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Large deviation asymptotics for Anosov flows

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ABSTRACT. – We derive precise asymptotic formulae for large deviation probabilities for suspensions of subshifts of finite type. As a corollary, we give a stronger version of the Central Limit Theorem. We apply our results to transitive Anosov flows, giving a result describing fluctuations in the volume of Bowen balls and an asymptotic large deviation formula of a homological nature involving Schwartzmann's winding cycle.

0. INTRODUCTION

Let M be a compact C^∞ Riemannian manifold and let $\phi_t : M \longrightarrow M$ be a C^1 flow. Let m be a fully supported, ϕ-invariant, ergodic Borel probability measure. According to Birkhoff's Ergodic Theorem, for any real-valued $F \in L^1_m(M)$,

$$\frac{1}{T}\int_0^T F(\phi_t y)dt \longrightarrow \int F dm \quad \text{as } T \to \infty$$

for m almost all $y \in M$. We shall be interested in studying the large deviations from this limit.

According to Ellis [E], such a process is said to satisfy the *large deviation property* if there exists a function $I : \mathbb{R} \longrightarrow [0, \infty]$, called an *entropy* *function*, such that

(*a*) I is lower semicontinuous on $\mathbb{R}$,
(*b*) I has compact level sets,

$$(c) \quad \limsup_{T\to\infty} \frac{1}{T} \log m\left\{ y \in M : \frac{1}{T}\int_0^T F(\phi_t y)dt \in K \right\} \leq -\inf_{a\in K} I(a)$$

for each non-empty closed subset K of $\mathbb{R}$, and

$$(d) \quad \liminf_{T\to\infty} \frac{1}{T} \log m\left\{ y \in M : \frac{1}{T}\int_0^T F(\phi_t y)dt \in G \right\} \geq -\inf_{a\in G} I(a)$$

for each non-empty open subset G of $\mathbb{R}$.

(If properties (*a*)-(*d*) are satisfied then the entropy function is uniquely determined).

The Large Deviation Property has been studied extensively in hyperbolic dynamics, both for flows and diffeomorphisms. (*See* for example [D], [Ki] and [Y], which also contain many further references). We shall be concerned with the case that ϕ is a transitive Anosov flow. We will show that, under certain conditions on ϕ, m and F, the entropy function satisfies higher regularity properties, (in fact it is real analytic) and statements (*c*) and (*d*) can be replaced by a much stronger asymptotic formula. These results for flows extend earlier results of Lalley for diffeomorphisms [La1]. Our main theorem also yields medium deviation results, such as the Central Limit Theorem of [Ra].

Using work of Bowen, we can reduce large deviation problems for Anosov flows to the level of symbolic dynamics. The asymptotic formulae we obtain are essentially based on a careful description of the spectrum of the Ruelle operator, [Ru1]. We introduce a new complex function in dynamics, which is the Laplace transform of the moment generating function of the process $\{F \circ \phi_t\}_{t\geq 0}$, with stationary probability m. Using the now well established 'zeta function' technique in dynamics, (*see* [PP], [Po2], etc.), information on the spectrum of the Ruelle operator is used to analyse the analytic domain of this complex function, and the asymptotic formulae are deduced by applying an appropriate Tauberian theorem.

We use our results to study fluctuations in the volume of Bowen balls and give a large deviation result of a homological nature involving Schwartzmann's winding cycle.

In section nine, we state a more general multidimensional large deviation result which can be proved by the same method.

1. PRESSURE AND THE RUELLE OPERATOR

Throughout this section, we let A be a $k \times k$, zero-one aperiodic matrix, and we define

$$\Sigma_A^+ = \left\{ x \in \prod_{n=0}^{\infty} \{1, \ldots, d\} : A(x_n, x_{n+1}) = 1 \text{ for all } n \geq 0 \right\}.$$

For any $\alpha \in (0,1)$, we can define a metric d^+ on Σ_A^+ by $d^+(x,y) = \alpha^n$, where n is the largest positive integer for which $x_i = y_i$, for $0 \leq i < n$. With respect to this metric, Σ_A^+ is a compact space. The continuous map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ given by $(\sigma x)_n = x_{n+1}$ is called a *(one-sided) subshift of finite type.* In fact, σ is a bounded-to-one, local homeomorphism.

For $g \in C(\Sigma_A^+; \mathbb{C})$, define

$$\mathrm{var}_n(g) = \sup\{|g(x) - g(y)| : x_i = y_i \quad \text{for } 0 \leq i \leq n\},$$

and for any $0 < \alpha < 1$, define a norm $|.|_\alpha$ by

$$|g|_\alpha = \sup\left\{ \frac{\mathrm{var}_n(g)}{\alpha^n} : n \geq 0 \right\}.$$

The space $\mathcal{F}_\alpha^+(\mathbb{C})$ defined by

$$\mathcal{F}_\alpha^+(\mathbb{C}) = \{g \in C(\Sigma_A^+; \mathbb{C}) : |g|_\alpha < \infty\}$$

is a Banach space when endowed with the norm $\|g\|_\alpha = |g|_\alpha + \|g\|_\infty$, where $\|.\|_\infty$ is the uniform norm. Let $\mathcal{F}_\alpha^+(\mathbb{R})$ denote the subspace $C(\Sigma_A^+; \mathbb{R}) \cap \mathcal{F}_\alpha^+(\mathbb{C})$ of $\mathcal{F}_\alpha^+(\mathbb{C})$.

Two functions $f, g \in \mathcal{F}_\alpha^+(\mathbb{C})$ are said to be *cohomologous* (written $f \sim g$) if there exists a continuous function w such that $f = g + w \circ \sigma - w$. A function f is called a *coboundary* if it is cohomologous to the function which is identically zero. Given $g \in C(\Sigma_A^+; \mathbb{R})$, we define a real number $P(g)$, called the *pressure* of g by

$$P(g) = \sup\left\{ h(\nu) + \int g d\nu : \nu \text{ is a } \sigma\text{-invariant Borel probability measure} \right\},$$

where $h(\nu)$ is the entropy of σ with respect to the measure ν. When $g \in \mathcal{F}_\alpha^+(\mathbb{R})$, the supremum is attained by a unique measure $\mu = \mu_g$, (*i.e.* $P(g) = h(\mu) + \int g d\mu$) called the *equilibrium state* or *Gibbs state* of g. (*See* [B2], page 31). If $f, g \in \mathcal{F}_\alpha^+(\mathbb{R})$ are functions such that $f - g$ is cohomologous to a constant function, then f, g have the same equilibrium state.

For $f \in \mathcal{F}_\alpha^+(\mathbb{C})$, we define the *Ruelle operator* $\mathcal{L}_f : \mathcal{F}_\alpha^+(\mathbb{C}) \to \mathcal{F}_\alpha^+(\mathbb{C})$ by

$$(\mathcal{L}_f g)(x) = \sum_{\sigma y = x} e^{f(y)} g(y),$$

where the summation is over the finite set $\{y \in \Sigma_A^+ : \sigma y = x\}$. The properties of the Ruelle operator which we shall require are summarised in Propositions 1 and 2.

PROPOSITION 1 ([Ru1] parts (i)-(iii), [Po1] part (iv)). – *Let $f = u + iv \in \mathcal{F}_\alpha^+(\mathbb{C})$ be given.*

(i) *There is a unique simple positive maximal eigenvalue $e^{P(u)}$ of $\mathcal{L}_u$ with corresponding strictly positive eigenfunction $h = h_u \in \mathcal{F}_\alpha^+(\mathbb{R})$. Further, the remainder of the spectrum of $\mathcal{L}_u : \mathcal{F}_\alpha^+(\mathbb{C}) \to \mathcal{F}_\alpha^+(\mathbb{C})$ is contained in a disc of radius strictly less than $e^{P(u)}$.*

(ii) *There is a unique probability measure $\nu = \nu_u$ such that $\mathcal{L}_u^* \nu_u = e^{P(u)} \nu_u$.*

(iii) $\frac{\mathcal{L}_u^n k}{e^{nP(u)}} \to h_u \int k d\nu_u$ *as $n \to \infty$ exponentially fast, uniformly for all $k \in \mathcal{F}_\alpha^+(\mathbb{C})$, and furthermore, $\int h_u d\nu_u = 1$.*

(iv) $\rho(\mathcal{L}_f) \le e^{P(u)}$ *and for $0 \le a < 2\pi$, $e^{ia + P(u)}$ is an eigenvalue of $\mathcal{L}_f$ if and only if $v - a + w \circ \sigma - w \in C(\Sigma_A^+; 2\pi\mathbb{Z})$, for some $w \in C(\Sigma_A^+; \mathbb{C})$. If $\mathcal{L}_f$ has no eigenvalues of modulus $e^{P(u)}$ then $\rho(\mathcal{L}_f) < e^{P(u)}$.*

For $u \in \mathcal{F}_\alpha^+(\mathbb{R})$, the equilibrium state μ_u and the measure ν_u given by Proposition 1 are related by the formula

$$\frac{d\mu_u}{d\nu_u} = h_u \tag{1.1}$$

where $\mathcal{L}_u h_u = e^{P(u)} h_u$, and $h_u \in \mathcal{F}_\alpha^+(\mathbb{R})$.

Let $L_\alpha^+(\mathbb{R})$ denote the set

$$\{f \in \mathcal{F}_\alpha^+(\mathbb{R}) : f + c \text{ is cohomologous to a function in } C(\Sigma_A^+; 2\pi\mathbb{Z}) \text{ for some } c \in [0, 2\pi)\}.$$

Let $L_\alpha^+(\mathbb{C})$ denote the set $\{f \in \mathcal{F}_\alpha^+(\mathbb{C}) : \operatorname{Im} f \in L_\alpha^+(\mathbb{R})\}$.

We also define a set $I_\alpha^+(\mathbb{R})$ by

$$I_\alpha^+(\mathbb{R}) = \mathcal{F}_\alpha^+(\mathbb{R}) \setminus L_\alpha^+(\mathbb{R}),$$

and similarly, let

$$I_\alpha^+(\mathbb{C}) = \{f \in \mathcal{F}_\alpha^+(\mathbb{C}) : \operatorname{Im} f \in I_\alpha^+(\mathbb{R})\}.$$

Note that there is a natural inclusion $\mathcal{F}_\alpha^+(\mathbb{R}) \subset L_\alpha^+(\mathbb{C})$.

We can extend the definition of pressure to $L_\alpha^+(\mathbb{C})$ by

$$P(g) = P(u) + c$$

whenever $g \in L_\alpha^+(\mathbb{C})$, $g = u + iv$, $u \in \mathcal{F}_\alpha^+(\mathbb{R})$ and v is cohomologous to $2\pi M + c$ for some $M \in C(\Sigma_A^+; \mathbb{Z})$ and $c \in [0, 2\pi)$. Similarly, we can define $h_g = h_u$ and $\nu_g = \nu_u$.

For each $g \in L_\alpha^+(\mathbb{C})$, there exists an open neighbourhood of g in $\mathcal{F}_\alpha^+(\mathbb{C})$, denoted by $N(g)$, such that the maps $g \mapsto e^{P(g)}$ and $g \mapsto h_g$ have analytic extensions to $N(g)$, such that $\mathcal{L}_f h_f = e^{P(f)} h_f$ holds for all $f \in N(g)$.

We define $P(f)$ to be the principal branch of $\log(e^{P(f)})$, for each $f \in N(g)$. Also the map $L_\alpha^+(\mathbb{C}) \to M(\Sigma_A^+)$ defined by $g \mapsto \nu_g$ can be extended to a weak-*-analytic map on a neighbourhood of $L_\alpha^+(\mathbb{C})$ in $\mathcal{F}_\alpha^+(\mathbb{C})$ by $f \mapsto \nu_f$ such that $\mathcal{L}_f^* \nu_f = e^{P(f)} \nu_f$ and $\int h_f d\nu_f = 1$ hold for all f in this neighbourhood. (*Weak-*-analytic* means that for each $v \in \mathcal{F}_\alpha^+(\mathbb{C})$, the map $N(g) \to \mathbb{C}$ given by $f \mapsto \int v d\nu_f$ is analytic).

The following proposition is a reformulation of Corollary 1 and Proposition 4 in [La1], or alternatively Propositions 5-7 in Appendix 1 of [La2].

PROPOSITION 2. – *Let $g \in L_\alpha^+(\mathbb{C})$ and let $B \subset \mathcal{F}_\alpha^+(\mathbb{C})$ be compact.*

(i) *Let $K_1 \subset N(g)$ be compact. Then there exists $\delta_1 > 0$ such that*

$$(1+\delta_1)^n \left\| \frac{\mathcal{L}_f^n k}{e^{nP(f)}} - h_f \int k d\nu_f \right\|_\alpha \longrightarrow 0 \quad \text{as } n \to \infty,$$

uniformly for $f \in K_1$ and $k \in B$.

(ii) *Let $K_2 \subset I_\alpha^+(\mathbb{C})$ be compact. Then there exists $\delta_2 > 0$ such that*

$$(1+\delta_2)^n \frac{\left\|\mathcal{L}_f^n k\right\|_\alpha}{e^{nP(Ref)}} \longrightarrow 0 \quad \text{as } n \to \infty,$$

uniformly for $f \in K_2$ and $k \in B$.

We now define the notion of σ-independence.

DEFINITION 1. – Two functions $f_1, f_2 \in \mathcal{F}_\alpha^+(\mathbb{R})$ are said to be σ-*independent* if whenever there are constants $t_1, t_2 \in \mathbb{R}$ such that $t_1 f_1 + t_2 f_2$ is cohomologous to an element of $C(\Sigma_A^+; 2\pi\mathbb{Z})$ then $t_1 = 0 = t_2$.

2. SUSPENDED FLOWS

As in section 1, we assume that A is a $d \times d$ aperiodic matrix with entries 0 or 1, and we define

$$\Sigma_A = \left\{ x \in \prod_{-\infty}^{\infty} \{1, \ldots, d\} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z} \right\}.$$

For this space, we define a metric d, (for a given $0 < \alpha < 1$), by $d(x, y) = \alpha^{2n}$, where n is the largest positive integer for which $x_i = y_i$, for $-n < i < n$. The homeomorphism $\sigma : \Sigma_A \to \Sigma_A$ defined by $(\sigma x)_n = x_{n+1}$ is called the *(two-sided) subshift of finite type.*

We shall denote the real and complex Banach spaces of Hölder continuous functions by $\mathcal{F}_\alpha(\mathbb{R})$ and $\mathcal{F}_\alpha(\mathbb{C})$ respectively, which are defined analogously to those for the one-sided shift. We can also define pressure and equilibrium states for functions in $C(\Sigma_A; \mathbb{R})$ in complete analogy with the one-sided shift. We refer the reader to [PP] for further details.

Hölder continuous functions defined on the one and two sided shift spaces are related as follows. If $f \in \mathcal{F}_\alpha(\mathbb{C})$, then there exist $g, w \in \mathcal{F}_{\alpha^{1/2}}(\mathbb{C})$ such that $f = g + w - w \circ \sigma$ and $g(x) = g(y)$ whenever $x_i = y_i$ for $i \geq 0$, (so that we may regard g as an element of $\mathcal{F}^+_{\alpha^{1/2}}(\mathbb{C})$).

For a strictly positive function $r \in \mathcal{F}_\alpha(\mathbb{R})$, we define a new space by

$$\Sigma_A^r = \{(x, t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq r(x)\} / \sim,$$

where the equivalence relation $\sim$ identifies the points $(x, r(x))$ and $(\sigma x, 0)$, for each $x \in \Sigma_A$. The space Σ_A^r inherits the product topology from Σ_A and $\mathbb{R}$. We define the *suspended flow* $\sigma^r : \Sigma_A^r \to \Sigma_A^r$ locally, by $\sigma_t^r(x, s) = (x, s + t)$, taking into account the identifications.

The flow $\sigma^r : \Sigma_A^r \to \Sigma_A^r$ is called *topologically weak mixing* if there is no non-trivial solution to $F \circ \sigma_t^r = e^{iat} F$ with $F \in C(\Sigma_A^r)$ and $a > 0$. The case that σ^r is not topologically weak mixing reduces to studying the technically much easier case the the shift map σ. Large deviations for subshifts of finite type were considered in [La1], so we will not make further reference to them here.

For $F \in C(\Sigma_A^r; \mathbb{C})$ or $C(\Sigma_A; \mathbb{R})$, let $\mathcal{F}_\alpha^r(\mathbb{C})$ and $\mathcal{F}_\alpha^r(\mathbb{R})$ denote the respective spaces of functions which are Lipschitz continuous. Define $f \in C(\Sigma_A; \mathbb{C})$ by

$$f(x) = \int_0^{r(x)} F(x, t) dt.$$

Note that if $F \in \mathcal{F}_\alpha^r(\mathbb{C})$ then $f \in \mathcal{F}_\alpha(\mathbb{C})$.

We now define the notions of pressure and equilibrium states for flows. For $F \in C(\Sigma_A^r;\mathbb{R})$, define the *pressure* $P(F)$ of F by

$$P(F) = \sup\left\{h(m)+\int Fdm : m \text{ is a } \sigma^r \text{ invariant Borel probability measure}\right\},$$

(where $h(m)$ is the entropy of $\sigma_1^r : \Sigma_A^r \to \Sigma_A^r$ with respect to m). If $F \in \mathcal{F}_\alpha^r(\mathbb{R})$, there is a unique measure $m = m_F$ such that $P(F) = h(m) + \int Fdm$, and this measure is called the *equilibrium state* of F. Such a measure takes the form $m = \frac{\mu\times l}{\int rd\mu}$, where μ is the unique equilibrium state of $f - P(F)r \in \mathcal{F}_\alpha(\mathbb{R})$, (*i.e.* $\mu = \mu_{f-P(F)r}$), and l is Lebesgue measure on $\mathbb{R}$. More explicitly, this means that for any $H \in C(\Sigma_A^r;\mathbb{C})$,

$$\int Hdm = \frac{\displaystyle\int_{\Sigma_A}\int_0^{r(x)} H(x,t)dtd\mu(x)}{\displaystyle\int rd\mu}.$$

Further, $c = P(F)$ is the unique real number such that $P(f-cr) = 0$, [BR].

A continuous function $H \in C(\Sigma_A^r;\mathbb{C})$ is called *continuously differentiable* with respect to σ^r if for each $y \in \Sigma_A^r$,

$$H'(y) = \lim_{t\to 0}\frac{H(\sigma_t^r y) - H(y)}{t}$$

exists and is continuous. Two functions $F, G \in C(\Sigma_A^r;\mathbb{C})$ are *cohomologous* if there exists a continuously differentiable function $H \in C(\Sigma_A^r;\mathbb{C})$ such that $F - G = H'$. For functions $F, G \in \mathcal{F}_\alpha^r(\mathbb{R})$, if $F - G$ is cohomologous to a constant function then F, G have the same equilibrium state.

For any $F, G \in \mathcal{F}_\alpha^r(\mathbb{R})$, define a map $\beta : \mathbb{R} \to \mathbb{R}$ by $\beta(t) = P(G + tF) - P(G)$. It is not difficult to verify that β is real analytic. Furthermore,

$$\beta'(t) = \int Fdm_{G+tF},$$

and

$$\beta''(t) = \sigma^2_{m_{G+tF}}(F),$$

where

$$\sigma_m^2(F) = \lim_{T\to\infty}\frac{1}{T}\left\{\int_0^T F\circ\sigma_t^r dt - T\int Fdm\right\}^2 < \infty$$

by [La2], section 5. Furthermore, $\sigma_m^2(F) = 0$ if and only if F is cohomologous to a constant function, and otherwise $\sigma_m^2(F) > 0$.

We now assume that F is *not* cohomologous to a constant. Then the map $t \mapsto \beta'(t)$ is strictly increasing. Let Γ_F be defined by

$$\Gamma_F = \{\beta'(t) : t \in \mathbb{R}\}.$$

Then for each $a \in \Gamma_F$, there exists a unique $\rho(a) \in \mathbb{R}$ such that $\beta'(\rho(a)) = a$. The function $\rho : \Gamma_F \to \mathbb{R}$ is strictly increasing, surjective and real analytic. We let $\gamma : \Gamma_F \to \mathbb{R}$ be defined by

$$\gamma(a) = -\sup\{at - \beta(t) : t \in \mathbb{R}\}.$$

Standard properties of the Legendre transform then give

$$\gamma(a) = \begin{cases} \beta(\rho(a)) - a\rho(a) & \text{for } a \in \Gamma_F \\ -\infty & \text{for } a \notin \Gamma_F. \end{cases}$$

By the Inverse Function Theorem,

$$\rho'(a) = \frac{1}{\beta''(\rho(a))}.$$

Thus we have that

$$\gamma'(a) = \beta'(\rho(a))\rho'(a) - a\rho'(a) - \rho(a) = -\rho(a),$$

and in particular, $\gamma'(a) = 0$ if and only if $a = \int F dm_G$. Furthermore, $\gamma''(a) = -\rho'(a) < 0$ since ρ is strictly increasing. We conclude that γ is a strictly concave, non-positive function with a unique maximum at $a = \int F dm_G$.

Notational comments. – We shall adopt, wherever possible, the notational conventions of [PP]. In particular, for $g : \Sigma_A \to \mathbb{C}$, we let

$$g^n(x) = \begin{cases} g(x) + g(\sigma x) + g(\sigma^2 x) + \ldots + g(\sigma^{n-1} x) & \text{for } n \geq 1 \\ 0 & \text{for } n = 0. \end{cases}$$

We introduce an analogous notation for flows. For $F \in C(\Sigma_A^r; \mathbb{C})$, we define

$$F^T(y) = \int_0^T F(\sigma_t^r(y))dt.$$

We now extend the ideas of independence in section one to suspended flows. First we remark that there is an obvious analogue of σ-independence for functions in $\mathcal{F}_\alpha(\mathbb{R})$, in the sense of Definition 1. For suspended flows, we define a notion of flow independence in Definition 2.

First let $G \in \mathcal{F}_\alpha^r(\mathbb{R})$, and define a skew product flow S_t^G on $S^1 \times \Sigma_A^r$ by

$$S_t^G(e^{2\pi i\theta}, y) = \left(e^{2\pi i(\theta + G^t(y))}, \sigma_t^r(y)\right).$$

DEFINITION 2 [La2]. – Given a suspended flow σ^r and a function $F \in \mathcal{F}_\alpha^r(\mathbb{R})$, F and σ^r are *flow independent* if the following condition is satisfied. If $t_0, t_1 \in \mathbb{R}$ are constants such that the skew product flow $S_t^G : S^1 \times \Sigma_A^r \to S^1 \times \Sigma_A^r$, where $G = t_0 + t_1 F$, is not topologically ergodic, then $t_0 = 0 = t_1$.

The following proposition contains some useful simple observations.

PROPOSITION 3. – *If F and σ^r are flow independent then the functions f and r are σ-independent, where $f(x) = \int_0^{r(x)} F(x, u)du$. Further, if either of these two conditions hold then the following two statements are true.*

(i) *The flow σ^r is topologically weak mixing.*

(ii) *The function F is not cohomologous to a constant function.*

3. STATEMENT OF RESULTS FOR SUSPENDED FLOWS

Let $\sigma_t^r : \Sigma_A^r \to \Sigma_A^r$ be a suspended flow and let $F \in \mathcal{F}_\alpha^r(\mathbb{R})$. We suppose throughout this section that F and σ^r are flow independent. Fix $G \in \mathcal{F}_\alpha^r(\mathbb{R})$ and let $m = m_G$ denote the equilibrium state of G. Our main result for large deviations for suspended flows is the following.

THEOREM 1. – *For every $b > 0$ and $a \in \Gamma_F$,*

$$\begin{aligned} &m\{y : F^T(y) - Ta \in [0, b]\} \\ &\sim \left(\int_0^b e^{-\rho(a)t} dt\right) \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{T\gamma(a)}}{\sqrt{T}} \end{aligned} \tag{3.1}$$

as $T \to \infty$. The constant $C(a)$ is given by (4.12). *Furthermore, for any compact set $J \subset \Gamma_F$, the convergence in* (3.1) *is uniform in J.*

In the statement of Theorem 1, we have used the standard notation $A(t) \sim B(t)$ to mean

$$\frac{A(t)}{B(t)} \to 1 \quad \text{as } t \to \infty.$$

The proof of Theorem 1 will be given in section 5.

Remarks 1. – (i) In the terminology of large deviations, the function $I : \Gamma_F \to \mathbb{R}$ given by $I(a) = -\gamma(a)$ is called the *entropy function*, [E], [T1], and is therefore real analytic.

(ii) By the variational principal stated in section two, for any $a \in \Gamma_F$,

$$\begin{aligned}\gamma(a) &= P(G+\rho(a)F) - P(G) - a\rho(a)\\ &= h(m_{G+\rho(a)F}) + \int \big(G - P(G)\big) dm_{G+\rho(a)F}.\end{aligned}$$

(iii) In Theorem 1, we can replace the assumption that F, σ^r are flow independent by the assumption that f, r are σ-independent.

(iv) We can replace the interval $[0, b]$ in the statement of Theorem 1 by any compact interval $K \subset \mathbb{R}$, to give

$$m\{y : F^T(y) - Ta \in K\} \sim \left(\int_K e^{-\rho(a)t} dt\right) \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{T\gamma(a)}}{\sqrt{T}}$$

as $T \to \infty$. The convergence of uniform on compact subsets of Γ_F.

We now deduce some corollaries of Theorem 1.

COROLLARY 1. – *For every $b > 0$ and $a \in \Gamma_F$,*

$$\frac{1}{T} \log m\{y : F^T(y) - aT \in [0, b]\} \longrightarrow \gamma(a) \quad \textit{as } T \to \infty.$$

COROLLARY 2. – *If $\rho(a) > 0$ then*

$$m\{y : F^T(y) \geq Ta\} \sim \frac{C(a)}{\rho(a)} \frac{1}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{T\gamma(a)}}{\sqrt{T}}$$

as $T \to \infty$.

The proof of Corollary 2 will be given in section 6.

We can now formulate a result which more closely parallels the notation used in the introduction.

COROLLARY 3. – *Let $J \subset \mathbb{R}$ be a closed interval which does not contain the point $\int F dm$. Let $a \in \Gamma_F$ be the unique point for which*

$$I(a) = \inf_{b \in J} I(b).$$

Then

$$m\left\{y : \frac{F^T(y)}{T} \in J\right\} \sim \frac{C(a)}{\rho(a)} \frac{1}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{-TI(a)}}{\sqrt{T}}$$

as $T \to \infty$.

The following corollary extends part of Theorem 2 in [La1] for subshifts of finite type.

COROLLARY 4. – *For any* $c \in \mathbb{R}$ *and* $b > 0$,

$$m\left\{y : F^T(y) - T\int F dm - c\sqrt{T} \in [0, b]\right\} \sim \frac{b e^{-\frac{c^2}{2\beta''(0)}}}{\sqrt{2\pi\beta''(0)}} \frac{1}{\sqrt{T}} \quad \textit{as } T \to \infty.$$

Proof. – From section two, we have that

$$\gamma\left(\int F dm\right) = 0 = \gamma'\left(\int F dm\right),$$

and

$$\gamma''\left(\int F dm\right) = -\frac{1}{\beta''(0)}.$$

By the real analycity of γ,

$$\begin{aligned}\gamma\left(\int F dm + \frac{c}{\sqrt{T}}\right) &= \frac{\gamma''(\int F dm)}{2}\frac{c^2}{T} + O\left(\frac{1}{T^{3/2}}\right)\\ &= -\frac{c^2}{2\beta''(0)}\frac{1}{T} + O\left(\frac{1}{T^{3/2}}\right).\end{aligned}$$

By making the simple observation that

$$\begin{aligned}&m\left\{y : F^T(y) - T\int F dm - c\sqrt{T} \in [0, b]\right\}\\ &\quad = m\left\{y : F^T(y) - T\left(\int F dm + \frac{c}{\sqrt{T}}\right) \in [0, b]\right\},\end{aligned}$$

we can apply Theorem 1 to deduce the result. (This uses the uniformity on compact subsets of Γ_F). ⋈

The Central Limit Theorem now follows easily from Corollary 4.

COROLLARY 5 (Central Limit Theorem). [Ra], [DP]. – *For every* $c \in \mathbb{R}$,

$$m\left\{y : \frac{F^T(y) - T\int F dm}{\sqrt{\beta''(0)}\sqrt{T}} \le c\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{1}{2}u^2} du \qquad \textit{as } T \to \infty.$$

Remarks 2. – (i) It will be interesting to see whether it is possible to estimate the rate of convergence in the Central Limit Theorem using our techniques. Such results are known for discrete processes such as subshifts of finite type, [CP], [GH].

(ii) One can derive similar results to those in this section when F, σ^r are not flow independent, but where σ^r is still assumed to be topologically weak mixing. In particular, we obtain the Central Limit Theorem for any topologically weak mixing flow. We hope to analyse these other cases in a future article.

4. THE MOMENT GENERATING FUNCTION

We first give some basic notation and definitions. Let $F \in \mathcal{F}_\alpha^r(\mathbb{C})$, $G \in \mathcal{F}_\alpha^r(\mathbb{R})$, and let

$$f(x) = \int_0^{r(x)} F(x,t)dt \in \mathcal{F}_\alpha(\mathbb{C}),$$

and

$$g(x) = \int_0^{r(x)} G(x,t)dt \in \mathcal{F}_\alpha(\mathbb{R}).$$

Let $m = m_G$ denote the (unique) equilibrium state of G and let E_G denote the expectation operator

$$E_G(H) = \int H dm_G,$$

for each $H \in C(\Sigma_A^r)$. Define the *moment generating function*, $M = M_{G,F}$ by

$$M(T) = E_G(e^{F^T}).$$

The Laplace transform $Z(s) = Z_{G,F}(s)$ of M is given by

$$Z(s) = \int_0^\infty e^{-sT} M(T)dT \tag{4.1}$$

for $s \in \mathbb{C}$, whenever the integral converges.

Substituting $(-iw + \rho(a))(F - a)$, where $F \in \mathcal{F}_\alpha^r(\mathbb{R})$ and $a \in \Gamma_F$, we let

$$Z(s, w, a) = Z_{G,(-iw+\rho(a))(F-a)}(s) \tag{4.2}$$

where $w \in \mathbb{R}$, whenever this is well defined.

PROPOSITION 4. – *Suppose that $F \in \mathcal{F}_\alpha^r(\mathbb{R})$ and σ^r are flow independent. Let $a \in \Gamma_F$ be arbitrary. Then the following statements hold.*

(i) *$Z(s,w,a)$ is analytic for $(s,w) \in \{s : Re(s) > \gamma(a)\} \times \mathbb{R}$.*

(ii) *There exists an open neighbourhood U of $(\gamma(a), 0)$ in $\mathbb{C}^2$ such that for each $(s,w) \in U$,*

$$Z(s,w,a) = \frac{C(a)\int r d\mu}{1 - e^{P(g+(-iw+\rho(a))(f-ar)-(s+P(G))r)}} + J_1(s,w,a)$$

where $C(a) \neq 0$ depends only on a and $J_1(s,w,a)$ is analytic for all $(s,w) \in U$.

(iii) *$Z(s,w,a)$ is analytic for (s,w) in an open neighbourhood V of $\{s : Re(s) = \gamma(a), s \neq \gamma(a)\} \times \{0\}$.*

(iv) *For each $\omega \in \mathbb{R} \setminus \{0\}$, $Z(s,w,a)$ is analytic for (s,w) in an open neighbourhood W of $\{s : Re(s) = \gamma(a)\} \times \{\omega\}$.*

Moreover, if $J \subset \Gamma_F$ is any non-empty, compact set, and for any $a \in J$,

$$U' = U - \{(\gamma(a),0)\}, \quad V' = V - \{(\gamma(a),0)\}, \quad W' = W - \{(\gamma(a),0)\}$$

(where $X - \{y\} = \{x - y : x \in X\}$, for $X \subset \mathbb{C}^N$, $y \in \mathbb{C}^N$), then U', V', W' and ε can be chosen to depend only on J.

We will require the following elementary lemma in the proof of Proposition 4.

LEMMA 1. – *If $K \in \mathcal{F}_\alpha^r(\mathbb{C})$ then*

$$\tilde{k}(x) = \int_0^{r(x)} e^{K^\theta(x,0)} d\theta \in \mathcal{F}_\alpha(\mathbb{C}).$$

Proof of Lemma 1. – We will use the elementary inequality

$$|e^{z_1} - e^{z_2}| \leq |z_1 - z_2| e^{|z_1|+|z_2|} \qquad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Suppose that $K \in \mathcal{F}_\alpha^r(\mathbb{C})$, and note that

$$\begin{aligned}
&|\tilde{k}(x) - \tilde{k}(y)| \\
&= \left| \int_0^{r(x)} e^{K^\theta(x,0)} d\theta - \int_0^{r(y)} e^{K^\theta(y,0)} d\theta \right| \\
&\leq \int_0^{r(x)} |e^{K^\theta(x,0)} - e^{K^\theta(y,0)}| d\theta + \left| \int_{r(y)}^{r(x)} e^{K^\theta(y,0)} d\theta \right| \\
&\leq e^{2\|r\|_\infty \|K\|_\infty} \int_0^{r(x)} \int_0^\theta |K(x,u) - K(y,u)| du d\theta \\
&\quad + e^{\|K\|_\infty \|r\|_\infty} |r(x) - r(y)| \\
&\leq C d(x,y)
\end{aligned}$$

for a constant $C > 0$, since $r \in \mathcal{F}_\alpha(\mathbb{R})$. Thus $\tilde{k} \in \mathcal{F}_\alpha(\mathbb{C})$ as claimed. ⋈

Proof of Proposition 4. – Given real numbers $T, \theta \geq 0$, there exists a unique choice of $n \geq 0$ and $v \in [0, r^n(x))$ such that $T + \theta = v + r^n(x)$. For this choice of n, v we have the identity

$$e^{F^{T+\theta}(x,0)} = e^{F^{v+r^n(x)}(x,0)}.$$

An alternative, shorthand way of expressing this is to use a modified version of a technical device employed in [Po2], p. 418. This is the identity

$$e^{F^{T+\theta}(x,0)} = \sum_{n=0}^{\infty} \left(\int_0^{r(\sigma^n x)} e^{F^{v+r^n(x)}(x,0)} \delta\big(\theta + T - v - r^n(x)\big) dv \right), \tag{4.3}$$

where δ denotes the Dirac Delta Function. Note that only one term in the summation in (4.3) can be non-zero. Equation (4.3) can be interpreted rigorously as described above.

From section 2, the equilibrium state $m = m_G$ of G may be expressed as $m = \frac{\mu \times l}{\int r d\mu}$ where $\mu = \mu_{g-P(G)r}$ is the equilibrium state of $g - P(G)r$. Hence we have,

$$\begin{aligned} &Z_{G,F}(s) \\ &= \frac{1}{\int r d\mu} \int_0^{\infty} e^{-sT} \left(\int \int_0^{r(x)} e^{F^T(x,\theta)} d\theta d\mu(x) \right) dT \\ &= \frac{1}{\int r d\mu} \int_0^{\infty} e^{-sT} \left(\int \int_0^{r(x)} e^{F^{T+\theta}(x,0) - F^{\theta}(x,0)} d\theta d\mu(x) \right) dT. \end{aligned} \tag{4.4}$$

Substituting (4.3) into (4.4) gives

$$\begin{aligned} &Z_{G,F}(s) \\ &= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Bigg(\int \int_0^{r(x)} e^{-s(r^n(x)-\theta)} \left\{ \int_0^{r(\sigma^n x)} e^{-sv + F^{v+r^n(x)}(x,0)} dv \right\} \\ &\qquad \times e^{-F^{\theta}(x,0)} d\theta d\mu(x) \Bigg) \\ &= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Bigg(\int e^{-sr^n(x)} \Bigg\{ \int_0^{r(\sigma^n x)} e^{-sv + F^{v+r^n(x)}(x,0)} dv \\ &\qquad \times \int_0^{r(x)} e^{s\theta - F^{\theta}(x,0)} d\theta \Bigg\} d\mu(x) \Bigg) \end{aligned}$$

$$= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Big(\int e^{-sr^n(x)+f^n(x)} \Big\{ \int_0^{r(\sigma^n x)} e^{-sv+F^v(\sigma^n x,0)} dv \times \int_0^{r(x)} e^{s\theta - F^\theta(x,0)} d\theta \Big\} d\mu(x) \Big)$$

$$= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Big(\int e^{-sr^n(x)+f^n(x)} B_1(s, \sigma^n x) B_2(s,x) d\mu(x) \Big) \tag{4.5}$$

where

$$B_1(s,x) = \int_0^{r(x)} \exp\{-sv + F^v(x,0)\} dv$$

and

$$B_2(s,x) = \int_0^{r(x)} \exp\{s\theta - F^\theta(x,0)\} d\theta.$$

We can now legitimately make three further assumptions. For some $\alpha > 0$, these assumptions are as follows.

(*a*) By adding a coboundary to r, we can assume that $r \in \mathcal{F}_\alpha^+(\mathbb{R})$, (*cf.* section 2). In particular, this leaves invariant the topological conjugacy class of σ^r.

(*b*) By adding a coboundary to g, we can assume $g \in \mathcal{F}_\alpha^+(\mathbb{R})$.

(*c*) We can assume that $F \in \mathcal{F}_\alpha^r(\mathbb{R})$ does not depend on x_i for all $i < 0$ by first approximating F by functions depending on only finitely many Σ_A-coordinates. Thus we may also suppose that $B_1(s,.), B_2(s,.)$ depend only on future coordinates for all $s \in \mathbb{C}$, and that $f \in \mathcal{F}_\alpha^+(\mathbb{R})$.

By (1.1), we may write $d\mu = h d\nu$, where $h = h_{g-P(G)r} \in \mathcal{F}_\alpha^+(\mathbb{R})$ and $\nu = \nu_{g-P(G)r}$. Thus by Proposition 1(i),(ii), and (4.5),

$$Z_{G,F}(s) = \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Big(\int e^{-sr^n(x)+f^n(x)} B_1(s, \sigma^n x) B_2(s,x) h(x) d\nu(x) \Big)$$

$$= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \Big(\int \mathcal{L}_{g-P(G)r}^n \{ e^{-sr^n(x)+f^n(x)} \times B_1(s, \sigma^n x) B_2(s,x) h(x) \} d\nu(x) \Big)$$

$$= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \left(\int \mathcal{L}^n_{g-(s+P(G))r+f} \{B_1(s, \sigma^n x) B_2(s, x) h(x)\} d\nu(x) \right)$$

$$= \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \left(\int B_1(s, x) \mathcal{L}^n_{g-(s+P(G))r+f} \{B_2(s, x) h(x)\} d\nu(x) \right). \quad (4.6)$$

By Lemma 1, $B_2(s, x) \in \mathcal{F}^+_\alpha(\mathbb{C})$, for each $s \in \mathbb{C}$. By (4.2), we therefore have

$$Z(s, w, a) = \frac{1}{\int r d\mu} \sum_{n=0}^{\infty} \left(\int B_3(s, w, a, .) \mathcal{L}^n_{g+(-iw+\rho(a))(f-ar)-(s+P(G))r} \{h B_4(s, w, a, .)\} d\nu \right) \quad (4.7)$$

where

$$B_3(s, w, a, x) = \int_0^{r(x)} \exp\{-sv + (-iw + \rho(a))(F - a)^v(x, 0)\} dv$$

and

$$B_4(s, w, a, x) = \int_0^{r(x)} \exp\{s\theta - (-iw + \rho(a))(F - a)^\theta(x, 0)\} d\theta.$$

We now proceed to describe the analytic domain of $Z(s, w, a)$. It is useful here to regard (4.7) as the definition of $Z(s, w, a)$, as the expression in (4.2) may not have an extension beyond its domain of convergence. For notational convenience, we take

$$b(s, w, a) = g + (-iw + \rho(a))(f - ar) - (s + P(G))r. \quad (4.8)$$

First let

$$Y_4(s, w, a) = \sum_{n=0}^{\infty} \left(\int B_3(s, w, a, .) \mathcal{L}^n_{b(s,w,a)} \{h B_4(s, w, a, .)\} d\nu \right) \quad (4.9)$$

and

$$B_5(s, w, a) = \|B_3(s, w, a, .)\|_\infty \|h B_4(s, w, a, .)\|_\infty.$$

Proof of (i). – By Proposition 1(iii), we have for each fixed (s, w, a), for some $\alpha \in (0,1)$ depending on s, a,

$$\begin{aligned}
&|Y_4(s,w,a)| \\
&\quad \leq B_5(s,w,a) \sum_{n=0}^{\infty} \left\| \mathcal{L}^n_{g+\rho(a)(f-ar)-(Re(s)+P(G))r} 1 \right\|_{\infty} \\
&\quad \leq B_5(s,w,a) \sum_{n=0}^{\infty} \exp\{nP(g+\rho(a)(f-ar) \\
&\qquad - (Re(s)+P(G))r\}(1+\alpha^n) \\
&\quad < \infty
\end{aligned}$$

provided $P(g + \rho(a)(f - ar) - (\mathrm{Re}(s) + P(G))r) < 0$. By the strict monotonicity of pressure, this is true if Re(s) $> \gamma(a)$. Thus $Z(s, w, a)$ is analytic for $(s, w) \in \{s : \mathrm{Re}(s) > \gamma(a)\} \times \mathbb{R}$, for any $a \in \Gamma_F$, which proves (i).

Proof of (ii). – By Proposition 2(i), provided $\varepsilon > 0$ is sufficiently small, and U is a sufficiently small neighbourhood of $\gamma(a) \in \mathbb{C}$, there exists $\delta_1 > 0$ such that

$$\begin{aligned}
(1+\delta_1)^n \Bigg\| &\frac{\mathcal{L}^n_{b(s,w,a)}\{hB_4(s,w,a,.)\}}{\exp\{nP(b(s,w,a))\}} \\
&- h_{b(s,w,a)} \int hB_4(s,w,a,.)d\nu_{b(s,w,a)} \Bigg\|_{\alpha} \longrightarrow 0 \quad (4.10)
\end{aligned}$$

as $n \to \infty$, for each $s \in U$, and $w \in (-\varepsilon, \varepsilon)$. From (4.7) and (4.10), we therefore have that

$$Z(s,w,a) = \frac{B_6(s,w,a)}{1 - e^{P(b(s,w,a))}} + J_2(s,w,a)$$

where $J_2(s, w, a)$ is analytic for $(s, w) \in U_1$. Here, U_1 is an open neighbourhood of $(\gamma(a), 0) \in \mathbb{C}^2$. The function B_6 is defined by

$$\begin{aligned}
&B_6(s,w,a) \\
&= \frac{1}{\int r d\mu} \int B_3(s,w,a,.)h_{b(s,w,a)}d\nu . \int hB_4(s,w,a,.)d\nu_{b(s,w,a)}. \quad (4.11)
\end{aligned}$$

Calculation of the residue at $(s, w) = (\gamma(a), 0)$ gives

$$Z(s,w,a) = \frac{C(a) \int r d\mu}{1 - e^{P(b(s,w,a))}} + J_3(s,w,a)$$

where $J_3(s, w, a)$ is analytic for $(s, w) \in U_2$, where U_2 is an open neighbourhood of $(\gamma(a), 0)$. Further, the constant $C(a)$ is given by

$$C(a) = \frac{B_6(\gamma(a), 0, a)}{\int r d\mu} \tag{4.12}$$

where B_6 is defined by (4.11). In particular, $C(a) > 0$ for all $a \in \Gamma_F$. This proves (ii).

Proof of (iii). – Now suppose that $t \neq 0$. There are two cases to consider. Firstly suppose that

$$b(\gamma(a) + it, 0, a) \in L_\alpha^+(\mathbb{C}),$$

say $\mathrm{Im}(b(\gamma(a) + it, 0, a))$ is cohomologous to $2\pi M + c$ for some $c \in [0, 2\pi)$ and $M \in C(\Sigma_A; \mathbb{Z})$. By applying Proposition 2(i), as in part (ii), there exists an open neighbourhood U_3 of $(\gamma(a) + it, 0)$ in $\mathbb{C}^2$ such that

$$Z(s, w, a) = \frac{B_7(s, w, a)}{1 - e^{P(b(s,w,a))}} + J_4(s, w, a),$$

for $(s, w) \in U_3$, for an explicit function $B_7(s, w, a)$. Further, $J_4(s, w, a)$ is analytic for $(s, w) \in U_3$. By Proposition 3(i), σ^r is topologically weak mixing, and hence $c \neq 0$. Thus $P(b(s, w, a)) \neq 0$ for all $(s, w) \in U_3$, and so $Z(s, w, a)$ is analytic in U_3.

Secondly, suppose that

$$b(\gamma(a) + it, 0, a) \in I_\alpha^+(\mathbb{C}).$$

By Proposition 1(iv), there exists $\eta_1 \in (0, 1)$ such that

$$\rho(\mathcal{L}_{b(\gamma(a)+it,0,a)}) \leq 1 - \eta_1.$$

By the upper semicontinuity of the map $\mathcal{F}_\alpha^+(\mathbb{R}) \to \mathbb{C}$ defined by $g \mapsto \rho(\mathcal{L}_g)$, given $\eta_2 \in (\eta_1, 1)$, we can find $\varepsilon_1 > 0$ and $\xi_1 > 0$ such that

$$\rho(\mathcal{L}_{b(\gamma(a)+\lambda+it,w,a)}) \leq 1 - \eta_2$$

for all $|\lambda| \leq \xi_1, |w| \leq \varepsilon_1$. By Proposition 1(iv) again, the compact set

$$\{b(\gamma(a) + \lambda + it, w, a) : |\lambda| \leq \xi_1, |w| \leq \varepsilon_1\}$$

is contained in $I_\alpha^+(\mathbb{C})$. Therefore by Proposition 2(ii), there exists $\delta_2 > 0$ such that for $s = \gamma(a) + \lambda + it$,

$$(1 + \delta_2)^n \left\| \frac{\mathcal{L}_{b(s,w,a)}^n\{hB_4(s, w, a)\}}{\exp\{nP(\mathrm{Re}\ b(s, w, a))\}} \right\|_\alpha \longrightarrow 0 \quad \text{as } n \to \infty,$$

for all $|\lambda| < \xi_1$ and $w \in (-\varepsilon_1, \varepsilon_1)$. Thus by (4.7) again, $Z(s,w,a)$ is analytic for $(s,w) \in \big(\gamma(a) + i(t-\xi_1), \gamma(a) + i(t+\xi_1)\big) \times (-\varepsilon_1, \varepsilon_1)$.

Combining these two cases shows that, for each $a \in \Gamma_F$,the function $Z(s,w,a)$ is analytic in an open neighbourhood of $\{s : \mathrm{Re}(s) = \gamma(a), s \neq \gamma(a)\} \times \{0\}$, as required. This completes the proof of (iii).

Proof of (iv). – Let $\omega \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$ be fixed. Again there are two cases to consider. First suppose that

$$b(\gamma(a) + it, \omega, a) \in L_\alpha^+(\mathbb{C}),$$

say $\mathrm{Im}(b(\gamma(a)+it,\omega,a))$ is cohomologous to $2\pi M + c$ for some $c \in [0, 2\pi)$ and $M \in C(\Sigma_A; \mathbb{Z})$. By proceeding as in part (ii), there exists an open neighbourhood U_4 of $(\gamma(a) + it, \omega)$ in $\mathbb{C}^2$ such that

$$Z(s,w,a) = \frac{B_8(s,w,a)}{1 - e^{P(b(s,w,a))}} + J_5(s,w,a),$$

for $(s,w) \in U_4$, for an explicit function $B_8(s,w,a)$. Further, $J_5(s,w,a)$ is analytic for $(s,w) \in U_4$. By Proposition 3, f and r are σ-independent, and hence $c \neq 0$. Thus $P(b(s,w,a)) \neq 0$ for all $(s,w) \in U_4$. Hence $Z(s,w,a)$ is analytic in U_4.

Secondly, suppose that

$$b(\gamma(a) + it, \omega, a) \in L_\alpha^+(\mathbb{C}).$$

By duplicating part of the argument for case (iii), we can find $\varepsilon_2 > 0$ and $\xi_2 > 0$ such that the compact set

$$\{b(\gamma(a) + \lambda + it, w, a) : |\lambda| \leq \xi_2, |w - \omega| \leq \varepsilon_2\}$$

is contained in $I_\alpha^+(\mathbb{C})$. Thus by Proposition 2(ii), there exists $\delta_3 > 0$ such that for $s = \gamma(a) + \lambda + it$,

$$(1+\delta_3)^n \left\| \frac{\mathcal{L}^n_{b(s,w,a)}\{hB_4(s,w,a,.)\}}{\exp\{nP(\mathrm{Re}\ b(s,w,a))\}} \right\|_\alpha \longrightarrow 0 \quad \text{as } n \to \infty,$$

for $|\lambda| < \xi_2$ and $|w - \omega| < \varepsilon_2$. Thus by (4.7), $Z(s,w,a)$ is analytic for all

$$(s,w) \in \big(\gamma(a) + i(t-\xi_2), \gamma(a) + i(t+\xi_2)\big) \times (-\varepsilon_2, \varepsilon_2).$$

Combining these two cases shows $Z(s,w,a)$ is analytic in an open neighbourhood of $\{s : \mathrm{Re}(s) = \gamma(a)\} \times \{\omega\}$, for each $\omega \in \mathbb{R} \setminus \{0\}$ as required.

Proof of uniformity. – We consider the proofs of parts (ii)-(iv) again. Let $J \subset \Gamma_F$ be a non-empty, compact set.

(ii) By Proposition 2(i), by choosing U, ε sufficiently small, we may suppose that δ_1 in (4.10) depends only on J. Thus U_1', U_2' also depend only on J.

(iii) In the case $\mathrm{Im}(b(\gamma(a) + it, 0, a))$ is cohomologous to $2\pi M + c$, for some $c \in [0, 2\pi)$ and $M \in C(\Sigma_A; \mathbb{Z})$, we can apply Proposition 2(i), as in (ii), to deduce that $U_3' = U_3 - (\gamma(a), 0)$ depends only on J. In the case

$$b(\gamma(a) + it, 0, a) \in I_\alpha^+(\mathbb{C})$$

for all $a \in J$, we can choose $\varepsilon, \xi_1 > 0$ depending only on J, by the upper semicontinuity of the map $g \mapsto \rho(\mathcal{L}_g)$ and the compactness of J. Then, by Proposition 2(ii), we can choose $\delta_2 > 0$ to depend only on J. Combining these two cases allows us to choose a neighbourhood V of

$$\{s : \mathrm{Re}(s) = \gamma(a), s \neq \gamma(a)\} \times \{0\}$$

such that $V' = V - \{(\gamma(a), 0)\}$ depends only on J.

(iv) This is similar to part (iii). ⋈

5. PROOF OF THEOREM 1

Let Δ denote the space of C^∞ test functions on $\mathbb{R}$, *i.e.*

$$\Delta = \left\{u \in C^\infty(\mathbb{R}) : \sup_{t \in \mathbb{R}}\{(1 + |t|^m)|u^{(n)}(t)|\} < \infty, \text{ for all } m, n \geq 0\right\}.$$

For $u \in \Delta$, let $\hat{u}$ denote the Fourier-Laplace transform of u, defined by

$$\hat{u}(z) = \int_{-\infty}^{\infty} e^{zt} u(t) dt \tag{5.1}$$

for $z \in \mathbb{C}$, wherever the integral converges. If $u \in \Delta$, then the function $\mathbb{R} \to \mathbb{R}$ given by $w \mapsto \hat{u}(iw)$ is also in Δ.

Let $(k_N)_{N=1}^\infty$ denote the approximate identity whose Fourier transform is given by

$$\hat{k}_N(iw) = \begin{cases} \exp\left\{-\dfrac{w^2}{N^2 - w^2}\right\} & \text{if } |w| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

(One can obtain an explicit formula for k_N itself by Fourier inversion). We remark that the map $w \mapsto \hat{k}_N(iw)$ is in Δ, and that $\hat{k}_N(iw) \to 1$ as $N \to \infty$, uniformly for w in any compact subset of $\mathbb{R}$.

Let $J \subset \Gamma_F$ be a fixed non-empty, compact set. To prove Theorem 1, it suffices, by standard smoothing arguments, to show that for any non-negative $u \in \Delta$ with compact support,

$$E_G u(F^T - aT) \sim \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \hat{u}(-\rho(a)) \frac{e^{T\gamma(a)}}{\sqrt{T}} \quad \text{as } T \to \infty, \tag{5.2}$$

uniformly for all $a \in J$.

Let

$$H_T(t) = m_G\{y \in \Sigma_A^r : (F^T - aT)(y) \le t\},$$

and note that H_T has Fourier transform

$$\begin{aligned} \hat{H}_T(z) &= \int e^{z(F^T - aT)(y)} dm(y) \\ &= E_G(e^{z(F^T - aT)}). \end{aligned}$$

So we have

$$\begin{aligned} &E_G u(F^T - aT) \\ &= \int_{-\infty}^{\infty} u(t) dH_T(t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(iw) \hat{H}_T(-iw) dw \qquad \text{by Parseval's identity, [Ka], p. 132,} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(iw - \rho(a)) \hat{H}_T(-iw + \rho(a)) dw \\ &\qquad\qquad \text{by translating the path of integration,} \\ &= \int_{-\infty}^{\infty} u(t) e^{-\rho(a)t} d\Big(H_T(t) e^{\rho(a)t}\Big) \qquad \text{by the Parseval identity again,} \\ &= \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{T\gamma(a)}}{\sqrt{T}} \int_{-\infty}^{\infty} u(t) e^{-\rho(a)t} dq_{-\rho(a)}^{(T)}(t) \end{aligned} \tag{5.3}$$

where $q_{-\rho(a)}^{(T)}$ is a positive Borel measure on $\mathbb{R}$ given by

$$dq_{-\rho(a)}^{(T)}(t) = \frac{\sqrt{2\pi\beta''(\rho(a))}}{C(a)} \frac{\sqrt{T}}{e^{T\gamma(a)}} e^{\rho(a)t} dH_T(t). \tag{5.4}$$

To justify translating the contour of integration, note that the family $\{u(t)e^{ct}\}_{c \in K}$ is compact in $C(\mathrm{supp}\, u; \mathbb{R})$, for any compact set $K \subset \mathbb{R}$.

Thus the Fourier transforms satisfy $\hat{u}(iw + c) \to 0$ as $|w| \to \infty$, uniformly for $c \in K$, by the Riemann Lebesgue Lemma.

To prove (5.2), it therefore suffices to show that

$$\int_{-\infty}^{\infty} u(t)e^{-\rho(a)t} dq_{-\rho(a)}^{(T)}(t) \to \int_{-\infty}^{\infty} u(t)e^{-\rho(a)t} dt = \hat{u}(-\rho(a)) \tag{5.5}$$

as $T \to \infty$, uniformly for all $a \in J$.

We will use the following lemma which is a modified version of Lemma 2 in [La1], or alternatively Theorem G in [La2], for a continuous, rather than discrete, family of functions. The proofs are entirely analogous.

LEMMA 2. – *Let $(k_N)_{N=1}^{\infty}$ be an approximate identity and let $J \subset \mathbb{R}$ be a closed interval. Let $(q_\xi^{(T)})_{\xi \in J}$ be a collection of positive Borel measures on $\mathbb{R}$, having the property that, for each finite interval I of $\mathbb{R}$, there exists a constant K_I such that*

$$\sup_{T \geq 1} \sup_{\xi \in J} \sup_{t \in \mathbb{R}} q_\xi^{(T)}(I + t) \leq K_I. \tag{5.6}$$

Suppose that for each $T \geq 1$ and each $v \in \Delta$ with $v \geq 0$,

$$\lim_{T \to \infty} \sup_{\xi \in J} \left| \int (k_N * v) dq_\xi^{(T)} - \int (k_N * v) dq_\xi \right| = 0 \tag{5.7}$$

for certain positive Borel measures $(q_\xi)_{\xi \in J}$ on $\mathbb{R}$. Then for any family $(u_\xi)_{\xi \in J}$ of non-negative functions in Δ such that $\xi \mapsto u_\xi$ is continuous in the Δ-topology,

$$\lim_{T \to \infty} \sup_{\xi \in J} \left| \int u_\xi dq_\xi^{(T)} - \int u_\xi dq_\xi \right| = 0. \tag{5.8}$$

First note that, for a function $u \in \Delta$ with compact support, the map $\Gamma_F \to \Delta$ given by $a \mapsto u(t)e^{\rho(a)t}$ is continuous in the Δ-topology. To prove (5.5), it therefore suffices, by Lemma 2, to verify (5.6) and (5.7) for the measures $q_{-\rho(a)}^{(T)}$.

First we prove (5.7). Let $a \in \Gamma_F$ and let $v \in \Delta$ be non-negative. Define

$$Z_{N,v}(s, a) = \int_0^{\infty} e^{-sT} E_G\big(v_{N,a}(F^T - aT)\big) dT$$

where $v_{N,a} = (k_N e_{\rho(a)}) * (v e_{\rho(a)})$ and $e_{\rho(a)} : \mathbb{R} \to \mathbb{R}$ is given by $e_{\rho(a)}(t) = e^{\rho(a)t}$. Then by Fourier inversion, we have as in (5.3) that

$$\begin{aligned} Z_{N,v}(s,a) &= \int_0^\infty e^{-sT}\left(\frac{1}{2\pi}\int_{-\infty}^\infty E_G\left(e^{(-iw+\rho(a))(F^T-aT)}\right)\hat{v}_{N,a}(iw-\rho(a))dw\right)dT \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \hat{k}_N(iw)\hat{v}(iw)\left(\int_0^\infty e^{-sT}E_G\left(e^{(-iw+\rho(a))(F^T-aT)}\right)dT\right)dw \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \hat{k}_N(iw)\hat{v}(iw)Z(s,w,a)dw \\ &= \frac{1}{2\pi}\int_{-N}^N \hat{k}_N(iw)\hat{v}(iw)Z(s,w,a)dw \\ &= \frac{1}{2\pi}\int_{-\varepsilon}^\varepsilon \hat{k}_N(iw)\hat{v}(iw)Z(s,w,a)dw + J_1(s) \end{aligned} \tag{5.9}$$

where $J_1(s)$ is analytic in V, where V is an open neighbourhood of $\{s : \text{Re(s)} \geq \gamma(a)\}$, by Proposition 4(i),(iv), and $V - \{\gamma(a)\}$ depends only on J.

By Proposition 4(i),(ii) and (iii), there exists an open neighbourhood U_1 of $\{s : \text{Re}(s) \geq \gamma(a)\} \times \{0\}$ such that for $(s,w) \in U_1$,

$$Z(s,w,a) = \frac{C(a)\int r d\mu}{1 - e^{P(g+(-iw+\rho(a))(f-ar)-(s+P(G))r)}} + J_2(s,w,a) \tag{5.10}$$

where $J_2(s,w,a)$ is analytic for $(s,w) \in U_1$, and $U_1' = U_1 - (\gamma(a),0)$ depends only on J.

LEMMA 3. – *For any* $a \in \Gamma_F$,

(i) *there exists a function* $s(w,a)$*, well defined and real analytic in* w *for* $|w| < \varepsilon$*, for all* ε *sufficiently small, such that* $s(w,a)$ *is the unique simple pole of* $Z(s,w,a)$ *in an open neighbourhood of* $(s,w) = (\gamma(a),0)$,

(ii) $\frac{\partial}{\partial w}\text{Re } s(w,a)|_{w=0} = 0$,

(iii) $\frac{\partial}{\partial w}\text{Im } s(w,a)|_{w=0} = 0$,

(iv) $\frac{\partial^2}{\partial w^2}\text{Re } s(w,a)|_{w=0} = -\beta''(\rho(a)) < 0$,

(v) $\frac{\partial^2}{\partial w^2}\text{Im } s(w,a)|_{w=0} = 0$,

(vi) Re $s(w,a)$ *is an even function of* w,

(vii) $Z(s,w,a)$ *has no poles in* U_1 *except* $s(w,a)$.

Proof. – From section 2, we may assume that $r, f, g \in \mathcal{F}_\alpha^+(\mathbb{R})$ for some $\alpha > 0$. Let

$$\lambda(s, w, a) = \exp\{P(g + (-iw + \rho(a))(f - ar) - (s + P(G))r)\}$$

which is the unique maximal eigenvalue of the Ruelle operator

$$\mathcal{L}_{g+(-iw+\rho(a))(f-ar)-(s+P(G))r)}$$

for $|w| < \varepsilon$. Furthermore, we have $\lambda(\gamma(a), 0, a) = 1$. By the Perturbation Theory for the Ruelle operator, (*see* section one), λ is analytic for (s, w) in an open neighbourhood of $(\gamma(a), 0)$ for each $a \in \Gamma_F$, and

$$\left.\frac{\partial\lambda}{\partial s}\right|_{(s,w,a)=(\gamma(a),0,a)} < 0. \tag{5.11}$$

By (5.11), we can apply the Implicit Function Theorem to find $s = s(w, a)$ which satisfies

$$\lambda(s(w, a), w, a) = 1, \qquad s(0, a) = \gamma(a). \tag{5.12}$$

Part (i) follows from the representation of $Z(s, w, a)$ given in (5.10).

By differentiating (5.12) with respect to w, we have

$$\frac{\partial\lambda}{\partial s}\frac{\partial s}{\partial w} + \frac{\partial\lambda}{\partial w} = 0. \tag{5.13}$$

Thus

$$\begin{aligned}
-\left.\frac{\partial\lambda}{\partial w}\right|_{(s,w,a)=(\gamma(a),0,a)} &= i\int r d\mu \int (F - a) dm_{G+\rho(a)(F-a)} \\
&= i\int r d\mu \int (F - a) dm_{G+\rho(a)F} \\
&= i\int r d\mu \quad \big(\beta'(\rho(a)) - a\big) \\
&= i\int r d\mu \quad (a - a) \\
&= 0.
\end{aligned} \tag{5.14}$$

Also, by differentiation of (5.12) with respect to s,

$$\left.\frac{\partial\lambda}{\partial s}\right|_{(s,w,a)=(\gamma(a),0,a)} = -\int r d\mu \neq 0. \tag{5.15}$$

Together, (5.13)-(5.15) prove (ii) and (iii).

To prove (iv),(v), we use the identity

$$\frac{\partial^2}{\partial w^2} s(0,a) = -\sigma^2_{m_{G+\rho(a)F}}(F),$$

a similar version of which occurs in [KS], Proposition 2.1. By hypothesis, F is not cohomologous to a constant function, and so

$$\sigma^2_{m_{G+\rho(a)F}}(F) > 0.$$

By definition, we have the identity

$$\beta''(\rho(a)) = \sigma^2_{m_{G+\rho(a)F}}(F).$$

To prove (vi), since $\overline{Z(s,w,a)} = Z(\overline{s},-w,a)$, it follows immediately that $\overline{s(w,a)} = s(-w,a)$, and hence Re $s(w,a) =$ Re $s(-w,a)$.

Finally, (vii) follows immediately from Proposition 4(iii). ⋈

Now we remark that

$$\lim_{(s,w)\to(\gamma(a),0)} \frac{1 - e^{P(g+(-iw+\rho(a))(f-ar)-(s+P(G))r)}}{s-\gamma(a)} = \int r\,d\mu,$$

where $\int r\,d\mu \neq 0$, (*cf.* [PP], page 75, for a similar calculation).

Thus, by applying Lemma 3 to (5.10), there exists an open neighbourhood U_2 of $\{s : \text{Re}(s) \geq \gamma(a)\} \times \{0\}$ such that for $(s,w) \in U_2$,

$$Z(s,w,a) = \frac{C(a)}{(s - s(w,a))} + J_3(s,w,a) \tag{5.16}$$

where $J_3(s,w,a)$ is analytic for $(s,w) \in U_2$, and U_2 depends only on J.

By applying the Morse Lemma to Re $s(w,a)$, we have by Lemma 3 that there exists a function $y = y(w,a)$ defined for $|w| < \varepsilon$ and $a \in \Gamma_F$ such that

$$\text{Re } s(w,a) = \gamma(a) - y^2 \tag{5.17}$$

(A statement of the Morse Lemma in the precise form in which we require it is given in [KS], Lemma 3.4).

By combining (5.16),(5.17) and the observations concerning $s(w,a)$ in Lemma 3, we can rewrite (5.9) as

$$Z_{N,v}(s,a) = \frac{C(a)}{2\pi} \int_{-\varepsilon}^{\varepsilon} \frac{\hat{k}_N(iw)\hat{v}(iw)}{s - \gamma(a) + y(w,a)^2 + iq(w,a)} dw + J_4(s) \tag{5.18}$$

where $J_4(s)$ is analytic in an open neighbourhood U_3 of $\mathrm{Re}(s) \geq \gamma(a)$, depending on ε, and where $U_3 - \{\gamma(a)\}$ depends only on J. Also, $q(w,a) = \mathrm{Im}\ s(w,a)$ satisfies

$$\frac{\partial q}{\partial w} = 0 = \frac{\partial^2 q}{\partial w^2}$$

by Lemma 3(iii),(v).

By a change of variables, we may rewrite (5.18) as

$$Z_{N,v}(s,a) = \frac{C(a)}{2\pi} \frac{\sqrt{2}\hat{v}(0)}{\sqrt{\beta''(\rho(a))}} \int_{-y(\varepsilon,a)}^{y(\varepsilon,a)} \frac{1 + P(y,a)}{s - \gamma(a) + y^2 + iQ(y,a)} dy + J_4(s) \quad (5.19)$$

where $P(y,a)$ is defined by

$$\frac{\sqrt{2}\hat{v}(0)}{\sqrt{\beta''(\rho(a))}}(1 + P(y,a)) = \hat{k}_N(iw(y,a))\hat{v}(iw(y,a))\frac{\partial y}{\partial w}(w,a).$$

Further, $Q(y,a) = \mathrm{Im}\ s(w(y,a),a)$ is an odd function of y and satisfies

$$Q(0,a) = \frac{\partial Q}{\partial y}(0,a) = \frac{\partial^2 Q}{\partial y^2}(0,a) = 0 \qquad \text{for all } a \in \Gamma_F.$$

(*See* [Sh1], page 278, for a similar calculation).

The integral in (5.19) occurs in a calculation in [KS], section three. By duplicating this analysis, we have the following proposition. The uniformity on compact subsets of Γ_F follows from (5.19), and a careful examination of the calculation in [KS]. Intuitively, the uniformity is obvious as the convergence in Proposition 5 is determined by the convergence in (5.19), which depends uniformly on compact subsets of Γ_F.

PROPOSITION 5. – *Let $a \in \Gamma_F$ be arbitrary. Then the limit*

$$\lim_{\sigma \to \gamma(a)} \left(Z_{N,v}(\sigma + it, a) - \frac{\hat{v}(0)C(a)}{\sqrt{2\beta''(\rho(a))}} \frac{1}{(\sigma + it - \gamma(a))^{1/2}} \right) \quad (5.20)$$

exists for almost every $t \in \mathbb{R}$, and is in the space $W^{1,1}_{loc}(\mathbb{R})$ of locally integrable functions with locally integrable first derivatives. Further, there exists a locally integrable function $h(t)$ such that

$$\left| Z_{N,v}(\sigma + it, a) - \frac{\hat{v}(0)C(a)}{\sqrt{2\beta''(\rho(a))}} \frac{1}{(\sigma + it - \gamma(a))^{1/2}} \right| \leq h(t)$$

for $\sigma > \gamma(a)$.

If $a \in J \subset \Gamma_F$, *with* J *compact, then the convergence in* (5.20) *is uniform in* J, *and* h *can be chosen depending only on* J.

Now we define a function $\tilde{Z}_{N,v}(s,a)$ by a change of variables, namely

$$\begin{aligned}\tilde{Z}_{N,v}(s,a) &= Z_{N,v}(s+\gamma(a)-1,a)\\ &= \int_0^\infty e^{-(s+\gamma(a)-1)T} E_G v_{N,a}\big(F^T - aT\big)dt\\ &= \int_0^\infty e^{-sT} d\alpha_{N,v,a}(T)\end{aligned} \tag{5.21}$$

where

$$\alpha_{N,v,a}(T) = \int_0^T e^{(-\gamma(a)+1)t} E_G v_{N,a}\big(F^t - at\big)dt.$$

The function $\tilde{Z}_{N,v}$ is well defined for $\mathrm{Re}(s) > 1$.

We now require the following Tauberian Theorem, part (i) of which is stated in [KS], (Proposition 4.2). We have made two slight modifications. Firstly, we have made the change of variables $\frac{\alpha(T)}{\sqrt{T}} \to \alpha(T)$. (A result in this precise form was proved by Delange in [De], but with stronger hypotheses). Secondly, uniformity on compact sets follows immediately from the proof given in [KS]. Once more, the uniformity statement is intuitively clear in the same way as in Proposition 5. Part (ii) requires a minor modification to the proof of part (i).

PROPOSITION 6. – *Let* $J \subset \mathbb{R}$ *be compact.*

(i) *Let* $\alpha : J \times \mathbb{R} \to \mathbb{R}$ *be continuous and write* $\alpha_a(T) = \alpha(a,T)$. *Let* $\alpha_a(0) = 0$ *and suppose that* $\alpha_a(T)$ *is monotonic non-decreasing, for all* $a \in J$. *Let* $f_a(s)$ *be a family of functions, depending continuously on* $a \in J$, *such that* $f_a(s)$ *is analytic for all Re*$(s) > 1$ *and all* $a \in J$. *Let* $a \mapsto A_a$ *be a continuous map* $J \to \mathbb{R} \setminus \{0\}$. *Suppose that the identity*

$$f_a(s) = \frac{A_a}{(s-1)^{1/2}} - \int_0^\infty e^{-sT} d\alpha_a(T) \tag{5.22}$$

holds, for all Re$(s) > 1$ *and all* $a \in J$. *Further suppose that*

$$\lim_{\delta \to 0} f_a\big(1+\delta+it\big) \tag{5.23}$$

exists for almost every $t \in \mathbb{R}$, *and is in* $W^{1,1}_{loc}(\mathbb{R})$, *independent of* $a \in J$. *Further suppose that there is a locally integrable function* $h(t)$, *depending only on* J, *such that*

$$\big|f\big(1+\delta+it\big)\big| \le h(t) \tag{5.24}$$

for $\delta > 0$. *Then*

$$\alpha_a(T) \sim \frac{A_a}{\sqrt{\pi}} \frac{e^T}{\sqrt{T}} \quad \textit{as } T \to \infty, \tag{5.25}$$

uniformly for $a \in J$.

(ii) *Let* $\alpha : J \times \mathbb{R} \to \mathbb{R}$ *be continuous and suppose that* $\alpha_a(T)$ *is non-negative for all* $a \in J$. *Suppose that* f_a, A_a *are as in part* (i), *and that assumptions* (5.22)-(5.24) *hold. Then*

$$\alpha_a(T) = O\left(\frac{e^T}{\sqrt{T}}\right) \quad \textit{as } T \to \infty,$$

where the implied constant depends only on J.

Applying Propositions 5,6(i) to the function $\tilde{Z}_{N,v}$, as defined in (5.21), we obtain

$$\alpha_{N,v,a}(T) \sim \frac{\hat{v}(0)C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^T}{\sqrt{T}} \quad \text{as } T \to \infty. \tag{5.26}$$

uniformly for $a \in J$. In order to obtain an asymptotic formula for $E_G v_{N,a}(F^T - aT)$, we require the following lemma, which is easily deduced from [W], Theorem 15, page 223.

LEMMA 4. – *Let* $J \subset \mathbb{R}$ *be compact, and let* $g : J \times [0, \infty) \to [0, \infty)$ *be continuous and write* $g_a(T) = g(a, T)$. *Suppose that* g_a *is* C^2 *and satisfies*

$$g_a''(T) = O\left(\frac{e^T}{\sqrt{T}}\right) \quad \textit{as } T \to \infty \tag{5.27}$$

for every $a \in J$, *where the implied constant depends only on* J. *Suppose there exists a continuous map* $J \to \mathbb{R} \setminus \{0\}$, $a \mapsto A_a$, *such that*

$$g_a(T) \sim A_a \frac{e^T}{\sqrt{T}} \quad \textit{as } T \to \infty \tag{5.28}$$

uniformly for $a \in J$. *Then*

$$g_a'(T) \sim A_a \frac{e^T}{\sqrt{T}} \quad \textit{as } T \to \infty$$

uniformly for $a \in J$.

We will apply Lemma 4 to $\alpha_{N,v,a}$. First note that $T \mapsto \alpha_{N,v,a}(T)$ is C^2. Furthermore, we may find a non-negative function $\omega_{N,a} \in \Delta$ for which the map $w \mapsto \hat{\omega}_{N,a}(iw)$ has support $[-N, N]$, and a constant C_J depending only on J, such that

$$|\alpha''_{N,v,a}(T)| \leq e^{(-\gamma(a)+1)T} C_J E_G \omega_{N,a}(F^T - aT). \tag{5.29}$$

The function

$$(s, a) \mapsto \int_0^\infty e^{-sT} E_G \omega_{N,a}(F^T - aT) dT$$

shares the properties of $Z_{N,v}(s, a)$ stated in Proposition 5. By integration by parts,

$$\begin{aligned}\int_0^\infty & e^{-sT} d\big(e^{(-\gamma(a)+1)T} E_G \omega_{N,a}(F^T - aT)\big) \\ &= s \int_0^\infty e^{-sT} e^{(-\gamma(a)+1)T} E_G \omega_{N,a}(F^T - aT) dT - \omega_{N,a}(0)\end{aligned} \tag{5.30}$$

for Re$(s) > 1$. Thus we may apply Proposition 6(ii) to the left hand integral in (5.30). Together with (5.29), this yields

$$\alpha''_{N,v,a}(T) = O\left(\frac{e^T}{\sqrt{T}}\right) \quad \text{as } T \to \infty, \tag{5.31}$$

where the implied constant depends only on J. This verifies (5.27). Condition (5.28) follows from (5.26), so by Lemma 4, we have

$$E_G\big((k_N e_{\rho(a)}) * (v e_{\rho(a)})(F^T - aT)\big) \sim \frac{\hat{v}(0) C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \frac{e^{T\gamma(a)}}{\sqrt{T}} \tag{5.32}$$

as $T \to \infty$, uniformly for $a \in J$. Thus from (5.3), we deduce that

$$\int_{-\infty}^\infty (k_N * v)(t) dq^{(T)}_{-\rho(a)}(t) \longrightarrow \hat{v}(0) \quad \text{as } T \to \infty, \tag{5.33}$$

uniformly for $a \in J$. This proves condition (5.7) in Lemma 2.

To complete the proof of Theorem 1, we need also to verify condition (5.6) in Lemma 2. We use an analogous argument to [La1], applying our 'zeta function' analysis at the appropriate point. We include all the details for completeness.

Let I be a closed interval in $\mathbb{R}$, and let $J \subset \Gamma_F$ be compact and non-empty. Choose a non-negative function $v \in \Delta$ such that $v > 1$ on I,

$\hat{v}(iw) \geq 0$ and such that the map $w \mapsto \hat{v}(iw)$ has support $[-N, N]$, for N sufficiently large. It suffices to show that

$$\sup_{T\geq 1}\sup_{a\in J}\sup_{t\in\mathbb{R}}\int_{-\infty}^{\infty} v(\tau+t)dq^{(T)}_{-\rho(a)}(\tau) < \infty.$$

By Parseval's identity,

$$\begin{aligned}&\int_{-\infty}^{\infty} v(\tau+t)dq^{(T)}_{-\rho(a)}(\tau)\\ &\quad= \frac{\sqrt{2\pi\beta''(\rho(a))}}{C(a)}\frac{\sqrt{T}}{e^{T\gamma(a)}}\int_{-N}^{N}\hat{v}(iw)e^{itw}E_G\left(e^{(-iw+\rho(a))(F^T-aT)}\right)dw.\end{aligned}$$

By defining formally $\zeta_t(s,a)$ by

$$\zeta_t(s,a) = \int_0^{\infty} e^{-sT}\left(\int_{-N}^{N}\hat{v}(iw)e^{itw}E_G\left(e^{(-iw+\rho(a))(F^T-aT)}\right)dw\right)dT$$

for $(s,a) \in \mathbb{C}\times\Gamma_F$. By analysing $\zeta_t(s,a)$ in a similar way to $Z_{N,v}(s,a)$, we have

$$\begin{aligned}&\int_{-N}^{N}\hat{v}(iw)e^{itw}E_G\left(e^{(-iw+\rho(a))(F^T-aT)}\right)dw\\ &\qquad\leq K_I\hat{v}(0)\frac{e^{T\gamma(a)}}{\sqrt{T}}\frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}}\end{aligned}$$

for all T, where K_I is a constant depending only on I. This proves (5.6).

6. PROOF OF COROLLARY 2

Fix an $a \in \Gamma_F$ such that $\rho(a) > 0$. Since the map $\gamma : \Gamma_F \to \mathbb{R}$ is real analytic, we may choose $\varepsilon_1 > 0$ so that the power series representation of γ at a converges in the interval $(a-\varepsilon_1, a+\varepsilon_1)$. Also, since $\beta' : \mathbb{R} \to \Gamma_F$ is continuous, surjective and strictly increasing, the set Γ_F is open. Thus we may choose $\varepsilon_2 > 0$ so that $(a, a+\varepsilon_2) \subset \Gamma_F$. We will choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

We will use the inequality

$$\begin{aligned}m\{y : F^T(y) \geq Ta\} =& \sum_{j=0}^{[T\varepsilon]} m\{y : F^T(y) - aT \in [j, j+1)\}\\ &+ m\{y : F^T(y) - aT \geq [T\varepsilon]+1\}. \quad (6.1)\end{aligned}$$

First we consider the first term on the right hand side of (6.1). For each $0 \leq j \leq [T\varepsilon]$, we may write

$$\begin{aligned} m\{y : F^T(y) - aT \in [j, j+1)\} \\ = m\left\{y : F^T(y) - \left(a + \frac{j}{T}\right)T \in [0, 1)\right\}. \end{aligned} \tag{6.2}$$

By our choice of ε, we may write

$$\begin{aligned} \gamma\left(a + \frac{j}{T}\right) &= \gamma(a) + \gamma'(a)\frac{j}{T} + O\left(\frac{1}{T^2}\right) \\ &= \gamma(a) - \rho(a)\frac{j}{T} + O\left(\frac{1}{T^2}\right) \end{aligned} \tag{6.3}$$

for $0 \leq j \leq [T\varepsilon]$. Using Theorem 1, (and in particular the uniformity in a), together with (6.2), (6.3), we have

$$\begin{aligned} &\sum_{j=0}^{[T\varepsilon]} m\{y : F^T(y) - aT \in [j, j+1)\} \\ &\sim \frac{e^{T\gamma(a)}}{\sqrt{T}} \left(\sum_{j=0}^{[T\varepsilon]} e^{-j\rho(a)}\right) \left(\int_0^1 e^{-\rho(a)t} dt\right) \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \\ &\sim \frac{e^{T\gamma(a)}}{\sqrt{T}} \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}} \left(\int_0^\infty e^{-\rho(a)t} dt\right) \quad \text{as } T \to \infty. \end{aligned} \tag{6.4}$$

For the second term on the right hand side of (6.1), note that

$$\begin{aligned} m\{y : F^T(y) - aT \geq [T\varepsilon] + 1\} &\leq m\{y : F^T(y) - (a+\varepsilon)T \geq 0\} \\ &\leq E_G\left(e^{\rho(b)(F^T - bT)}\right) \end{aligned} \tag{6.5}$$

where $b = a + \varepsilon$.

By Proposition 4, $Z(s, 0, b)$ is analytic for $\text{Re}(s) > \gamma(b)$. As in section 5, we now define a function

$$\begin{aligned} \tilde{Z}(s, 0, b) &= Z(s + \gamma(b) - 1, 0, b) \\ &= \int_0^\infty e^{-sT} \alpha_b(T) dT \end{aligned} \tag{6.6}$$

where

$$\alpha_b(T) = e^{(-\gamma(b)+1)T} E_G\left(e^{\rho(b)(F^T - bT)}\right)$$

which is analytic for $\mathrm{Re}(s) > 1$. Integrating (6.6) by parts yields

$$\int_0^\infty e^{-sT} d\alpha_b(T) = s\int_0^\infty e^{-sT}\alpha_b(T)dT - \alpha_b(0)$$

for $\mathrm{Re}(s) > 1$. Hence, by [W] Theorem 2.2a, page 39, for any $\delta > 0$,

$$\frac{\alpha_b(T)}{e^{(1+\delta)T}} \longrightarrow 0 \quad \text{as } T \to \infty$$

and thus

$$E_G\left(e^{\rho(b)(F^T - bT)}\right)e^{-(\gamma(b)+\delta)T} \longrightarrow 0 \quad \text{as } T \to \infty. \tag{6.7}$$

Combining (6.1), (6.4), (6.5) and (6.7) and the fact that $\gamma(b) = \gamma(a+\varepsilon) < \gamma(a)$ completes the proof of Corollary 2.

7. APPLICATION I: SRB MEASURES FOR ANOSOV FLOWS

Let M be a compact C^∞ Riemannian manifold, and let $\phi_t : M \to M$ be a C^1 flow. A point $y \in M$ is called *wandering* if there exists an open neighbourhood U of y such that $\phi_t U \cap U = \emptyset$ for all $t > 0$ sufficiently large. The *non-wandering set* Ω is the complement of the union of the set of wandering points, and is closed and ϕ-invariant.

The flow ϕ is called *Anosov* if TM can be written as a Whitney sum of three $D\phi_t$-invariant continuous subbundles

$$TM = E \oplus E^s \oplus E^u$$

where E is the one-dimensional bundle tangent to the flow, and there are constants $C, \lambda > 0$ such that

(*a*) $\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|$ for $v \in E^s, t \geq 0$,

(*b*) $\|D\phi_{-t}(v)\| \leq Ce^{-\lambda t}\|v\|$ for $v \in E^u, t \geq 0$.

We make a further assumption that $\Omega = M$, *i.e.* the flow ϕ is assumed to be transitive.

For $G \in C(M;\mathbb{R})$, we can define the *pressure* $P(G)$ of G in an analogous way to suspended flows by

$$P(G) = \sup\left\{h(m) + \int G dm : m \text{ is a } \phi\text{-invariant Borel probability measure}\right\}.$$

If G is Hölder continuous, then there is a unique measure at which the supremum is attained, called the *equilibrium state* of G.

Analyticity of pressure and formulae for its derivatives are completely analogous to those for suspended flows. Thus, we can also define β, γ, ρ and Γ_F as before. There are also analogous notions of topological weak mixing, cohomology and flow independence for ϕ.

The connection between suspended flows and transitive Anosov flows is described by the following result of Bowen.

PROPOSITION 7 [B1]. – *Let ϕ be a transitive Anosov flow. Then there exists a suspended flow $\sigma_t^r : \Sigma_A^r \to \Sigma_A^r$ and a Lipschitz continuous, surjective, bounded-to-one map $p : \Sigma_A^r \to M$ such that $p\sigma_t^r = \phi_t p$. If m is the (unique) equilibrium state for the Hölder continuous function $G \in C(M; \mathbb{R})$ then $p^* m$ is the (unique) equilibrium state of $G \circ p$. Furthermore, p is a measure-theoretic isomorphism between these two measures. In particular, ϕ is weak mixing if and only if σ^r weak mixing.*

We remark that Anosov flows are examples of Axiom A attractors (see [PP] chapter 11 for example). Thus, following [BR], for any $x \in M$ and $t > 0$, the map

$$D\phi_t : T_x M \to T_{\phi_t x} M$$

and its restriction

$$D\phi_t|_{E_x^u} : E_x^u \to E_{\phi_t x}^u$$

are well defined. We define a function $\phi^u : M \to \mathbb{R}$ by

$$\phi^u(y) = \lim_{t \to 0} \frac{1}{t} \log \left| \mathrm{Jac}\left(D\phi_t | E_y^u \right) \right|.$$

We now make the additional assumption that ϕ is C^2. Then the splitting $x \mapsto E \oplus E_x^s \oplus E_x^u$ is known to be Hölder continuous, and hence the map $y \mapsto \phi^u(y)$ is Hölder continuous. We also have that $P(-\phi^u) = 0$.

The *Sinai-Ruelle-Bowen (SRB) measure* $\mathfrak{m}$ is defined to be the equilibrium state of $-\phi^u$, *i.e.* $\mathfrak{m} = m_{-\phi^u}$, and is therefore unique and supported on M.

Let $\mathfrak{v}$ denote the normalised Riemannian volume on M. The Bowen-Ruelle ergodic theorem for C^2 transitive Anosov flows is given by the following proposition.

PROPOSITION 8 [BR]. – *For any $F \in C(M; \mathbb{R})$,*

$$\frac{1}{T} \int_0^T F(\phi_t y) dt \longrightarrow \int F d\mathfrak{m} \quad \textit{as } T \to \infty,$$

for $\mathfrak{v}$ *almost all* $y \in M$.

The SRB measure is also an ergodic measure for ϕ, and so we also have the following ergodic theorem.

PROPOSITION 9. – *For any* $F \in C(M; \mathbb{R})$,

$$\frac{1}{T}\int_0^T F(\phi_t y)dt \longrightarrow \int F d\mathfrak{m} \quad \text{as } T \to \infty,$$

for $\mathfrak{m}$ *almost all* $y \in M$.

A Borel probability measure is called *smooth* if it is absolutely continuous with respect to the volume measure $\mathfrak{v}$. If ϕ has a smooth invariant Borel probability measure $\mathfrak{u}$, then by Proposition 8 and Birkhoff's Ergodic Theorem, we have $\mathfrak{m} = \mathfrak{u}$. In this case, Proposition 9 is a corollary of Proposition 8.

We now consider large deviations from the limit in Proposition 9. Using the modelling theory for Anosov flows, described in Proposition 7, we can reformulate Theorem 1 as follows.

THEOREM 2. – *Let* ϕ_t *be a* C^2 *transitive Anosov flow. Let* $F \in C(M; \mathbb{R})$ *be Hölder continuous and suppose that* F *and* ϕ *are flow independent. Then for every* $b > 0$ *and* $a \in \Gamma_F$,

$$\begin{aligned}&\mathfrak{m}\{y \in M : F^T(y) - Ta \in [0, b]\}\\ &\quad\sim \left(\int_0^b e^{-\rho(a)t}dt\right)\frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}}\frac{e^{T\gamma(a)}}{\sqrt{T}}\end{aligned} \tag{7.1}$$

as $T \to \infty$. *The constant* $C(a)$ *is given by* (4.12) *and*

$$\gamma(a) = P\big(-\phi^u + \rho(a)(F - a)\big)$$

Remarks 4. – (i) Analogous statements of Corollaries 1-5 can also be given in this context.

(ii) In general it does not seem possible, using our techniques, to give precise large deviation formulae for the limit in Proposition 8. Some less precise results are given in [K].

(iii) The results of this section could equally well be formulated in the more general setting of C^2 Axiom A attractors, [BR].

(iv) The constant $C(a)$ in Theorem 2 must be independent of the choices of Markov sections that we used in the proof, since all the other terms in

(6.1) are independent of such choices. It may be possible to use ideas in [Ru2] to give an explicit expression for $C(a)$ in terms of ϕ.

For any $\varepsilon > 0, T \geq 0$ and $y \in M$, we define the *Bowen ball* $B(y;\varepsilon,T)$ by

$$B(y;\varepsilon,T) = \{z \in M : d(\phi_t y, \phi_t z) \leq \varepsilon \text{ for all } 0 \leq t \leq T\}.$$

PROPOSITION 10 (Volume Lemma) [BR]. – *For every $\varepsilon > 0$, there exists a real number $D = D(\varepsilon) > 1$ such that*

$$\frac{1}{D} \leq \frac{\mathfrak{v}(B(y;\varepsilon,T))}{e^{-\int_0^T \phi^u(\phi_t y)dt}} \leq D$$

for all $y \in M, T \geq 0$.

The following theorem is a large deviation result for Bowen balls.

THEOREM 3. – *Let ϕ_t be a C^2 transitive Anosov flow. Suppose ϕ^u, ϕ are flow independent. Let $\varepsilon > 0$ be given. Then for any $c \in (0, \frac{1}{D(\varepsilon)^2})$ and $a \in \Gamma_{\phi^u}$, we have*

$$\mathfrak{m}\{y \in M : \mathfrak{v}(B(y;\varepsilon,T)) \in [ce^{-Ta}, e^{-Ta}]\} \asymp \frac{e^{T\gamma(a)}}{\sqrt{T}} \quad \textit{as } T \to \infty,$$

where the implied upper and lower bounds are given by

$$\left(\int_{-\log D}^{\log(D/c)} e^{-\rho(a)t} dt\right) \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}},$$

and

$$\left(\int_{\log D}^{-\log(cD)} e^{-\rho(a)t} dt\right) \frac{C(a)}{\sqrt{2\pi\beta''(\rho(a))}}$$

respectively. The constant $D = D(\varepsilon)$ is given by Proposition 10. *Further,*

$$\gamma(a) = P\big((\rho(a) - 1)\phi^u\big) - a\rho(a),$$

and $C(a)$ can be computed from equation (4.12).

In the statement of Theorem 3, we have used the standard notation $A(t) \asymp B(t)$ to mean that there exist constants K_1, K_2 such that

$$K_1 \leq \frac{A(t)}{B(t)} \leq K_2,$$

for all t sufficiently large.

Proof of Theorem 3. – Let $\varepsilon > 0$ be given. For any $y \in M$ and $T \geq 0$, we have by Proposition 10 that there exists $D = D(\varepsilon) > 1$ such that

$$-\log D + (\phi^u)^T(y) \leq -\log \mathfrak{v}(B(y;\varepsilon,T)) \leq (\phi^u)^T(y) + \log D.$$

Hence, for any $b > 2\log D$,

$$\begin{aligned}&\mathfrak{m}\{y \in M : (\phi^u)^T(y) - Ta \in [\log D, b - \log D]\}\\&\quad\leq \mathfrak{m}\{y \in M : \mathfrak{v}(B(y;\varepsilon,T)) \in [e^{-b-Ta}, e^{-Ta}]\}\\&\quad\leq \mathfrak{m}\{y \in M : (\phi^u)^T(y) - Ta \in [-\log D, b + \log D]\}.\end{aligned}$$

Taking $c = e^{-b}$, and applying Theorem 2 gives the result. $\bowtie$

Remark 5. – Some less precise results comparing the size of Bowen balls of flows, for two given measures, were proved in [T2].

8. APPLICATION II: WINDING CYCLES FOR ANOSOV FLOWS

In this section, we continue in a similar setting to section six. In particular, let $\phi_t : M \to M$ be a C^1 transitive Anosov flow and let $m = m_G$ be the (unique) equilibrium state of a real valued, Hölder continuous function $G : M \to \mathbb{R}$.

We define a linear functional $\Phi_G : H^1(M;\mathbb{R}) \to \mathbb{R}$, called the *winding cycle*, by

$$\Phi_G(\omega) = \int_M < \omega, Z > dm_G,$$

where ω is a closed 1-form, and Z is the vector field generated by the flow ϕ. (This was introduced by Schwartzmann, [Sc]). If ω is an exact 1-form then $\Phi_G(\omega) = 0$, so Φ_G yields a homology class in $H_1(M;\mathbb{R})$, that is

$$\Phi_G \in \mathrm{Hom}\big(H^1(M;\mathbb{R}), \mathbb{R}\big) \cong H_1(M;\mathbb{R}).$$

The ergodicity of the flow gives

$$\Phi_G(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_0^T < \omega, Z > (\phi_t y) dt$$

for m almost all $y \in M$. We shall give a large deviation result for this limit.

We define the *covariance form* $\delta_G : H^1(M;\mathbb{R}) \times H^1(M;\mathbb{R}) \to \mathbb{R}$ by

$$\delta_G(\omega,\omega) = \lim_{T\to\infty} \frac{1}{T} \int \left(\int_0^T < \omega, Z > (\phi_t y) dt - T\Phi_G(\omega) \right)^2 dm_G,$$

which is a positive, semidefinite quadratic form on $H^1(M;\mathbb{R})$. In particular, $\delta(\omega,\omega) = 0$ if and only if $< \omega, Z >: M \to \mathbb{R}$ is cohomologous to a constant function.

We now suppose that, for some fixed non-trivial $\omega \in H^1(M;\mathbb{R})$, $< \omega, Z >$ and ϕ are flow independent. In particular, this implies that ϕ is topologically weak mixing and that $< \omega, Z >$ is not cohomologous to a constant function, (*cf.* Proposition 3). Let

$$\Gamma_\omega = \{\Phi_{G+t<\omega,Z>}(\omega) : t \in \mathbb{R}\},$$

where

$$\Phi_{G+t<\omega,Z>}(\omega) = \int < \omega, Z > dm_{G+t<\omega,Z>}.$$

Since $\delta_{G+t<\omega,Z>}(\omega,\omega) > 0$ for all $t \in \mathbb{R}$, the map $t \mapsto \Phi_{G+t<\omega,Z>}(\omega)$ is strictly increasing. Thus we may define an inverse map $\rho : \Gamma_\omega \to \mathbb{R}$.

The following theorem is the analogue of Theorem 1 in this situation.

THEOREM 4. – *Let $\phi_t : M \to M$ be a C^1 transitive Anosov flow and let $\omega \in H^1(M;\mathbb{R})$ be non-trivial. Suppose that $< \omega, Z >$ and ϕ are flow independent, where Z is the vector field generated by ϕ. Then for any $a \in \Gamma_\omega$,*

$$m\{y \in M :< \omega, Z >^T -Ta \in [0,b]\}$$
$$\sim \left(\int_0^b e^{-\rho(a)t} dt \right) \frac{C(a)}{\sqrt{2\pi\delta_{G+\rho(a)<\omega,Z>}(\omega,\omega)}} \frac{e^{T\gamma(a)}}{\sqrt{T}}$$

as $T \to \infty$, where

$$\gamma(a) = P\big(G + \rho(a)(< \omega, Z > -a)\big) - P(G),$$

and $C(a)$ is a constant, (given by (4.12)).

Remarks 6. – (i) Analogous statements of Corollaries 1-5 can also be given in this context.

(ii) The winding cycle can be interpreted intuitively as measuring the 'average homological direction' of orbits of the flow. Our result therefore quantifies the deviations from this average.

Remarks 7. – If $\Phi_m \equiv 0$ then ϕ and $< \omega, Z >$ are flow independent, for every nonexact, closed 1-form ω. (*See* [Sh2] for details). For example, if ϕ is a geodesic flow on the unit tangent bundle of a negatively curved surface, which is of Anosov type, and m is the measure of maximal entropy (*i.e.* $G \equiv 0$), then $\Phi_m \equiv 0$. (This fact is proved in [KS], section one).

9. MULTIDIMENSIONAL LARGE DEVIATIONS

In this section, we state a multidimensional analogue of Theorem 1 for suspended flows. We refer the reader to [La1], [La2] for the unproved results about pressure.

First let $\sigma^r : \Sigma_A^r \to \Sigma_A^r$ be a suspended flow and let $F = (F_1, \ldots, F_k) \in (\mathcal{F}_\alpha^r(\mathbb{R}))^k$. For $G \in \mathcal{F}_\alpha^r(\mathbb{R})$, define a skew product flow S_t^G as in Definition 2. We say that F, σ^r are *flow independent* if there are constants $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ and $t_0 \in \mathbb{R}$, such that the skew product flow S_t^G, with $G = t_0 + < t, F >$ is not topologically ergodic, then $t = 0$ and $t_0 = 0$. In particular, this condition implies that $< t, F >$ is not cohomologous to a constant for any $t \in \mathbb{R}^k$.

Define a map $\beta : \mathbb{R}^k \to \mathbb{R}$ by

$$\beta(t) = P\big(G + < t, F >\big) - P(G)$$

which is again real analytic. Further,

$$\nabla\beta(t) = \int F dm_{G+<t,F>} = \left(\int F_1 dm_{G+<t,F>}, \ldots, \int F_k dm_{G+<t,F>}\right)$$

and the Hessian matrix $\nabla^2\beta(t)$ is positive definite at every $t \in \mathbb{R}^k$. Let

$$\Gamma_F = \{\nabla\beta(t) : t \in \mathbb{R}^k\},$$

which is a subset of $\mathbb{R}^k$. The map $t \mapsto \nabla\beta(t)$ is real analytic. For each $a \in \Gamma_F$, there exists a unique $\rho(a) \in \mathbb{R}^k$ such that $\nabla\beta(\rho(a)) = a$. The function $\rho : \Gamma_F \to \mathbb{R}$ is also real analytic and surjective. We define $\gamma : \Gamma_F \to \mathbb{R}$ by

$$\gamma(a) = P(G + < \rho(a), F - a >) - P(G).$$

The following theorem can be proved in a totally analogous way to Theorem 1.

THEOREM 5. – *Let $\sigma^r : \Sigma_A^r \to \Sigma_A^r$ be a suspended flow. Let $m = m_G$ be the equilibrium state of $G \in \mathcal{F}_\alpha^r(\mathbb{R})$. Suppose that $F = (F_1, \ldots, F_k) \in (\mathcal{F}_\alpha^r(\mathbb{R}))^k$ and σ^r are flow independent. Then for any $b = (b_1, \ldots, b_k) \in (0, \infty)^k$ and $a = (a_1, \ldots, a_k) \in \Gamma_F$,*

$$\begin{aligned} &m\{y : F_i^T(y) - Ta_i \in [0, b_i] \textit{ for } i = 1, \ldots, k\} \\ &\sim \frac{1}{(2\pi)^{k/2}} \frac{C(a)}{\left(\det \nabla^2\beta(\rho(a))\right)^{1/2}} \\ &\times \left(\int_0^{b_1} \cdots \int_0^{b_k} e^{-<t,\rho(a)>} dt_k \ldots dt_1\right) \frac{e^{T\gamma(a)}}{T^{k/2}} \end{aligned} \tag{9.1}$$

as $T \to \infty$, *for a computable constant* $C(a)$. *The convergence is uniform on compact subsets of* Γ_F. *Further, if* $(\rho(a))_i > 0$ *for some* $i \in \{1, \ldots, k\}$, *then (8.1) holds with* b_i *replaced by* ∞.

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