

ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 12, n° 1 (1995), p. 61-73

http://www.numdam.org/item?id=AIHPC_1995__12_1_61_0

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Connectivity properties of the range of a weak diffeomorphism (*)

by

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ABSTRACT. – We discuss connectivity properties of the range of Sobolev maps, and of weak diffeomorphisms or equivalently of elastic deformations.

Key words: Calculus of variations, finite elasticity.

RÉSUMÉ. – On présente des résultats sur la connexion de l'image d'un difféomorphisme faible.

In this paper we shall study some properties of the range $u(\Omega)$ of a weak diffeomorphism u , where Ω is an open connected domain in \mathbf{R}^n . We denote by $\hat{\mathbf{R}}^n$ another copy of \mathbf{R}^n and we recall, compare [3] [2], that a

Classification A.M.S.: 49 Q 20, 73 C 50.

(*) This work has been partially supported by the Ministero dell'Università e della Ricerca Scientifica, by C.N.R. contract n. 91.01343.CT01, by the European Research Project GADGET, and by Grant n.11957 of the Czech Acad. of Sciences. The second author would like to acknowledge hospitality of the Mathematisches Institut of ETH Zürich during the preparation of this work.

function $u \in W^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ is said to belong to $\text{cart}^1(\Omega, \widehat{\mathbf{R}}^n)$ if

$$\int_{\Omega} |M(Du)| dx < \infty,$$

where $M(Du)$ denotes the n -vector in $\Lambda_n \mathbf{R}^n \times \widehat{\mathbf{R}}^n$

$$M(Du) := (e_1 + D_1u) \wedge (e_2 + D_2u) \wedge \dots \wedge (e_n + D_nu)$$

e_1, \dots, e_n being the canonical basis of $\widehat{\mathbf{R}}^n$, and moreover

$$\partial G_u \llcorner \Omega \times \widehat{\mathbf{R}}^n = 0,$$

where G_u denotes the n -dimensional rectifiable current in $\mathbf{R}^n \times \widehat{\mathbf{R}}^n$ carried by the graph of u . If

$$\mathcal{L}_u := \{x \in \Omega \mid x \text{ is a Lebesgue point for } u\}$$

$$\mathcal{R}_u := \mathcal{L}_u \cap \mathcal{L}_{Du},$$

and

$$\tilde{u}(x) := \begin{cases} \text{Lebesgue value of } u \text{ at } x, & \text{if } x \in \mathcal{R}_u \\ \text{constant,} & \text{if } x \notin \mathcal{R}_u \end{cases}$$

then

$$G_u = \tau(\mathcal{M}, 1, \xi)$$

where $\mathcal{M} := \{(x, \tilde{u}(x)) \mid x \in \mathcal{R}_u\}$ and $\xi(x, \tilde{u}(x))$ is the n -vector orienting \mathcal{M}

$$\xi(x, \tilde{u}(x)) := \frac{M(Du(x))}{|M(Du(x))|}.$$

compare [5].

The class of *weak diffeomorphisms* is then defined as the class of $u \in \text{cart}^1(\Omega, \widehat{\mathbf{R}}^n)$ such that

- (i) $\det Du > 0$ a.e. in Ω ,
- (ii) u is *globally invertible* in the sense that

$$\int_{\Omega} \phi(x, u) \det Du dx \leq \int_{\widehat{\mathbf{R}}^n} (\sup_{x \in \Omega} \phi(x, y)) dy$$

holds for all $\phi \in C_c^0(\Omega \times \widehat{\mathbf{R}}^n)$ with $\phi \geq 0$.

In the physical case $n = 3$ such a class arises naturally in finite elasticity, and in fact characterizes *elastic deformations* in the sense of [4].

In order to discuss properties of the range of u , of course we should first identify $u(\Omega)$. First we consider Sobolev maps $u \in W^{1,p}(\Omega, \widehat{\mathbf{R}}^n)$, $p \geq 1$, and we do it in terms of *approximate continuity*, compare e.g. [1]. Given a point $x \in \Omega$ at which Ω has positive density, $\theta(\Omega, x) > 0$, one says that $z \in \widehat{\mathbf{R}}^n$ is the approximate limit of $u(y)$ for $y \rightarrow x$,

$$z = \operatorname{ap} \lim_{y \rightarrow x} u(y),$$

if for every $r > 0$ the point x is a zero density point for the set $u^{-1}(\widehat{\mathbf{R}}^n \setminus B(z, r))$. Here the density is defined in terms of Lebesgue measure

$$\theta(A, x) := \lim_{\rho \rightarrow 0} \frac{|B(x, \rho) \cap A|}{|B(x, \rho)|}.$$

It turns out that for a.e. $x \in \Omega$ the approximate limit of $u(y)$ for $y \rightarrow x$ exists and agrees with Lebesgue's value of u . More precisely, setting

$$\begin{aligned} \mathcal{A}_u &:= \{x \in \Omega \mid \operatorname{ap} \lim_{y \rightarrow x} u(x) \text{ exists}\} \\ \bar{u}(x) &:= \operatorname{ap} \lim_{y \rightarrow x} u(x), \quad x \in \mathcal{A}_u \end{aligned}$$

we have

$$\begin{aligned} \mathcal{R}_u \subset \mathcal{L}_u \subset \mathcal{A}_u, \quad \mathcal{H}^n(\Omega \setminus \mathcal{R}_u) = 0 \\ \bar{u}(x) = \lim_{r \rightarrow 0^+} \int_{B(x, r)} u(y) dy \quad \text{for } x \in \mathcal{L}_u, \end{aligned}$$

and finally, compare [7], [13]

$$\dim_{\mathcal{H}}(\Omega \setminus \mathcal{L}_u) \leq n - p, \quad \mathcal{H}^{n-1}(\Omega \setminus \mathcal{L}_u) = 0$$

consequently

$$\dim_{\mathcal{H}}(\Omega \setminus \mathcal{A}_u) \leq n - p, \quad \mathcal{H}^{n-1}(\Omega \setminus \mathcal{A}_u) = 0.$$

We shall now work with the approximately continuous representative \bar{u} on \mathcal{A}_u of u and prove that the image $\bar{u}(\Omega)$ of a connected set Ω by a $W^{1,p}$ map is connected.

DEFINITION 1. – We say that $\Lambda \subset \mathbf{R}^n$ is d -open if and only if $\theta(\Lambda, x) = 1$ for all $x \in \Lambda$.

The collection of d -open sets defines the so-called d -topology; it is in fact easily seen that unions and finite intersections of d -open sets are d -open sets. Such a topology is studied for instance in [8] [9] and [10], where the notion of d_0 -connectedness is introduced.

LEMMA 1. – *Let $\Omega \subset \mathbf{R}^n$ be open and connected and let $A \subset \Omega$ be a set of positive measure with finite perimeter in Ω for which $|A| > 0$ and $P(A, \Omega) = 0$. Then $|\Omega \setminus A| = 0$.*

Proof. – From

$$\begin{aligned} P(A, \Omega) &= |D\chi_A|(\Omega) \\ &= \sup \left\{ \int \chi_A \operatorname{div} \varphi \, dx \mid \varphi \in C_c^1(\Omega, \mathbf{R}^n), |\varphi| \leq 1 \right\} = 0 \end{aligned}$$

we infer that the distribution χ_A has zero derivatives in Ω , hence χ_A is constant in Ω , as Ω is connected. \square

We can now state

THEOREM 1. – *Let Ω be an open connected set in \mathbf{R}^n and let $u \in W^{1,p}(\Omega, \mathbf{R}^N)$, $p \geq 1$. Then the set $\bar{u}(\Omega \cap \mathcal{A}_u)$ is connected.*

Proof. – Suppose that $B := \bar{u}(A \cap \mathcal{A}_u)$ is not connected. Then there are disjoint open sets \mathcal{U}_1 and \mathcal{U}_2 in \mathbf{R}^N such that

$$B \subset \mathcal{U}_1 \cup \mathcal{U}_2, \quad B \cap \mathcal{U}_i \neq \emptyset \quad i = 1, 2.$$

Since \bar{u} is approximately continuous on \mathcal{A}_u the sets

$$\Lambda_i := \bar{u}^{-1}(\mathcal{U}_i) \quad i = 1, 2$$

are contained in \mathcal{A}_u and d -open in \mathbf{R}^n .

From $B \subset \mathcal{U}_1 \cup \mathcal{U}_2$ we then deduce that $\mathcal{A}_u = \Lambda_1 \cup \Lambda_2$, hence

$$\mathcal{H}^{n-1}(\Omega \setminus (\Lambda_1 \cup \Lambda_2)) = 0.$$

As $B \cap \mathcal{U}_i \neq \emptyset$ we have $\Lambda_i \neq \emptyset$, hence $|\Lambda_i| > 0$, since the Λ_i 's are d -open. Also

$$\widehat{\Lambda}_2 := \{x \in \Omega \mid \theta(\Lambda_1, x) = 0\} \supset \Lambda_2,$$

Consider now the reduced boundary $\partial_{\Omega}^- \Lambda_1$ of Λ_1 in Ω . We have

$$\partial_{\Omega}^- \Lambda_1 \subset \{x \in \Omega \mid \text{neither } \theta(\Lambda_1, x) = 1 \text{ nor } \theta(\Lambda_1, x) = 0\}$$

hence

$$\partial_{\Omega}^{-} \Lambda_1 \subset \Omega \setminus (\Lambda_1 \cup \widehat{\Lambda}_2) \subset \Omega \setminus (\Lambda_1 \cup \Lambda_2) ,$$

so that

$$\mathcal{H}^{n-1}(\partial_{\Omega}^{-} \Lambda_1) = 0 .$$

From Lemma 1 we therefore deduce $|\Omega \setminus \Lambda_1| = 0$ which contradicts $|\Lambda_2| > 0$. \square

Remark 1. – It is easily seen that in fact Theorem 1 holds for any representative v of u which is approximately continuous in \mathcal{A}_v , with $\mathcal{L}^n(\Omega \setminus \mathcal{A}_v) = 0$, replacing \mathcal{A}_u and \bar{u} respectively with \mathcal{A}_v and v .

Remark 2. – The choice of a representative of u in Theorem 1 is essential. For example consider the function

$$u : B(0,1) \subset \mathbf{R}^2 \longrightarrow \mathbf{R}$$

$$u(x_1, x_2) := \begin{cases} x_1 & \text{if } x_1 \neq 0 \\ 2 & \text{if } x_1 = 0 . \end{cases}$$

Then $u(B(0,1)) = (-1,0) \cup (0,1) \cup \{2\}$, i.e. $u(B(0,1))$ is not connected. Only carefully redefining u on \mathcal{H}^{n-1} -a.e. point of the line $\{0\} \times (-1,1) \subset B(0,1)$ in terms of an approximately continuous representative v we can infer that $v(B(0,1))$ is connected.

We are now ready to discuss connectivity properties of the range of weak diffeomorphisms $u \in \widetilde{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Since $\tilde{u}|_{\mathcal{R}_u}$ has the double Lusin's property, from [4] we know that there exists a null set $N_u \subset \Omega$, $|N_u| = 0$, such that \tilde{u} is one to one on $\mathcal{R}_u \setminus N_u$ and

$$\tilde{u}(\mathcal{R}_u \setminus N_u) \cap \tilde{u}(N_u) = \emptyset .$$

In fact this holds for any Lusin's representative of u , and, moreover, we can assume that \tilde{u} be defined in all of Ω .

We shall now introduce a variant of the notion of connectivity which is suited for our purposes. Given two subsets A and B of \mathbf{R}^n we define their *essential distance* by

$$d_{\text{ess}}(A, B) := \sup\{\text{dist}(A \setminus N_1, B \setminus N_2) \mid |N_1| = |N_2| = 0\} . \quad (1)$$

We then set

DEFINITION 2. – A set $A \subset \mathbf{R}^n$ is said to be essentially connected, *ess-connected*, iff

$$A = A_1 \cup A_2 \text{ with } |A_1| > 0 \text{ and } |A_2| > 0 \text{ implies } d_{\text{ess}}(A_1, A_2) = 0 . \quad (2)$$

We have

THEOREM 2. – *Let $\Omega \subset \mathbf{R}^n$ be a bounded connected domain and let $u \in \widetilde{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Then $\tilde{u}(\Omega)$ is ess-connected for any Lusin representative \tilde{u} of u .*

Proof. – Suppose $\tilde{u}(\Omega)$ is not ess-connected. Then we can find $B_1, B_2 \subset \tilde{u}(\Omega)$ such that

$$|B_1|, |B_2| > 0, \quad |\tilde{u}(\Omega) \setminus (B_1 \cup B_2)| = 0, \quad \text{dist}(B_1, B_2) > 0. \quad (3)$$

Consider now an open set $\mathcal{U} \supset B_1$ with smooth boundary $\partial\mathcal{U}$ such that $\text{dist}(\mathcal{U}, B_2) > 0$, and the function

$$\lambda(y) := \text{dist}(y, \mathcal{U}) \quad y \in \widehat{\mathbf{R}}^n.$$

$\lambda(y)$ is Lipschitz-continuous and for some $\delta_0 > 0$

$$0 < \lambda(y) < \delta_0 \quad \text{implies} \quad y \in \widehat{\mathbf{R}}^n \setminus (\overline{\mathcal{U}} \cup \overline{B_2}). \quad (4)$$

Finally we extend λ to all of $\mathbf{R}^n \times \widehat{\mathbf{R}}^n$ by setting

$$\lambda(x, y) = \lambda(y) \quad \text{for } (x, y) \in \mathbf{R}^n \times \widehat{\mathbf{R}}^n. \quad (5)$$

Setting

$$A_i := \tilde{u}^{-1}(B_i) \quad i = 1, 2 \quad (6)$$

we see as consequence of the double Lusin's property of \tilde{u} (in fact from $|\tilde{u}(A)| = 0$ if $|A| = 0$) that

$$|A_1|, |A_2| > 0 \quad \text{and} \quad |\Omega \setminus (A_1 \cup A_2)| = 0. \quad (7)$$

Now we slice the current G_u by λ , compare [11]. For a.e. δ , $0 < \delta < \delta_0$ the slice $\langle G_u, \lambda, \delta \rangle$ exists as a $(n-1)$ -rectifiable current and

$$\langle G_u, \lambda, \delta \rangle = \partial(G_u \llcorner \{(x, y) \mid \lambda(x, y) < \delta\})$$

on $(n-1)$ -forms with compact support in $\Omega \times \widehat{\mathbf{R}}^n$. On the other hand

$$G_u \llcorner \{(x, y) \mid \lambda(x, y) < \delta\} = G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n$$

and

$$G_u \llcorner \{(x, y) \mid \lambda(x, y) > \delta_0\} = G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n$$

so that

$$\begin{aligned} \text{spt } \partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) &\subset \overline{\Omega} \times \{(x, y) \mid \lambda(x, y) \leq \delta\} \\ \text{spt } \partial(G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n) &\subset \overline{\Omega} \times \{(x, y) \mid \lambda(x, y) \geq \delta_0\} \end{aligned}$$

Since $\partial G_u \llcorner \Omega \times \widehat{\mathbf{R}}^n = 0$, we get

$$\partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = -\partial(G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n)$$

and therefore

$$\partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = 0 .$$

This implies

$$P(A_1, \Omega) = \mathbf{M}_\Omega(\partial[A_1]) = 0 \tag{8}$$

as $\partial[A_1] = \pi_\# \partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = 0$, where π is the map $\pi : (x, y) \rightarrow x$. This gives a contradiction since by Lemma 1 we then have $|A_2| = 0$. \square

We shall now show that maps in $\widehat{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ do *not* cavitate. Results of these type are relatively simple if one has some control on the boundary. Indeed we have

PROPOSITION 1. – *Let Ω be a bounded open connected set in \mathbf{R}^n and let $u \in \widehat{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Suppose that*

$$K := \text{spt } \widehat{\pi}_\# \partial G_u$$

is compact and let

$$\widehat{\mathbf{R}}^n \setminus K = \bigcup_{i=0}^m \mathcal{U}_i \quad m \in \mathbf{N}, \text{ or } m = \infty$$

the decomposition of the complement of K into connected components, where \mathcal{U}_0 is the unbounded component. Then

$$|\tilde{u}(\Omega) \cap \mathcal{U}_0| = 0$$

and for each $i = 1, \dots, m$ we have either $|\tilde{u}(\Omega) \cap \mathcal{U}_i| = 0$ or $|\mathcal{U}_i \setminus \tilde{u}(\Omega)| = 0$. In particular, if $m = 1$, then

$$|\tilde{u}(\Omega) \Delta \mathcal{U}_1| = 0 .$$

Proof. – It is an easy application of the constancy theorem, see e.g. [11]. In fact

$$\widehat{\pi}_{\#} G_u = \llbracket \tilde{u}(\Omega) \rrbracket$$

is an n -dimensional rectifiable current in $\widehat{\mathbf{R}}^n$ with multiplicity one and $\llbracket \tilde{u}(\Omega) \cap \mathcal{U}_i \rrbracket$ is boundaryless in \mathcal{U}_i . The last statement follows then at once as $|\tilde{u}(\Omega)| > 0$ because of $\det Du > 0$ a.e. \square

Remark 3. – A simple consequence of Proposition 1 is the result in [12]. In fact one easily sees that the class $\mathcal{A}_{p,q}$ introduced in [12] is a subclass of $\widetilde{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ and, if ∂G_u is the current carried by the graph of a continuous map u , then $K \subset \tilde{u}(\partial\Omega)$.

Our last theorem concerns the regularity, in the sense of non-cavitation, of weak diffeomorphisms without any control on the boundary. We consider a bounded and simply connected domain Ω in \mathbf{R}^n . We denote by $d : \Omega \rightarrow (0, \infty)$ a *Whitney's regularized distance* function, i.e., a smooth function $d(x)$ which is equivalent to $\text{dist}(x, \partial\Omega)$ for $x \in \Omega$. We then assume that

(S) For some $\delta_0 > 0$ the open sets

$$\Omega_\delta := \{x \in \Omega \mid d(x) > \delta\} \quad \text{and} \quad \Omega \setminus \Omega_\delta$$

are connected for all δ with $0 < \delta < \delta_0$.

THEOREM 3. – Let $\Omega \subset \mathbf{R}^n$ be a connected domain satisfying (S) and let $u \in \widetilde{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Then for almost every δ , $0 < \delta < \delta_0$, the image $\tilde{u}(\Omega_\delta)$ of Ω_δ and $\widehat{\mathbf{R}}^n \setminus \tilde{u}(\Omega_\delta)$ are both essentially connected.

Proof. – Set $\Delta_\delta := \tilde{u}(\Omega_\delta)$, $0 < \delta < \delta_0$. We know that $\tilde{u}(\Omega_\delta)$ is essentially connected. Assume on the contrary that $\widehat{\mathbf{R}}^n \setminus \Delta_\delta$ is not essentially connected for a non zero measure set of δ 's. For almost every such δ 's

$$\partial(G_u \llcorner \Omega_\delta \times \widehat{\mathbf{R}}^n)$$

is of course a rectifiable current with finite mass, consequently Δ_δ is a Caccioppoli set, as

$$\llbracket \Delta_\delta \rrbracket = \widehat{\pi}_{\#}(G_u \llcorner \Omega_\delta \times \widehat{\mathbf{R}}^n),$$

$\widehat{\pi}$ being the map $(x, y) \rightarrow y$. Fix now one such a δ . We claim that for almost every δ_1 with $0 < \delta_1 < \delta$ we have

$$\partial(G_u \llcorner \Omega_{\delta_1} \times \widehat{\mathbf{R}}^n) \quad \text{is a rectifiable current} \quad (9)$$

and moreover

$$\mathcal{H}^{n-1}(\partial^- \Delta_{\delta_1} \cap \partial^- \Delta_\delta) = 0 . \tag{10}$$

In order to prove (10) we first observe that

$$\mathcal{H}^n(E) = 0$$

where

$$E := \{ x \in (\Omega \setminus \Omega_\delta) \cap \mathcal{R}_u \mid \tilde{u}(x) \in \partial^- \Delta_\delta \}$$

because \tilde{u} satisfies the double Lusin property. From this we easily infer

$$\mathcal{H}^{n-1}(E_{\delta_1}) = 0 \quad \text{for a.e. } \delta_1 \tag{11}$$

where

$$E_{\delta_1} := \{ x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_u \mid \tilde{u}(x) \in \partial^- \Delta_\delta \} .$$

On the other hand the trace \bar{u} of u on $\partial\Omega_{\delta_1}$ belongs for a.e. δ_1 to $W^{1,1}(\partial\Omega_{\delta_1}, \widehat{\mathbf{R}}^n)$, \mathcal{H}^{n-1} almost every point in $\partial\Omega_{\delta_1}$ is a regular point for u and \bar{u} , and

$$\bar{u}(x) = \tilde{u}(x) \quad \mathcal{H}^{n-1} - \text{ a.e. } x \in \partial\Omega_{\delta_1} ;$$

moreover

$$\partial(G_u \llcorner \Omega_{\delta_1} \times \widehat{\mathbf{R}}^n) = G_{\bar{u}|_{\partial\Omega_{\delta_1}}} ,$$

compare e.g. [6]. Denoting by $\tilde{\bar{u}}$ the Lusin representative of \bar{u} , we then infer that

$$\partial^- \Delta_{\delta_1} \subset \tilde{\bar{u}}(\partial\Omega_{\delta_1}) \quad \mathcal{H}^{n-1} - \text{ a.e.}$$

consequently

$$\partial^- \Delta_{\delta_1} \cap \partial^- \Delta_\delta \subset \tilde{\bar{u}}(\{x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_{\bar{u}} \mid \tilde{\bar{u}}(x) \in \partial^- \Delta_\delta\})$$

Since the set $\{x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_{\bar{u}} \mid \tilde{\bar{u}}(x) \in \partial^- \Delta_\delta\}$ differs from E_{δ_1} by a null set and $\tilde{\bar{u}}$ has the H^{n-1} Lusin property on $\partial\Omega_{\delta_1}$, we finally infer (10) from (11).

We now choose δ_1 in such a way that (9) and (10) hold. Setting

$$\begin{aligned} B &:= \tilde{u}(\Omega_\delta) = \Delta_\delta \\ C &:= \tilde{u}(\Omega_{\delta_1} \setminus \Omega_\delta) = \Delta_{\delta_1} \setminus \Delta_\delta \end{aligned}$$

(10) yields

$$\partial^- \mathcal{B} \subset \partial^- C \quad \mathcal{H}^{n-1} \text{ a.e.} \quad (12)$$

On account of our assumption, $R := \widehat{\mathbf{R}}^n \setminus \tilde{u}(\Omega_\delta)$ is not *ess*-connected. We can then find disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \subset \widehat{\mathbf{R}}^n$ such that

$$R \subset \mathcal{U}_1 \cup \mathcal{U}_2 \text{ a.e.}, |R \cap \mathcal{U}_1| > 0, |R \cap \mathcal{U}_2| > 0, \text{dist}(\mathcal{U}_1, \mathcal{U}_2) > 0. \quad (13)$$

Being \tilde{u} one-to-one in Ω we have

$$|\mathcal{B} \cap C| = 0 \quad (14)$$

and

$$C = (C \cap \mathcal{U}_1) \cup (C \cap \mathcal{U}_2) \quad \text{a.e.} \quad (15)$$

if we take into account (13) and (14).

From Theorem 2 we know that C is *ess*-connected, hence one of the addenda of (15), say $C \cap \mathcal{U}_1$, must be a null set

$$|C \cap \mathcal{U}_1| = 0, \quad \text{i.e.,} \quad C \subset \mathcal{U}_2 \text{ a.e.,}$$

consequently

$$\partial^- C \subset \overline{\mathcal{U}}_2 \quad \mathcal{H}^{n-1} \text{ a.e. .}$$

From (12) we therefore obtain

$$\partial^- \mathcal{B} \subset \overline{\mathcal{U}}_2 \quad \mathcal{H}^{n-1} \text{ a.e. .} \quad (16)$$

Now we define

$$\tilde{\mathcal{B}} := \mathcal{B} \cup \overline{\mathcal{U}}_2 \quad (17)$$

and observing that $\mathcal{B}, \mathcal{U}_1, \mathcal{U}_2$ is a covering of $\widehat{\mathbf{R}}^n$, we get

$$\partial^- \tilde{\mathcal{B}} = \partial^- \mathcal{B} \cap \overline{\mathcal{U}}_1,$$

consequently $\partial^- \tilde{\mathcal{B}} = \emptyset$, if we take into account (16). We therefore conclude that either $\tilde{\mathcal{B}} = \widehat{\mathbf{R}}^n$ a.e. or $|\tilde{\mathcal{B}}| = 0$. Because of (13) we finally infer $\tilde{\mathcal{B}} = \widehat{\mathbf{R}}^n$ a.e. and, because of (17), that

$$\mathcal{B} \supset \widehat{\mathbf{R}}^n \setminus \overline{\mathcal{U}}_2 \supset \overline{\mathcal{U}}_1$$

which contradicts $|R \cap \mathcal{U}_1| > 0$ in (13). \square

We shall now show by means of a few examples that Theorems 2 and 3 are in some sense optimal. Our first example shows that $\tilde{u}(\Omega)$ may be disconnected.

Example 1. – Let $\Omega = (-1, 1)^2 \subset \mathbf{R}^2$ and let $\alpha \in (0, 1)$. Consider the map $u : \Omega \rightarrow \mathbf{R}^2$ defined by

$$u^1(x) := x_1, \quad u^2(x) := x_2 |x_1|^\alpha.$$

We have $u \in \widetilde{\text{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. In fact $G_{u_\varepsilon} \rightarrow G_u$ where u_ε is the family of maps defined by

$$u_\varepsilon^1 := u^1 \quad \text{in } \Omega$$

$$u_\varepsilon^2 := \begin{cases} u^2 & \text{in } \Omega \cap \{x \mid |x_1| \geq \varepsilon\} \\ x_2 \varepsilon^\alpha & \text{in } \Omega \cap \{x \mid |x_1| < \varepsilon\}. \end{cases}$$

It is immediately seen that

$$\tilde{u}(\Omega \cap \mathcal{R}_u) = \{y \in \widehat{\mathbf{R}}^2 \mid |y_2| < |y_1|^\alpha, |y_1| < 1\}$$

is *not* connected, but it is *ess*-connected. Notice that instead for the approximately continuous representative \bar{u} of u we have

$$\bar{u}(\Omega \cap \mathcal{A}_u) = \{y \mid |y_2| < |y_1|^\alpha, |y_1| < 1\} \cup \{(0, 0)\}.$$

In fact $\{0\} \times (-1, 1)$ is mapped to zero and by the Hausdorff estimates of the singular set of u we cannot have $\{0\} \times (-1, 1) \subset \Omega \setminus \mathcal{L}_u$. We therefore see that $\bar{u}(\Omega)$ is connected according to Theorem 1.

Finally we notice that Theorem 2 does not hold for Cartesian maps as it is shown by the following variant of the previous example. Set

$$\Omega = (-2, 2) \times (-1, 1) \subset \mathbf{R}^2$$

and

$$u^1(x) = x_1, \quad u^2(x) = x_2 (\max(|x_1| - 1, 0))^\alpha.$$

Then it is easily seen that

$$\bar{u}(\Omega \cap \mathcal{A}_u) = \tilde{u}(\Omega \cap \mathcal{R}_u)$$

$$= \{y \mid |y_2| < (|y_1| - 1)^2, 1 < |y_1| < 2\} \cup ([-1, 1] \times \{0\})$$

is *not ess*-connected while it is connected in the usual sense.

Example 2. – We identify \mathbf{R}^2 with the complex plane \mathbf{C} , we set

$$\Omega := \left\{ z \in \mathbf{C} \mid \frac{1}{2} < |z| < 1, \operatorname{Im} z > 0 \right\},$$

and we consider $u(z) := z^2$ for $z \in \Omega$. In the a.e. sense we have

$$\tilde{u}(\Omega \cap \mathcal{R}_u) = \widehat{\Omega} := \left\{ z \in \mathbf{C} \mid \frac{1}{4} < |z| < 1 \right\}.$$

Clearly $\mathbf{R}^2 \setminus \tilde{u}(\Omega \cap \mathcal{R}_u)$ is not essentially connected, while $\mathbf{R}^2 \setminus \tilde{u}(\Omega_\delta \cap \mathcal{R}_u)$ is essentially connected for all $\delta > 0$.

The map u is of course of class C^1 in Ω and its *exact image* (not the a.e. image) is given by

$$\begin{aligned} & \bar{u}(\Omega \cap \mathcal{A}_u) \\ &= \widehat{\Omega}' := \left\{ z \in \mathbf{C} \mid \frac{1}{4} < |z| < 1 \right\} \setminus \{z \in \mathbf{C} \mid \operatorname{Im} z = 0, \operatorname{Re} z > 0\} \quad (18) \end{aligned}$$

which is simply connected in the usual sense.

The next example of a discontinuous weak diffeomorphism shows that in general $\mathbf{R}^2 \setminus \bar{u}(\Omega \cap \mathcal{A}_u)$ need not be connected (in the usual sense).

Example 3. – Let $\Omega = B(0, 2) \subset \mathbf{R}^2$, $Q = \{(x_1, x_2) \mid |x_1| + |x_2| < 1\}$, and $u = (u^1, u^2) : B(0, 2) \rightarrow \widehat{\mathbf{R}}^2$ given by

$$u^1(x) := \begin{cases} \frac{x_1}{|x_1| + |x_2|}, & \text{if } x \in Q \\ x_1, & \text{if } x \in \Omega \setminus Q \end{cases} \quad u^2(x) := x_2, \quad \text{if } x \in \Omega.$$

One can show that $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^2)$ (in fact $u \in \widetilde{\operatorname{dif}}^{p,q}(\Omega, \widehat{\mathbf{R}}^2)$ for all $p, q < 2$), by approximating u by a sequence of smooth maps u_ε , for instance

$$u_\varepsilon^1(x) := \begin{cases} \frac{x_1}{\varepsilon}, & \text{if } |x_1| + |x_2| < \varepsilon \\ u^1, & \text{otherwise} \end{cases}, \quad u^2(x) := u^2.$$

One then sees that

$$\begin{aligned} \mathcal{L}_u &= B(0, 2) \setminus \{0\} \\ \bar{u}(x_1, 0) &= (1, 0) \quad \text{for } 0 < x_1 < 1 \\ \bar{u}(x_1, 0) &= (-1, 0) \quad \text{for } -1 < x_1 < 0; \end{aligned}$$

while u is not approximately continuous at $(0, 0)$. Thus we have

$$\bar{u}(\Omega \cap \mathcal{L}_u) = B(0, 2) \setminus ((-1, 1) \times \{0\})$$

which is not simply connected, but $\bar{u}(\Omega \cap \mathcal{L}_u)$ and $\mathbf{R}^2 \setminus \bar{u}(\Omega \cap \mathcal{L}_u)$ are both essentially connected.

Example 3 shows that weak diffeomorphisms may create holes of zero measure.

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(Manuscript received November 18, 1993.)