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Solutions of Ginzburg-Landau equations and critical points of the renormalized energy

by

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ABSTRACT. – Here we establish some connection between solutions of Ginzburg-Landau equations and critical points of the renormalized energy W_g introduced in the recent works of Bethuel-Brezis and Helein. In particular, local nondegenerate minimums of W_g have their associated solutions of GL-equations.

RÉSUMÉ. – On établit une relation entre les solutions de l'équation de Ginzburg-Landau et les points critiques de l'énergie renormalisée W_g introduite dans les travaux récents de Bethuel-Brezis et Helein. En particulier, tout minimum local non dégénéré de W_g est associé à une famille de solutions des équations de G-L.

1. INTRODUCTION

The aim of this article is to establish some connection between the solutions of Ginzburg-Landau Equations and critical points of the renormalized energy introduced in the recent work of Bethuel-Brezis-Helein [BBH]. We shall use the following Ginzburg-Landau heat flow as a basic

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link in proving such connections

$$(1.1) \quad u_t = \Delta u + \frac{1}{\epsilon^2}(1 - |u|^2)u, \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(1.2) \quad u(x, t) = g(x), \quad \text{for } x \in \partial\Omega, \quad t > 0,$$

$$(1.3) \quad u(x, 0) = u_0(x), \quad \text{for } x \in \Omega.$$

Here Ω is a two-dimensional, smooth, bounded domain, ϵ is a positive parameter, $u : \Omega \rightarrow \mathbb{R}^2, g : \partial\Omega \rightarrow \mathbb{R}^2$ is smooth and $|g| = 1$. We also assume that $u_0(x) = g(x)$ on $\partial\Omega$ and $|u_0(x)| \leq 1$. The motion of vortices of solutions u of (1.1)-(1.3) will be our main concern. Based on formal asymptotics, various authors have arrived at the same conclusion that vortices move in the time scale of size $\log \frac{1}{\epsilon}$, see, e.g., [N], [E], and [PR]. Under some special assumptions, it was shown in [RS] that it will take at least $0\left(\log \frac{1}{\epsilon}\right)$ time for a vortex to move a distance 1. On the other hand, for the steady state solutions $u_\epsilon(x)$ of (1.1)-(1.2) that is, the critical points of the functional

$$(1.4) \quad E_\epsilon(u) = \frac{1}{2} \int_\Omega \left[|\nabla u|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2 \right] dx$$

with $u|_{\partial\Omega} = g$, a complete characterization of asymptotic behavior (as $\epsilon \rightarrow 0^+$) for vortices of u_ϵ is given in the recent book [BBH].

Let us recall one of the main results shown in [BBH] under the additional hypothesis that Ω is star-shaped. (Cf. also [St] when Ω is not star-shaped.)

THEOREM [BBH]. – *Let Ω, g be as above and let u_{ϵ_n} be a sequence of steady state solutions of (1.1)-(1.2) (with $\epsilon = \epsilon_n$). Then there is a subsequence $\{u_{\epsilon_n}\}$ such that $u_{\epsilon_n}(x) \rightarrow u^*(x)$ in $C_{\text{loc}}^{1,\alpha}(\bar{\Omega}/\{a_1, a_2, \dots, a_k\}), k \leq k(\Omega, g)$. Here $u^*(x)$ is given by*

$$(1.5) \quad u^*(x) = \prod_{j=1}^k \left(\frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{ih(x)}, \quad z = x \in \Omega \subset \mathbb{C},$$

$$\sum_{j=1}^k d_j = d = \text{degree of } g$$

and

$$\Delta h = 0 \text{ in } \Omega, \quad u^*|_{\partial\Omega} = g.$$

For conveniences, we shall always assume $d \geq 0$.

If, in the addition, that $\{u_{\epsilon_n}\}$ are minimizers of (1.4), then in the above formula for $u^*, k = d, d_j = 1, \text{ for } j = 1, \dots, k$. Moreover, the point $a = (a_1, \dots, a_d) \in \Omega^d$ is a global minimum point of the renormalized energy $W_g(b)$ defined on $\bar{\Omega}^d$ where for $b = (b_1, \dots, b_d) \in \bar{\Omega}^d$,

$$(1.6) \quad W_g(b) = -\pi \sum_{i \neq j} \log |b_i - b_j| + \frac{1}{2} \int_{\partial\Omega} \Phi(g \wedge g_T) - \pi \sum_{j=1}^d R(b_j),$$

and Φ is the solution of

$$(1.7) \quad \begin{aligned} \Delta \Phi &= 2\pi \sum_{j=1}^d \delta_{b_j} \text{ in } \Omega, \\ \frac{\partial \Phi}{\partial \nu} &= g \wedge g_T \text{ on } \partial\Omega, \end{aligned}$$

R is given by

$$(1.8) \quad R(x) = \Phi(x) - \sum_{j=1}^d \log |x - b_j|.$$

From the above theorem, it is then natural to ask if there is a relation between the critical points of the function (1.4) and those of the renormalized energy $W_g(b)$, cf. the open problems in [BBH]. For this we have the following result.

THEOREM A. - *Let $a = (a_1, \dots, a_d) \in \Omega^d$ be a non-degenerate local minimum point for the renormalized energy W_g . Then there is a family $u_\epsilon, 0 < \epsilon < \epsilon_0$, of steady state solutions of (1.1)-(1.2) such that*

$$u_\epsilon(x) \rightarrow u^*(x) \text{ in } C_{\text{loc}}^{1,\alpha}(\bar{\Omega}/\{a_1, \dots, a_d\})$$

where

$$u^*(x) = \prod_{j=1}^d \frac{x - a_j}{|x - a_j|} e^{ih(x)}, \Delta h = 0 \text{ in } \Omega$$

and $u^* = g$ on $\partial\Omega$.

Here we say $a = (a_1, \dots, a_d)$ is a nondegenerate local minimum of W_g if

$$(1.9) \quad \nabla W_g(a) = 0 \quad \text{and} \quad \nabla^2 W_g(a)$$

is positive definite. In fact, we shall prove a somewhat more general result in which the hypothesis $\nabla^2 W_g(a)$ being positive definite may be relaxed a bit.

The strategy of proving the above theorem is, for all sufficiently small ϵ , to construct suitable initial data $u_0^\epsilon(x)$ and then solve (1.1)-(1.3) with $u(x, 0) = u_0^\epsilon(x)$. Let us denote $u_\epsilon(x)$ the asymptotic limit of $u(x, t)$ as $t \rightarrow \infty$, then $u_\epsilon(x)$ is the desired family for $0 < \epsilon < \epsilon_0$. The key point in verifying the later statement is to control the locations of vortices of $u(x, t)$ for all $t \in (0, \infty)$. In general, we let $u_\epsilon(x, t)$ to denote the solution of (1.1)-(1.3). We consider the following two limits.

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} u_\epsilon(x, t)$$

and

$$\lim_{t \rightarrow \infty} \lim_{\epsilon \rightarrow 0} u_\epsilon(x, t).$$

For the initial data used in the proof of Theorem A, these two limits turn out to be the same. However, it is not the case for general initial data. In fact, these two limits may be characterized by the following.

THEOREM B.

(i) *The $\lim_{t \rightarrow \infty} u_\epsilon(x, t) = u_\epsilon(x)$ exists, and $u_\epsilon(x)$ is a steady state solution of (1.1)-(1.2).*

(ii) *$\lim_{\epsilon \rightarrow 0} u_\epsilon(x)$ exists (up to subsequences). Moreover, these limits are those $u^*(x)$ of form given in (1.5). If we choose the initial data $u_0^\epsilon(x)$ suitably, then $k = d$, and $d_j = 1$, for $j = 1, 2, \dots, d$, in the formula (1.5). The singular points a_1, \dots, a_d are so that the point $a = (a_1, \dots, a_d)$ is a critical point of W_g .*

(iii) *By choosing $u_0^\epsilon(x)$ properly, one has $\lim_{\epsilon \rightarrow 0} u_\epsilon(x, t) = u_0(x)$, and $u_0(x)$ is of form*

$$(1.10) \quad u_0(x) = \prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{ih_0(x)}$$

with $\Delta h_0(x) = 0$ in Ω . Here $b = (b_1, \dots, b_d)$ can be an arbitrary point (depending on u_0^ϵ) of Ω^d such that $b_i \neq b_j$, for $i \neq j$.

The paper is written as follows. In the next section we shall start the proof of Theorem B and we shall establish various preliminary facts. Theorem A

is proved in Section 3. Some of the arguments are employed to verify one statement needed in proving Theorem B, part (iii). This and some generalizations of Theorem A are discussed in the final section. Moreover, we shall also show the existence of mountain-pass type critical points of (1.4) whenever the renormalized energy W_g has a critical point of the mountain-pass type.

2. PROOF OF THEOREM B

Under the forementioned hypothesis on Ω , and $u_0(x)$ the global existence of a unique classical solution $u_\epsilon(x, t)$ for (1.1)-(1.3) follows from the standard parabolic theory, cf. [LSU]. Moreover, the solution $u_\epsilon(x, t)$ satisfies

$$(2.1) \quad \frac{d}{dt} \int_{\Omega} \left[|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2} (|u_\epsilon|^2 - 1)^2 \right] dx = -2 \int_{\Omega} \left| \frac{d}{dt} u_\epsilon \right|^2 dx.$$

(2.1) implies, in particular, that

$$\sup_{t>0} \int_0^t \int_{\Omega} \left| \frac{d}{dt} u_\epsilon(x, \tau) \right|^2 dx d\tau + E_\epsilon(u_\epsilon(t)) \leq E_\epsilon(u_0).$$

Therefore, there is a sequence $\{t_n\}, t_n \rightarrow \infty$, such that $u_\epsilon(x, t_n)$ converges, say in $C^{1,\alpha}$ norm, to a steady state solution $u_\epsilon(x)$ of (1.1)-(1.2). It follows from a deep theorem of L. Simon [S] that $u_\epsilon(x, t)$ converges to $u_\epsilon(x)$ as $t \rightarrow \infty$ by the analyticity of $(|u|^2 - 1)^2$. This proves (i). The statement (ii) was proved in [BBH, Chapter 2]. In order to have $k = d, d_j = 1, j = 1, 2, \dots, d$ in (1.5), it suffices to construct the initial data $u_0^\epsilon(x)$ so that $E_\epsilon(u_0^\epsilon(x)) \leq \pi d \log \frac{1}{\epsilon} + C_0$, for some constant C_0 .

Indeed, in such a case, one also has $E_\epsilon(u_\epsilon(x)) \leq \pi d \log \frac{1}{\epsilon} + C_0$. The fact that $k = d$, and each d_j is 1 for $j = 1, \dots, d$, follows from the arguments in [St] or [BBH]. To show (iii), we let b_1, \dots, b_d be arbitrary d -distinct points in Ω , and let $u_0^\epsilon(x)$ be a function as $w(x)$ in the proof of [BBH, Lemma VIII.1]. That is, $u_0^\epsilon(x) = g(x)$ on $\partial\Omega$, and for some $\rho > 0$,

$$(2.2) \quad E_\epsilon(u_0^\epsilon) \leq dI(\epsilon, \rho) + W_g(b) + \pi d \log \frac{1}{\rho} + 0(\rho)$$

where

$$0 < \rho < \rho_0 = \min \{|b_i - b_j|, \text{dist}(b_i, \partial\Omega), i \neq j, \text{ for } i, j = 1, 2, \dots, d\}.$$

For the definition of $I(\epsilon, \rho)$, see (3.4) below (cf. [BBH]).

In particular, as $\epsilon \rightarrow 0$ and $\rho \rightarrow 0$, one has

$$(2.3) \quad u_0^\epsilon(x) \rightarrow \prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{ih_0(x)} \quad \text{in } C_{\text{loc}}^1(\bar{\Omega} \setminus \{b_1, \dots, b_d\}).$$

Here $\Delta h_0(x) = 0$ in Ω . Let $u_\epsilon(x, t)$ be the solution of (1.1)-(1.3) with the initial data $u_0^\epsilon(x)$ as above. Then we have the following estimates,

$$(2.4) \quad \int_0^\infty \int_\Omega \left| \frac{d}{dt} u_\epsilon \right|^2 dx dt \leq C; \text{ (a constant)}$$

$$(2.5) \quad \int_\Omega \phi(x) \left[|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2} (|u_\epsilon|^2 - 1)(t)^2 \right] dx \leq C(\phi) e^t, \quad t > 0.$$

Whose $\phi(x)$ in (2.5) should be chosen suitably, and here all the constants may depend on g and Ω but they are independent of ϵ .

The estimates (2.4) follow from the energy identity (2.1), the choice of $u_0^\epsilon(x)$ (so that (2.2) is true) and the lower energy bound discovered in [BBH]. In the estimates (2.5), ϕ can be any of the following functions, for $x_0 \in \Omega \setminus \{b_1, \dots, b_d\}$, let $2r = \rho(x_0) = \min_{1 \leq j \leq d} |x_0 - b_j|$, and $\phi \in C_0^1(B_r(x_0))$ with $\phi \equiv 1$ on $B_{r/2}(x)$. In fact,

$$(2.6) \quad \begin{aligned} & \frac{d}{dt} \int_\Omega \phi^2 \left[|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2} (|u_\epsilon|^2 - 1)^2 \right] dx \\ &= -2 \int_\Omega \phi^2 \left| \frac{d}{dt} u_\epsilon \right|^2 dx + 2 \int_\Omega \phi \nabla u_\epsilon \frac{du_\epsilon}{dt} dx \\ &\leq \int_\Omega \phi^2 |\nabla u_\epsilon|^2 dx + C(\phi) \int_\Omega \left| \frac{d}{dt} u_\epsilon \right|^2 dx. \end{aligned}$$

where $C(\phi)$ is simply $\|\nabla \phi\|_{L^\infty(B_r)}$. (2.5) then follows from (2.6) by a simple integration and by using the estimate (2.4).

From (2.4) and (2.5), we obtain, for all $0 < \epsilon < 1$, that

$$(2.7) \quad \int_{\Omega \setminus \bigcup_{j=1}^d B_\delta(b_j)} \left[|\nabla u_\epsilon|^2(\cdot, T) + \frac{1}{2\epsilon^2} (|u_\epsilon|^2 - 1)^2(\cdot, T) \right] dx \leq C(\delta, T)$$

for any $\delta > 0, T \in (0, \infty)$.

We can now prove (1.10). First we observe that

$$(2.8) \quad u_\epsilon \wedge \frac{du_\epsilon}{dt} = \operatorname{div}(u_\epsilon \wedge \nabla u_\epsilon) \quad \text{in } \Omega \times R_+.$$

Since (2.4), (2.7), we see, for any sequence $\epsilon_n \downarrow 0$, there is a subsequence (still denoted by ϵ_n) so that $u_{\epsilon_n}(x, t) \rightarrow u_0(x, t)$ weakly in $H'_{\text{loc}}(\bar{\Omega}/\{b_1, b_2, \dots, b_d\} \times R_+)$ and strongly in $L^2_{\text{loc}}(\bar{\Omega} \times R_+)$. The later is because that $|u_\epsilon(x, t)| \leq 1$. Moreover, $|u_0(x, t)| = 1$ a.e. in $\Omega \times R_+$. We deduce from (2.8) and (1.1)-(1.3) that

$$(2.9) \quad \begin{cases} \frac{\partial u_0}{\partial t} = \Delta u_0 + |\nabla u_0|^2 u_0 & \text{in } \Omega/\{b_1, \dots, b_d\} \times R_+ \\ u_0(x, 0) = \prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{ih_0(x)} \\ u_0 = g & \text{on } \partial\Omega \times R_+ \end{cases}$$

Note that $u_0(x, t)$ is smooth in $\bar{\Omega}/\{b_1, \dots, b_d\} \times \mathbb{R}_+$. Indeed, by (2.4), (2.7), one may deduce that, for any $t > 0, 0 < \rho < \rho_0, \deg(u_0(\cdot, t), \partial B_\rho(b_j)) = 1$, for all $j = 1, \dots, d$. Hence, we may write

$$(2.10) \quad \begin{aligned} u_0(x, t) &= \prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{ih(x, t)}, \\ &\text{with } h(x, t) \in H^1_{\text{loc}}(\bar{\Omega}/\{b_1, \dots, b_d\} \times R_+) \end{aligned}$$

It is then clear that $h(x, t)$ satisfies the heat equation (1.10) away from $\{b_1, \dots, b_d\} \times R_+$, and thus u_0 is smooth in $\Omega/\{b_1, \dots, b_d\} \times R_+$. Next, we observe that

$$(2.11) \quad \begin{aligned} \frac{d}{dt} \int_{B_R(b_j)} |x - b_j|^2 \left[\frac{1}{2} |\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2} (|u_\epsilon|^2 - 1)^2 \right] dx \\ = - \int_{B_R(b_j)} \left[|x - b_j|^2 \left| \frac{d}{dt} u_\epsilon \right|^2 + 2(x - b_j) \cdot \nabla u_\epsilon \frac{d}{dt} u_\epsilon \right] dx \\ + \int_{\partial B_R(b_j)} |x - b_j|^2 \frac{\partial u_\epsilon}{\partial t} \cdot \frac{\partial u_\epsilon}{\partial \nu} \quad \text{for } j = 1, 2, \dots, d. \end{aligned}$$

Here $0 < R < \rho_1 = \min\{\rho_0, 1\}$. Since

$$(2.12) \quad \begin{aligned} \int_{B_R(b_i)} (x - b_j) \nabla u_\epsilon \frac{du_\epsilon}{dt} dx + \int_{\partial B_R(b_j)} (x - b_j) \nu e_\epsilon(u) \\ - \int_{\partial B_R(b_j)} |x - b_j| \left(\frac{\partial u}{\partial \nu} \right)^2 = \frac{1}{2\epsilon^2} \int_{B_R(b_j)} (1 - |u_\epsilon|^2)^2 dx. \end{aligned}$$

where $e_\epsilon(u) = \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2}(1 - |u_\epsilon|^2)^2$ and ν is the exterior unit normal vector of $\partial B_R(b_j)$, we conclude from (2.11) that

$$\begin{aligned}
 (2.13) \quad & \frac{d}{dt} \int_{B_R(b_j)} |x - b_j|^2 e_\epsilon(u) dx = - \int_{B_R(b_j)} |x - b_j|^2 \left| \frac{d}{dt} u_\epsilon \right|^2 dx \\
 & - \int_{B_R(b_j)} \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2)^2 dx \\
 & + \int_{\partial B_R(b_j)} |x - b_j|^2 \frac{\partial u_\epsilon}{\partial \nu} \frac{d}{dt} u_\epsilon \\
 & + \int_{\partial B_R(b_j)} |x - b_j| \left(\frac{|\nabla_T u_\epsilon|^2}{2} + \frac{(|u_\epsilon|^2 - 1)^2}{4\epsilon^2} \right) \\
 & - \int_{\partial B_R} \frac{|x - b_j|}{2} \left| \frac{\partial u_\epsilon}{\partial \nu} \right|^2 \\
 & \leq \frac{-1}{\epsilon^2} \int_{B_R(b_j)} (1 - |u_\epsilon|^2)^2 dx \\
 & + R \int_{\partial B_R(b_j)} \left[e_\epsilon(u) + \left| \frac{d}{dt} u_\epsilon \right|^2 \right]
 \end{aligned}$$

Here we have used the fact $|x - b_j|^2 \leq |x - b_j|$ as $R \leq 1$. By Fubini's Theorem, (2.7) and (2.13), we deduce, for each $j = 1, 2, \dots, d$, that

$$\begin{aligned}
 (2.14) \quad & \int_{B_R(b_j)} |x - b_j|^2 e_\epsilon(u) dx + \int_0^T \int_{B_R(b_j)} \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2)^2 dx \\
 & \leq R \int_0^T \int_{\partial B_R(b_j)} \left[e_\epsilon(u) + \left| \frac{d}{dt} u_\epsilon \right|^2 \right] \leq C(R, T).
 \end{aligned}$$

Here $R \in [\rho_1/2, \rho_1]$ is chosen so that

$$\begin{aligned}
 (2.15) \quad & \int_0^T \int_{\partial B_R(b_j)} \left[e_\epsilon(u) + \left| \frac{d}{dt} u_\epsilon \right|^2 \right] \\
 & \leq \frac{4}{R} \int_0^T \int_{B_{\rho_1/2}(b_j)} \left[e_\epsilon(u) + \left| \frac{d}{dt} u_\epsilon \right|^2 \right] dx dt.
 \end{aligned}$$

Next we choose a small $\delta > 0$ (its precise value will be determined later, see also discussion in the next section). Then there is $t \geq \epsilon_0 \delta$ so that

whenever $0 \leq t < t_0$, the set $\left\{x \in \Omega : |u_\epsilon(x, t)| \leq \frac{1}{2}\right\}$ is contained in $U_{j=1}^d B_\delta(b_j) \times [0, t_0]$. This follows from the small energy regularity theorems in [CS], [CL] and our energy estimates.

In the final section we should use arguments in the next section and estimates (2.4), (2.7) to show $h(x, t) \equiv h_0(x)$. The conclusion (1.10) then follows.

Remark. – It will be interesting to investigate also the case that $h_0(x)$ is not harmonic in Ω .

3. PROOF OF THEOREM A

We first consider the case that $a = (a_1, \dots, a_d)$ is a global nondegenerate minimum point for W_g defined on $\bar{\Omega}^d$. Hence, in particular, one has

$$(3.1) \quad \begin{aligned} W_g(b) - W_g(a) &\geq \lambda_0 |b - a|^2 \\ \text{whenever } |b - a| &\leq \delta_0, \text{ for some } \delta_0, \lambda_0 > 0. \end{aligned}$$

By [BBH, Lemma VIII.1] and its proof, there is some $\rho_0 > 0$ (depending only on a and Ω) such that, for every $0 < \rho \leq \rho_0$ and every $\epsilon > 0$, one may find some $w_\epsilon(x)$ defined on Ω with $w_\epsilon = g$ on $\partial\Omega$ and

$$(3.2) \quad E_\epsilon(w_\epsilon) \leq dI(\epsilon, \rho) + W_g(a) + \pi d \log \frac{1}{\rho} + 0(\rho).$$

On the other hand, Lemma VIII 2 of [BBH] shows that

$$(3.3) \quad \begin{aligned} \min\{E_\epsilon(U) : U|_{\partial\Omega} = g\} &\geq dI(\epsilon, \rho) + W_g(a) \\ &+ \pi d \log \frac{1}{\rho} + 0(\rho^2), \text{ whenever } \epsilon \leq \epsilon(\rho). \end{aligned}$$

In inequalities (3.2) and (3.3), the quantity $I(\epsilon, \rho)$ is defined by

$$(3.4) \quad \begin{aligned} I(\epsilon, \rho) = \min \left\{ \int_{B_\rho} \frac{1}{2} \left[|\nabla U|^2 + \frac{1}{2\epsilon^2} (|U|^2 - 1)^2 \right] dx \right. \\ \left. u(x) = \frac{x}{|x|} \text{ on } |x| = \rho \right\}. \end{aligned}$$

Next, we consider the heat-flow (1.1)-(1.3) with the initial data given by $w_\epsilon(x)$. Let us denote the solution of (1.1)-(1.3) by $u_\epsilon(x, t)$. Then the following estimates hold:

$$(I) \quad -E_\epsilon(u_\epsilon(\cdot, t)) + E_\epsilon(w_\epsilon) = \int_0^t \int_\Omega \left| \frac{d}{dt} u_\epsilon(x, t) \right|^2 dx dt \leq 0(\rho);$$

$$(II) \quad |\nabla u_\epsilon(x, t)|^2 + \left| \frac{\partial}{\partial t} u_\epsilon(x, t) \right| \leq \frac{C}{\epsilon^2}, \quad \text{for all } (x, t) \in \bar{\Omega} \times \mathbb{R}^+.$$

Here the number ρ is suitably small but fixed, $0 < \epsilon < \epsilon(\rho)$. The first estimate (I) follows easily from (3.2) and (3.3). The second estimate (II) can be easily obtained by consider the function $u_\epsilon(\epsilon x, \epsilon^2 t)$ and the usual parabolic estimates. Of course, the initial data $w_\epsilon(x)$ should be chosen properly so that the estimate (II) will be valid for $0 \leq t \leq \epsilon$. This can be done and we leave it to the reader. In fact, from discussions in the previous section, we shall use (II) only for those $t \geq t_0$ (for some t_0 depending on δ_0).

We let $\delta = \frac{1}{4d} \min\{\delta_0, |a_j - a_i|, i \neq j, i, j = 1, \dots, d\}$ and let $T_\epsilon > 0$ be the first time that the set $\left\{ x \in \Omega : |u_\epsilon(x, T_\epsilon)| \leq \frac{1}{2} \right\}$ intersects the set $\bigcup_{j=1}^d \partial B_\delta(a_j)$. Suppose such time T_ϵ does not exist, then the time asymptotic limit $u_\epsilon(x) = \lim_{t \rightarrow \infty} u_\epsilon(x, t)$ will possess the property that the set

$$(3.5) \quad \deg(u_\epsilon, \partial B_\delta(a_j)) = 1, \quad j = 1, 2, \dots, d.$$

If (3.5) is true for all sufficiently small $\epsilon \in (0, \epsilon_0)$, $\epsilon_0 < \epsilon(\rho)$, then we are done. In fact, from the *Theorem [BBH]* stated in the introduction we conclude that $u^*(x) = \lim_{\epsilon \rightarrow 0} u_\epsilon(x)$ exists (up to subsequence). Moreover, each $u^*(x)$ is of form

$$u^*(x) = \prod_{j=1}^d \frac{x - \bar{a}_j}{|x - \bar{a}_j|} e^{ih(x)},$$

for some

$$(3.6) \quad \bar{a} = (\bar{a}_1, \dots, \bar{a}_d), \quad \text{and} \quad |a_j - \bar{a}_j| \leq \delta.$$

The last inequality follows from (3.5). By our assumption (3.1), we have $a = \bar{a}$ and therefore the conclusion of Theorem A.

Suppose now such time $T_\epsilon \in (0, \infty)$ exists, for a sufficiently small ϵ . We want to derive a contradiction from it. To avoid confusions that may arise from the sizes of various constants, we point out the order of these constants

$$(3.7) \quad 0 < \epsilon \leq \epsilon_0 \leq \epsilon(\rho) \ll \rho \ll \delta < \delta_0.$$

We assume, without loss of the generality, that $\partial B_\delta(a_1) \cap \left\{ x \in \Omega : |u_\epsilon(x, T_\epsilon)| \leq \frac{1}{2} \right\} \neq \emptyset$. Then there is an $x_1^\epsilon \in \Omega$ such that

$$(3.8) \quad |u_\epsilon(x_1^\epsilon, T_\epsilon)| = \frac{1}{2}, \quad |x_1^\epsilon - a_1| = \delta.$$

We choose $x_j^\epsilon, j = 2, \dots, d$, so that $|x_j^\epsilon - a_j| \leq \delta$ and $|u_\epsilon(x_j^\epsilon, T_\epsilon)| \leq \frac{1}{2}$. For each j , we may find $\alpha_j \in [\alpha, 2\alpha], \alpha = \frac{1}{16d + 1}$, such that

$$(3.9) \quad \epsilon^{\alpha_j} \int_{\partial B_j} \left[|\nabla V_\epsilon|^2 + \frac{1}{2\epsilon^2} (|V_\epsilon|^2 - 1)^2 \right] \leq C(\alpha)$$

where $B_j = \{x : |x - x_j^\epsilon| \leq \epsilon^{\alpha_j}\}, \epsilon(x) = u_\epsilon(x, T_\epsilon)$, and $C(\alpha)$ is a constant depending only on α, g and Ω . The estimates (3.9) is an easy consequence of the Fubini's Theorem (3.2), and the energy estimate (I), see also [St]. Next, we let $\tilde{U}_\epsilon(x)$ be such that it minimizes

$$\int_{\Omega_\epsilon} \left[|\nabla u|^2 + \frac{1}{2\epsilon^2} (|u|^2 - 1)^2 \right] dx$$

with $\tilde{U}_\epsilon = V_\epsilon$ on $\partial\Omega_\epsilon$. Where $\Omega_\epsilon = \Omega / \bigcup_{j=1}^d B_j$. Now we claim the following two statements are true:

$$(3.10) \quad |\tilde{U}_\epsilon(x)| \geq \frac{1}{2} \quad \text{on } \Omega_\epsilon;$$

$$(3.11) \quad \deg(V_\epsilon, \partial B_j) = 1, \quad \text{for } j = 1, \dots, d.$$

These two statements should be proved later. Suppose, for the moment, that they are both true. Then, as $\epsilon \rightarrow 0$, we may assume, by taking subsequences if necessary, that

$$x_j^\epsilon \rightarrow \bar{a}_j, \quad \text{for } j = 1, \dots, d, \quad \text{and } |\bar{a}_1 - a_1| = \delta.$$

Moreover, the corresponding subsequence of $\tilde{U}_\epsilon(x)$ converges to $\bar{U}(x) = \prod_{j=1}^d \frac{x - \bar{a}_j}{|x - \bar{a}_j|} e^{iH(x)}$ in $C_{\text{loc}}^{1,\alpha}(\bar{\Omega}/\{\bar{a}_1, \dots, \bar{a}_d\})$ with $\Delta H(x) = 0$. The later statement follows easily from the assertions (3.10), (3.11).

In fact, for each $x \in \Omega_\epsilon$, we may find a disc D of radius ϵ^β , for some $\beta \in [3\alpha, 4\alpha]$, such that

$$x \in D \cap \Omega_\epsilon, \quad \epsilon^\beta \int_{\partial D \cap \Omega_\epsilon} \left[|\nabla \tilde{u}_\epsilon|^2 + \frac{1}{2\epsilon^2} (|\tilde{u}_\epsilon|^2 - 1) \right] \leq c(\alpha)$$

(see (3.9)). Moreover, $\frac{1}{\epsilon^\beta}(D \cap \Omega_\epsilon)$ is a Lipschitz domain with Lipschitz character independent of ϵ . On $\frac{1}{\epsilon^\beta}(D \cap \Omega_\epsilon) = D_\epsilon$, the function $\tilde{u}_\epsilon(\epsilon^\beta x)$ minimizes the functional of form

$$\int_{D_\epsilon} \left[|\nabla u|^2 + \frac{1}{2\epsilon^{2(1-\beta)}} (|u|^2 - 1) \right] dx$$

and with boundary data g_ϵ on ∂D_ϵ satisfying

$$(3.12) \quad \int_{\partial D_\epsilon} \left[|\nabla_T g_\epsilon|^2 + \frac{1}{2\epsilon^{2(1-\beta)}} (|g_\epsilon|^2 - 1)^2 \right] \leq C(\alpha).$$

Since $|\tilde{u}_\epsilon| \geq \frac{1}{2}$ on $D \cap \Omega_\epsilon$, one has also that

$$\text{deg}(g_\epsilon, \partial D_\epsilon) = 0.$$

We write $\tilde{u}_\epsilon(\epsilon^\beta x) = \rho_\epsilon e^{i\psi_\epsilon}$ on D_ϵ . Then it is clear that ψ_ϵ minimizes $\int_{D_\epsilon} \rho_\epsilon^2 |\nabla \psi_\epsilon|^2 dx$ and with $\psi_\epsilon \in H^1(\partial D_\epsilon)$. Thus, $\nabla \psi_\epsilon \in L^p(D_\epsilon)$ for some $p > 2$.

Then the function $f_\epsilon(x) = 1 - \rho_\epsilon(x)$ satisfies

$$(3.13) \quad -\Delta f_\epsilon(x) + \frac{c_\epsilon(x)}{\epsilon^{2(1-\beta)}} f_\epsilon(x) + p_\epsilon(x) = 0$$

in D_ϵ with $0 \leq f_\epsilon \leq \frac{1}{2}$, $\frac{1}{4} \leq c_\epsilon(x) \leq 4$, and $p_\epsilon(x) \in L^q(D_\epsilon)$, for some $q > 1$. Moreover, on ∂D_ϵ , $f_\epsilon \leq c\epsilon^{\frac{1}{4}}$, by (3.12). Therefore, elliptic estimates yield $f_\epsilon \leq c\epsilon^{\frac{1}{4}} + c\epsilon^{q-1}$.

We now write $\tilde{u}_\epsilon(x)$ on Ω_ϵ as

$$\tilde{u}_\epsilon(x) = \prod_{j=1}^d \frac{x - x_j^\epsilon}{|x - x_j^\epsilon|} e^{ih_\epsilon(x)} \cdot \tilde{\rho}_\epsilon(x)$$

with $1 - c\epsilon^{\frac{1}{4}} - c\epsilon^{q-1} \leq \tilde{\rho}_\epsilon(x) \leq 1$. Let $\Theta_\epsilon(x)$ be such that $e^{i\Theta_\epsilon(x)} = \prod_{j=1}^d \frac{x - x_j^\epsilon}{|x - x_j^\epsilon|}$, then Θ_ϵ is a multivalued harmonic function in Ω_ϵ , and $h_\epsilon(x)$ minimizes the integral

$$(3.14) \quad \int_{\Omega_\epsilon} [\tilde{\rho}_\epsilon^2 |\nabla h_\epsilon|^2 + 2\tilde{\rho}_\epsilon^2 \nabla h_\epsilon \cdot \nabla \Theta_\epsilon] dx.$$

We want to show $\int_{\Omega_\epsilon} \tilde{\rho}_\epsilon^2 |\nabla h_\epsilon|^2 dx \leq c < \infty$. It then follows that $\nabla h_\epsilon \in L^p_{loc}(\Omega)$ for some $p > 2$. The convergence of \tilde{u}_ϵ to $\bar{u}(x)$ follows easily as in [BBH, Chapter VI].

To show $\int_{\Omega_\epsilon} |\nabla h_\epsilon|^2 dx \leq c$, we first choose a function $\tilde{h}_\epsilon(x)$ on Ω_ϵ such that $\tilde{h}_\epsilon(x) \equiv 0$ on $\Omega \setminus \left(\bigcup_{j=1}^d 2B_j \right)$ where $\Omega_\epsilon = \Omega \setminus \bigcup_{j=1}^d B_j$, and $2B_j$ is the double of B_j (with the same center), $j = 1, \dots, d$. Moreover, $\int_{\Omega_\epsilon} |\nabla \tilde{h}_\epsilon|^2 \leq c(\alpha, g)$, $|\tilde{h}_\epsilon| \leq c(\alpha, g)$. The later can be achieved due to the conformal invariant of the Dirichlet integral and the fact that

$$\int_{\partial\Omega_\epsilon} |\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2} (1 - |u_\epsilon|^2)^2 \leq c(\alpha, g).$$

Therefore, by minimality, one has

$$(*) \quad \begin{aligned} & \int_{\Omega_\epsilon} [\tilde{\rho}_\epsilon^2 |\nabla h_\epsilon|^2 + 2\tilde{\rho}_\epsilon^2 \nabla h_\epsilon \cdot \nabla \Theta_\epsilon] dx \\ & \leq \int_{\Omega_\epsilon} [\tilde{\rho}_\epsilon^2 |\nabla \tilde{h}_\epsilon|^2 + 2\tilde{\rho}_\epsilon^2 \nabla \tilde{h}_\epsilon \cdot \nabla \Theta_\epsilon] dx \end{aligned}$$

Next we wish to show $\int_{\Omega_\epsilon} \nabla h_\epsilon \cdot \nabla \Theta_\epsilon dx$ and $\int_{\Omega_\epsilon} \nabla \tilde{h}_\epsilon \cdot \nabla \Theta_\epsilon dx$ are uniformly bounded. If we show this, then we deduce from the above inequality * and the estimate $\tilde{\rho}_\epsilon \geq 1 - c\epsilon^{\frac{1}{4}} - c\epsilon^{q-1}$ that

$$\int_{\Omega_\epsilon} \tilde{\rho}_\epsilon^2 |\nabla h_\epsilon|^2 dx \leq c(\alpha, g) + \int_{\Omega_\epsilon} \tilde{\rho}_\epsilon^2 |\nabla \tilde{h}_\epsilon|^2 dx.$$

where we noticed that $|\nabla \Theta|(x)(1 - \tilde{\rho}(x)) \leq c(\epsilon^{\frac{1}{8}} + \epsilon^{q-1-\alpha})$ and we may assume $\alpha < q - 1$.

Finally, since Θ_ϵ is harmonic in Ω_ϵ , one has

$$\int_{\Omega_\epsilon} \nabla h_\epsilon \cdot \nabla \Theta_\epsilon dx = \sum_{j=1}^d \int_{\partial B_j} h_\epsilon \frac{\partial \Theta_\epsilon}{\partial \nu} + \int_{\partial \Omega} h_\epsilon \frac{\partial \Theta_\epsilon}{\partial \nu}.$$

As we may always assume $h_\epsilon|_{\partial \Omega}$ is bounded (depending only on g) the last term in the above equation is clearly bounded.

Next, for each j , we write $\Theta_\epsilon(x) = \theta_j(x) + \psi_j^\epsilon(x)$, $\theta_j(x)$ is the argument of $(x - x_\epsilon^j)$. Then, since $\frac{\partial \theta_j}{\partial \nu} = 0$ on ∂B_j , and $\psi_j^\epsilon(x)$ is harmonic inside B_j , we have

$$\int_{\partial B_j} h_\epsilon \frac{\partial \Theta_\epsilon}{\partial \nu} = \int (h_\epsilon - \bar{h}_\epsilon) \frac{\partial \psi^\epsilon}{\partial \nu}$$

where \bar{h}_ϵ is the average of h_ϵ on ∂B_j .

Now, via the fact that $osc h_\epsilon$ on ∂B_j is uniformal bounded, we see each $\int_{\partial B_j} h_\epsilon \frac{\partial \Theta_\epsilon}{\partial \nu} \rightarrow 0$ as $\epsilon \rightarrow 0$. This completes the proof of our statement that $\int_{\Omega_\epsilon} |\nabla h_\epsilon|^2 dx \leq c(\alpha, g)$.

It then also follows from the definition of W_g in [BBH], that

$$\frac{1}{2} \int_{\Omega_\rho(\bar{a})} \left[|\nabla \tilde{U}_\epsilon|^2 + \frac{1}{2\epsilon^2} (|\tilde{U}_\epsilon|^2 - 1)^2 \right] dx = \pi d \log \frac{1}{\rho} + W_g(\bar{a}) + o(\rho) + o(1)$$

Here $o(1) \rightarrow 0$ as $\epsilon \rightarrow 0$, and $\rho < \delta$ is a fixed constant. Finally, to complete our arguments, we look at the comparison map $\bar{V}_\epsilon(x)$,

$$\bar{V}_\epsilon(x) = \begin{cases} V_\epsilon(x), & x \in \bigcup_{j=1}^d B_j, \\ \tilde{U}_\epsilon(x) & x \in \Omega_\epsilon. \end{cases}$$

Then, on the one hand, we have

$$(3.15) \quad E_\epsilon(\bar{V}_\epsilon) \leq E_\epsilon(V_\epsilon) \leq E_\epsilon(w_\epsilon) \leq \pi d \log \frac{1}{\rho} + \min W_g + dI(\epsilon, \rho) + o(\rho),$$

and, on the other hand

$$(3.16) \quad E_\epsilon(\bar{V}_\epsilon) = \pi d \log \frac{1}{\rho} + W_g(\bar{a}) + o(\rho) + o(1) + \sum_{j=1}^d \frac{1}{2} \int_{B_\rho(\bar{a}_j)} \left[|\nabla \bar{V}_\epsilon|^2 + \frac{1}{2\epsilon^2} (|\bar{V}_\epsilon|^2 - 1)^2 \right] dx.$$

By the convergence of \tilde{U}_ϵ to $\bar{U}(x)$ and by the definition of $I(\epsilon, \rho)$, we see the last term on the right hand side of (3.16) is larger than $dI(\epsilon, \rho) + 0(\rho)$, That is

$$(3.17) \quad E_\epsilon(\bar{V}_\epsilon) \geq \pi d \log \frac{1}{\rho} + W_g(\bar{a}) + 0(\rho) + o(1) + dI(\epsilon, \rho).$$

We observe that $|\bar{a} - a| \cong \delta < \delta_0$, thus (3.1) yields $W_g(\bar{a}) \geq W_g(a) + C\delta^2$ for some $C > 0$.

(3.16) and (3.17) lead to a contradiction if we choose ρ is much small than δ^2 to start with. This proves Theorem A, in the case a is a global nondegenerate minimum point for W_g , modulo the statements (3.10) and (3.11).

Remark. – Under the assumption (3.1), we first choose $\delta = \frac{1}{4d} \min\{\delta_0, |a_i - a_j|, i \neq j\}$. Then we fix $\rho \ll \delta^2$ so that various $0(\rho)$ terms can be bounded by $\frac{\lambda_0}{4d}|a - \bar{a}|^2$. Finally we choose $\epsilon_0 < \epsilon(\rho)$ to make all of the arguments go through.

Remark. – The assumption that a is a global minimum point of W_g is used only to conclude (3.10) and (3.11). When a is a local nondegenerate minimum point, we should introduce another argument to obtain the similar statements as (3.10) and (3.11).

Proof of (3.10) and (3.11).

We start with the following lemmas.

LEMMA 1. – *Let u be a minimizer of the functional*

$$E_\epsilon(u) = \frac{1}{2} \int_B \left[|\nabla u|^2 + \frac{1}{2\epsilon^2} (|u|^2 - 1)^2 \right] dx$$

in the unit disc B with $u = g$ on ∂B . Suppose that

$$(3.18) \quad \int_{\partial B} \left[|\nabla_T g|^2 + \frac{1}{2\epsilon^2} (|g|^2 - 1)^2 \right] \leq C_1,$$

for a constant C_1 . Then, for all sufficiently small $\epsilon > 0$ (depending only on C_1) we have

$$(3.19) \quad E_\epsilon(u) \leq C_2 \text{ (depending only on } C_1)$$

whenever $\deg(g, \partial B) = 0$, and

$$(3.20) \quad E_\epsilon(u) \geq \pi |d| \log \frac{1}{\epsilon} + C_3$$

if $\deg(g, \partial B) = d \neq 0$.

Proof. – From (3.18) one easily deduces that $g \in C^{1/2}(\partial B)$ and $|g| \geq 1 - C\epsilon^{1/4}$ on ∂B , for a constant C depending on C_1 . In particular, the $\text{deg}(g, \partial B)$ is well-defined. In the case $\text{deg}(g, \partial B) = 0$, we may write $g(\theta) = \rho(\theta)e^{i\psi(\theta)}$ on ∂B . Let ψ^* be the harmonic extension of ψ into B , and $\rho^*(r, \theta) \equiv 1$ on $B_{1-\epsilon}$, $\rho^*(r, \theta)$ is linear in $r \in [1 - \epsilon, 1]$ for each θ . Then let $U^* = \rho^*e^{i\psi^*}$, we have

$$E_\epsilon(U^*) \leq \frac{1}{2} \int_B |\nabla \psi^*|^2 dx + C\epsilon = C_2.$$

This is (3.19). The estimate (3.20) is much hard to prove but it can be found in [BBH].

LEMMA 2. – *With the same hypothesis as Lemma 1, and suppose $\text{deg}(g, \partial B) = 0$ Then $|u(x)| \geq \frac{3}{4}$ in B whenever $0 < \epsilon \leq \epsilon_1$, for some ϵ_1 .*

Proof. – Suppose not, we would have a sequence $\epsilon_n \downarrow 0$, and a sequence of minimizers $u_{\epsilon_n} = u_n$ with boundary data g_n satisfying (3.18) and $\text{deg}(g_n, \partial B) = 0$. Moreover, $\inf |u_n| \leq \frac{3}{4}$.

Since $|g_n| \rightarrow 1, \|g_n\|_{C^{1/2}(\partial B)} \leq C$, we see $|u|(x) \geq \frac{4}{5} > \frac{3}{4}$ whenever $1 - |x| \leq C_0\epsilon_n$, for some C_0 . Indeed, the function $V_n(x) = u_n(\epsilon_n x)$ satisfies (a) $|V_n(x) - V_n(y)| \leq C|x - y|^{1/2}$ whenever $|x - y| \leq 1$ and $x, y \in \frac{1}{\epsilon_n}B$; and (b) $|\nabla V_n(x)| \leq \frac{C}{R}$ for $R \in (0, 1)$ and $|x| \leq \frac{1}{\epsilon_n} - R$. Both (a) and (b) follow from the standard elliptic estimates.

Hence, if $|u_n(x_n)| \leq \frac{3}{4}$, then there is a ball $\{x : |x - x_n| \leq \eta\epsilon_n\} \subset B$, for some $\eta > 0$ with $|u_n(x)| \leq \frac{4}{5}$ for all $x, |x - x_n| \leq \eta\epsilon_n$. Therefore

$$(3.21) \quad \int_B \frac{1}{\epsilon_n^2} (|u_n|^2 - 1)^2 dx \geq C(\eta) > 0.$$

By Lemma 1, $E_{\epsilon_n}(u_n) \leq E_{\epsilon_n}(U_n^*) \leq C_2$. Since $g_n \rightarrow g = e^{i\psi}$ weakly in $H^1(\partial B)$, $\int_B |\nabla \psi_n^*|^2 dx$ converges to $\int_B |\nabla \psi^*|^2 dx$, where ψ^* is the harmonic extension of ψ . Thus $\overline{\lim} E_{\epsilon_n}(U_n^*) \leq \frac{1}{2} \int_B |\nabla \psi^*|^2 dx$. On the other hand, $u_n \rightarrow u$ weakly in $H^1(B)$ with $u = g$ on ∂B and $|u| = 1$ a.e. in B . We have

$$\underline{\lim} E_{\epsilon_n}(u_n) \geq C(\eta) + \lim_n \frac{1}{2} \int_B |\nabla u_n|^2 \geq C(\eta) + \frac{1}{2} \int_B |\nabla \psi^*|^2.$$

We therefore obtain a contradiction.

Remark. – Both lemma 1 and lemma 2 remain true when we replace B by a bounded Lipschitz domain.

To prove (3.10) and (3.11), we first show $|\text{deg}(V_\epsilon, \partial B_j)| = 1$, for $j = 1, 2, \dots, d$. Indeed, $|\text{deg}(V_\epsilon, \partial B_j)| \geq 1$, for $j = 1, 2, \dots, d$, for otherwise we may replace V_ϵ in one of the ball B_j by a new map \tilde{V}_ϵ that minimizes the $E_\epsilon(\cdot)$ on B_j and $\tilde{V}_\epsilon = V_\epsilon$ on ∂B_j . Since $|V_\epsilon(x_j^\epsilon)| \leq \frac{1}{2}$ and since $|\nabla V_\epsilon| \leq \frac{C}{\epsilon}$, we conclude as in the Lemma 2 that

$$\begin{aligned} & \int_{B_j} \left[|\nabla V_\epsilon|^2 + \frac{1}{2\epsilon^2} (|V_\epsilon|^2 - 1)^2 \right] dx \\ & \geq \int_{B_j} \left[|\nabla \tilde{V}_\epsilon|^2 + \frac{1}{2\epsilon^2} (|\tilde{V}_\epsilon|^2 - 1)^2 \right] + C_0 \end{aligned}$$

for some $C_0 > 0$ (see (3.21)). This will contradict the fact that (3.2), (3.3) and the estimate (I).

By (3.20), we have

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^d \int_{B_j} \left[|\nabla V_\epsilon|^2 + \frac{1}{2\epsilon^2} (|V_\epsilon|^2 - 1)^2 \right] dx \\ & \geq \pi \sum_{j=1}^d (1 - \alpha_j) |\text{deg}(V_\epsilon, \partial B_j)| \log \frac{1}{\epsilon} + 0(1) \end{aligned}$$

(by our choice of α_j) $\geq \pi \left(d - \frac{1}{8} \right) \log \frac{1}{\epsilon} + 0(1)$. This combined with (3.2) implies each $|\text{deg}(V_\epsilon, \partial B_j)| = 1$. Next, we want to show (3.10). If for some $x_\epsilon \in \Omega_\epsilon$, $|\tilde{U}_\epsilon(x_\epsilon)| \leq \frac{1}{2}$, then there are two possibilities: either $\text{dist}(x_\epsilon, \partial\Omega_\epsilon) \geq \epsilon^{3\alpha}$ or $\text{dist}(x_\epsilon, \partial\Omega_\epsilon) < \epsilon^{3\alpha}$. In the first case, we find a ball D around x_ϵ of radius ϵ^β , $\beta \in (3\alpha, 4\alpha)$ such that the similar estimate as (3.9) is true. Then if $\text{deg}(\tilde{U}_\epsilon, \partial D) = 0$, we obtain a contradiction from Lemma 2 and the fact that \tilde{U}_ϵ is already a minimizer on D . If $\text{deg}(\tilde{U}_\epsilon, \partial D) \neq 0$, then the energy of \tilde{U}_ϵ on D will be larger than $\pi(1 - 4\alpha) \log \frac{1}{\epsilon} + 0(1)$. This contradicts the fact that the total energy of \tilde{U}_ϵ on Ω_ϵ is not larger than $\frac{\pi}{8} \log \frac{1}{\epsilon} + 0(1)$. In the second case, we may find a point $y_\epsilon \in \partial\Omega_\epsilon$, $|x_\epsilon - y_\epsilon| < \epsilon^{3\alpha}$. We again find a ball D around y_ϵ

of radius ϵ^β , for some $\beta \in \left(\frac{5}{2}\alpha, 3\alpha\right)$ so that

$$\epsilon^\beta \int_{\partial D \cap \Omega_\epsilon} \left[|\nabla \tilde{U}_\epsilon|^2 + \frac{1}{2\epsilon^2} (|\tilde{U}_\epsilon|^2 - 1)^2 \right] dx \leq C(\alpha).$$

Since the domain $\frac{1}{\epsilon^\beta}(D \cap \Omega_\epsilon)$ is clearly a bounded Lipschitz domain as $\beta \geq \frac{5\alpha}{2}$, we apply Lemma 1 and Lemma 2 again to obtain a contradiction as in the first case. We have therefore completed the proof of Theorem A under the assumption that a is also a global minimum point.

We turn now to the case that a is only a nondegenerate minimum point. We need to modify the part showing the statements (3.20) and (3.11). The key point is that we may not have

$$(3.22) \quad \deg(V_\epsilon, \partial B_j) \neq 0, \text{ for all } j = 1, 2, \dots, d.$$

Indeed, if we have (3.22), then above arguments apply to conclude that $|\tilde{U}_\epsilon(x)| \geq \frac{1}{2}$ on Ω_ϵ and by (3.2), we must have $\deg(V_\epsilon, \partial B_j) = 1$, for $j = 1, 2, \dots, d$. Here one notices also that, since

$$|\tilde{U}_\epsilon(x)| \geq \frac{1}{2}, d = \sum_{j=1}^d \deg(V_\epsilon, \partial B_j) \leq \sum_{j=1}^d |\deg(V_\epsilon, \partial B_j)| \leq d.$$

Then as $\deg(V_\epsilon, \partial B_j) = 1$, and $|\tilde{U}_\epsilon|(x) \geq \frac{1}{2}, x \in \Omega_\epsilon$, we may apply the same arguments as before to obtain

$$(3.23) \quad W_g(\bar{a}) \leq W_g(a) + o(\rho) + o(1)$$

where $\epsilon \leq \epsilon_0$. Here $|a_1 - \bar{a}_1| = \delta, |a_j - \bar{a}_j| \leq \delta$. (cf. (3.15), (3.16) and (3.17)). (3.23) will contradict to (3.1) whenever ρ is much smaller than δ^2 and ϵ is sufficiently small.

Therefore, (3.22) can not be true for all j . Note that by our choice of T_ϵ , we have $\deg(V_\epsilon, B_\delta(a_j)) = 1$, for $j = 1, 2, \dots, d$.

From now on, the first ball B_1 with center lies at distance δ form a_1 will be no longer used. We may assume, without loss of the generality, that $\deg(V_\epsilon, \partial B_1) = 0$. We then change V_ϵ inside B_1 by a minimizer of $E_\epsilon(\cdot)$ with the same boundary value as V_ϵ . Since the estimate (II), Lemma 2, (3.21) we see the energy of the new map (which is V_ϵ outside B_1 and is the minimizer inside B_1), which we call V'_ϵ , will satisfy $E_\epsilon(V'_\epsilon) \leq E_\epsilon(V_\epsilon) - C_0$,

for some positive constant C_0 . We then pick up another point, say x_1 , inside $B_\delta(a_1)$ such that $|V'_\epsilon(x_1)| \leq \frac{1}{2}$. Note $x_1 \notin B_1$, as $|V'_\epsilon| \geq \frac{3}{4}$ on B_1 . We repeat the same procedure for V'_ϵ as for V_ϵ , i.e., we choose a suitable ball centered at x_1 , and of radius $\epsilon^\beta, \beta \in (\alpha, 2\alpha)$ such that the corresponding inequality (3.9) is valid. We call this ball D_1 , suppose $\text{deg}(V'_\epsilon, \partial D_1) \neq 0$, we stop. Otherwise we claim that we may replace V'_ϵ inside D_1 by a minimizer with the same boundary value as V'_ϵ on ∂D_1 and such that the energy will again drop by at least C_0 . To see this, it suffices to observe that $V'_\epsilon(x_1) = V_\epsilon(x_1)$ and, by (II), there is a ball of size $\cong \epsilon$ on which $|V_\epsilon(x_1)| < \frac{3}{4}$. This ball will not interest any point y for which $|V'_\epsilon(y)| \geq \frac{3}{4}$. In particular it will not interest the first ball B_1 . Then we apply Lemma 2, (3.21) to conclude the energy dropping fact. We repeat this process, and after a finite number of steps, we arrive at a ball D_{N_1} with $\text{deg}(V_\epsilon^{N_1}, \partial D_{N_1}) \neq 0$. Note the centers of all these balls D_j are required to lie in side $B_\delta(a_1)$. We apply the same procedure to each $B_\delta(a_j), j = 2, \dots, d$. Eventually, we find d -balls which we still call B_1, \dots, B_d and a new map \bar{V}_ϵ as before with $\text{deg}(\bar{V}_\epsilon, \partial B_j) = 1, j = 1, 2, \dots, d$, and $E_\epsilon(\bar{V}_\epsilon) \leq E_\epsilon(V_\epsilon) - c_0$, for a definite constant $c_0 > 0$.

We go back to the estimates (3.15), (3.16), and (3.17) to obtain the following: there is a point $b \in \Omega^d, s.t. |b_j - a_j| \leq \delta, j = 1, \dots, d$, and such that

$$(3.24) \quad W_g(b) \leq W_g(a) - c_0 + 0(\rho).$$

By choosing $\rho \leq \delta^2 \ll 1$, (3.24) contradicts to the continuity of W_g at $a \in \Omega^d$. This ends the proof of Theorem A.

4. FINAL REMARKS

Let a be a nondegenerate local minimum point for W_g , and let $w_\epsilon(x)$ be as chosen in the previous section. In particular w_ϵ satisfies (3.2). Then, as a consequence of the proof of Theorem A, we see the solution $u_\epsilon(x, t)$ of (1.1)-(1.3) with $u_\epsilon(x, 0) = w_\epsilon(x)$ has the following property: The set $\left\{ (x, t) \in \Omega \times R^+ : |u_\epsilon(x, t)| \leq \frac{1}{2} \right\}$ is contained in $\bigcup_{j=1}^d B_\delta(a_j) \times R^+$ whenever ϵ is sufficiently small (depending only on δ, g, Ω and ρ in the construction of w_ϵ).

Now we consider the case that the set $K = \{b \in \Omega^d : W_g(b) = W_g(a)\}$ is not isolated points. From the property of W_g , K is obvious a compact set in Ω^d . We make the following assumption.

(4.1) The connected component K_a of K , which contains the point $a \in \Omega^d$, is a smooth k -dimensional manifold, $1 \leq k \leq 2d - 1$. Moreover, for each point $b \in K_a$, $\nabla^2 W_g(b)$ has $(n - k)$ -positive eigenvalues.

From (4.1), we immediately see each $b \in K_a$ is a local minimum of W_g . Moreover, we have $\lambda_0 = \inf\{\lambda_b, \lambda_b \neq 0 \text{ is an eigenvalue of } \nabla^2 W_g(b), b \in K_a\}$ is positive. Suppose $w_\epsilon(x)$ is chosen as before so that (3.2) is valid. Then we conclude from the proof of Theorem A that the solution $u_\epsilon(x, t)$ of (1.1)-(1.3) with the initial condition $u_\epsilon(x, 0) = w_\epsilon(x)$ has the property that the set

$$\left\{ (x, t) \in \Omega \times \mathbb{R}^+ : |u_\epsilon(x, t)| \leq \frac{1}{2} \right\}$$

$\subseteq K_a^\delta \times \mathbb{R}^+$ whenever ϵ is sufficiently small. Here $K_a^\delta = \{x \in \Omega, \text{dist}(x, K_a) \leq \delta\}$. In particular, there is a point a^* in K_a and a sequence $\{u_\epsilon\}$ of steady state solutions of (1.1)-(1.2) such that

$$(4.2) \quad \lim_{\epsilon \rightarrow 0} u_\epsilon = \prod_{j=1}^d \frac{x - a_j^*}{|x - a_j^*|} e^{ih(x)}$$

for some harmonic function h in Ω .

Open Problem. – Can every point $b \in K_a$ have the property (4.2)? (Cf. also [BBH, problem 6]).

On the other hand, one can follow the proof of Theorem A to show the following.

COROLLARY. – Let K be a compact subset of Ω^d , and $K_\delta = \{b \in \Omega^d : \text{dist}(b, K) < \delta\}, \delta > 0$. Suppose that $\inf\{W_g(b) : b \in \partial K_\delta\} > \sup\{W_g(b) : b \in K\}$. Then there is a family of steady state solutions $\{u_\epsilon\}, 0 < \epsilon < \epsilon_0$ of (1.1)-(1.2) such that, for any sequence of $\epsilon_n \downarrow 0$, the sequence u_{ϵ_n} (up to subsequences) converges to $u^*(x) = \prod_{j=1}^d \frac{x - a_j}{|x - a_j|} e^{ih(x)}, \Delta h = 0$ in $\Omega, u^* = g$ on $\partial\Omega$, for some $a \in K_\delta$.

It is clear then $\nabla W_g(a) = 0$. Next we consider the critical points of mountain-pass type.

We assume that

(H) there are two local minimum points a and b of W_g such that

$$\max \{W_g(a), W_g(b)\} < \inf_{\gamma \in \Gamma} \max_{y \in \gamma([0,1])} W_g(y) = c < \infty$$

where

$$\Gamma = \{ \gamma \in C([0, 1], \Omega^d) : \gamma(0) = a, \gamma(1) = b \}.$$

Then, it is well-known (cf. [R]) that there is $\mathbf{a} \in \Omega^d, W_g(\mathbf{a}) = c$ and $\nabla W_g(\mathbf{a}) = 0$. For $a, b \in \Omega^d$ as above, we can find two maps u_a, u_b (as for (3.2)) such that

$$E_\epsilon(u_a) \leq dI(\epsilon, \rho) + W_g(a) + \pi d \log \frac{1}{\rho} + 0(\rho).$$

$$E_\epsilon(u_b) \leq dI(\epsilon, \rho) + W_g(b) + \pi d \log \frac{1}{\rho} + 0(\rho).$$

Here $\epsilon < \epsilon(\rho), \rho \ll \frac{1}{4} \min\{|b - a|, c_0\}, c_0 = c - \max\{W_g(a), W_g(b)\}$.

We let $\Gamma = \{ \gamma \in C([0, 1], H'_g(\Omega)), \gamma(0) = u_a, \gamma(1) = u_b \}$, where

$$H'_g(\Omega) = \{ u \in H'(\Omega) : u|_{\partial\Omega} = g \}.$$

We claim, for all sufficiently small $\epsilon > 0$, that

$$(4.3) \quad \inf_{\gamma \in \Gamma, u \in \gamma([0,1])} \max E_\epsilon(u) = \mathbf{c} > \max \{E_\epsilon(u_a), E_\epsilon(u_b)\}.$$

Suppose for the moment that (4.5) is true, then, as the functional $E_\epsilon(\cdot)$ clearly satisfies the Palais-Smale condition (see [R]) and C^1 on $H'_g(\Omega)$, we conclude the following.

THEOREM C. – *Suppose W_g has a mountain-pass critical point in Ω^d so that (H) is valid. Then there is a family $\{u_\epsilon\}, 0 < \epsilon < \epsilon_0$ of critical points of (1.4) of mountain-pass type.*

To show (4.3), we let $\gamma \in \Gamma$, and assume, without loss of generality, that $E_\epsilon(u_b) \leq E_\epsilon(u_a)$, and $W_g(b) \leq W_g(a)$. For any $v \in \gamma([0, 1])$, we solve (1.1)-(1.3) with $u_0(x) = v(x)$. Denote $\mathbf{v}(x) = u(x, 1, v)$, where $u(x, t, v)$ is the solution of (1.1)-(1.3) with initial data v .

Then we find a new path connecting u_a and u_b as u_a is connected to \mathbf{u}_a , u_b is connected to \mathbf{u}_b by the heat flow, and \mathbf{u}_a is connected to \mathbf{u}_b by the path $u(x, 1, v), v \in \gamma([0, 1])$. We still denote this new path by γ . For any $s \in [0, 1], \gamma(s)$ will satisfies, in the addition, that $|\nabla_x \gamma(s)| \leq \frac{C_0}{\epsilon}$, for some

constant C_0 . This follows from our choices of u_a, u_b and the estimate (II). In fact, C_0 will only depend on g and Ω . It is also clear that $\max E_\epsilon(u)$ on this new path will be only smaller than $\max E_\epsilon(u)$ on the original path.

Let K_a be the connected component of $\{\mathbf{b} \in \Omega^d : W_g(\mathbf{b}) = W_g(a)\}$ containing a , and let K^δ be the set $\{\mathbf{b} \in \Omega^d : \text{dist}(\mathbf{b}, K_a) < \delta\}, \delta > 0$. By choosing $\delta > 0$ so small that we have $K_a^\delta \subset \Omega^d$ and $\min\{W_g(\mathbf{b}), \mathbf{b} \in \partial K_a^\delta\} \geq W_g(a) + C(\delta)$ for some $C(\delta) > 0$.

Now we look at vortices of $\gamma(s), s \in [0, 1]$. Let s_0 be the minimum of those $s \in [0, 1]$ that there is a point $\mathbf{x} = (x_1, \dots, x_d) \in \partial K_a^\delta$ with x_1, \dots, x_d belonging to the set $\left\{y \in \Omega : |\gamma(s_0)(y)| \leq \frac{1}{2}\right\}$.

Then we claim $E_\epsilon(\gamma(s_0)) - E_\epsilon(u_a) \geq C(\delta) - 0(\rho) > C(\delta) > 0$ for all sufficiently small $\epsilon, 0 < \epsilon < \epsilon_0 < \epsilon(\rho)$, and ρ . Indeed, following the constructions in the proof of Theorem A, we have *either*

$$E_\epsilon(\gamma(s_0)) \geq dI(\epsilon, \rho) + W_g(\mathbf{a}) + \pi d \log \frac{1}{\rho} + 0(\rho) + o(1), \text{ for some } \mathbf{a} \in \partial K_a^\delta$$

or

$$E_\epsilon(\gamma(s_0)) \geq dI(\epsilon, \rho) + W_g(\mathbf{a}) + \pi d \log \frac{1}{\rho} + c_0 + 0(\rho) + o(1).$$

for some definite constant $c_0 > 0$ and some $\mathbf{a} \in K$. In either case, we have (4.5). This completes the proof.

Complete the proof of Theorem B

Finally we wish to verify $h(x, t) = h_0(x)$ which is needed in the proof of Theorem B. For a fixed time $t > 0$, we look at the set $G(t) = \left\{x \in \Omega : |u_\epsilon(x, t)| \leq \frac{1}{2}\right\}$. Since $u_\epsilon \rightarrow u_0(x, t)$ weakly in $H'_{\text{loc}}(\bar{\Omega}/\{b_1, \dots, b_d\} \times R_+)$ and strongly in $L^2_{\text{loc}}(\bar{\Omega}/\{b_1, \dots, b_d\} \times R_+)$. We set that, for any $\delta > 0, B_\delta(b_j) \cap G(t) \neq \emptyset$, and that $\text{deg}(u_\epsilon(\cdot, t), \partial B_\delta(b_i)) = 1$ for $j = 1, \dots, d$.

Let $x_j \in B_\delta(b_j) \cap G(t)$, and choose a ball B_j of size $\epsilon^\beta, \alpha \leq \beta \leq 2\alpha$, centered at x_j so that the estimate similar to (3.9) is valid. We claim $\text{deg}(u_\epsilon(\cdot, t), \partial B_j) = 1$, for each $j = 1, \dots, d$. Otherwise, we simply follow the same argument as in the proof of Theorem A (for the case of local nondegenerate minimum) to obtain a new map $\bar{V}_\epsilon(x, t)$ such that

$$E_\epsilon(\bar{V}_\epsilon) \leq E_\epsilon(u_0^\epsilon) - C_0,$$

for a definite positive constant C_0 . At the same time,

$$(4.4) \quad E_\epsilon(\bar{V}_\epsilon) \geq dI(\epsilon, \rho) + W_g(\bar{b}) + \pi d \log \frac{1}{\rho} + 0(\rho) + o(1),$$

for some $\bar{b} \in \Omega^d$ with $|b_j - \bar{b}_j| \leq \delta$. We note that (4.3), (4.4), and (2.2) lead to a contradiction.

Thus we must have $\deg(u_\epsilon(\cdot, t), \partial B_j) = 1$ for all $j = 1, \dots, d$. Note that the estimate (2.14) will further imply that $x_j \rightarrow b_j$ as $\epsilon \rightarrow 0$, i.e., $\bar{b} = b$ in (4.4).

Similarly, we follow the previous arguments to obtain $|u_\epsilon(x, t)| \geq \frac{1}{2}$ on $\Omega_\epsilon = \Omega / \bigcup_{j=1}^d B_j$. (For otherwise (4.4) will be true as it was shown in

Section 3.) We write $u_\epsilon(x, t) = \rho_\epsilon(x, t) \prod_{j=1}^d \frac{x - x_j}{|x - x_j|} e^{ih_\epsilon(x, t)}$ on Ω_ϵ , with

$|\rho_\epsilon(x, t)| \geq \frac{1}{2}$. Note that $h_\epsilon(x, t)$ is uniquely determined up to $2\pi \pmod{}$ on each ∂B_j . Since $\Omega_\epsilon \supseteq \Omega_\delta = \Omega / \bigcup_{j=1}^d B_\delta(b_j)$ for any $\delta > 0$ whenever

$\epsilon \leq \epsilon(\delta)$, we may rewrite $u_\epsilon(x, t)$ as $\prod_{j=1}^d \frac{x - b_j}{|x - b_j|} \tilde{\rho}_\epsilon(x, t) e^{i\tilde{h}_\epsilon}$ in $\Omega_\delta \times [0, T]$.

We may also choose $\tilde{h}_\epsilon(x, t)$ so that it will be continuous in t in $\Omega_\delta \times [0, T]$. It is easy to see, for any $\delta > 0, \epsilon \leq \epsilon(\delta)$, that

$$\int_0^T \int_{\Omega_\delta} \left| \frac{\partial \tilde{h}_\epsilon}{\partial t} \right|^2 dx \leq 4 \left(\int_0^T \int_\Omega \left| \frac{\partial}{\partial t} u_\epsilon \right|^2 dx dt \right) \rightarrow 0$$

from our choice of the initial data. Since $h_\epsilon(x, 0) \rightarrow h_0(x)$ in $H'(\Omega)$ as $\epsilon \rightarrow 0$, we thus have $h_\epsilon(x, t) \rightarrow h_0(x)$ for a.e. $(x, t) \in \Omega \times [0, T]$.

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