that it is the union of a disc contained in a leaf of $\mathcal{F}^{su}$ containing D and a cylinder transversal to $\mathcal{W}^{ss}(s)$ that contains $C_{\overline{\mu}}$ (see Figure 3).

For $\mu \in \overline{I}$ we choose a continuous family of C^1 discs, D_μ, in $\mathcal{F}^{su}(x_0)$, and a continuous family of C^2 cylinders, C_μ, $C = C_{\overline{\mu}}$, such that $\cup_{\mu \in \overline{I}}(C_\mu \cup D_\mu) \times \{\mu\}$ is diffeomorphic to $(C \cup D) \times \overline{I}$ and it is a fundamental domain for the vector field X restricted to $\mathcal{W}^{cs}(s) = \cup_{\mu \in \overline{I}} \mathcal{W}_\mu^{cs}(s)$. Define a continuous family of fences $\Sigma_{i,\mu}^s$ such that $\Sigma_{i,\overline{\mu}}^s$ is the fence of above and for every μ, $\Sigma_{i,\mu}^s$ is the union of a disc B_μ contained in the leaf of $\mathcal{F}^{su}$ containing D_μ and a cylinder K_μ transversal to $\mathcal{F}_\mu^{su}$ containing C_μ. The same construction applies for families $\{\tilde{X}_\mu\}$.

As in [Ve] we define a μ-dependent compatible systems of unstable foliations $\{\mathcal{F}_{1,\mu}^u, \ldots, \mathcal{F}_{i-1,\mu}^u\}_\mu$ and $\{\widetilde{\mathcal{F}}_{1,\rho(\mu)}^u, \ldots, \widetilde{\mathcal{F}}_{i-1,\rho(\mu)}^u\}$ for the families $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$, respectively, and a compatible selection of leaves, $\mathcal{R}$, over these foliations, sending leaves of $\mathcal{F}_{j,\mu}^u$ into leaves of $\widetilde{\mathcal{F}}_{j,\rho(\mu)}^u$, where ρ is a parametrization as in [P-T].
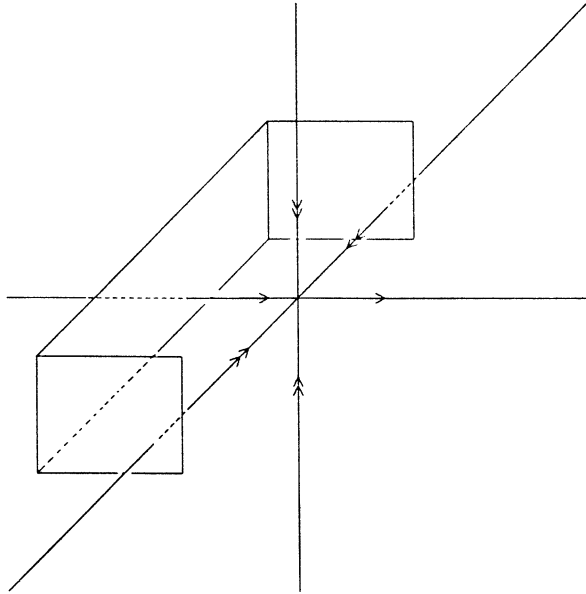
Fig. 3.

Now we define a μ-dependent one dimensional foliation $\mathcal{F}_{i,\mu}^u$ over $\Sigma_{i,\mu}^u$. To do this we first observe that we may assume that discs B_μ are contained in the unstable manifold of a trivial repellor (transversality condition). Define $\mathcal{F}_{i,\mu}^u$ over B_μ as the intersection between B_μ and the leaves of $\mathcal{F}_\mu^{su}$. To define $\mathcal{F}_{i,\mu}^u$ over K_μ take the intersection between K_μ and the X_μ-saturated of every leaf of $\mathcal{F}_{j,\mu}^u$, $1 \leq j \leq i-1$, and then extend to a μ-depedent one-dimensional foliation (same technique as in [Ve]). In the same way, we define $\widetilde{\Sigma}_{i,\mu}^s$ and $\widetilde{\mathcal{F}}_{i,\mu}^u$. To define $\mathcal{R}$ between the leaves of $\mathcal{F}_{i,\mu}^u$ and $\mathcal{F}_{i,\rho(\mu)}^u$ we proceed as in [P-T] for the leaves of $\mathcal{F}_{i,\mu}^u$ contained in the cylinder K_μ and as in [Ve] for the leaves of $\mathcal{F}_{i,\mu}^u$ contained in the disc B_μ. For the remaining indices, $i+1, \ldots, n$, we define the fences $\Sigma_{j,\mu}^s$ and $\widetilde{\Sigma}_{j,\mu}^s$, the foliations, $\mathcal{F}_{j,\mu}^u$ and $\widetilde{\mathcal{F}}_{j,\mu}^u$, and the selection $\mathcal{R}$, the same way as in [Ve]. A similar construction gives us compatible system of unstable foliations $\{\mathcal{F}_{1,\mu}^u, \ldots, \mathcal{F}_{i,\mu}^u, \ldots, \mathcal{F}_{n,\mu}^u\}$, $\{\widetilde{\mathcal{F}}_{1,\mu}^u, \ldots, \widetilde{\mathcal{F}}_{i,\mu}^u, \ldots, \widetilde{\mathcal{F}}_{n,\mu}^u\}$ and a compatible selection $\mathcal{R}$ between the leaves of these foliations. From these contructions the topological equivalence between the families follows as in [Ve], of course in a neighborhood of s we define the equivalence as in [P-T].

When $\dim \mathcal{W}^s(s) = 3$ we define $\Sigma_{i,\bar{\mu}}^s$ like the union of a cylinder, $C_{\bar{\mu}}$, transversal to the strong stable manifold $\mathcal{W}^{ss}(s)$ (*see* Figure 4), disjoint from all the connected components of $\mathcal{W}^u(\Omega_{j,\bar{\mu}})$ that no intersect $\mathcal{W}^{ss}(s)$,

$1 \leq j \leq i - 1$, and a disc D_μ contained in a leaf of $\mathcal{F}_{\bar{\mu}}^{su}$ that intersects $\mathcal{W}^{cs}(s)$.
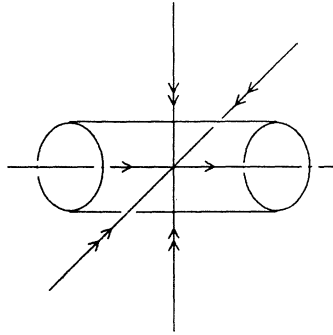

Fig. 4.

Take a continuous family of fences (cylinders) $\Sigma_{i,\mu}^s$ such that $\Sigma_{i,\bar{\mu}}^s$ is the fence of above and for every μ, $\Sigma_{i,\mu}^s$ is the union of a cylinder, C_μ, transversal to the strong stable manifold $\mathcal{W}^{ss}(s)$ and a disc D_μ contained in a leaf of $\mathcal{F}_\mu^{su}$ that intersect $\mathcal{W}^{cs}(s)$.

To construct a compatible system of unstable foliations and a compatible selection of leaves between the foliations of $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$ we proceed as above but for $j = i$ we only foliate the cylinder C_μ contained in $\Sigma_{i,\mu}^s$. To define the selection $\mathcal{R}$ we do it as above, and as in [Ve] the equivalence follows.

II. Isolated Hopf singularity

Without loss of generality we suppose that the basic set $s_{\bar{\mu}} = \Omega_{i,\bar{\mu}}$ is the isolated Hopf singularity and that the bifurcation occurs as it is shown in Figure 5, (a), (b) and (c)
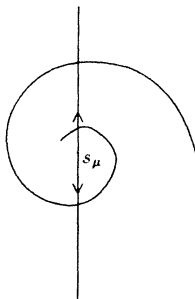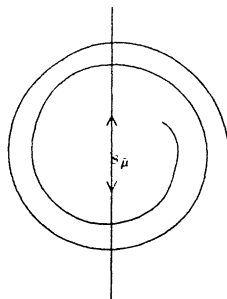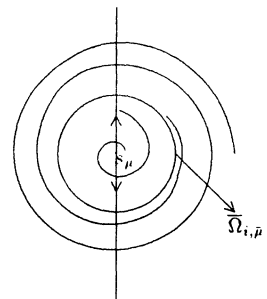

Fig. 5a   Fig. 5b   Fig. 5c.

that is, for

$\mu < \bar{\mu}$: s_μ is a hyperbolic singularity of saddle-type with complex weakest contraction. Let $\lambda_\mu = a + ib$, $a < 0$ and $\bar{\lambda}_\mu = a - bi$ the eigenvalues of $DX_\mu(s_\mu)|_{\mathcal{W}^s(s_\mu)}$,

$\mu = \bar{\mu}$: $s_{\bar{\mu}}$ is a nonhyperbolic singularity, and $\lambda_{\bar{\mu}} = ib$, $\bar{\lambda}_{\bar{\mu}} = -ib$ are the eigenvalues of $DX_{\bar{\mu}}(s_{\bar{\mu}})|_{\mathcal{W}^s(s_{\bar{\mu}})}$, and

$\mu > \bar{\mu}$: s_μ is a hyperbolic singularity (source) and $\overline{\Omega}_{i,\mu}$ is an hyperbolic attracting periodic orbit of saddle-type.

Notation. $- \Omega_{i,\mu} = s_\mu$, $\mu < \bar{\mu}$, $\Omega_{i,\bar{\mu}} = s$, and $\overline{\Omega}_{i,\mu} = \sigma$, $\mu > \bar{\mu}$.

Let $\{X_\mu\}$, $\{\tilde{X}_\mu\} \in \Gamma_2$ be nearby families which have a bifurcation due to a Hopf singularity s and $\tilde{s}$, respectively, for nearby parameter values $\bar{\mu}$ and $\tilde{\mu}$. As before we assume that each time that we make a construction for $\{X_\mu\}$ then the similar construction is made for each family $\{\tilde{X}_\mu\}$ close to $\{X_\mu\}$.

In this case, the construction of a topological equivalence between close families $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$ is really more simple than case treated before, isolated saddle-node singularity, because at present case, if $\mathcal{W}^u(\Omega_{j,\bar{\mu}}) \cap \mathcal{W}^s(\Omega_{\ell,\bar{\mu}}) \neq \emptyset$ for $j < i - 1$ and $\ell \geq i + 1$, then this intersection is non empty for all $\mu \in \bar{I}$, ($\bar{I} \subset I$ a small neighborhood of $\bar{\mu}$). Furthermore, the topological equivalence $H : M \times \bar{I} \to M \times \tilde{I}$, ($\tilde{I} \subseteq I$ a small neighborhood of $\tilde{\mu}$), $H(x, \mu) = (h_\mu(x), \rho(\mu))$ where $\rho : (\bar{I}, \bar{\mu}) \to (\tilde{I}, \tilde{\mu})$ is a reparametrization and for each $\mu \in \bar{I}$, the restriction $H_\mu = H|_{M \times \{\mu\}} : M \times \{\mu\} \to M \times \{\rho(\mu)\}$ is a topological equivalence between the vector fields $X_\mu|_{M \times \{\mu\}}$ and $\tilde{X}_{\rho(\mu)}|_{M \times \{\rho(\mu)\}}$, is made as in [Ve], for this we must note that the fundamental domain for $\mathcal{W}^s(s)$, say C, gives us only a part of the fundamental domain for a small neighborhood of σ, the other part is constructed as in Figure 6.

III. Isolated flip periodic orbit

Without loss of generality we suppose that the bifurcation occurs as follows:

III(i) Case $\dim \mathcal{W}^s(q) = 2$, $q \in \sigma = \Omega_{i,\bar{\mu}}$ the isolated flip periodic orbit, (*i.e.*, $\dim \mathcal{W}^s(\sigma) = 3$) (*see* Figure 7)

Let $P_\mu : (\Sigma_q, q) \to (\Sigma_q, q)$ be the Poincaré map associated to the flip periodic orbit σ. In this case we have, for $\mu > \bar{\mu}$, $P_\mu(q_{1,\mu}) = q_{2,\mu}$, $P_\mu(q_{2,\mu}) = q_{1,\mu}$ and $P_\mu(q_\mu) = q_\mu$, that is, $\{q_{1,\mu}, q_{2,\mu}\}$ is a periodic orbit of period two and it is a sink for P_μ.

III(ii) $\dim \mathcal{W}^s(q) = 1$, (*i.e.*, $\dim \mathcal{W}^s(\sigma) = 2$).

Fig. 6.

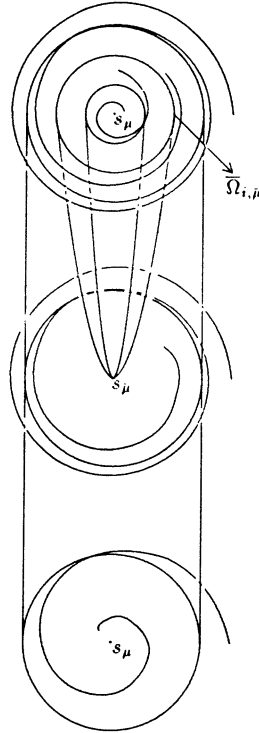$\mu < \bar{\mu}$ $\qquad\qquad \mu = \bar{\mu}$ $\qquad\qquad\qquad \mu > \bar{\mu}$
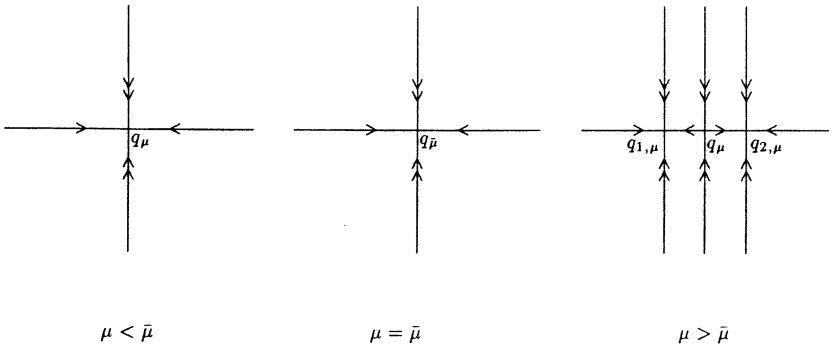
Fig. 7.

In this case, $\{q_{1,\mu}, q_{2,\mu}\}$ is a periodic orbit of period two for P_μ and it is a hyperbolic saddle.

Let $\{X_\mu\} \in \Gamma_2$ and $\bar{\mu} \in B(\{X_\mu\})$ its first bifurcation value such that $X_{\bar{\mu}}$ has an isolated flip periodic orbit. Then it is well known that there exists a neighborhood $\mathcal{U}$ of $\{X_\mu\}$ in $\chi_1^\infty(M)$ such that for $\{\tilde{X}_\mu\} \in \mathcal{U}$ there exists
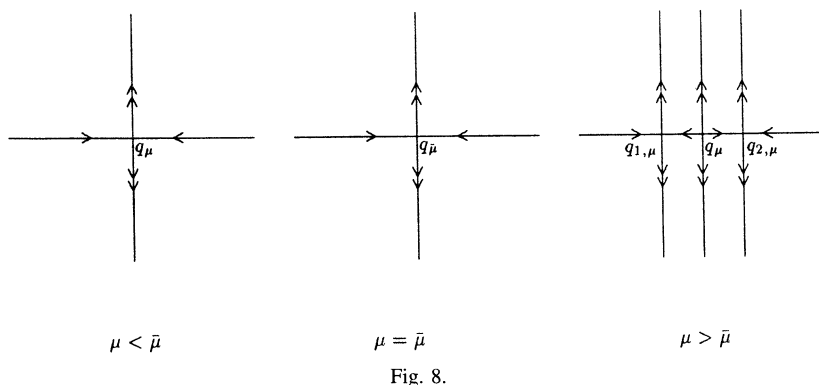
$$\mu < \bar{\mu} \qquad\qquad \mu = \bar{\mu} \qquad\qquad \mu > \bar{\mu}$$

Fig. 8.

a bifurcation value (the first bifurcation value) $\tilde{\mu}$ near $\bar{\mu}$ for which $\tilde{X}_{\tilde{\mu}}$ has an isolated flip periodic orbit. As before each time that we make some construction for the family $\{X_\mu\}$ we assume that the similar construction is maked for each family $\{\tilde{X}_\mu\} \in \mathcal{U}$.

Case III(i). For the construction of a topological equivalence between two nearby families $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$ as above, we note that: if $\mathcal{W}^u(\Omega_{j,\bar{\mu}}) \cap \mathcal{W}^s(\Omega_{\ell,\bar{\mu}}) \neq \emptyset$, then the same holds for $\mu \in \bar{I}$ ($\bar{I} \subseteq I$ a small neighborhood of $\bar{\mu}$). In the present case a fundamental domain for the associated Poincaré map is constructed as in [N-P-T], and following the method of the s-behavior as in [Ve], it is easy construct a topological equivalence between two nearby families $\{X_\mu\}$ and $\{\tilde{X}_\mu\}$ as above.

Case III(ii) This case is treated in a similar way as case III(i), with the obvious modifications.

Remark. – The fundamental fact in the above construction is that there are not mistake between the unstable and the stable manifolds of $\{X_\mu\}$ for all $\mu \in \bar{I}$ when this fact does not happen for $\mu < \bar{\mu}$, that is, if we have control of the behavior of the unstable and the stable manifolds for $\mu < \bar{\mu}$, then we have control of the behavior of these manifolds for all $\mu \in \bar{I}$.

IV. Isolated saddle-node periodic orbit

Let $\{X_\mu\} \in \chi_1^\infty(M)$ and $\bar{\mu} \in B(\{X_\mu\})$ such that $X_{\bar{\mu}}$ has an isolated saddle-node periodic orbit, say $\sigma = \Omega_{i,\bar{\mu}}$. We will suppose that for $\mu < \bar{\mu}$ there are two hyperbolic periodic orbits, say $\sigma_{1,\mu}$ and $\sigma_{2,\mu}$, collapsing for the parameter value $\mu = \bar{\mu}$ creating the saddle-node periodic orbit and disappearing for $\mu > \bar{\mu}$. Let $q \in \sigma$ and let $\Sigma_q \subseteq M$ be a transversal section to $X_{\bar{\mu}}$ at q. Without loss of generality, we may assume that $\Sigma = \cup_{\mu \in \bar{I}} \Sigma_q \times \{\mu\}$ is a transversal section to the vector field $X(x,\mu) = (X_\mu(x), 0)$ at $(q, \bar{\mu})$, where $\bar{I} \subseteq I$ is a small neighborhood of $\bar{\mu}$.

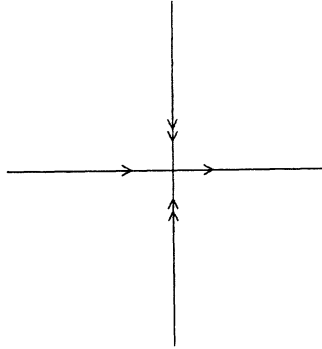We will assume that $\dim \mathcal{W}^s(q) = 2$ (the other case is analogous) (*see* Figure 9)

Fig. 9.

It is well known that there exists a center manifold $\mathcal{W}^c(q)$ of $q \in \sigma$, $\dim \mathcal{W}^c(q) = 2$, and in $\mathcal{W}^c(q)$ there exists a $C^r, r \geq 5$, adapted vector field and at $\mu = \bar{\mu}$ in $\mathcal{W}^c_{\bar{\mu}}(q) = \mathcal{W}^c(q) \cap (U \times \{\bar{\mu}\})$ (U a small neighborhood of q in Σ_q) there exist a unique vector field, Z, such that if $P^c = P|_{\mathcal{W}^c}$ denotes the restriction of the Poincaré map to the center manifold and for each $\mu \in \bar{I}$, $P^c_\mu = P_\mu|_{\mathcal{W}^c_\mu}$, $\mathcal{W}^c_\mu(q) = \mathcal{W}^c(q) \cap U \times \{\mu\}$, is the arc of Poincaré maps, then $P^c = \{P^c_\mu\}$ is an arc of saddle-node diffeomorphisms (*see* [N-P-T]) and we have that $P^c_{\bar{\mu}} = Z_{t=1}$ (Z_t is the respective flow of the vector field Z). Moreover, if $\{\tilde{X}_\mu\}$ is another family of vector fields close to $\{X_\mu\}$, we denote by $\tilde{\sigma} = \tilde{\Omega}_{i,\tilde{\mu}}$, $\tilde{q} \in \tilde{\sigma}$, and $\tilde{P}^c = \{\tilde{P}^c_\mu\}$ the corresponding (close) saddle-node periodic orbit, the point in the intersection of $\tilde{\sigma}$ with the corresponding transversal section, and the corresponding arc of saddle-node diffeomorphisms. Then if $h : \mathcal{W}^c(q) \to \mathcal{W}^c(\tilde{q})$, $h(x, \mu) = (h_\mu(x), \rho(\mu))$, where $\rho : (\bar{I}, \bar{\mu}) \to (\tilde{I}, \tilde{\mu})$ ($\tilde{I} \subseteq I$ a small neighborhood of $\tilde{\mu}$) is a reparametrization, and h is a conjugacy between the arcs of saddle-node diffeomorphisms P^c and $\tilde{P}^c$, that is, for each $\mu \in \bar{I}$, $h_\mu : \mathcal{W}^c_\mu(q) \to \mathcal{W}^c_{\rho(\mu)}(\tilde{q})$ is a conjugacy between the diffeomorphisms P_μ and $\tilde{P}_{\rho(\mu)}$, then $h_{\bar{\mu}} \circ Z_t = \tilde{Z}_t \circ h_{\bar{\mu}}$, where $\tilde{Z}$ is the corresponding unique C^r vector field defined in $\mathcal{W}^c_\mu(\tilde{q})$ such that $\tilde{P}_{\tilde{\mu}} = \tilde{Z}_{t=1}$, thus $h_{\bar{\mu}}$ is a conjugacy between the vector fields Z and $\tilde{Z}$ in an small neighborhood of q, respectively of $\tilde{q}$ (*see* [Ta]).

On the other hand, a conjugacy between the arcs $P|_{\mathcal{W}^{cs}(\sigma)} = \{P_\mu|_{(\mathcal{W}^{cs}(\sigma) \times \{\mu\})}\}$ and $\tilde{P}|_{\mathcal{W}^{cs}(\tilde{\sigma})} = \{\tilde{P}_\mu|_{(\mathcal{W}^{cs}(\tilde{\sigma}) \times \{\mu\})}\}$ at the parameter value $\mu = \bar{\mu}$ must send leaves of the strong stable foliation $\mathcal{F}^{ss}|_{\mathcal{W}^s(q)}$ into leaves of the strong stable foliation $\mathcal{F}^{ss}|_{\mathcal{W}^s(\tilde{q})}$ in a compatible way with $h_{\bar{\mu}}$.

Now it is well known that can not exist orbit of s-criticality between the unstable foliation of a nontrivial basic set or a periodic orbit or a singularity with complex weakest contraction (*see* [Pℓ]). Furthermore, if there exists an orbit of s-criticality between the unstable foliation of $\Omega_{\ell,\bar{\mu}}$, $\ell \leq i - 1$ then $\Omega_{\ell,\bar{\mu}} = \alpha$ is a singularity with real weakest contraction and the orbit of s-criticality must be generic and there is a center unstable manifold of $\Omega_{\ell,\bar{\mu}}$ which is transversal to $\mathcal{F}^{ss}$ in a neighborhood of the s-criticality (*see* [Pℓ]) (*see* Figure 10).
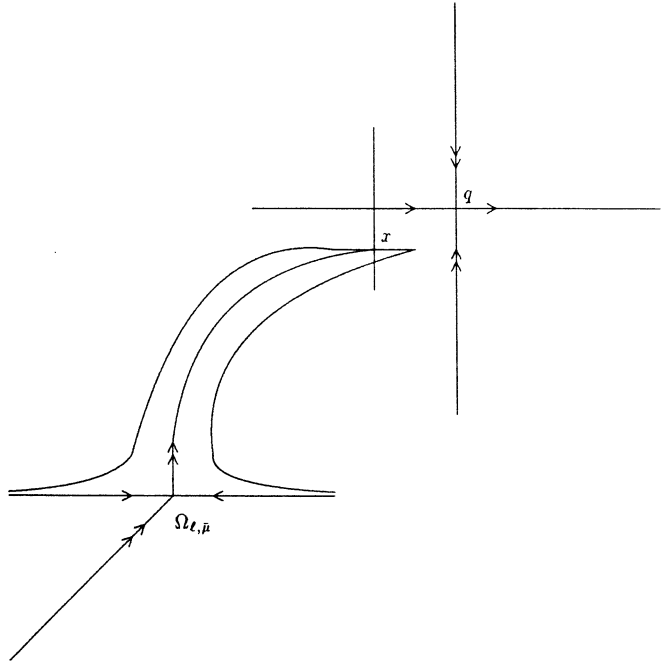

Fig. 10.

Note that for $\mu > \bar{\mu}$ we can have orbits of quasi-transversal intersection, for example if α and β ($\alpha \preceq \sigma$ is a singularity with real weakest contraction and $\sigma \preceq \beta$ is a singularity with real weakest expansion) are such that $\mathcal{W}^u(\alpha)$ has a generic orbit of s-criticality with $\mathcal{F}^{ss}$ and $\mathcal{W}^s(\beta)$ intersects transversally $\mathcal{W}^c(q)$ as in Figure 10.

It is clear that the generic condition *"there is a center unstable manifold of α which is transversal to $\mathcal{F}^{ss}$ in a neighborhood of the s-criticality"* implies that the orbits of quasi-transversal intersection that appear for $\mu > \bar{\mu}$ are generic.

Now as it is done in [N-P-T] we construct a fundamental domain in the stable manifold of the vector field $X(x, \mu) = (X_\mu(x), 0)$ at the nonhyperbolic periodic orbit $(\sigma, \bar{\mu})$ and a fundamental domain in the center manifold $\mathcal{W}^c(q) = \cup_{\mu \in \bar{I}} \mathcal{W}^c_{\bar{\mu}}(q) \times \{\mu\}$, without loss of generality we may suppose that, $\mathcal{W}^s(\beta) \cap \Sigma_q$ are leaves of the strong stable foliation $\mathcal{F}^{ss}$ of $\mathcal{W}^{cs}(q)$, for this note that, $\mathcal{F}^{ss}$ is uniquely determined in $\mathcal{W}^s(q)$, and we may extend it to all $\mathcal{W}^{cs}(q)$, for this the unique condition required is that such foliation is P-invariant. Now let π^{ss} denote the projection along leaves of $\mathcal{F}^{ss}$. Let θ be the orbit of s-criticality between $\mathcal{W}^u(\alpha)$ and $\mathcal{F}^{ss}$, and let x denote the point of the orbit θ which is in the fundamental domain, $D^s(q)$, of $P|_{\mathcal{W}^s(q)}$. Then we choose the fundamental domain in such a way that $x^c = \pi^{ss}(x)$ is in the interior of the fundamental domain, $D^c(q) = \mathcal{W}^c(q) \cap D^s(q)$, of $P|_{\mathcal{W}^c(q)}$. (*See* Figure 11).

We denote by y^c the point of intersection of $\mathcal{W}^c(q)$ and $\mathcal{W}^s(\beta)$, and in this part of $\mathcal{W}^c(q)$ we choose a fundamental domain for $P|_{\mathcal{W}^c(q)}$ in such a way that y^c is in the interior of this domain (*see* Figure 11).
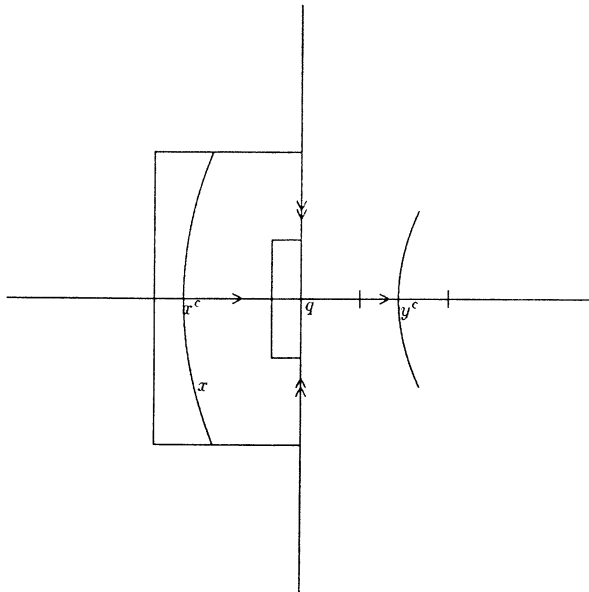

Fig. 11.

In the above condition, we have the following

LEMMA 1. – ([Ma-P]). *Under the above conditions, there are $\varepsilon > 0$ and a strictly monotone sequence $(\bar{\mu}_n)_{n \in \mathbb{N}}$ of parameter values such that*

(1) $\bar{\mu} < \bar{\mu}_n < \varepsilon$, for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \bar{\mu}_n = \bar{\mu}$

(2) for each $\mu \in [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon]$ we have that

(a) if $\mu \neq \bar{\mu}_n$, for all $n \in \mathbb{N}$ then X_μ has not quasi-transversal intersection orbits.

(b) for each $n \in \mathbb{N}$, $\bar{\mu}_n \in B(\{X_\mu\})$ and $X_{\bar{\mu}_n}$ has a unique quasi-transversal intersection orbit $\gamma_{\bar{\mu}_n}$ between $\mathcal{W}^u(\alpha_{\bar{\mu}_n})$ and $\mathcal{W}^s(\beta_{\bar{\mu}_n})$.

Proof. – Note that $\pi^{ss} \circ P_\mu = P_\mu^c \circ \pi^{ss}$, and then from Lemma 1 (p. 21) of [Ma-P] the result follows.

In what follows, we will take as neighborhood of $\bar{\mu}$ the interval $\bar{I} = [\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon]$ given by the above lemma.

LEMMA 2. – ([Ma-P]). *Given* $\{X_\mu\} \in \Gamma_2$ *and* $\bar{\mu} \in B(\{X_\mu\})$ *as above. Let* x_1, $y_1 \in \mathcal{W}^c(q) \cap D^c(q)$ *be two points near* q, *such that* $x_1 < x^c < P_\mu^c(x_1) < q < y_1 < y^c < P_\mu^c(y_1)$, *and let* Z *be a* C^5 *vector field adapted to the arc of saddle-node diffeomorphisms,* $\{P_\mu^c\}$, *defined in a neighborhood of* $(q, \bar{\mu})$ *in* $\mathcal{W}^c(q)$. *Then there exists a local conjugacy,* $H = (h, \eta)$, *between* $\{P_\mu^c\}$ *and* $Z_{t=1}$, *such that*

(1) $h_\mu(x^c(\mu)) = x^c(\mu)$, $h_\mu(y^c(\mu)) = y^c(\mu)$, *where* $x^c(\mu)$ *(resp.* $y^c(\mu)$*) is the point of s-criticality (resp. the intersection point of* $\mathcal{W}^c(q) \times \{\mu\}$ *and* $\mathcal{W}^c(\beta_\mu)$*) is in the corresponding fundamental domain for* P_μ.

(2) $(P_\mu^c)^n(x^c(\mu)) = y^c(\mu)$ *if and only if* $Z_1^n(x^c(\mu), \eta(\mu)) = y^c(\mu)$.

Now we have.

LEMMA 3. – ([Pℓ]). *Let* $\{X_\mu\}$, $\{\widetilde{X}_\mu\} \in \Gamma_2$ *be close families and* $\bar{\mu}, \tilde{\mu} \in I$ *its corresponding first bifurcation values, such that* $X_{\bar{\mu}}$ *and* $\widetilde{X}_{\tilde{\mu}}$ *have a saddle-node periodic orbit* σ, *respectively* $\tilde{\sigma}$. *Let* Z, $\tilde{Z}$ *denote adapted vector fields to the arcs of diffeomorphisms* $\{P_\mu^c\}$ *and* $\{\tilde{P}_\mu^c\}$ *associated to the families* $\{X_\mu\}$ *and* $\{\widetilde{X}_\mu\}$, *respectively. Then there exists a conjugacy* $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\tilde{q})$ $(q \in \sigma$, $\tilde{q} \in \tilde{\sigma})$, $h^c(x, \mu) = (h_\mu^c(x), \rho(\mu))$, *where* $\rho : (\bar{I}, \bar{\mu}) \to (\tilde{I}, \tilde{\mu})$ *is a reparametrization and for each* $\mu \in \bar{I}$, $h_\mu^c : \mathcal{W}^c(q) \cap \Sigma_q \times \{\mu\} \to \mathcal{W}^c(\tilde{q}) \cap \Sigma_{\tilde{q}} \times \{\rho(\mu)\}$ *is a conjugacy between* P_μ^c *and* $\tilde{P}_{\rho(\mu)}^c$, *and* h_μ^c *varies continuously with* μ.

Note that ρ sends the parameter values $\bar{\mu}_n$ (given by the above Lemma 1) into the corresponding ones $\tilde{\mu}_n$, that is, values $\bar{\mu}_n$ for which there exists an orbit of quasi-transversal intersection between $\mathcal{W}^u(\alpha_{\bar{\mu}_n})$ and $\mathcal{W}^s(\beta_{\bar{\mu}_n})$ into the corresponding parameter values $\tilde{\mu}_{m(n)}$ for which there exists an orbit of quasi-transversal intersection between $\mathcal{W}^u(\tilde{\alpha}_{\tilde{\mu}_{m(n)}})$ and $\mathcal{W}^s(\tilde{\beta}_{\tilde{\mu}_{m(n)}})$.

Proof. – Under the above conditions for our arcs in Γ_2, we have:

(a) There exists a conjugacy $h : \mathcal{W}^c(q) \to \mathcal{W}^c(\tilde{q})$ between the vector fields Z and $\tilde{Z}$, which may be choosen to be differentiable with respect to the variable x outside the singular points of Z. This conjugacy provides us a conjugacy between the diffeomorphisms $Z_{t=1}$ and $\tilde{Z}_{t=1}$, which we will denote by $h_2 = (h_{2,\mu}, k)$, where $k : (\bar{I}, \bar{\mu}) \to (\tilde{I}, \tilde{\mu})$ is a reparametrization and for each $\mu \in \bar{I}$, $h_{2,\mu} : \mathcal{W}^c(q) \cap (\Sigma_q \times \{\mu\}) \to \mathcal{W}^c(\tilde{q}) \cap (\Sigma_{\tilde{q}} \times \{k(\mu)\})$ is a conjugacy between $Z_{t=1}|_{\mathcal{W}^c_\mu(q)}$ and $\tilde{Z}_{t=1}|_{\mathcal{W}^c_{k(\mu)}(\tilde{q})}$ (see [N-P-T]). $(\mathcal{W}^c_\mu(q) = \mathcal{W}^c(q) \cap \Sigma_q \times \{\mu\})$

(b) By the Lemma 2, there are conjugacies

(i) $h_1 : \mathcal{W}^c(q) \to \mathcal{W}^c(q)$ between $\{P^c_\mu\}$ and $Z_{t=1}$, $h_1 = (h_{1,\mu}, \eta)$, where $\eta : (\bar{I}, \bar{\mu}) \to (\bar{I}, \bar{\mu})$ is a reparametrization and for each $\mu \in \bar{I}$, $h_{1,\mu} : \mathcal{W}^c_\mu(q) \to \mathcal{W}^c_{\eta(\mu)}(q)$ is a conjugacy between $P^c_\mu|_{\mathcal{W}^c_\mu(q)}$ and $Z_{t=1}|_{\mathcal{W}^c_{\eta(\mu)}(q)}$.

(ii) $h_3 : \mathcal{W}^c(\tilde{q}) \to \mathcal{W}^c(\tilde{q})$ between $\{\tilde{P}^c_\mu\}$ and $\tilde{Z}_{t=1}$, $h_3 = (h_{3,\mu}, \tilde{\eta})$, where $\tilde{\eta} : (\tilde{I}, \tilde{\mu}) \to (\tilde{I}, \tilde{\mu})$ is a reparametrization and for each $\mu \in \tilde{I}$, $h_{3,\mu} : \mathcal{W}^c_\mu(\tilde{q}) \to \mathcal{W}^c_{\tilde{\eta}(\mu)}(\tilde{q})$ is a conjugacy between $\tilde{P}^c_\mu|_{\mathcal{W}^c_\mu(\tilde{q})}$ and $\tilde{Z}_{t=1}|_{\mathcal{W}^c_{\tilde{\eta}(\mu)}(\tilde{q})}$.

Finally define $h^c : \mathcal{W}^c(q) \to \mathcal{W}^c(\tilde{q})$ by $h^c(x, \mu) = (h_{3,\mu}^{-1} \circ h_{2,\mu} \circ h_{1,\mu}(x), \tilde{\eta}^{-1} \circ k \circ \eta(\mu))$. Then h^c satisfies the properties required.

Remark. – It is clear that the above constructions are equivalent to the similar constructions in a fundamental domain of the vector field X_μ instead of the Poincaré map P_μ.

Taking in account the above remark and the s-behavior it is easy to see that the construction made in [Pℓ] well glues with those made in [Ve] for the s-behavior. Therefore the global stability of our families follows from the above constructions.

Case V. Flip periodic orbit inside a nontrivial basic set

Example. – A flip periodic orbit is created inside a horseshoe. Let $\{P_\mu\}$ be a family of two-dimensional diffeomorphisms with a horseshoe for $\mu < \bar{\mu}$, as in Figure 12. Let p_μ and q_μ be the fixed points of P_μ, $\mu < \bar{\mu}$, and let λ_μ be the negative eigenvalue of $DP_\mu(q_\mu)$, $\mu < \bar{\mu}$. We will suppose that λ_μ decreases with the parameter μ, *i.e.*, for $0 \le \mu < \bar{\mu}$, $\lambda_\mu \in]-1, 0[$, for $\mu = \bar{\mu}$, $\lambda_{\bar{\mu}} = -1$ and for $\mu > \bar{\mu}$, $\lambda_\mu < -1$.

Taking the suspension of $\{P_\mu\}$ we construct a family of vector fields for which at the parameter value $\mu = \bar{\mu}$, there exists a flip periodic orbit σ contained in a nontrivial basic set, say $\Lambda_{\bar{\mu}}$. Furthermore, if the flip fixed point, $q_{\bar{\mu}}$ of $\{P_\mu\}$ unfolds generically then the same holds for the flip periodic orbit σ of $\{X_\mu\}$.

Now we will suppose that for $\mu < \bar{\mu}$ the basic set Λ_μ is a hyperbolic basic set for the vector field X_μ and it is of the saddle type.
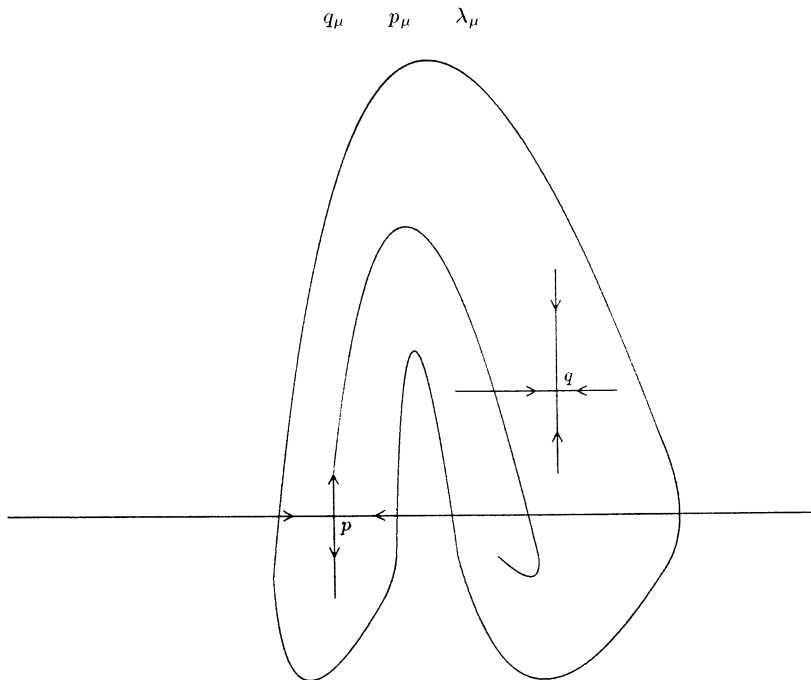
$$q_\mu \qquad p_\mu \qquad \lambda_\mu$$
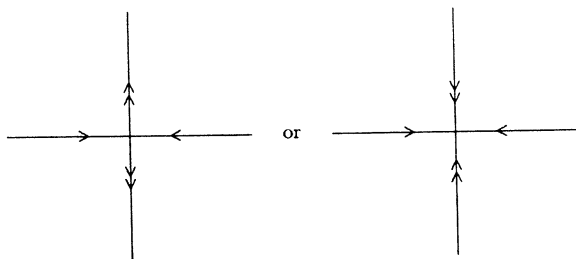

Fig. 12.

Fig. 13.

Remark. – Since $\dim M = 3$, every one dimensional basic set for an hyperbolic flow is d-separated (*see* [N] for the definition).

Note that in this case, for $\mu > \bar{\mu}$, X_μ is a hyperbolic vector field and its basic sets close to $\Lambda_{\bar{\mu}}$ consist of a repellor periodic orbit (a source) running through q_μ and a nontrivial basic set $\overline{\Lambda}_\mu$ containing the orbit $q_{1,\mu}$ (= orbit of $q_{2,\mu}$) (*see* [N]). Note that for $\mu = \bar{\mu}$, $\Lambda_{\bar{\mu}}$ is nonhyperbolic since the point $q_{\bar{\mu}}$ is in $\Lambda_{\bar{\mu}}$ and it is nonhyperbolic, but even in this case we have a $DX_{t,\bar{\mu}}$-invariant descomposition, $T_{\Lambda_{\bar{\mu}}}M = E^u \oplus E^c \oplus E^s$, of the fiber bundle, and there are constants $c > 0$ and $\lambda > 0$, such that:

1) $|DX_{-t,\bar{\mu}}(q)v| \leq c\,e^{-\lambda t}|v|, \quad v \in E_q^u, \ t \geq 0$

2) $E_q^c = \langle X_{\bar{\mu}}(q) \rangle$ (subspace of $T_q M$ spanned by the vector $X_{\bar{\mu}}(q)$, $q = q_{\bar{\mu}}$).

3) $|DX_{t,\bar{\mu}}(q)v| \leq c\,e^{-\lambda t}|v|, \quad v \in E_q^s, \ t \geq 0$, large enough, $q \notin \Theta(p_{\bar{\mu}})$. $(\Theta(x) = \text{orbit of the point } x)$.

On the other hand we have that, for all $\mu \in \bar{I}$ there exists an unstable foliation, $\mathcal{F}^u$, on which the derivative of X_μ has exponential expansion, but for the stable foliation $\mathcal{F}^s$, the orbit $\Theta(q_{\bar{\mu}})$ is the unique orbit along which there exists a nonexponential contraction, using this fact we have a stable invariant foliation at the parameter value $\mu = \bar{\mu}$. From these facts, the global stability of this bifurcation follows. The proof is similar to the case of isolated flip periodic orbit, because the existence of the stable and unstable foliations permit us the use of the s-behavior.

Remark. – Let $\Omega_{i,\mu}$ be a basic set of $\{X_\mu\}$ so that for $\mu < \bar{\mu}$, $\Omega_{i,\mu}$ is hyperbolic and for $\mu = \bar{\mu}$ a saddle-node periodic orbit of $X_{\bar{\mu}}$ appears inside $\Omega_{i,\bar{\mu}}$ then $\{X_\mu\}$ is nonstable at $\bar{\mu}$. This follows from the fact that the periodic orbits are dense in $\Omega_{i,\mu}$, and therefore there are infinitely many periodic orbits whose unstable manifolds intersect the center manifold of $p_{\bar{\mu}}$ ($p_{\bar{\mu}}$ is a point in the saddle-node periodic orbit), and then applying the results of [Ma-P] or [Pℓ] the proof of the claim is done.

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