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Some sufficient conditions for the existence of positive solutions to the equation $-\Delta u + a(x)u = u^{2^*-1}$ in bounded domains

by

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ABSTRACT. – This paper is concerned with the problem

$$(*) \qquad \begin{cases} -\Delta u + a(x)u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u > 0 \ \text{ in } \ \Omega; \ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $a(x)$ is a nonnegative function in Ω. We give some conditions on the function $a(x)$, sufficient to guarantee the existence and multiplicity of solutions for the considered problem without any assumption on the shape of Ω.

Key words: Nonlinear elliptic equations. Critical Sobolev exponent. Positive solutions.

RÉSUMÉ. – On considère le problème $(*)$ où Ω est un ouvert borné de $\mathbb{R}^n$ avec $n \geq 3$ et $a(x)$ une fonction non-négative dans Ω.

On établit des conditions sur la fonction $a(x)$ suffisantes pour assurer l'existence et la multiplicité de solutions du problème considéré sans aucune condition sur la forme de Ω.

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1. INTRODUCTION

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. In this paper we are concerned with the problem

$$(1.1) \qquad \begin{cases} -\Delta u + a(x)u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $a(x) \in L^{n/2}(\Omega)$ is a given nonnegative function, and $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$.

The aim of our investigation is to give conditions on $a(x)$ sufficient to guarantee existence and multiplicity of solutions for (1.1). Notice that, through a lemma of Brezis and Kato (*see* [5]), the assumption $a(x) \in L^{n/2}$ ensures that the solutions u to the problem are in $C^{1,\lambda}, \forall \lambda \in (0,1)$.

The first contribution to the study of this problem is the well known Pohozaev nonexistence result: in [20] he proved that a solution u of Problem (1.1) must satisfy the identity

$$(1.2) \qquad \int_{\partial\Omega} (x \cdot \nu)(Du \cdot \nu)^2 d\sigma + \int_\Omega \left[a(x) + \frac{1}{2}(x \cdot Da(x)) \right] u^2 dx = 0$$

(where ν denotes the outward normal on $\partial\Omega$), and this implies that (1.1) has no solution if Ω is starshaped and $a(x)$ is a nonnegative constant function.

The main feature of the considered problem is the lack of compactness due to the presence of the critical exponent: in fact, solutions of (1.1) correspond to critical points of the functional

$$(1.3) \qquad f(u) = \int_\Omega [|Du|^2 + a(x)u^2] dx$$

constrained on the manifold

$$(1.4) \qquad V(\Omega) = \left\{ u \in H_0^{1,2}(\Omega) : \int_\Omega |u|^{2^*} dx = 1 \right\},$$

and, since the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, the well known Palais–Smale compactness condition does not hold.

Therefore the classical variational methods cannot be applied in a straightforward way. In particular critical points cannot be obtained by minimizing f on $V(\Omega)$; in fact, f does not achieve its infimum on $V(\Omega)$ if $a(x) \geq 0$, as shown in [4].

On the contrary, if $a(x)$ is negative somewhere, Brezis and Nirenberg proved that the infimum of f on $V(\Omega)$ is achieved if $n \geq 4$ (*see* [6], [4]).

On the other hand, if Ω is an annulus and $a(x)$ is radially symmetric, it is not difficult to prove that (1.1) has solutions even if $a(x) \geq 0$ (*see* [11] for example).

Moreover several results show that, when $a(x) = 0$, the existence of solutions of (1.1) is strictly related to the shape of Ω. Firstly Coron in [7] proved the existence of a positive solution in domains Ω having a "small hole"; then, in [2] this result was extended by Bahri and Coron to every domain having nontrivial topology (in a suitable sense). More recently, multiplicity results related to the shape of Ω have been stated, for instance, in [21], [13], [16], [19], [18], [17]; furthermore existence results have been obtained also in some contractible bounded domains (*see* [8], [9], [13]).

In [4] Brezis pointed out that in every bounded domain Ω (even starshaped) one can easily exhibit a positive function u that solves (1.1) when $a(x)$ is a positive function suitably chosen: in fact, if $g \not\equiv 0$ is a positive function with compact support in Ω and h satisfies $-\Delta h = g$ in Ω, $h = 0$ on $\partial\Omega$, then the pair (u, a) with $u = \lambda h$ and $a = \frac{(\lambda h)^{2^*-1} - \lambda g}{\lambda h}$ solves the problem, and $a > 0$ in Ω for λ large enough. So he focused the attention of the mathematicians on the problem of giving some conditions on $a(x) \geq 0$, sufficient for the solvability of (1.1) in general domains Ω (even starshaped).

A first contribution to this question was given by Benci and Cerami in [3]. They considered the case $\Omega = \mathbb{R}^n$ (their method does not apply when Ω is a bounded domain) and proved that the problem

$$\begin{cases} -\Delta u + a(x)u = u^{2^*-1} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} |Du|^2 dx < +\infty \end{cases}$$

has at least one solution if $a(x)$ is a nonnegative function, strictly positive somewhere, having $L^{\frac{n}{2}}$ norm suitably bounded and belonging to $L^p(\mathbb{R}^n)$ for every p in a suitable neighbourhood of $\frac{n}{2}$

Multiplicity results concerning a related problem in $\mathbb{R}^n$ have been obtained in [15].

In this paper we consider the case of a general bounded domain Ω and give an answer to the question posed by Brezis. The main results (already announced in [14]) are stated in Theorems 2.1, 3.1, 3.2 and 3.3.

We consider, in section 2, functions $a(x)$ of the form:

$$a(x) = \bar{\alpha}(x) + \lambda^2 \alpha[\lambda(x - x_0)] \tag{1.5}$$

where $\bar{\alpha}(x)$ is a given nonnegative function in $L^{n/2}(\Omega)$, x_0 is a fixed point in Ω (the concentration point), $\lambda > 0$ is a "concentration parameter", and α is a given nonnegative function in $L^{n/2}(\mathbb{R}^n)$ with $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$.

We prove, in Theorem 2.1, that Problem (1.1) has a solution if λ is large enough; moreover we show that there are at least two solutions (for λ large enough) if the additional assumption

$$\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} < S(2^{2/n} - 1) \tag{1.6}$$

is satisfied (S is the best Sobolev constant: *see* (2.3)).

We notice that our assumptions seem fairly general. In fact, if we assume for example that in (1.3) $x_0 = 0$, $\bar{\alpha} \equiv 0$ in Ω, and

$$\alpha(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ |x|^{-\beta} & \text{if } |x| \geq 1, \end{cases}$$

then, if $\beta \leq 2$ (*i.e.* $\alpha \notin L^{n/2}(\mathbb{R}^n)$) and Ω is a bounded domain starshaped with respect to zero, Pohozaev identity (1.2) implies that Problem (1.1), with $a(x) = \lambda^2 \alpha(\lambda x)$, has no solution for any $\lambda > 0$; on the contrary, if $\beta > 2$ (*i.e.* $\alpha \in L^{n/2}(\mathbb{R}^n)$), Theorem 2.1 guarantees the existence of a solution for λ large enough, without any assumption on the shape of Ω (if λ is small and Ω is starshaped, no solution can exist, also in this last case, because of the Pohozaev identity).

The assumption (1.6) is strictly related to the method we use in the proof and it is very reasonable that it might be weakened arguing like in [2]; however, unlike [3], here we need it only to prove the existence of a second solution.

More general results can be obtained considering functions $a(x)$ of the form:

$$a(x) = \bar{\alpha}(x) + \sum_{i=1}^{r} \lambda_i^2 \mu_i \alpha_i[\lambda_i(x - x_i)] + \sum_{i=r+1}^{h} \lambda_i^2 \alpha_i[\lambda_i(x - x_i)] \tag{1.7}$$

where $x_1, \ldots, x_h$ are given points in $\Omega, \bar{\alpha} \in L^{n/2}(\Omega)$ and $\alpha_1 \ldots \alpha_h \in L^{n/2}(\mathbb{R}^n)$ are nonnegative functions, and λ_i, μ_j (with $i = 1 \ldots h$, $j = 1 \ldots r$, $0 \leq r \leq h$) are positive parameters.

This case is studied in section 3; Theorems 3.1, 3.2, 3.3 show that, for a suitable choice of λ_i and μ_j, Problem (1.1) has at least $(r + h)$ distinct solutions.

Remark also that it is not necessary to choose distinct concentration points $x_1, \ldots, x_h$ in order to obtain distinct solutions.

2. AN EXISTENCE AND MULTIPLICITY RESULT

The aim of this section is to prove the following existence results for Problem (1.1).

THEOREM 2.1. – *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 3$ and x_0 be a fixed point in Ω. Let $\bar{\alpha} \in L^{n/2}(\Omega)$ and $\alpha \in L^{n/2}(\mathbb{R}^n)$ be two nonnegative functions and assume that $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$.*

Then there exists $\bar{\lambda} > 0$ such that for every $\lambda > \bar{\lambda}$ Problem (1.1) *with*

$$a(x) = \bar{\alpha}(x) + \lambda^2 \alpha[\lambda(x - x_0)]$$

has at least one solution u_λ. Moreover

$$\lim_{\lambda \to +\infty} f\left(\frac{u_\lambda}{\|u_\lambda\|_{L^{2^*}}}\right) = S. \tag{2.1}$$

If we also assume that

$$\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} < S(2^{2/n} - 1), \tag{2.2}$$

then Problem (1.1) *has at least another solution $\hat{u}_\lambda$ and*

$$f\left(\frac{u_\lambda}{\|u_\lambda\|_{L^{2^*}}}\right) < f\left(\frac{\hat{u}_\lambda}{\|\hat{u}_\lambda\|_{L^{2^*}}}\right).$$

In order to prove this theorem we need to introduce some notations, to recall some known facts and to state some preliminary lemmas.

In what follows, as usual $L^p(\Omega)$, $1 \leq p \leq \infty$, denote Lebesgue spaces, $H_0^{1,2}(\Omega)$ $(H_0^{1,2}(\mathbb{R}^n))$ denotes the Sobolev spaces, closure of $C_0^\infty(\Omega)$ $(C_0^\infty(\mathbb{R}^n))$ with respect to the norm $\|u\| = (\int_\Omega |Du|^2 dx)^{\frac{1}{2}}$.

From now on, also, for any function $u \in H_0^{1,2}(\Omega)$ we denote by the same symbol its extension to $\mathbb{R}^n$, obtained setting $u \equiv 0$ outside Ω.

A function u in $H_0^{1,2}(\Omega)$ is a weak solution of Problem (1.1) if and only if $u \geq 0$ in Ω,

$$\int_\Omega |Du|^2 dx + \int_\Omega a(x) u^2 dx = \int_\Omega |u|^{2^*} dx \neq 0,$$

and $\frac{u}{\|u\|_{L^{2^*}}}$ is a critical point for the functional f (defined in (1.3)), constrained on the manifold $V(\Omega)$ (defined in (1.4)). Thus, solving Problem (1.1) is equivalent to looking for constrained critical points for f on $V(\Omega)$.

But, since the pair $(f, V(\Omega))$ does not verify the well known Palais–Smale compactness condition, the critical points cannot be obtained by applying directly the classical variational methods.

A very important role in this type of problems is plaied by the best Sobolev constant S for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$:

$$(2.3) \qquad S \stackrel{\text{def}}{=} \inf \left\{ \int_\Omega |Du|^2 dx : u \in H_0^{1,2}(\Omega),\ \int_\Omega |u|^{2^*} dx = 1 \right\}.$$

Its main properties can be summarized in the following

PROPOSITION 2.2. – *a) S is independent of $\Omega \subseteq \mathbb{R}^n$; it depends only on the dimension n;*

b) S is never achieved when $\Omega \subset \mathbb{R}^n$ is bounded;

c) when $\Omega = \mathbb{R}^n$, S is achieved by the function

$$(2.4) \qquad \bar{\psi} = \frac{\psi}{\|\psi\|_{L^{2^*}}} \quad \textit{with} \quad \psi(x) = \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}};$$

moreover every minimizing function has the form

$$\bar{\psi}_{\sigma,x_0} = \frac{\psi_{\sigma,x_0}}{\|\psi_{\sigma,x_0}\|_{L^{2^*}}} \quad \textit{where} \quad \psi_{\sigma,x_0}(x) = \psi\left(\frac{x-x_0}{\sigma}\right)$$

with $\sigma > 0$ and $x_0 \in \mathbb{R}^n$;

d) if $u \in H_0^{1,2}(\mathbb{R}^n)$, $u \geq 0$, is a critical point of the functional $\int_{\mathbb{R}^n} |Du|^2 dx$, constrained on $V(\mathbb{R}^n) = \{u \in H_0^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u|^{2^*} dx = 1\}$, then $u = \bar{\psi}_{\sigma,x_0}$ for suitable $\sigma > 0$ and x_0 in $\mathbb{R}^n$.

The proof of properties a), b), c) can be found, for instance, in [6] or in [23]; for d) we refer to [10].

The following proposition describes the behaviour of the minimizing sequences for the Sobolev constant S; for its proof see, for example, [12], [22].

PROPOSITION 2.3. – *Let $(u_i)_i$ be a sequence in $H_0^{1,2}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} |u_i|^{2^*} dx = 1 \quad \forall i \in \mathbb{N}; \ \lim_{i\to\infty} \int_{\mathbb{R}^n} |Du_i|^2 dx = S.$$

Then there exist a sequence $(y_i)_i$ in $\mathbb{R}^n$ and a sequence of positive numbers $(\sigma_i)_i$ such that the sequence $(\tilde{u}_i)_i$ in $H_0^{1,2}(\mathbb{R}^n)$, defined by

$$\tilde{u}_i(x) = \sigma_i^{-\frac{n}{2^*}} u_i\left(\frac{x+y_i}{\sigma_i}\right),$$

is relatively compact in $L^{2^*}(\mathbb{R}^n)$.

So $\tilde{u}_i \to \tilde{u}$ *in* $L^{2^*}(\mathbb{R}^n)$ *(up to a subsequence) and*

$$\int_{\mathbb{R}^n} |D\tilde{u}|^2 dx = S.$$

If, in particular, $u_i \in H_0^{1,2}(\Omega)$ *and* Ω *is bounded, then* $\lim_{i\to+\infty} \sigma_i = +\infty$ *and the sequences* $(|u_i|^{2^*})_i$ *and* $(|Du_i|^2)_i$ *concentrate near a point of* $\bar{\Omega}$ *(like a Dirac mass).*

We recall now a nonexistence result which can be found in [4].

PROPOSITION 2.4. – *Let* Ω *be a bounded domain of* $\mathbb{R}^n$, $n \geq 3$, *and* $a(x)$ *be a nonnegative function in* $L^{n/2}(\Omega)$. *Then it results:*

$$\inf_{V(\Omega)} f = S \tag{2.5}$$

and the infimum is not achieved.

The proof is obtained (*see* [4] or [3]) by testing f on the functions introduced in (2.4), suitably cut off Ω, and using the estimates given in [6]. Moreover, the proof evidences that the minimizing sequences for f on $V(\Omega)$, in booth cases, when $a(x) = 0$ and when $a(x) \geq 0$, are exactly the same.

The following proposition and the subsequent corollary describe the behaviour of the Palais–Smale sequences, giving useful informations about the compactness properties of f on $V(\Omega)$.

PROPOSITION 2.5. – *Let* Ω *and* $a(x)$ *be as in Proposition 2.4. Let* $(u_i)_i$ *be a Palais–Smale sequence for the functional* f *constrained on* $V(\Omega)$, i.e.:

$$\sup_{i\in\mathbb{N}} f(u_i) < +\infty \quad \textit{and} \quad \operatorname{grad} f_{|V(\Omega)}(u_i) \to 0 \ \textit{ in } \ H^{-1,2}(\Omega).$$

Then one of the following two cases happens: either the sequence $(u_i)_i$ *is relatively compact in* $H_0^{1,2}(\Omega)$, *or there exist k solutions* $\bar{u}_1, \ldots \bar{u}_k$ $(k \geq 1)$ *of*

$$\begin{cases} \Delta u + |u|^{2^*-2}u = 0 & \textit{in } \mathbb{R}^n \\ u \in H^{1,2}(\mathbb{R}^n),\ u \not\equiv 0 & \textit{in } \mathbb{R}^n \end{cases}$$

and a solution $\bar{u}_0$ *of*

$$\begin{cases} \Delta u - a(x)u + |u|^{2^*-2}u = 0 & \textit{in } \Omega \\ u \in H_0^{1,2}(\Omega) \end{cases}$$

such that $(u_i)_i$ (up to a subsequence) verifies

$$u_i \rightharpoonup \bar{u}_0 \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |\bar{u}_j|^{2^*} dx \right]^{-\frac{1}{2^*}} \quad \textit{weakly in} \quad H_0^{1,2}(\Omega);$$

$$\lim_{i\to\infty} \int_\Omega |Du_i|^2 dx = \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |D\bar{u}_j|^2 dx \right] \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |\bar{u}_j|^{2^*} dx \right]^{-\frac{2}{2^*}}.$$

The proof can be obtained by the same arguments used in [22].

COROLLARY 2.6. – *Let Ω and $a(x)$ be as in Proposition 2.4. Let $(u_i)_i$ in $V(\Omega)$ satisfies*

$$\begin{cases} \lim_{i\to\infty} f(u_i) \in]S, 2^{2/n}S[\\ \operatorname{grad} f_{|V(\Omega)}(u_i) \to 0 \quad \textit{in } H^{-1,2}(\Omega). \end{cases}$$

Then $(u_i)_i$ is relatively compact in $H_0^{1,2}(\Omega)$.

The following lemma gives a lower bound to the energy of the functions changing sign, that are critical points for f on $V(\Omega)$.

LEMMA 2.7. – *Let Ω and $a(x)$ be as in Proposition 2.4. Let $u \in H_0^{1,2}(\Omega)$ be a critical point for f on $V(\Omega)$. If $f(u) < 2^{2/n}S$, then the function u has a constant sign.*

Proof. – Assume, by contradiction, that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. Since u is a critical point for f on $V(\Omega)$, u solves $\Delta u - a(x)u + \mu|u|^{2^*-2}u = 0$ in Ω with $\mu = f(u)$. Thus

$$f(u)\int_\Omega |u^\pm|^{2^*} dx = \int_\Omega [|Du^\pm|^2 + a(x)|u^\pm|^2]dx \geq S\left(\int_\Omega |u^\pm|^{2^*} dx\right)^{2/2^*}.$$

Then we obtain

$$\int_\Omega |u^\pm|^{2^*} dx \geq \left(\frac{S}{f(u)}\right)^{\frac{n}{2}},$$

that implies $f(u) \geq 2^{2/n}S$, contradicting our assumption. □

Let us now introduce some useful tools. We define two continuous maps:

$$\beta : V(\mathbb{R}^n) \to \mathbb{R}^n \quad \text{and} \quad \gamma : V(\mathbb{R}^n) \to \mathbb{R}^+$$

by

$$\beta(u) = \int_{\mathbb{R}^n} \frac{x}{1+|x|} |u(x)|^{2^*} dx \tag{2.6}$$

$$\gamma(u) = \int_{\mathbb{R}^n} \left|\frac{x}{1+|x|} - \beta(u)\right| |u(x)|^{2^*} dx. \tag{2.7}$$

We notice that β is a "barycenter" type function, while γ measures the concentration of the function u near its barycenter $\beta(u)$.

The following remark, also, will be helpful in the sequel: a function u solves the equation

$$-\Delta u + \alpha(x)u = u^{\frac{n+2}{n-2}} \text{ in } \Omega$$

if and only if the function u_λ defined by $u_\lambda(x) = \lambda^{\frac{n-2}{2}} u[\lambda(x-x_0)]$ solves the equation

$$-\Delta u_\lambda + \lambda^2 \alpha[\lambda(x-x_0)]u_\lambda = u_\lambda^{\frac{n+2}{n-2}}$$

in $\Omega_\lambda = x_0 + \frac{1}{\lambda}\Omega$ (notice that $\|u_\lambda\|_{L^{2^*}(\Omega_\lambda)} = \|u\|_{L^{2^*}(\Omega)}$). Moreover, setting $\alpha_\lambda(x) = \lambda^2\alpha[\lambda(x-x_0)]$, we have for every $\epsilon > 0$:

$$\int_{B(x_0,\epsilon)} \alpha_\lambda^{n/2}(x)dx = \int_{B(0,\lambda\epsilon)} \alpha^{n/2}(x)dx,$$

that implies

$$\lim_{\lambda\to\infty} \int_\Omega \alpha_\lambda^{n/2} dx = \int_{\mathbb{R}^n} \alpha^{\frac{n}{2}} dx.$$

Let α be a nonnegative function in $L^{\frac{n}{2}}(\mathbb{R}^n)$; we set

(2.8)

$$c(\alpha) \stackrel{\text{def}}{=} \inf\left\{\int_{\mathbb{R}^n} [|Du|^2 + \alpha(x)u^2]dx : u \in V(\mathbb{R}^n),\ \beta(u) = 0,\ \gamma(u) = \frac{1}{3}\right\}.$$

The following inequality holds.

LEMMA 2.8. – *Let $\alpha \geq 0$, $\alpha \in L^{n/2}(\mathbb{R}^n)$, satisfy $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$. Then*

$$c(\alpha) > S. \tag{2.9}$$

Proof. – Clearly $c(\alpha) \geq S$, thus we must show that the equality cannot hold. If this were the case, we could find a sequence $(u_i)_i \in V(\mathbb{R}^n)$ such that

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} [|Du_i|^2 + \alpha(x)u_i^2]dx = S \tag{2.10}$$

$$\beta(u_i) = 0 \quad \text{and} \quad \gamma(u_i) = \int_{\mathbb{R}^n} \frac{|x|}{1+|x|}|u_i(x)|^{2^*}dx = \frac{1}{3} \quad \forall i \in \mathbb{N}. \tag{2.11}$$

Since $\alpha(x) \geq 0$, it follows

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} |Du_i|^2 dx = S. \tag{2.12}$$

Then there exist a sequence of points $(y_i)_i$ in $\mathbb{R}^n$, a sequence of positive numbers $(\sigma_i)_i$ and a sequence $(w_i)_i$ in $H_0^{1,2}(\mathbb{R}^n)$ such that

$$u_i = w_i + \bar{\psi}_{\sigma_i, y_i}$$

where $\bar{\psi}_{\sigma_i,y_i}$ are the functions (2.4), and $w_i \to 0$ strongly in $L^{2^*}(\mathbb{R}^n)$.

We claim that the sequences $(y_i)_i$ and $(\sigma_i)_i$ are bounded. In fact suppose, first, $\lim_{i\to\infty} |y_i| = +\infty$ (up to a subsequence) and set

$$\Sigma_i = \{x \in \mathbb{R}^n : ((x - y_i) \cdot y_i) \geq 0\};$$

since $\frac{|x|}{1+|x|} \geq \frac{|y_i|}{1+|y_i|}$ $\forall x \in \Sigma_i$ and $\lim_{i\to\infty} \int_{\Sigma_i} |u_i|^{2^*}dx = \frac{1}{2}$, we should have

$$\gamma(u_i) \geq \int_{\Sigma_i} \frac{|x|}{1+|x|}|u_i(x)|^{2^*}dx \geq \frac{|y_i|}{1+|y_i|}\int_{\Sigma_i} |u_i|^{2^*}dx$$

that implies

$$\liminf_{i\to\infty} \gamma(u_i) \geq \frac{1}{2}$$

contradicting (2.11).

Assume now that, up to a subsequence, $\lim_{i\to\infty} \sigma_i = +\infty$. Then

$$\lim_{i\to\infty} \sup_{x\in\mathbb{R}^n} |\bar{\psi}_{\sigma_i,y_i}(x)| = 0$$

and so

$$\lim_{i\to\infty} \int_{B(0,r)} |u_i|^{2^*}dx = 0 \quad \forall r > 0.$$

From

$$\gamma(u_i) = \int_{\mathbb{R}^n} \frac{|x|}{1+|x|}|u_i(x)|^{2^*}dx \geq \int_{\mathbb{R}^n \setminus B(0,r)} \frac{|x|}{1+|x|}|u_i(x)|^{2^*}dx$$
$$\geq \frac{r}{1+r}\int_{\mathbb{R}^n \setminus B(0,r)} |u_i|^{2^*}dx \quad \forall r > 0,$$

it follows

$$\liminf_{i\to\infty} \gamma(u_i) \geq \frac{r}{1+r} \qquad \forall r > 0$$

that implies $\lim_{i\to\infty} \gamma(u_i) = 1$, contradicting again (2.11).

Thus the claim holds and we can assume, passing eventually to a subsequence, $y_i \to \bar{y} \in \mathbb{R}^n$ and $\sigma_i \to \bar{\sigma} \geq 0$.

We have $\bar{\sigma} > 0$: otherwise we should have

$$\lim_{i\to\infty} \beta(u_i) = \frac{\bar{y}}{1+|\bar{y}|} \quad \text{and} \quad \beta(u_i) = 0 \ \forall i \in \mathbb{N}$$

that implies $\bar{y} = 0$. On the other hand, if $\bar{\sigma} = 0$, we have

$$\lim_{i\to\infty} \gamma(u_i) = \lim_{i\to\infty} \int_{\mathbb{R}^n} \frac{|x|}{1+|x|}|u_i(x)|^{2^*}dx = \frac{|\bar{y}|}{1+|\bar{y}|} = 0$$

that contradicts (2.11).

Thus, $u_i \to \bar{\psi}_{\bar{\sigma},\bar{y}}$ strongly in $L^{2^*}(\mathbb{R}^n)$ with $\bar{\sigma} > 0$. Therefore we can deduce

$$\int_{\mathbb{R}^n} \alpha(x)\bar{\psi}^2_{\bar{\sigma},\bar{y}}dx > 0 \tag{2.13}$$

because $\bar{\psi}_{\bar{\sigma},\bar{y}}(x) > 0 \ \forall x \in \mathbb{R}^n$ and $\alpha(x) \in L^{n/2}(\mathbb{R}^n)$ is nonnegative and satisfies $\int_{\mathbb{R}^n} \alpha^{n/2}(x)dx > 0$.

So, using (2.12) and (2.13), we obtain

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} [|Du_i|^2 + \alpha(x)u_i^2]dx = S + \int_{\mathbb{R}^n} \alpha(x)\bar{\psi}^2_{\bar{\sigma},\bar{y}}dx > S$$

contradicting (2.10). □

Fix now $\epsilon > 0$ so small that $S + \epsilon < \min\{c(\alpha), 2^{2/n}S\}$.

In what follows φ denotes a function belonging to $H_0^{1,2}(B(0,1))$, satisfying the properties:

$$(2.14)\quad \begin{cases} \varphi \in C^\infty(B(0,1)), \qquad \varphi(x) > 0 \quad \forall x \in B(0,1) \\ \varphi \quad \textit{is radially symmetric and} \quad |x_1| < |x_2| \Rightarrow \varphi(x_1) > \varphi(x_2) \\ \int_{B(0,1)} \varphi^{2^*} dx = 1; \qquad S < \int_{B(0,1)} |D\varphi|^2 dx < S + \epsilon. \end{cases}$$

The existence of a φ fulfilling (2.14) is a consequence of the properties of S. For every $\sigma > 0$ and $y \in \mathbb{R}^n$, we define

$$T_{\sigma,y} : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$$

by

$$(2.15)\qquad T_{\sigma,y}(u) = \frac{u_{\sigma,y}}{\|u_{\sigma,y}\|_{L^{2^*}}} \qquad \text{where} \qquad u_{\sigma,y}(x) = u\left(\frac{x-y}{\sigma}\right).$$

LEMMA 2.9. – *Let β, γ, φ, $T_{\sigma,y}$ be the objects defined in* (2.6), (2.7), (2.14), (2.15) *respectively. The following relations hold*

$$(2.16)\quad \begin{cases} a) \lim\limits_{\sigma\to 0} \sup\{\gamma \circ T_{\sigma,y}(\varphi) : y \in \mathbb{R}^n\} = 0 \\ b) \ (\beta \circ T_{\sigma,y}(\varphi) \cdot y) > 0 \quad \forall y \in \mathbb{R}^n \setminus \{0\} \quad \textit{and} \quad \forall \sigma > 0 \\ c) \lim\limits_{\sigma\to +\infty} \inf\{\gamma \circ T_{\sigma,y}(\varphi) : y \in \mathbb{R}^n,\ |y| \le r\} = 1 \quad \forall r \ge 0. \end{cases}$$

Proof. – To prove (2.16) a) we argue by contradiction. So we assume that there exist a sequence $(y_i)_i$ in $\mathbb{R}^n$ and a sequence of positive numbers $(\sigma_i)_i$ such that:

$$(2.17)\qquad \lim_{i\to\infty} \sigma_i = 0;$$

$$(2.18)\qquad \lim_{i\to\infty} \gamma \circ T_{\sigma_i,y_i}(\varphi) > 0.$$

By (2.7) we have

$$(2.19)\quad \gamma \circ T_{\sigma_i,y_i}(\varphi) = \int_{B(y_i,\sigma_i)} \left| \frac{x}{1+|x|} - \beta \circ T_{\sigma_i,y_i}(\varphi) \right| T^{2^*}_{\sigma_i,y_i}(\varphi) dx$$

$$\le \int_{B(y_i,\sigma_i)} \left| \frac{x}{1+|x|} - \frac{y_i}{1+|y_i|} \right| T^{2^*}_{\sigma_i,y_i}(\varphi) dx + \left| \frac{y_i}{1+|y_i|} - \beta \circ T_{\sigma_i,y_i}(\varphi) \right|.$$

Now, since

$$(2.20)\qquad \left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|\leq |x-y| \quad \forall x,y \text{ in } \mathbb{R}^n,$$

we infer

$$(2.21)\qquad \int_{B(y_i,\sigma_i)}\left|\frac{x}{1+|x|}-\frac{y_i}{1+|y_i|}\right|T^{2^*}_{\sigma_i,y_i}(\varphi)dx \leq \int_{B(y_i,\sigma_i)}|x-y_i|T^{2^*}_{\sigma_i,y_i}(\varphi)dx\leq \sigma_i.$$

On the other hand, using (2.6) and (2.21), we deduce

$$(2.22)\qquad \left|\frac{y_i}{1+|y_i|}-\beta\circ T_{\sigma_i,y_i}(\varphi)\right| = \left|\int_{B(y_i,\sigma_i)}\left(\frac{x}{1+|x|}-\frac{y_i}{1+|y_i|}\right)T^{2^*}_{\sigma_i,y_i}(\varphi)dx\right| \leq \int_{B(y_i,\sigma_i)}\left|\frac{x}{1+|x|}-\frac{y_i}{1+|y_i|}\right|T^{2^*}_{\sigma_i,y_i}(\varphi)dx\leq\sigma_i.$$

Thus, taking account of (2.21), (2.22) and (2.17), we obtain from (2.19)

$$\lim_{i\to\infty}\gamma\circ T_{\sigma_i,y_i}(\varphi)\leq 2\lim_{i\to\infty}\sigma_i=0$$

contradicting (2.18).

In order to obtain (2.16) b), let us observe that (2.14) imply $\forall\sigma>0$

$$T_{\sigma,y}(\varphi)(x)\geq T_{\sigma,y}(\varphi)(-x)\quad \forall x\in\mathbb{R}^n \quad \text{such that}\quad (x\cdot y)\geq 0$$

and, if $y\neq 0$,

$$T_{\sigma,y}(\varphi)(x)> T_{\sigma,y}(\varphi)(-x)\quad \forall x\in B(y,\sigma) \quad \text{such that}\quad (x\cdot y)> 0.$$

Therefore

$$(\beta\circ T_{\sigma,y}(\varphi)\cdot y)=\int_{\mathbb{R}^n}\frac{(x\cdot y)}{1+|x|}T^{2^*}_{\sigma,y}(\varphi)dx>0\quad \forall y\in\mathbb{R}^n\setminus\{0\}.$$

To prove (2.16) c) we show that both the relations

$$\limsup_{\sigma\to+\infty} \inf\{\gamma\circ T_{\sigma,y}(\varphi) : y\in\mathbb{R}^n,\ |y|\le r\}\le 1 \quad \forall r\ge 0 \tag{2.23}$$

$$\liminf_{\sigma\to+\infty} \inf\{\gamma\circ T_{\sigma,y}(\varphi) : y\in\mathbb{R}^n,\ |y|\le r\}\ge 1 \quad \forall r\ge 0 \tag{2.24}$$

hold.

If (2.23) were not true, there would exist a sequence $(\sigma_i)_i$ of positive numbers and a sequence $(y_i)_i$ in $\mathbb{R}^n$ such that

$$\lim_{i\to\infty}\sigma_i = +\infty, \qquad |y_i|\le r \quad \forall i\in\mathbb{N} \tag{2.25}$$

$$\lim_{i\to\infty}\gamma\circ T_{\sigma_i,y_i}(\varphi) > 1. \tag{2.26}$$

By definition of γ, we have

$$\gamma\circ T_{\sigma_i,y_i}(\varphi) = \int_{\mathbb{R}^n}\left|\frac{x}{1+|x|} - \beta\circ T_{\sigma_i,y_i}(\varphi)\right| T^{2^*}_{\sigma_i,y_i}(\varphi)dx \tag{2.27}$$

$$\le \int_{\mathbb{R}^n}\frac{|x|}{1+|x|}T^{2^*}_{\sigma_i,y_i}(\varphi)dx + |\beta\circ T_{\sigma_i,y_i}(\varphi)| \le 1 + |\beta\circ T_{\sigma_i,y_i}(\varphi)|.$$

Now, taking account that $\beta\circ T_{\sigma_i,0}(\varphi) = 0 \quad \forall i\in\mathbb{N}$, we write

$$|\beta\circ T_{\sigma_i,y_i}(\varphi)| = \left|\int_{\mathbb{R}^n}\frac{x}{1+|x|}[T^{2^*}_{\sigma_i,y_i}(\varphi) - T^{2^*}_{\sigma_i,0}(\varphi)]dx\right| \tag{2.28}$$

$$\le \int_{\mathbb{R}^n}|T^{2^*}_{\sigma_i,y_i}(\varphi) - T^{2^*}_{\sigma_i,0}(\varphi)|dx = \int_{\mathbb{R}^n}|T^{2^*}_{1,\frac{y_i}{\sigma_i}}(\varphi) - T^{2^*}_{1,0}(\varphi)|dx$$

and from this we deduce

$$\lim_{i\to\infty}\beta\circ T_{\sigma_i,y_i}(\varphi) = 0 \tag{2.29}$$

because, by (2.25), $\frac{y_i}{\sigma_i}\to 0$ as $i\to+\infty$. So (2.27) and (2.29) imply

$$\lim_{i\to\infty}\gamma\circ T_{\sigma_i,y_i}(\varphi)\le 1$$

that contradicts (2.26); so (2.23) is proved.

If (2.24) does not hold, then there exist a sequence $(\sigma_i)_i$ of positive numbers and a sequence $(y_i)_i$ in $\mathbb{R}^n$, satisfying (2.25) and

$$\lim_{i\to\infty} \gamma \circ T_{\sigma_i,y_i}(\varphi) < 1. \tag{2.30}$$

We have

$$\begin{aligned}\gamma \circ T_{\sigma_i,y_i}(\varphi) &= \int_{\mathbb{R}^n} \left| \frac{x}{1+|x|} - \beta \circ T_{\sigma_i,y_i}(\varphi) \right| T^{2^*}_{\sigma_i,y_i}(\varphi)dx \\ &\geq \int_{\mathbb{R}^n} \frac{|x|}{1+|x|} T^{2^*}_{\sigma_i,y_i}(\varphi)dx - |\beta \circ T_{\sigma_i,y_i}(\varphi)|.\end{aligned} \tag{2.31}$$

Now for every $\rho > 0$ we have, as $i \to +\infty$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|x|}{1+|x|} T^{2^*}_{\sigma_i,y_i}(\varphi)dx &\geq \int_{\mathbb{R}^n \setminus B(0,\rho)} \frac{|x|}{1+|x|} T^{2^*}_{\sigma_i,y_i}(\varphi)dx \\ &\geq \frac{\rho}{1+\rho} \int_{\mathbb{R}^n \setminus B(0,\rho)} T^{2^*}_{\sigma_i,y_i}(\varphi)dx = \frac{\rho}{1+\rho} + o(1)\end{aligned} \tag{2.32}$$

because, for every $\rho > 0$, $\lim_{i\to\infty} \sigma_i = +\infty$ implies

$$\lim_{i\to\infty} \int_{B(0,\rho)} T^{2^*}_{\sigma_i,y_i}(\varphi)dx = 0.$$

Thus, using (2.29) and (2.32) in (2.31), we obtain

$$\lim_{i\to\infty} \gamma \circ T_{\sigma_i,y_i}(\varphi) \geq \frac{\rho}{1+\rho} \quad \forall \rho > 0$$

that gives, as $\rho \to +\infty$,

$$\lim_{i\to\infty} \gamma \circ T_{\sigma_i,y_i}(\varphi) \geq 1$$

contradicting (2.30). So (2.24), and then (2.16) c), is proved. □

LEMMA 2.10. – *Let $\alpha(x)$ be a nonnegative function in $L^{n/2}(\mathbb{R}^n)$. Let φ, $T_{\sigma,y}$ be as in Lemma 2.9. Then we have:*

$$\begin{cases} a)\ \lim_{\sigma\to 0} \sup\left\{\int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma,y}(\varphi)dx : y \in \mathbb{R}^n\right\} = 0 \\ b)\ \lim_{\sigma\to+\infty} \sup\left\{\int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma,y}(\varphi)dx : y \in \mathbb{R}^n\right\} = 0 \\ c)\ \lim_{r\to+\infty} \sup\left\{\int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma,y}(\varphi)dx : \sigma > 0,\ |y| = r\right\} = 0. \end{cases} \tag{2.33}$$

Proof. – Firstly, let us suppose (2.33) a) not true. Then there exist a sequence $(y_i)_i$ in $\mathbb{R}^n$ and a sequence $(\sigma_i)_i$ of positive numbers, such that $\lim_{i\to\infty} \sigma_i = 0$ and

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx > 0. \tag{2.34}$$

Then, taking account that $\lim_{i\to\infty} \sigma_i = 0$ implies

$$\lim_{i\to\infty} \int_{B(\sigma_i,y_i)} \alpha^{\frac{n}{2}}(x)dx = 0,$$

we obtain

$$\begin{aligned}\lim_{i\to\infty} \int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx &= \lim_{i\to\infty} \int_{B(\sigma_i,y_i)} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx \\ &\le \lim_{i\to\infty} \left(\int_{B(\sigma_i,y_i)} \alpha^{\frac{n}{2}}(x)dx\right)^{2/n} \left(\int_{B(\sigma_i,y_i)} T^{2^*}_{\sigma_i,y_i)}(\varphi)dx\right)^{2/2^*} \\ &= \lim_{i\to\infty} \left(\int_{B(\sigma_i,y_i)} \alpha^{\frac{n}{2}}(x)dx\right)^{2/n} = 0\end{aligned}$$

contradicting (2.34).

To prove (2.33) b), we argue by contradiction and we assume that there exist a sequence $(y_i)_i$ in $\mathbb{R}^n$ and a sequence of positive numbers $(\sigma_i)_i$, with $\lim_{i\to\infty} \sigma_i = +\infty$, satisfying (2.34) as before.

Let us observe that $\forall\rho > 0$, $\forall\sigma_i > 0$, $\forall(y_i)_i \in \mathbb{R}^n$

$$\begin{aligned}\int_{\mathbb{R}^n} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx &= \int_{B(0,\rho)} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx \\ &+ \int_{\mathbb{R}^n\setminus B(0,\rho)} \alpha(x) T^2_{\sigma_i,y_i}(\varphi)dx \\ &\le \left(\int_{B(0,\rho)} \alpha^{n/2}(x)dx\right)^{2/n} \left(\int_{B(0,\rho)} T^{2^*}_{\sigma_i,y_i}(\varphi)dx\right)^{2/2^*} \\ &+ \left(\int_{\mathbb{R}^n\setminus B(0,\rho)} \alpha^{n/2}(x)dx\right)^{2/n} \left(\int_{\mathbb{R}^n\setminus B(0,\rho)} T^{2^*}_{\sigma_i,y_i}(\varphi)dx\right)^{2/2^*}.\end{aligned} \tag{2.35}$$

Now, when $\sigma_i \to +\infty$, we have

$$\lim_{i\to\infty} \int_{B(0,\rho)} T^{2^*}_{\sigma_i,y_i}(\varphi)dx = 0 \quad \forall\rho > 0.$$

So

$$\limsup_{i\to\infty}\int_{\mathbb{R}^n}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx \le \left(\int_{\mathbb{R}^n\setminus B(0,\rho)}\alpha^{n/2}(x)dx\right)^{2/n};$$

but, clearly, (since $\alpha(x)\in L^{n/2}(\mathbb{R}^n)$)

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^n\setminus B(0,\rho)}\alpha^{n/2}(x)dx = 0;$$

thus

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx = 0,$$

contradicting our assumption.

In order to prove (2.33) c), let us assume, by contradiction, that there exist a sequence $(\sigma_i)_i$ of positive numbers and a sequence $(y_i)_i$ in $\mathbb{R}^n$, with $\lim_{i\to\infty}|y_i| = +\infty$, such that

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx > 0. \tag{2.36}$$

This implies (because of (2.33) a), b)) that

$$0 < \liminf_{i\to\infty}\sigma_i \le \limsup_{i\to\infty}\sigma_i < +\infty.$$

Then, up to a subsequence, $\lim_{i\to\infty}\sigma_i = \bar\sigma$ with $\bar\sigma\in]0,+\infty[$ and, since $\lim_{i\to\infty}|y_i| = +\infty$ and $\alpha(x)\in L^{n/2}(\mathbb{R}^n)$, we deduce

$$\lim_{i\to\infty}\int_{B(y_i,\sigma_i)}\alpha^{n/2}(x)dx = 0.$$

Thus, from

$$\begin{aligned}\int_{\mathbb{R}^n}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx &= \int_{B(y_i,\sigma_i)}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx\\ &\le \left(\int_{B(y_i,\sigma_i)}\alpha^{n/2}(x)dx\right)^{2/n}\left(\int_{B(y_i,\sigma_i)}T^{2^*}_{\sigma_i,y_i}(\varphi)dx\right)^{2/2^*}\\ &= \left(\int_{B(y_i,\sigma_i)}\alpha^{n/2}(x)dx\right)^{2/n}\end{aligned}$$

we infer

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\alpha(x)T^2_{\sigma_i,y_i}(\varphi)dx = 0$$

contradicting (2.36): so (2.33) c) is proved too. □

COROLLARY 2.11. – *Let* $\alpha(x)$, β, γ, φ, $T_{\sigma,y}$ *be as in Lemmas* 2.9, 2.10 *and suppose also* $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$. *Then, there exist* $r > 0$ *and* σ_1, σ_2 *satisfying* $0 < \sigma_1 < \frac{1}{3} < \sigma_2$, *such that*

$$(2.37)\quad \sup\left\{\int_{\mathbb{R}^n} [|DT_{\sigma,y}(\varphi)|^2 + \alpha(x)T^2_{\sigma,y}(\varphi)]dx : (y,\sigma) \in \partial K\right\} < S + \epsilon < c(\alpha),$$

where

$$K = K(\sigma_1, \sigma_2, r) = \{(y,\sigma) \in \mathbb{R}^n \times \mathbb{R} : |y| \leq r,\ \sigma_1 \leq \sigma \leq \sigma_2\}.$$

Moreover the map $\Theta : \partial K \to \mathbb{R}^n \times \mathbb{R}$, *defined by*

$$\Theta(y,\sigma) = (\beta \circ T_{\sigma,y}(\varphi),\ \gamma \circ T_{\sigma,y}(\varphi)),$$

is homotopically equivalent to the identity map in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, \frac{1}{3})\}$.

Proof. – By (2.16) a) and (2.33) a) there exists $\sigma_1 \in]0, \frac{1}{3}[$ such that

$$\gamma \circ T_{\sigma_1,y}(\varphi) < \frac{1}{3} \quad \forall y \in \mathbb{R}^n$$

and the relation

$$(2.38)\qquad \int_{\mathbb{R}^n} [|DT_{\sigma,y}(\varphi)|^2 + \alpha(x)T^2_{\sigma,y}(\varphi)]dx < S + \epsilon$$

holds, when $\sigma = \sigma_1$, for any $y \in \mathbb{R}^n$. Furthermore (2.33) c) allows to choose $r > 0$ such that, if $|y| = r$, (2.38) is satisfied whatever $\sigma > 0$ is. Lastly, fixed r, as before chosen, it is possible by (2.16) c) and (2.33) b) to find $\sigma_2 > \frac{1}{3}$ for which $\gamma \circ T_{\sigma_2,y}(\varphi) > \frac{1}{3}$ if $|y| \leq r$, and such that (2.38) holds, when $\sigma = \sigma_2$, for any $y \in \mathbb{R}^n$.

Clearly the set $K = K(\sigma_1, \sigma_2, r)$, with σ_1, σ_2, r chosen as before, is the wanted set satisfying (2.37).

To achieve the second part of the assertion, consider the map

$$\vartheta : \partial K \times [0,1] \to \mathbb{R}^n \times \mathbb{R} \setminus \left\{\left(0, \frac{1}{3}\right)\right\},$$

defined by

$$\vartheta(y,\sigma,t) = (1-t)(y,\sigma) + t\Theta(y,\sigma) \quad \forall (y,\sigma) \in \partial K, \quad \forall t \in [0,1].$$

Note that $\vartheta(y,\sigma,t) \neq (0,\frac{1}{3})$ $\forall(y,\sigma) \in \partial K$, $\forall t \in [0,1]$: in fact, if $|y| \leq r$,

$$(1-t)\sigma_1 + t\gamma \circ T_{\sigma_1,y}(\varphi) < \frac{1}{3} \quad \forall t \in [0,1]$$

and

$$(1-t)\sigma_2 + t\gamma \circ T_{\sigma_2,y}(\varphi) > \frac{1}{3} \quad \forall t \in [0,1];$$

if $|y| = r$ and $\sigma_1 \leq \sigma \leq \sigma_2$, by (2.16) b),

$$([(1-t)y + t\beta \circ T_{\sigma,y}(\varphi)] \cdot y) > 0 \quad \forall t \in [0,1].$$

Then ϑ is the required homotopy between the continuous function Θ and the identity map in ∂K. □

Let $x_0 \in \Omega$, $\bar{\alpha} \in L^{n/2}(\Omega)$, $\alpha \in L^{n/2}(\mathbb{R}^n)$ be as in Theorem 2.1; for every $\lambda > 0$, set

$$\beta_\lambda = \beta \circ T_{\lambda,-\lambda x_0} \qquad\qquad \gamma_\lambda = \gamma \circ T_{\lambda,-\lambda x_0}$$

and define $f_\lambda : H_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$f_\lambda(u) = \int_\Omega \{|Du|^2 + [\bar{\alpha}(x) + \lambda^2\alpha(\lambda(x-x_0))]u^2\}dx.$$

LEMMA 2.12. – *Let Ω, x_0, $\bar{\alpha}$, α be as in Theorem 2.1. Let $c(\alpha)$ be the number defined in* (2.8). *Then, for every $\lambda > 0$ the relations*

$$(2.39)\begin{cases} a) & \inf\{f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) = \frac{1}{3}\} \geq c(\alpha) > S \\ b) & \inf\{f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) \geq \frac{1}{3}\} > S \end{cases}$$

hold.

Proof. – Set, for any $u \in H_0^{1,2}(\Omega)$, $u_\lambda = T_{\lambda,-\lambda x_0}(u)$ and observe that $\beta_\lambda(u) = 0$ and $\gamma_\lambda(u) = \frac{1}{3}$ if and only if $\beta(u_\lambda) = 0$ and $\gamma(u_\lambda) = \frac{1}{3}$.

Then, since $\bar{\alpha}(x) \geq 0$, we have $\forall u \in H_0^{1,2}(\Omega)$ (u is extended by zero outside Ω):

$$f_\lambda(u) \geq \int_{\mathbb{R}^n} [|Du|^2 + \lambda^2\alpha(\lambda(x-x_0))u^2]dx = \int_{\mathbb{R}^n} [|Du_\lambda|^2 + \alpha(x)u_\lambda^2]dx.$$

So, for any $u \in H_0^{1,2}(\Omega)$ having $\beta_\lambda(u) = 0$ and $\gamma_\lambda(u) = \frac{1}{3}$, we deduce

$$f_\lambda(u) \geq \inf\left\{ \int_{\mathbb{R}^n} [|Du|^2 + \alpha(x)u^2]dx : u \in V(\mathbb{R}^n),\ \beta(u) = 0,\ \gamma(u) = \frac{1}{3} \right\} = c(\alpha)$$

that implies (2.39) a).

In order to prove (2.39) b), assume, by contradiction, that there exists a sequence $(u_i)_i$ in $V(\Omega)$ such that

$$(2.40)\qquad \beta_\lambda(u_i)=0,\ \gamma_\lambda(u_i)\geq\frac{1}{3}\quad\forall i\in\mathbb{N}$$
$$\lim_{i\to\infty} f_\lambda(u_i)=S.$$

Thus, since $\bar{\alpha}$ and α are nonnegative functions, it follows

$$\lim_{i\to\infty}\int_\Omega |Du_i|^2dx=S.$$

This implies (by Propositions 2.2 and 2.3) that there exist a sequence $(\delta_i)_i$ of positive numbers, a sequence $(x_i)_i$ in $\mathbb{R}^n$ and a sequence $(w_i)_i$ in $V(\mathbb{R}^n)$ such that

$$u_i=w_i+T_{\delta_i,x_i}(\bar{\psi})$$

where $\bar{\psi}$ is a minimizing function for the Sobolev constant S, $w_i\to 0$, strongly, in $L^{2^*}(\mathbb{R}^n)$ and $\delta_i\to 0$.

Now, setting $v_i=T_{\lambda,-\lambda x_0}(u_i)$, we have $\beta(v_i)=0$, $\gamma(v_i)\geq\frac{1}{3}$ $\forall i\in\mathbb{N}$. Moreover from

$$v_i=T_{\lambda,-\lambda x_0}(w_i)+T_{\lambda,-\lambda x_0}\circ T_{\delta_i,x_i}(\bar{\psi})=T_{\lambda,-\lambda x_0}(w_i)+T_{\lambda\delta_i,\lambda(x_i-x_0)}(\bar{\psi}),$$

taking in account that $T_{\lambda,-\lambda x_0}(w_i)\to 0$ in $L^{2^*}(\mathbb{R}^n)$ and $\delta_i\to 0$, we deduce

$$(2.41)\qquad \lim_{i\to\infty}\int_{B(\lambda x_i-\lambda x_0,\rho)} v_i^{2^*}dx=1\quad\forall\rho>0.$$

Using the fact that $\beta(v_i)=0$ and the relation (2.20), we can write, whatever ρ is:

$$\begin{aligned}
\frac{\lambda|x_i-x_0|}{1+\lambda|x_i-x_0|}&=\left|\int_{\mathbb{R}^n}\left[\frac{\lambda(x_i-x_0)}{1+\lambda|x_i-x_0|}-\frac{x}{1+|x|}\right]v_i^{2^*}dx\right|\\
&\leq\int_{\mathbb{R}^n}\left|\frac{x}{1+|x|}-\frac{\lambda(x_i-x_0)}{1+\lambda|x_i-x_0|}\right|v_i^{2^*}(x)dx\\
&=\int_{B(\lambda x_i-\lambda x_0,\rho)}\left|\frac{x}{1+|x|}-\frac{\lambda(x_i-x_0)}{1+\lambda|x_i-x_0|}\right|v_i^{2^*}(x)dx\\
&\quad+\int_{\mathbb{R}^n\setminus B(\lambda x_i-\lambda x_0,\rho)}\left|\frac{x}{1+|x|}-\frac{\lambda(x_i-x_0)}{1+\lambda|x_i-x_0|}\right|v_i^{2^*}(x)dx\\
&\leq\int_{B(\lambda x_i-\lambda x_0,\rho)}|x-\lambda(x_i-x_0)|v_i^{2^*}(x)dx+2\int_{\mathbb{R}^n\setminus B(\lambda x_i-\lambda x_0,\rho)}v_i^{2^*}(x)dx\\
&\leq\rho\int_{B(\lambda x_i-\lambda x_0,\rho)}v_i^{2^*}(x)dx+2\int_{\mathbb{R}^n\setminus B(\lambda x_i-\lambda x_0,\rho)}v_i^{2^*}(x)dx
\end{aligned}$$

that, together with (2.41), gives

$$\limsup_{i\to\infty} \frac{\lambda|x_i - x_0|}{1 + \lambda|x_i - x_0|} \le \rho \quad \forall \rho > 0.$$

Thus $\lim_{i\to\infty} \frac{\lambda|x_i - x_0|}{1+\lambda|x_i - x_0|} = 0$ and so $\lim_{i\to\infty} x_i = x_0$. On the other hand, since $\beta(v_i) = 0$, we have for every $\rho > 0$

$$\begin{aligned}
\gamma(v_i) &= \int_{\mathbb{R}^n} \frac{|x|}{1+|x|} v_i^{2^*}(x)dx \\
&= \int_{B(\lambda x_i - \lambda x_0,\rho)} \frac{|x|}{1+|x|} v_i^{2^*}(x)dx + \int_{\mathbb{R}^n \setminus B(\lambda x_i - \lambda x_0,\rho)} \frac{|x|}{1+|x|} v_i^{2^*}(x)dx \\
&\le \int_{B(\lambda x_i - \lambda x_0,\rho)} |x| v_i^{2^*}(x)dx + \int_{\mathbb{R}^n \setminus B(\lambda x_i - \lambda x_0,\rho)} v_i^{2^*}(x)dx \\
&\le (\rho + \lambda|x_i - x_0|) \int_{B(\lambda x_i - \lambda x_0,\rho)} v_i^{2^*}(x)dx + \int_{\mathbb{R}^n \setminus B(\lambda x_i - \lambda x_0,\rho)} v_i^{2^*}(x)dx.
\end{aligned}$$

Therefore we deduce

$$\limsup_{i\to\infty} \gamma(v_i) \le \rho \quad \forall \rho > 0$$

that implies $\lim_{i\to\infty} \gamma(v_i) = 0$, contradicting (2.40). □

LEMMA 2.13. – *Let* Ω, x_0, $\bar{\alpha}$, α *be as in Theorem* 2.1 *and* K *be the set introduced in Corollary* 2.11. *Then there exists* $\bar{\lambda} > 0$ *such that for every* $\lambda > \bar{\lambda}$ *it results:*

$$(2.42) \quad \begin{cases} a) \text{ the function } T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) \text{ has its support in } \Omega \quad \forall \quad (y,\sigma) \in K; \\ b) \ \sup\{f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\} < S + \epsilon < c(\alpha). \end{cases}$$

Proof. – The existence of $\bar{\lambda}_1$, such that (2.42) a) is satisfied for every $\lambda > \bar{\lambda}_1$, follows from the fact that φ has compact support and K is a bounded subset of $\mathbb{R}^n \times \mathbb{R}$.

In order to prove (2.42) b), let us remark that for every $\lambda > \bar{\lambda}_1$ it results

$$(2.43) \quad f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) = \int_{\mathbb{R}^n} [|DT_{\sigma,y}(\varphi)|^2 + \alpha(x)T^2_{\sigma,y}(\varphi)]dx + \int_\Omega \bar{\alpha}(x)[T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi)]^2 dx$$

and that, by Corollary 2.11,

$$\sup\left\{\int_{\mathbf{R}^n}[|DT_{\sigma,y}(\varphi)|^2+\alpha(x)T^2_{\sigma,y}(\varphi)]dx : (y,\sigma)\in\partial K\right\} < S+\epsilon.$$

So, to get (2.42) b), it suffices to show that

$$\lim_{\lambda\to+\infty}\sup\left\{\int_\Omega \bar{\alpha}(x)[T_{\frac{1}{\lambda},x_0}\circ T_{\sigma,y}(\varphi)]^2dx : (y,\sigma)\in K\right\}=0. \tag{2.44}$$

Now, $\forall(y,\sigma)\in K$ we have

$$\int_\Omega \bar{\alpha}[T_{\frac{1}{\lambda},x_0}\circ T_{\sigma,y}(\varphi)]^2dx=\int_{B(x_0+\frac{y}{\lambda},\frac{\sigma}{\lambda})}\bar{\alpha}(x)[T_{\frac{1}{\lambda},x_0}\circ T_{\sigma,y}(\varphi)]^2dx$$
$$\leq\left[\int_{B(x_0+\frac{y}{\lambda},\frac{\sigma}{\lambda})}\bar{\alpha}^{\frac{n}{2}}(x)dx\right]^{\frac{2}{n}}\leq\left[\int_{\Omega\cap B(x_0,\frac{r+\sigma_2}{\lambda})}\bar{\alpha}^{\frac{n}{2}}(x)dx\right]^{\frac{2}{n}}$$

where the last term goes to zero as $\lambda\to+\infty$, because $\bar{\alpha}\in L^{n/2}(\Omega)$. Then (2.44) is proved. □

Proof of Theorem 2.1. – Let $c(\alpha)$ be the number defined in (2.8); let K and Θ be as in Corollary 2.11. Let us choose $\epsilon>0$ so small that $S+\epsilon<\min\{c(\alpha),2^{\frac{2}{n}}S\}$, and φ satisfying (2.14); moreover consider $\lambda>\bar{\lambda}$ with $\bar{\lambda}$ fixed in such a way that the claim of Lemma 2.13 is true.

Let ϑ be the homotopy between Θ and the identity map in ∂K, used in the proof of Corollary 2.11. Then we have

$$\vartheta(y,\sigma,t)\neq\left(0,\frac{1}{3}\right)\quad\forall(y,\sigma)\in\partial K,\quad\forall t\in[0,1] \tag{2.45}$$

that implies the existence of $(\bar{y},\bar{\sigma})\in\partial K$ such that

$$\beta\circ T_{\bar{\sigma},\bar{y}}(\varphi)=\beta_\lambda\circ T_{\frac{1}{\lambda},x_0}\circ T_{\bar{\sigma},\bar{y}}(\varphi)=0,$$

$$\gamma\circ T_{\bar{\sigma},\bar{y}}(\varphi)=\gamma_\lambda\circ T_{\frac{1}{\lambda},x_0}\circ T_{\bar{\sigma},\bar{y}}(\varphi)\geq\frac{1}{3}$$

and the existence of $(y',\sigma')\in K$ for which $\Theta(y',\sigma')=(0,\frac{1}{3})$ that is

$$\beta_\lambda\circ T_{\frac{1}{\lambda},x_0}\circ T_{\sigma',y'}(\varphi)=0\quad\text{and}\quad\gamma_\lambda\circ T_{\frac{1}{\lambda},x_0}\circ T_{\sigma',y'}(\varphi)=\frac{1}{3}.$$

Therefore, using also (2.39) a) - b) and (2.42) b), we obtain

$$\begin{aligned}(2.46)\quad S &< \inf\left\{f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) \geq \frac{1}{3}\right\}\\ &\leq f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\bar{\sigma},\bar{y}}(\varphi) \leq \sup\{f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\}\\ &< S + \epsilon < c(\alpha) \leq \inf\left\{f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) = \frac{1}{3}\right\}\\ &\leq f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma',y'}(\varphi) \leq \sup\{f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in K\}.\end{aligned}$$

We want to prove that there exists a critical point v_λ for f_λ constrained on $V(\Omega)$ such that

$$\begin{aligned}c_1 &\overset{\text{def}}{=} \inf\left\{f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) \geq \frac{1}{3}\right\}\\ &\leq f_\lambda(v_\lambda) \leq \sup\{f_\lambda \circ T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\} \overset{\text{def}}{=} c_2.\end{aligned}$$

Assume, by contradiction, that no critical value lies in $[c_1, c_2]$. Then, since $S < c_1 \leq c_2 < S + \epsilon < 2^{2/n} S$ and the Palais–Smale condition holds in $f_\lambda^{-1}(]S, 2^{2/n}S[)$, there exists $c_1' \in]S, c_1[$ such that the sublevel $f_\lambda^{c_1'} = \{u \in V(\Omega) : f_\lambda(u) \leq c_1'\}$ is a deformation retract of the sublevel $f_\lambda^{c_2} = \{u \in V(\Omega) : f_\lambda(u) \leq c_2\}$; namely a continuous function $\Gamma : f_\lambda^{c_2} \times [0,1] \to f_\lambda^{c_2}$ exists such that

$$\Gamma(u, 0) = u \quad \text{and} \quad \Gamma(u,1) \in f_\lambda^{c_1'} \quad \forall u \in f_\lambda^{c_2}.$$

Since $\{T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\} \subseteq f_\lambda^{c_2}$, it follows that

$$(2.47)\qquad f_\lambda \circ \Gamma[T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi), 1] \leq c_1' < c_1 \quad \forall (y,\sigma) \in \partial K.$$

Now, let us define a continuous function $\eta : \partial K \times [0,1] \to \mathbb{R}^n \times \mathbb{R}$ by

$$\eta(y,\sigma,t) = \vartheta(y,\sigma,2t) \quad \forall (y,\sigma) \in \partial K, \quad \forall t \in \left[0, \frac{1}{2}\right]$$

$$\eta(y,\sigma,t) = \Big(\beta_\lambda \circ \Gamma[T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi), 2t-1], \gamma_\lambda \circ \Gamma[T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi), 2t-1]\Big)$$
$$\forall (y,\sigma) \in \partial K, \quad \forall t \in \left[\frac{1}{2}, 1\right].$$

Notice that η is well defined because $\{T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\} \subseteq f_\lambda^{c_2}$; moreover, by (2.45) and (2.46),

$$\eta(y,\sigma,t) \neq \left(0, \frac{1}{3}\right) \quad \forall (y,\sigma) \in \partial K, \quad \forall t \in [0,1].$$

Then a point $(\bar{x}, \bar{\delta}) \in \partial K$ must exist such that

$$\beta_\lambda \circ \Gamma[T_{\frac{1}{\lambda}, x_0} \circ T_{\bar{\delta}, \bar{x}}(\varphi), 1] = 0 \quad \text{and} \quad \gamma_\lambda \circ \Gamma[T_{\frac{1}{\lambda}, x_0} \circ T_{\bar{\delta}, \bar{x}}(\varphi), 1] \geq \frac{1}{3}$$

and this implies

$$f_\lambda \circ \Gamma[T_{\frac{1}{\lambda}, x_0} \circ T_{\bar{\delta}, \bar{x}}(\varphi), 1]$$
$$\geq \inf\left\{ f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) \geq \frac{1}{3} \right\} = c_1 > c_1'$$

that contradicts (2.47).

So it is proved, for any $\lambda \geq \bar{\lambda}$, the existence of a constrained critical point v_λ satisfying the energy estimate

$$S < c_1 \leq f_\lambda(v_\lambda) \leq c_2 < S + \epsilon$$

and, since $\epsilon > 0$ can be taken arbitrarily small, we derive

$$\lim_{\lambda \to +\infty} f_\lambda(v_\lambda) = S.$$

Remark also that, since $S + \epsilon < 2^{\frac{2}{n}} S$, by Lemma 2.7, v_λ must have constant sign.

Let us now prove the second part of the claim of Theorem 2.1.

First of all observe that, if $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} < S(2^{2/n} - 1)$, then it is possible to find φ, K and $\bar{\lambda}$ so that

$$\sup\{f_\lambda \circ T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma, y}(\varphi) : (y, \sigma) \in K\} < 2^{2/n} S \quad \forall \lambda > \bar{\lambda}. \tag{2.48}$$

In fact, since in this case $2^{2/n} S - \|\alpha\|_{L^{n/2}(\mathbb{R}^n)} > S$, φ can be chosen verifying, in addition to (2.14),

$$\int_{B(0,1)} |D\varphi|^2 dx < 2^{2/n} S - \|\alpha\|_{L^{n/2}(\mathbb{R}^n)}. \tag{2.49}$$

Now we have for every $(y, \sigma) \in K$

$$f_\lambda \circ T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma, y}(\varphi) = \int_{B(0,1)} |D\varphi|^2 dx + \int_\Omega \bar{\alpha}(x) \left[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma, y}(\varphi) \right]^2 dx$$
$$+ \int_{B(y,\sigma)} \alpha(x) [T_{\sigma, y}(\varphi)]^2 dx$$
$$\leq \int_{B(0,1)} |D\varphi|^2 dx + \|\alpha\|_{L^{n/2}(\mathbb{R}^n)} + \int_\Omega \bar{\alpha}(x) \left[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma, y}(\varphi) \right]^2 dx$$

and from this, using (2.49) and (2.44), we deduce the existence of $\bar{\lambda} > 0$ for which (2.48) is satisfied.

We shall prove that for every $\lambda > \bar{\lambda}$ there exists a constrained critical point $\hat{v}_\lambda$ for f_λ on $V(\Omega)$ such that

$$\hat{c}_1 \stackrel{\text{def}}{=} \inf\left\{ f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) = \frac{1}{3} \right\}$$
$$\leq f_\lambda(\hat{v}_\lambda) \leq \sup\{ f_\lambda \circ T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in K\} \stackrel{\text{def}}{=} \hat{c}_2.$$

We remark that, in this case, by (2.48) and Lemma 2.7, $\hat{v}_\lambda$ will have constant sign, and moreover $\hat{v}_\lambda \neq v_\lambda$, because $c_1 \leq f_\lambda(v_\lambda) \leq c_2 < c(\alpha) \leq \hat{c}_1 \leq f_\lambda(\hat{v}_\lambda) \leq \hat{c}_2$.

Assume, by contradiction, that no critical value lies in $[\hat{c}_1, \hat{c}_2]$.

Then, since $S < \hat{c}_1 \leq \hat{c}_2 < 2^{2/n} S$ and the Palais–Smale condition holds in $f_\lambda^{-1}(]S, 2^{2/n} S[)$, there exists $\hat{c}_1' \in]c_2, \hat{c}_1[$ such that the sublevel $f_\lambda^{\hat{c}_1'} = \{u \in V(\Omega) : f_\lambda(u) \leq \hat{c}_1'\}$ is a deformation retract of the sublevel $f_\lambda^{\hat{c}_2} = \{u \in V(\Omega) : f_\lambda(u) \leq \hat{c}_2\}$; namely there exists a continuous function $\hat{\Gamma} : f_\lambda^{\hat{c}_2} \times [0,1] \to f_\lambda^{\hat{c}_2}$ such that:

$$\hat{\Gamma}(u,0) = u \quad \text{and} \quad \hat{\Gamma}(u,1) \in f_\lambda^{\hat{c}_1'} \quad \forall u \in f_\lambda^{\hat{c}_2}$$
$$\hat{\Gamma}(u,t) = u \quad \forall t \in [0,1], \quad \forall u \in f_\lambda^{\hat{c}_1'}.$$

So, from

$$\{T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in \partial K\} \subseteq f_\lambda^{c_2} \subseteq f_\lambda^{\hat{c}_1'}$$
$$\{T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in K\} \subseteq f_\lambda^{\hat{c}_2},$$

it follows

$$\hat{\Gamma}[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi), t] = T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi) \quad \forall (y,\sigma) \in \partial K, \ \forall t \in [0,1] \tag{2.50}$$

$$\sup\{ f_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi), 1] : (y,\sigma) \in K\} \leq \hat{c}_1' < \hat{c}_1. \tag{2.51}$$

Now let us define a continuous function $\hat{\eta} : K \times [0,1] \to \mathbb{R}^n \times \mathbb{R}$ by

$$\hat{\eta}(y,\sigma,t) = ((1-2t)y + 2t\beta \circ T_{\sigma,y}(\varphi), \quad (1-2t)\sigma + 2t\gamma \circ T_{\sigma,y}(\varphi))$$
$$\forall (y,\sigma) \in K, \quad \forall t \in \left[0, \frac{1}{2}\right],$$
$$\hat{\eta}(y,\sigma,t) = (\beta_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi), 2t-1], \gamma_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda}, x_0} \circ T_{\sigma,y}(\varphi), 2t-1])$$
$$\forall (y,\sigma) \in K, \quad \forall t \in \left[\frac{1}{2}, 1\right].$$

Notice that $\hat{\eta}$ is well defined because $\{T_{\frac{1}{\lambda},x_0} \circ T_{\sigma,y}(\varphi) : (y,\sigma) \in K\} \subset f_\lambda^{\hat{c}_2}$; moreover we deduce from (2.45)

$$\hat{\eta}(y,\sigma,t) = \vartheta(y,\sigma,2t) \neq \left(0,\frac{1}{3}\right) \quad \forall (y,\sigma) \in \partial K, \quad \forall t \in \left[0,\frac{1}{2}\right]$$

and, using (2.45) and (2.50),

$$\hat{\eta}(y,\sigma,t) = \hat{\eta}\left(y,\sigma,\frac{1}{2}\right) \neq \left(0,\frac{1}{3}\right) \quad \forall (y,\sigma) \in \partial K, \quad \forall t \in \left[\frac{1}{2},1\right].$$

Then a point $(x',\delta') \in K$ must exist such that

$$\beta_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda},x_0} \circ T_{\delta',x'}(\varphi),1] = 0 \qquad \gamma_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda},x_0} \circ T_{\delta',x'}(\varphi),1] = \frac{1}{3}$$

and this implies

$$\begin{aligned} &f_\lambda \circ \hat{\Gamma}[T_{\frac{1}{\lambda},x_0} \circ T_{\delta',x'}(\varphi),1] \\ &\quad \geq \inf\left\{ f_\lambda(u) : u \in V(\Omega),\ \beta_\lambda(u) = 0,\ \gamma_\lambda(u) = \frac{1}{3}\right\} = \hat{c}_1 > \hat{c}'_1 \end{aligned}$$

contradicting (2.51).

Then we have proved the existence of two distinct critical points v_λ and $\hat{v}_\lambda$ of f_λ on $V(\Omega)$. These functions have constant sign, that we can assume positive; so they give rise to two positive solutions

$$u_\lambda = [f_\lambda(v_\lambda)]^{\frac{n-2}{4}} v_\lambda \quad \text{and} \quad \hat{u}_\lambda = [f_\lambda(\hat{v}_\lambda)]^{\frac{n-2}{4}} \hat{v}_\lambda$$

of Problem (1.1). □

Remark 2.14 *(radial symmetry).* – If $\Omega = B(0,\rho) = \{x \in \mathbb{R}^n : |x| < \rho\}$, and we assume $x_0 = 0$ and $\bar{\alpha}$, α radially symmetric functions, , then it is natural looking for the solutions of Problem (1.1) in the subspace of $H_0^{1,2}(\Omega)$ made up the functions having radial symmetry.

In this case the proof of Theorem (2.9) can be simplified. In particular, the solution u_λ corresponds to a local minimum point among the radial functions. In fact, if we denote by $V_r(\Omega)$ the subset of $V(\Omega)$ made up the radial functions, we have for $\lambda > \bar{\lambda}$

$$\gamma_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma_2,0}(\varphi) > \frac{1}{3}$$

and

$$\begin{aligned} S &< \inf\left\{ f_\lambda(u) : u \in V_r(\Omega),\ \gamma_\lambda(u) \geq \frac{1}{3} \right\} \\ &\leq f_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma_2,0}(\varphi) < S + \epsilon < c(\alpha) \\ &\leq \inf\left\{ f_\lambda(u) : u \in V_r(\Omega),\ \gamma_\lambda(u) = \frac{1}{3} \right\} \end{aligned}$$

with $S + \epsilon < 2^{2/n} S$.

Thus the existence of a minimum point v_λ of f_λ on the subset

$$\left\{ u \in V_r(\Omega) : \gamma_\lambda(u) > \frac{1}{3} \right\}$$

can be proved.

Moreover, under the additional assumption $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)} < S(2^{2/n} - 1)$, another solution $\hat{u}_\lambda$ can be obtained by a variant of the well known Mountain Pass Theorem by Ambrosetti–Rabinowitz [1]. In fact, in this case we have for $\lambda > \bar{\lambda}$

$$\gamma_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma_1,0}(\varphi) < \frac{1}{3} < \gamma_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma_2,0}(\varphi)$$

and

$$\begin{aligned} S &< \max\{ f_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma,0}(\varphi) : x \in \{\sigma_1, \sigma_2\}\} \\ &< \inf\left\{ f_\lambda(u) : u \in V_r(\Omega),\ \gamma_\lambda(u) = \frac{1}{3} \right\} \\ &\leq \sup\{ f_\lambda \circ T_{\frac{1}{\lambda},0} \circ T_{\sigma,0}(\varphi) : \sigma \in [\sigma_1, \sigma_2]\} < 2^{\frac{2}{n}} S. \end{aligned}$$

Remark 2.15. – The solutions u_λ and $\hat{u}_\lambda$ found in Theorem 2.1 have a different behaviour as $\lambda \to +\infty$. In fact, as we have before seen,

$$\lim_{\lambda \to +\infty} f_\lambda\left(\frac{u_\lambda}{\|u_\lambda\|_{L^{2^*}}} \right) = S,$$

while

$$\liminf_{\lambda \to +\infty} f_\lambda\left(\frac{\hat{u}_\lambda}{\|\hat{u}_\lambda\|_{L^{2^*}}} \right) \geq c(\alpha) > S.$$

Thus one cannot say that $\hat{u}_\lambda$ concentrates near a point as $\lambda \to +\infty$, like u_λ does.

It is only possible to remark that for λ large enough $f_\lambda\left(\frac{\hat{u}_\lambda}{\|\hat{u}_\lambda\|_{L^{2^*}}} \right)$ is close to S provided that $\|\alpha\|_{L^{n/2}(\mathbb{R}^n)}$ is small enough.

Remark 2.16. – The solution $\hat{u}_\lambda$ given by Theorem 2.1 corresponds, in some sense, to the solution obtained by Benci and Cerami [3] in the case $\Omega = \mathbb{R}^n$.

On the contrary u_λ is a solution of new type, whose existence is just related to the fact that Ω is a bounded domain.

Let us also remark that in Theorem 2.1 we do not require the stronger assumption $\alpha \in L^p(\mathbb{R}^n)$ $\forall p \in [p_1, p_2]$ with $p_1 < \frac{n}{2} < p_2$, used in [3].

3. MULTIPLICITY OF POSITIVE SOLUTIONS IN PRESENCE OF SEVERAL CONCENTRATIONS

This section is devoted to the study of Problem (1.1) when the function $a(x)$ has the form

$$a(x) = \bar{\alpha}(x) + \sum_{i=1}^{h} \lambda_i^2 \alpha_i(\lambda_i(x - x_i)) \tag{3.1}$$

or also

$$a(x) = \bar{\alpha}(x) + \sum_{i=1}^{r} \lambda_i^2 \mu_i \alpha_i(\lambda_i(x - x_i)) + \sum_{i=r+1}^{h} \lambda_i^2 \alpha_i(\lambda_i(x - x_i)) \tag{3.2}$$

where $x_1, \ldots, x_h$ are given points in Ω, $\bar{\alpha}$, $\alpha_1 \ldots \alpha_h$ are nonnegative functions, and λ_i, μ_j are positive parameters.

It is very natural to think that several concentrations in the function $a(x)$ can guarantee the existence of several distinct solutions. Indeed, we show that it is possible to choose the parameters λ_i and μ_j in such a way to obtain several distinct critical values of the functional f constrained on $V(\Omega)$.

Theorems 3.1, 3.2 and 3.3 describe some possible way to realize this choice. We point out that, when we exploit the parameter λ_i and μ_j in order to obtain several critical values, we do not need to require that the concentration points $x_1 \ldots x_h$ are necessarily distinct.

THEOREM 3.1. – *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 3$ and $x_1 \ldots x_h$ be given points in Ω (not necessarily distinct). Let $\bar{\alpha}$ in $L^{n/2}(\Omega)$ and $\alpha_1 \ldots \alpha_h$ in $L^{n/2}(\mathbb{R}^n)$ be nonnegative functions such that $\|\alpha_i\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$ $\forall i = 1 \ldots h$.*

Then, there exist $\bar{\lambda}_1 > 0$, $\bar{\lambda}_2 = \bar{\lambda}_2(\lambda_1) > 0$, $\bar{\lambda}_3 = \bar{\lambda}_3(\lambda_1, \lambda_2) > 0 \ldots$ $\bar{\lambda}_i = \bar{\lambda}_i(\lambda_1 \ldots \lambda_{i-1}) > 0 \ldots \bar{\lambda}_h = \bar{\lambda}_h(\lambda_1 \ldots \lambda_{h-1}) > 0$ such that Problem (1.1) with $a(x)$ of the form (3.1) has at least h distinct solutions $u_1 \ldots u_h$ for every choice of $\lambda_1 \ldots \lambda_h$ such that $\lambda_i > \bar{\lambda}_i$, $i = 1 \ldots h$.

Moreover

$$(3.3)\quad 2^{2/n}S > f\left(\frac{u_1}{\|u_1\|_{L^{2^*}}}\right) > f\left(\frac{u_2}{\|u_2\|_{L^{2^*}}}\right) > \ldots > f\left(\frac{u_h}{\|u_h\|_{L^{2^*}}}\right) > S,$$

$$(3.4)\qquad \lim_{\lambda_i \to +\infty} f\left(\frac{u_i}{\|u_i\|_{L^{2^*}}}\right) = S \quad \forall i = 1\ldots h.$$

THEOREM 3.2. – *Let Ω, $x_1\ldots x_h$, $\bar{\alpha}$, $\alpha_1\ldots\alpha_h$ be as in Theorem 3.1.*

Then there exist $\bar{\mu}_1 > 0$, $\bar{\lambda}_1 = \bar{\lambda}_1(\mu_1) > 0$, $\bar{\mu}_2 = \bar{\mu}_2(\lambda_1, \mu_1) > 0$, $\bar{\lambda}_2 = \bar{\lambda}_2(\mu_1, \lambda_1, \mu_2) > 0\ldots$ $\bar{\mu}_r = \bar{\mu}_r(\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_{r-1}, \lambda_{r-1}) > 0$, $\bar{\lambda}_r = \bar{\lambda}_r(\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_{r-1}, \lambda_{r-1}, \mu_r) > 0$, (with $r \le h$) and $\bar{\lambda}_{r+1} = \bar{\lambda}_{r+1}(\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_r, \lambda_r) > 0\ldots$ $\bar{\lambda}_h = \bar{\lambda}_h(\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_r, \lambda_r, \lambda_{r+1}\ldots\lambda_{h-1}) > 0$ such that Problem (1.1) *with $a(x)$ of the form* (3.2) *has at least $(r+h)$ distinct solutions $\hat{u}_1, u_1, \hat{u}_2, u_2, \ldots, \hat{u}_r, u_r, u_{r+1}, \ldots, u_h$ for every choice of $\lambda_1\ldots\lambda_h$, $\mu_1\ldots\mu_r$ such that*

$$\lambda_i > \bar{\lambda}_i \ \ \forall i = 1\ldots h \ \ \textit{and} \ \ 0 < \mu_j < \bar{\mu}_j \ \ \forall j = 1\ldots r.$$

Moreover

$$(3.5)\quad \begin{aligned} 2^{2/n}S &> f\left(\frac{\hat{u}_1}{\|\hat{u}_1\|_{L^{2^*}}}\right) > f\left(\frac{u_1}{\|u_1\|_{L^{2^*}}}\right) > f\left(\frac{\hat{u}_2}{\|\hat{u}_2\|_{L^{2^*}}}\right) \\ &> f\left(\frac{u_2}{\|u_2\|_{L^{2^*}}}\right) > \ldots > f\left(\frac{\hat{u}_r}{\|\hat{u}_r\|_{L^{2^*}}}\right) > f\left(\frac{u_r}{\|u_r\|_{L^{2^*}}}\right) \\ &> f\left(\frac{u_{r+1}}{\|u_{r+1}\|_{L^{2^*}}}\right) > \ldots > f\left(\frac{u_h}{\|u_h\|_{L^{2^*}}}\right) > S, \end{aligned}$$

$$(3.6)\quad \begin{aligned} &\lim_{\lambda_i \to +\infty} f\left(\frac{u_i}{\|u_i\|_{L^{2^*}}}\right) = S \quad \forall i = 1\ldots h, \\ &\lim_{\mu_j \to 0} f\left(\frac{\hat{u}_j}{\|\hat{u}_j\|_{L^{2^*}}}\right) = S \quad \forall j = 1\ldots r. \end{aligned}$$

THEOREM 3.3. – *Let $\Omega, x_1\ldots x_h$, $\bar{\alpha}, \alpha_1\ldots\alpha_h$ be as in Theorem* 3.1.

Then there exist $\bar{\mu}_1 > 0, \bar{\mu}_2 = \bar{\mu}_2(\mu_1) > 0$, $\bar{\mu}_3 = \bar{\mu}_3(\mu_1, \mu_2) > 0, \ldots, \bar{\mu}_r = \bar{\mu}_r(\mu_1, \mu_2\ldots\mu_{r-1}) > 0$ (with $r \le h$) and $\bar{\lambda}_1 = \bar{\lambda}_1(\mu_1\ldots\mu_r) > 0, \bar{\lambda}_2 = \bar{\lambda}_2(\mu_1\ldots\mu_r, \lambda_1) > 0\ldots\bar{\lambda}_h = \bar{\lambda}_h(\mu_1\ldots\mu_r, \lambda_1\ldots\lambda_{h-1}) > 0$ such that Problem (1.1) *with $a(x)$ of the form* (3.2) *has at least $(r+h)$ distinct* *solutions* $\hat{u}_1, \hat{u}_2 \ldots \hat{u}_r, u_1, u_2 \ldots u_h$ *for every choice of* $\lambda_1 \ldots \lambda_h$, $\mu_1 \ldots \mu_r$ *such that*

$$\lambda_i > \bar{\lambda}_i \quad \forall i = 1 \ldots h \quad \textit{and} \quad 0 < \mu_j < \bar{\mu}_j \quad \forall j = 1 \ldots r.$$

Moreover we have

(3.7)
$$2^{2/n} S > f\left(\frac{\hat{u}_1}{\|\hat{u}_1\|_{L^{2^*}}}\right) > f\left(\frac{\hat{u}_2}{\|\hat{u}_2\|_{L^{2^*}}}\right) > \ldots > f\left(\frac{\hat{u}_r}{\|\hat{u}_r\|_{L^{2^*}}}\right)$$
$$> f\left(\frac{u_1}{\|u_1\|_{L^{2^*}}}\right) > f\left(\frac{u_2}{\|u_2\|_{L^{2^*}}}\right) > \ldots > f\left(\frac{u_h}{\|u_h\|_{L^{2^*}}}\right) > S$$

and the relations (3.6) *hold.*

In what follows we denote by $f_{\lambda_1 \ldots \lambda_s}$ the functional f when

$$a(x) = \bar{\alpha}(x) + \sum_{i=1}^{s} \lambda_i^2 \alpha_i(\lambda_i(x - x_i)) \quad (\text{where } \; s \leq h)$$

and by $f_{\lambda_1 \ldots \lambda_s}^{\mu_1 \ldots \mu_t}$ the functional f when

$$a(x) = \bar{\alpha}(x) + \sum_{i=1}^{t} \lambda_i^2 \mu_i \alpha_i(\lambda_i(x - x_i))$$
$$+ \sum_{i=t+1}^{s} \lambda_i^2 \alpha_i(\lambda_i(x - x_i)) \; (\text{with } t \leq s \leq h).$$

Moreover we put

$$\beta_{\lambda_i} = \beta \circ T_{\lambda_i, -\lambda_i x_i} \quad \text{and} \quad \gamma_{\lambda_i} = \gamma \circ T_{\lambda_i, -\lambda_i x_i}.$$

Proof of Theorem 3.1. – The idea of the proof is the following: first we remark that Theorem 2.1 implies the existence of a critical value for f_{λ_1} on $V(\Omega)$ if λ_1 is large enough; moreover, fixed $\lambda_1 > 0$, the same theorem implies that for $\lambda_2 > 0$ large enough there exists a critical value for f_{λ_1, λ_2}, that goes to S as $\lambda_2 \to +\infty$.

Then, the crucial step is to prove that the previous critical value of f_{λ_1} persists in the sense that f_{λ_1, λ_2} has also another critical value, which is close to the one of f_{λ_1}, if $\lambda_2 > 0$ is large enough. Iterating this argument, we obtain h distinct critical values for $f_{\lambda_1 \ldots \lambda_h}$ for suitable choices of the parameters $\lambda_1 \ldots \lambda_h$.

For every $i = 1\ldots h$, let us set

$$c(\alpha_i) \stackrel{\text{def}}{=} \inf\left\{ \int_{\mathbb{R}^n} [|Du|^2 + \alpha_i(x)u^2]dx : u \in V(\mathbb{R}^n), \right.$$
$$\left. \beta(u) = 0,\ \gamma(u) = \frac{1}{3} \right\}.$$

By Lemma 2.8, since $\|\alpha_i\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$, we have that $c(\alpha_i) > S$ $\forall i = 1\ldots h$.

For every $\epsilon_1 > 0$ such that

$$S + \epsilon_1 < \min\{c(\alpha_1), \ldots, c(\alpha_h), 2^{2/n}S\},$$

we find, arguing as in section 2, with analogous notations, a constant $\bar{\lambda}_1 > 0$, a function $\varphi_1 \in H_0^{1,2}(B(0,1))$ and a subset K_1 of $\mathbb{R}^n \times \mathbb{R}$, with $(0, \frac{1}{3})$ in its interior, having the properties described in Corollary 2.11 and such that for every $\lambda_1 > \bar{\lambda}_1$ the relations

$$(3.8) \qquad \begin{cases} T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) \in H_0^{1,2}(\Omega) \quad \forall (y,\sigma) \in K_1 \\ \text{and} \\ S < \inf\{f_{\lambda_1}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) \geq \frac{1}{3}\} \\ \quad \leq \sup\{f_{\lambda_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1\} \\ \quad < S + \epsilon_1 < c(\alpha_1) \end{cases}$$

hold.

Let us fix $\lambda_1 > \bar{\lambda}_1$. Then, for every $\epsilon_2 > 0$ such that

$$S + \epsilon_2 < \inf\left\{ f_{\lambda_1}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) \geq \frac{1}{3} \right\},$$

there exist $\bar{\lambda}_2 = \bar{\lambda}_2(\lambda_1) > 0$, a function $\varphi_2 \in H_0^{1,2}(B(0,1))$, a subset K_2 in $\mathbb{R}^n \times \mathbb{R}$, with $(0, \frac{1}{3})$ in its interior, having the properties described in Corollary 2.11, so that for every $\lambda_2 > \bar{\lambda}_2$ it results:

$$T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) \in H_0^{1,2}(\Omega) \quad \forall (y,\sigma) \in K_2$$

and

$$S < \inf\left\{ f_{\lambda_1,\lambda_2}(u) : u \in V(\Omega),\ \beta_{\lambda_2}(u) = 0,\ \gamma_{\lambda_2}(u) \geq \frac{1}{3} \right\}$$
$$\leq \sup\{f_{\lambda_1,\lambda_2} \circ T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in \partial K_2\} < S + \epsilon_2 < c(\alpha_2).$$

Now, let us prove that

$$(3.9)\quad \lim_{\lambda_2\to+\infty} \sup\left\{\int_\Omega \lambda_2^2\alpha_2(\lambda_2(x-x_2))[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^2dx :\right.$$
$$\left.(y,\sigma)\in K_1\right\}=0.$$

In fact we have

$$(3.10)\qquad \sup\{T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)(x) : x\in\Omega,\ (y,\sigma)\in K_1\}<+\infty$$

because $\sup_{B(0,1)} \varphi_1 < +\infty$.

Moreover, it is easy to verify that

$$\lim_{\lambda_2\to+\infty}\int_{B(x_2,\rho)}[\lambda_2^2\alpha_2(\lambda_2(x-x_2))]^{\frac{n}{2}}dx = \int_{\mathbb{R}^n}\alpha_2^{\frac{n}{2}}(x)dx \quad \forall\rho>0$$

that is

$$\lim_{\lambda_2\to+\infty}\int_{\Omega\setminus B(x_2,\rho)}[\lambda_2^2\alpha_2(\lambda_2(x-x_2))]^{\frac{n}{2}}dx = 0 \quad \forall\rho>0.$$

Therefore we have

$$\begin{aligned}
&\int_\Omega \lambda_2^2\alpha_2(\lambda_2(x-x_2))[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^2dx\\
&\quad=\int_{B(x_2,\rho)}\lambda_2^2\alpha_2(\lambda_2(x-x_2))[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^2dx\\
&\qquad+\int_{\Omega\setminus B(x_2,\rho)}\lambda_2^2\alpha_2(\lambda_2(x-x_2))[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^2dx\\
&\quad\le\left(\int_{B(x_2,\rho)}[\lambda_2^2\alpha_2(\lambda_2(x-x_2))]^{n/2}dx\right)^{2/n}\\
&\qquad\left(\int_{B(x_2,\rho)}[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^{2^*}dx\right)^{\frac{n-2}{n}}\\
&\qquad+\left(\int_{\Omega\setminus B(x_2,\rho)}[\lambda_2^2\alpha_2(\lambda_2(x-x_2))]^{n/2}dx\right)^{2/n}\\
&\qquad\left(\int_{\Omega\setminus B(x_2,\rho)}[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^{2^*}dx\right)^{\frac{n-2}{n}}.
\end{aligned}$$

So from (3.10), for a suitable choice of $\bar{c}$, it follows

$$\limsup_{\lambda_2\to+\infty} \sup\left\{\int_\Omega \lambda_2^2\alpha_2(\lambda_2(x-x_2))[T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1)]^2dx : (y,\sigma)\in K_1\right\} \leq \bar{c}\rho^{n-2} \quad \forall\rho>0,$$

that implies (3.9), as $\rho \to 0$.

Thus we infer from (3.8) and (3.9) that

$$\lim_{\lambda_2\to+\infty} \sup\{f_{\lambda_1,\lambda_2}\circ T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1) : (y,\sigma)\in\partial K_1\}$$
$$= \sup\{f_{\lambda_1}\circ T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1) : (y,\sigma)\in\partial K_1\} < S+\epsilon_1.$$

Moreover, since $f_{\lambda_1,\lambda_2}(u)\geq f_{\lambda_1}(u)\ \ \forall u\in V(\Omega)$, we can assume that for every $\lambda_2>\bar{\lambda}_2$ the following inequalities hold:

$$S+\epsilon_2 < \inf\left\{f_{\lambda_1,\lambda_2}(u) : u\in V(\Omega),\ \beta_{\lambda_1}(u)=0,\ \gamma_{\lambda_1}(u)\geq\frac{1}{3}\right\}$$
$$\leq \sup\{f_{\lambda_1,\lambda_2}\circ T_{\frac{1}{\lambda_1},x_1}\circ T_{\sigma,y}(\varphi_1) : (y,\sigma)\in\partial K_1\} < S+\epsilon_1 < c(\alpha_1).$$

Iterating this argument for $i=3\ldots h$, we obtain that for every $\epsilon_i>0$, such that

$$S+\epsilon_i < \inf\left\{f_{\lambda_1\ldots\lambda_{i-1}}(u) : u\in V(\Omega),\ \beta_{\lambda_{i-1}}(u)=0,\ \gamma_{\lambda_{i-1}}(u)\geq\frac{1}{3}\right\},$$

there exist $\bar{\lambda}_i=\bar{\lambda}_i(\lambda_1\ldots\lambda_{i-1})>0$, functions $\varphi_i\in H_0^{1,2}(B(0,1))$, subsets K_i in $\mathbb{R}^n\times\mathbb{R}$, with $(0,\frac{1}{3})$ in their interior and satisfying the properties described in Corollary 2.11, so that, if $\lambda_i>\bar{\lambda}_i$, then

$$T_{\frac{1}{\lambda_i},x_i}\circ T_{\sigma,y}(\varphi_i)\in H_0^{1,2}(\Omega)\quad \forall(y,\sigma)\in K_i,\quad \forall i=1\ldots h$$

and, for every $i=2\ldots h$, it results:

$$S<\inf\left\{f_{\lambda_1\ldots\lambda_h}(u) : u\in V(\Omega),\ \beta_{\lambda_i}(u)=0,\ \gamma_{\lambda_i}(u)\geq\frac{1}{3}\right\}$$
$$\leq \sup\{f_{\lambda_1\ldots\lambda_h}\circ T_{\frac{1}{\lambda_i},x_i}\circ T_{\sigma,y}(\varphi_i) : (y,\sigma)\in\partial K_i\} < S+\epsilon_i$$
$$< \inf\left\{f_{\lambda_1\ldots\lambda_h}(u) : u\in V(\Omega),\ \beta_{\lambda_{i-1}}(u)=0,\ \gamma_{\lambda_{i-1}}(u)\geq\frac{1}{3}\right\}$$
$$\leq \sup\{f_{\lambda_1\ldots\lambda_h}\circ T_{\frac{1}{\lambda_{i-1}},x_{i-1}}\circ T_{\sigma,y}(\varphi_{i-1}) : (y,\sigma)\in\partial K_{i-1}\} < S+\epsilon_{i-1}$$

where $S+\epsilon_i < c(\alpha_i)\ \ \forall i=1\ldots h$.

It is easy to verify that for every choice of the positive constants $\lambda_1 \dots \lambda_h$ the relation

$$c(\alpha_i) \leq \inf\left\{ f_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\} \quad \forall i = 1 \dots h$$

is satisfied. Therefore the following inequalities hold, if $\lambda_i > \bar{\lambda}_i\ \forall i = 1 \dots h$:

(3.11)
$$\begin{aligned} S &< \inf\left\{ f_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} < S + \epsilon_i \\ &< c(\alpha_i) \leq \inf\left\{ f_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\} \end{aligned}$$

with $S + \epsilon_i < 2^{2/n} S\ \forall i = 1 \dots h$.

Arguing as in the proof of Theorem 2.1, using (3.11) and the properties of K_i, it is not difficult to prove that, for every $i = 1 \dots h$, the functional $f_{\lambda_1 \dots \lambda_h}$ admits a critical value v_i verifying

$$\begin{aligned} \inf\left\{ f_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} &\leq f_{\lambda_1 \dots \lambda_h}(v_i) \\ \leq \sup\{ f_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} &< S + \epsilon_i. \end{aligned}$$

Thus, the solutions $u_i = [f_{\lambda_1 \dots \lambda_h}(v_i)]^{\frac{n-2}{4}} v_i \ (i = 1 \dots h)$ of Problem (1.1) verify the relations (3.3); moreover, since $\epsilon_i > 0$ can be taken arbitrarily small, (3.4) holds. □

Proof of Theorem 3.2. – Like in the proof of Theorem 3.1, we use an iterative procedure: we find consecutively the parameters $\mu_1, \lambda_1, \mu_2, \lambda_2, \dots, \mu_r, \lambda_r, \lambda_{r+1} \dots \lambda_h$ in such a way that the functional $f_{\lambda_1 \dots \lambda_h}^{\mu_1 \dots \mu_r}$ constrained on $V(\Omega)$ has at least $(r+h)$ distinct critical values.

Let us choose $\bar{\mu}_1 > 0$ in such a way that

$$\|\bar{\mu}_1 \alpha_1\|_{L^{n/2}(\mathbb{R}^n)} < S(2^{2/n} - 1).$$

Arguing as in Theorem 2.1, we deduce that for every $\epsilon_1 > 0$ there exist $\bar{\lambda}_1 = \bar{\lambda}_1(\mu_1) > 0$, $\varphi_1 \in H_0^{1,2}(B(0,1))$, and a subset K_1 of $\mathbb{R}^n \times \mathbb{R}$ (having the properties of Corollary 2.11) such that the following inequalities hold for $0 < \mu_1 < \bar{\mu}_1$ and $\lambda_1 > \bar{\lambda}_1(\mu_1)$:

$$\begin{aligned} S &< \inf\left\{ f_{\lambda_1}^{\mu_1}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) \geq \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1 \} \\ &< \inf\left\{ f_{\lambda_1}^{\mu_1}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) = \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in K_1 \} < 2^{\frac{2}{n}} S \end{aligned}$$

and, moreover,

$$\begin{aligned} &\sup\{ f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1 \} < S + \epsilon_1 \\ &\sup\{ f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in K_1 \} < S + 2\mu_1 \|\alpha_1\|_{L^{n/2}(\mathbb{R}^n)}. \end{aligned}$$

Let us fix $\mu_1 < \bar{\mu}_1$ and $\lambda_1 > \bar{\lambda}_1$. Then, as before, for every $\epsilon_2 > 0$ there exist $\bar{\mu}_2 = \bar{\mu}_2(\mu_1, \lambda_1) > 0$, $\bar{\lambda}_2 = \bar{\lambda}_2(\mu_1, \lambda_1, \mu_2) > 0$, $\varphi_2 \in H_0^{1,2}(B(0,1))$ and a subset K_2 of $\mathbb{R}^n \times \mathbb{R}$ (satisfying the properties of Corollary 2.11) such that, if $0 < \mu_2 \leq \bar{\mu}_2$ and $\lambda_2 \geq \bar{\lambda}_2$, it results:

$$\begin{aligned} S &< \inf\left\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2}(u) : u \in V(\Omega),\ \beta_{\lambda_2}(u) = 0,\ \gamma_{\lambda_2}(u) \geq \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in \partial K_2 \} \\ &< \inf\left\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2}(u) : u \in V(\Omega),\ \beta_{\lambda_2}(u) = 0,\ \gamma_{\lambda_2}(u) = \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in K_2 \} \end{aligned}$$

and moreover

$$\begin{aligned} &\sup\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in \partial K_2 \} < S + \epsilon_2 \\ &\sup\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2}, x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in K_2 \} < S + 2\mu_2 \|\alpha_2\|_{L^{n/2}(\mathbb{R}^n)}. \end{aligned}$$

As in the proof of Theorem 3.1, we can also assume that $\bar{\mu}_2(\mu_1, \lambda_1)$ is so small and $\bar{\lambda}_2 = \bar{\lambda}_2(\mu_1, \lambda_1, \mu_2)$ is so large that the following inequalities hold:

$$\begin{aligned}
&S + 2\mu_2\|\alpha_2\|_{L^{n/2}(\mathbb{R}^n)}\\
&\le \inf\left\{ f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) \ge \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2} \circ T_{\frac{1}{\lambda_1},x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1\}\\
&< \inf\left\{ f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2}(u) : u \in V(\Omega),\ \beta_{\lambda_1}(u) = 0,\ \gamma_{\lambda_1}(u) = \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2} \circ T_{\frac{1}{\lambda_1},x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in K_1\} < 2^{2/n}S
\end{aligned}$$

and moreover

$$\begin{aligned}
&\sup\{f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2} \circ T_{\frac{1}{\lambda_1},x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1\} < S + \epsilon_1\\
&\sup\{f^{\mu_1,\mu_2}_{\lambda_1,\lambda_2} \circ T_{\frac{1}{\lambda_1},x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in K_1\} < S + 2\mu_1\|\alpha_1\|_{L^{n/2}(\mathbb{R}^n)}.
\end{aligned}$$

Repeating this procedure for $i = 3\dots r$, for every $\epsilon_i > 0$ we find $\bar{\mu}_i = \bar{\mu}_i(\mu_1, \lambda_1 \dots \mu_{i-1}, \lambda_{i-1}) > 0$, $\bar{\lambda}_i = \bar{\lambda}_i(\mu_1, \lambda_1 \dots \mu_{i-1}, \lambda_{i-1}, \mu_i)$, $\varphi_i \in H^{1,2}_0(B(0,1))$, $K_i \subset \mathbb{R}^n \times \mathbb{R}$ such that, if $0 < \mu_i \le \bar{\mu}_i$ and $\lambda_i \ge \bar{\lambda}_i$ $\forall i = 1\dots r$, the following inequalities hold for $i = 2\dots r$:

$$\begin{aligned}
S &< \inf\left\{ f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \ge \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i\}\\
&< \inf\left\{ f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in K_i\}\\
&< \inf\left\{ f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_{i-1}}(u) \ge \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_{i-1}},x_{i-1}} \circ T_{\sigma,y}(\varphi_{i-1}) : (y,\sigma) \in \partial K_{i-1}\}\\
&< \inf\left\{ f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_{i-1}}(u) = \frac{1}{3}\right\}\\
&\le \sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_{i-1}},x_{i-1}} \circ T_{\sigma,y}(\varphi_{i-1}) : (y,\sigma) \in K_{i-1}\} < 2^{2/n}S
\end{aligned}$$

and, for $i = 1\dots r$,

$$\begin{aligned}
&\sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i\} < S + \epsilon_i\\
&\sup\{f^{\mu_1\dots\mu_r}_{\lambda_1\dots\lambda_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in K_i\} < S + 2\mu_i\|\alpha_i\|_{L^{n/2}(\mathbb{R}^n)}.
\end{aligned}$$

For $i = r+1, \ldots, h$, the same arguments used in the proof of Theorem 3.1 allow to state that for every $\epsilon_i > 0$ there exist

$$\bar{\lambda}_i = \bar{\lambda}_i(\mu_1, \lambda_1 \ldots \mu_r, \lambda_r, \lambda_{r+1} \ldots \lambda_{i-1}),\ \ \varphi_i \in H_0^{1,2}(B(0,1)),\ \ K_i \subset \mathbb{R}^n \times \mathbb{R}$$

(with the properties described in Corollary 2.11) such that, if $\lambda_i \geq \bar{\lambda}_i$ for $i = 1 \ldots h$ and $0 < \mu_i \leq \bar{\mu}_i$ for $i = 1 \ldots r$, then the previous inequalities hold with $f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}$ instead of $f_{\lambda_1 \ldots \lambda_r}^{\mu_1 \ldots \mu_r}$, and, moreover, for $i = r+1, \ldots, h$, we have

$$\begin{aligned} S &< \inf\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} \\ &\leq \sup\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} \\ &< \inf\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_{i-1}}(u) \geq \frac{1}{3} \right\} \\ &\leq \sup\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_{i-1}}, x_{i-1}} \circ T_{\sigma,y}(\varphi_{i-1}) : (y,\sigma) \in \partial K_{i-1} \right\}; \end{aligned}$$

$$\begin{aligned} &\sup\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} \\ &\quad < \inf\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\}; \\ &\sup\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} < S + \epsilon_i. \end{aligned}$$

Using these inequalities and the properties of the subsets K_i, arguing as in the proof of Theorem 2.1, it is not difficult to see that the functional $f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}$ constrained on $V(\Omega)$ has, for every $i = 1 \ldots r$, a critical point $\hat{v}_i$ such that

$$\begin{aligned} &\inf\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\} \leq f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(\hat{v}_i) \\ &\leq \sup\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in K_i \} \leq S + 2\mu_i \|\alpha_i\|_{L^{n/2}(\mathbb{R}^n)}. \end{aligned}$$

Moreover for every $i = 1 \ldots h$ it is possible to prove the existence of another critical point $v_i \in V(\Omega)$ such that

$$\begin{aligned} &\inf\left\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} \leq f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(v_i) \\ &\leq \sup\{ f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} \leq S + \epsilon_i. \end{aligned}$$

Thus, we obtain the solutions of (1.1)

$$\begin{aligned} u_i &= \left[f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(v_i) \right]^{\frac{n-2}{4}} v_i \quad (i = 1 \ldots h) \\ \hat{u}_i &= \left[f_{\lambda_1 \ldots \lambda_h}^{\mu_1 \ldots \mu_r}(\hat{v}_i) \right]^{\frac{n-2}{4}} \hat{v}_i \quad (i = 1 \ldots r) \end{aligned}$$

that, clearly, verify (3.5) and (3.6). □

Proof of Theorem 3.3. – In analogy to what done in the proof of Theorems 3.1 and 3.2, we choose the positive parameters λ_i and μ_i consecutively, in such a way that the corresponding functional $f_{\lambda_1\dots\lambda_h}^{\mu_1\dots\mu_r}$ constrained on $V(\Omega)$ has at least $(r+h)$ distinct critical values. Here we choose these parameters in the following order: $\mu_1, \mu_2 \dots \mu_r, \lambda_1, \lambda_2 \dots \lambda_h$.
Let us choose $\bar{\mu}_1 > 0$ such that

$$S + \|\bar{\mu}_1 \alpha_1\|_{L^{n/2}(\mathbb{R}^n)} < 2^{2/n} S.$$

Since $\|\alpha_1\|_{L^{n/2}(\mathbb{R}^n)} \neq 0$, Lemma 2.8 implies that $c(\mu_1\alpha_1) > S$ for every $\mu_1 \in]0, \bar{\mu}_1[$.

Therefore there exists $\bar{\mu}_2 = \bar{\mu}_2(\mu_1) > 0$ such that

$$S + \|\bar{\mu}_2 \alpha_2\|_{L^{n/2}(\mathbb{R}^n)} < c(\mu_1\alpha_1).$$

Notice that $c(\mu_1\alpha_1) < S + \|\bar{\mu}_1\alpha_1\|_{L^{n/2}(\mathbb{R}^n)} \quad \forall \mu_1 \in]0, \bar{\mu}_1[$, because

$$\int_{\mathbb{R}^n} |Du|^2 dx + \mu_1 \int_{\mathbb{R}^n} \alpha_1(x) u^2 dx \leq \int_{\mathbb{R}^n} |Du|^2 dx + \|\mu_1\alpha_1\|_{L^{n/2}(\mathbb{R}^n)} \qquad \forall u \in V(\mathbb{R}^n).$$

Iterating this procedure, we obtain, for every $i = 2 \dots r$,

$$\bar{\mu}_i = \bar{\mu}_i(\mu_1 \dots \mu_{i-1}) > 0 \text{ such that, if } \mu_i \in]0, \bar{\mu}_i[,$$

$$\begin{aligned} S < c(\mu_i\alpha_i) \leq S + \|\bar{\mu}_i\alpha_i\|_{L^{n/2}(\mathbb{R}^n)} < c(\mu_{i-1}\alpha_{i-1}) \\ \leq S + \|\bar{\mu}_{i-1}\alpha_{i-1}\|_{L^{n/2}(\mathbb{R}^n)} < 2^{\frac{2}{n}} S. \end{aligned}$$

Then arguing as in section 2, we find, for every $\epsilon_1 > 0$, $\bar{\lambda}_1 = \bar{\lambda}_1(\mu_1 \dots \mu_r)$, $\varphi_1 \in H_0^{1,2}(B(0,1))$, $K_1 \subset \mathbb{R}^n \times \mathbb{R}$ (satisfying the properties described in Corollary 2.11) such that, if $0 < \mu_1 < \bar{\mu}_1$ and $\lambda_1 > \bar{\lambda}_1$,

$$\begin{aligned} &\sup\{f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in \partial K_1\} < S + \epsilon_1, \\ &\sup\{f_{\lambda_1}^{\mu_1} \circ T_{\frac{1}{\lambda_1}, x_1} \circ T_{\sigma,y}(\varphi_1) : (y,\sigma) \in K_1\} \\ &\qquad < S + \|\alpha_1\|_{L^{n/2}(\mathbb{R}^n)} \min(\bar{\mu}_1, 2\mu_1) < 2^{\frac{2}{n}} S. \end{aligned} \tag{3.12}$$

In particular, we choose $\epsilon_1 > 0$ such that $S + \epsilon_1 < c(\mu_r\alpha_r)$.

Repeating the same procedure, for any $\epsilon_2 > 0$ we prove that there exist

$$\bar{\lambda}_2 = \bar{\lambda}_2(\mu_1 \ldots \mu_r, \lambda_1), \ \varphi_2 \in H_0^{1,2}(B(0,1)), \ K_2 \subset \mathbb{R}^n \times \mathbb{R},$$

such that, if

$$0 < \mu_1 < \bar{\mu}_1, \lambda_1 > \bar{\lambda}_1, \ 0 < \mu_2 < \bar{\mu}_2, \lambda_2 > \bar{\lambda}_2,$$

then the inequalities (3.12) hold with $f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2}$ instead of $f_{\lambda_1}^{\mu_1}$ and, moreover,

$$\begin{aligned}
&\sup\{f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2},x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in \partial K_2\} < S + \epsilon_2,\\
&\sup\{f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2} \circ T_{\frac{1}{\lambda_2},x_2} \circ T_{\sigma,y}(\varphi_2) : (y,\sigma) \in \partial K_2\}\\
&\quad < S + \|\alpha_2\|_{L^{n/2}(\mathbb{R}^n)} \min(\bar{\mu}_2, 2\mu_2) < c(\mu_1\alpha_1).
\end{aligned}$$

In particular, choose $\epsilon_2 > 0$ such that

$$\begin{aligned}
S + \epsilon_2 &< \inf\left\{ f_{\lambda_1}^{\mu_1}(u) : u \in V(\Omega), \ \gamma_{\lambda_1}(u) = 0, \ \beta_{\lambda_1}(u) \geq \frac{1}{3}\right\}\\
&\leq \inf\left\{ f_{\lambda_1,\lambda_2}^{\mu_1,\mu_2}(u) : u \in V(\Omega), \ \beta_{\lambda_1}(u) = 0, \ \gamma_{\lambda_1}(u) \geq \frac{1}{3}\right\}.
\end{aligned}$$

Arguing in the same way for $i = 3 \ldots r$, for every $\epsilon_i > 0$ we find $\bar{\lambda}_i = \bar{\lambda}_i(\mu_1 \ldots \mu_r, \lambda_1 \ldots \lambda_{i-1})$, $\varphi_i \in H_0^{1,2}(B(0,1))$, $K_i \subset \mathbb{R}^n \times \mathbb{R}$, such that, if $0 < \mu_i < \bar{\mu}_i$ and $\lambda_i > \bar{\lambda}_i$ $\forall i = 1 \ldots r$, then

$$\begin{aligned}
&\sup\{f_{\lambda_1 \ldots \lambda_r}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i\} < S + \epsilon_i,\\
&\sup\{f_{\lambda_1 \ldots \lambda_r}^{\mu_1 \ldots \mu_r} \circ T_{\frac{1}{\lambda_i},x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in K_i\}\\
&\quad < S + \|\alpha_i\|_{L^{n/2}(\mathbb{R}^n)} \min(\bar{\mu}_i, 2\mu_i) \quad \forall i = 1 \ldots r.
\end{aligned} \tag{3.13}$$

In particular, we choose $\epsilon_i > 0$ such that

$$S + \epsilon_i < \inf\left\{ f_{\lambda_1 \ldots \lambda_r}^{\mu_1 \ldots \mu_r}(u) : u \in V(\Omega), \ \ \beta_{\lambda_{i-1}}(u) = 0, \ \gamma_{\lambda_{i-1}}(u) \geq \frac{1}{3}\right\} \quad \forall i = 2 \ldots r.$$

For $i = r+1, \ldots, h$, by arguing as in the proof of Theorem (3.1), we find for every $\epsilon_i > 0$,

$$\bar{\lambda}_i = \bar{\lambda}_i(\mu_1 \ldots \mu_r, \lambda_1 \ldots \lambda_{i-1}), \quad \varphi_i \in H_0^{1,2}(B(0,1)), \ K_i \subset \mathbb{R}^n \times \mathbb{R}$$

(satisfying the properties of Corollary 2.11) such that, if

$$0 < \mu_i < \bar{\mu}_i \quad \forall i = 1 \dots r \quad \text{and} \quad \lambda_i > \bar{\lambda}_i \quad \forall i = 1 \dots h,$$

then the inequalities (3.13) hold when we replace the functionals by $f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}$, and, moreover, it results for every $i = r+1, \dots, h$:

$$\begin{aligned}
&\sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi_i) : (y, \sigma) \in \partial K_i\} < S + \epsilon_i, \\
&\sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi_i) : (y, \sigma) \in \partial K_i\} \\
&\quad < \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_{i-1}}(u) \geq \frac{1}{3} \right\}.
\end{aligned}$$

We also assume that

$$S + \epsilon_i < c(\alpha_i) \leq \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\}.$$

Thus, the topology of the sublevels of the functional $f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}$ constrained on $V(\Omega)$ can be described, if $0 < \mu_i < \bar{\mu}_i \ \forall i = 1 \dots r$ and $\lambda_i > \bar{\lambda}_i \forall i = 1 \dots h$, by the following inequalities (that hold for $i = 2 \dots h$):

$$\begin{aligned}
S &< \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} \\
&\leq \sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi)_i : (y, \sigma) \in \partial K_i\} < S + \epsilon_i \\
&< \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_i - 1}(u) \geq \frac{1}{3} \right\} \\
&\leq \sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi)_i : (y, \sigma) \in \partial K_{i-1}\} < S + \epsilon_{i-1} \\
&< \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_r}(u) = 0,\ \gamma_{\lambda_r}(u) = \frac{1}{3} \right\} < 2^{\frac{2}{n}} S;
\end{aligned}$$

for $i = 2 \dots r$ we have, in addition,

$$\begin{aligned}
S &< \sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi_i) : (y, \sigma) \in \partial K_i\} \\
&< \inf\left\{ f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\} \\
&\leq \sup\{f^{\mu_1 \dots \mu_r}_{\lambda_1 \dots \lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma, y}(\varphi_i) : (y, \sigma) \in K_i\}
\end{aligned}$$

$$
\begin{aligned}
&< S + \|\alpha_i\|_{L^{n/2}(\mathbf{R}^n)} \min(\bar{\mu}_i, 2\mu_i) < c(\mu_{i-1}\alpha_{i-1}) \\
&\leq \inf\left\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_{i-1}}(u) = 0,\ \gamma_{\lambda_{i-1}}(u) = \frac{1}{3} \right\} \\
&\leq \sup\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h} \circ T_{\frac{1}{\lambda_{i-1}}, x_{i-1}} \circ T_{\sigma,y}(\varphi_{i-1}) : (y,\sigma) \in K_{i-1} \} \\
&< S + \|\alpha_{i-1}\|_{L^{n/2}(\mathbf{R}^n)} \min(\bar{\mu}_{i-1}, 2\mu_{i-1}) < 2^{\frac{2}{n}} S.
\end{aligned}
$$

Then the above inequalities, and the properties of the subsets K_i, allow us to state that the functional $f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}$ constrained on $V(\Omega)$ has, for every $i = 1\ldots h$, a critical point v_i such that

$$
\begin{aligned}
&\inf\left\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) \geq \frac{1}{3} \right\} \leq f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(v_i) \\
&\leq \sup\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in \partial K_i \} < S + \epsilon_i.
\end{aligned}
$$

Furthermore for every $i = 1\ldots r$ there exists another critical point $\hat{v}_i \in V(\Omega)$ such that

$$
\begin{aligned}
&\inf\left\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(u) : u \in V(\Omega),\ \beta_{\lambda_i}(u) = 0,\ \gamma_{\lambda_i}(u) = \frac{1}{3} \right\} \leq f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(\hat{v}_i) \\
&\leq \sup\{ f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h} \circ T_{\frac{1}{\lambda_i}, x_i} \circ T_{\sigma,y}(\varphi_i) : (y,\sigma) \in K_i \} \\
&\leq S + 2\mu_i \|\alpha_i\|_{L^{n/2}(\mathbf{R}^n)}.
\end{aligned}
$$

Therefore the solutions of (1.1)

$$
\begin{aligned}
u_i &= \left[f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(v_i) \right]^{\frac{n-2}{4}} v_i \quad (i = 1\ldots h) \\
\hat{u}_i &= \left[f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}(\hat{v}_i) \right]^{\frac{n-2}{4}} \hat{v}_i \quad (i = 1\ldots r),
\end{aligned}
$$

corresponding to these critical points, satisfy (3.7) and, since ϵ_i and μ_i can be chosen arbitrarily small, the relations (3.6) hold. □

Remark 3.4. – In the proof of Theorems 3.2 and 3.3 we obtain $(r+h)$ distinct solutions of Problem (1.1) by choosing the parameter λ_i and μ_j in such a way that the corresponding functional $f^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_h}$ has at least $(r+h)$ distinct critical values; as already observed, we do not require that the concentration points $x_1\ldots x_h$ are distinct. On the other hand, it is very reasonable that, if we assume that the concentration points $x_1\ldots x_h$ are distinct, an assertion of the following type holds:

there exists $\epsilon > 0$ such that, if

$$
\lambda_i > \frac{1}{\epsilon} \quad \forall i = 1\ldots h \quad \text{and} \quad 0 < \mu_j < \epsilon \quad \forall j = 1\ldots r,
$$

then there exist at least $(r+h)$ distinct critical points for $f^{\mu_1 \ldots \mu_r}_{\lambda_1 \ldots \lambda_h}$ constrained on $V(\Omega)$, corresponding to critical values not necessarily distinct.

Moreover, in analogy with other multiplicity results on elliptic problems involving critical Sobolev exponents (*see* [21], for example), we can conjecture that h distinct concentration points $x_1 \ldots x_h$ guarantee the existence of at least $2^{(r+h)} - 1$ distinct positive solutions, if $\lambda_i > \frac{1}{\epsilon}$ $\forall i = 1 \ldots h$ and $0 < \mu_j < \epsilon$ $\forall j = 1 \ldots r$.

Remark 3.5 (concentration on subset of small capacity). – Theorems 2.1, 3.1, 3.2 and 3.3 associate the existence and the multiplicity of positive solutions for Problem (1.1) to the property that some parts of the nonnegative function $a(x)$ are concentrated near some points of Ω.

More in general, one can consider the case where $a(x)$ is a nonnegative function concentrated near some subsets of Ω, having small capacity: for example, we can consider functions $a(x)$ of the form:

$$a(x) = \begin{cases} \lambda_i & \text{if } x \in H_i \\ 0 & \text{otherwise,} \end{cases}$$

where $(H_i)_i$ is a sequence of subsets of Ω with $\lim_{i \to \infty} \text{cap}_\Omega H_i = 0$, and $\lim_{i \to \infty} \lambda_i = +\infty$.

In this case the study of the multiplicity of positive solutions become more interesting because the topological properties of the subsets H_i also intervene and contribute to increase the number of solutions.

Multiplicity results concerning functions $a(x)$ of this type will be reported in a paper in preparation.

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