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FRANK PACARD

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A priori regularity for weak solutions of some nonlinear elliptic equations

by

Frank PACARD

ENPC-CERGRENE, La Courtine, 93167 Noisy-le-Grand Cedex, France

ABSTRACT. – Let $f(u)$ be some positive regular function bounded from above by $u^{\frac{n}{n-2}}$, in $\mathbb{R}^+$. We derive some necessary and sufficient conditions, in order for all positive solutions to $-\Delta u = f(u) \in L^1_{\text{loc}}(\mathbb{R}^n)$ to be regular.

Key words: Regularity, Nonlinear Elliptic Equation.

RÉSUMÉ. – Soit $f(u)$ une fonction positive, suffisamment régulière, bornée par $u^{\frac{n}{n-2}}$ sur $\mathbb{R}^+$. On démontre qu'il existe un critère permettant de déterminer si toutes les solutions faibles positives de $-\Delta u = f(u) \in L^1_{\text{loc}}(\mathbb{R}^n)$ sont régulières.

1. INTRODUCTION

It is well known that positive weak solutions of $-\Delta u = u^\alpha$ defined in some domain of $\mathbb{R}^n$, $n \geq 3$, are all smooth if and only if $\alpha < n/(n-2)$. The fact that the last condition is sufficient follows easily from the result of G. Stampacchia [6] and a classical bootstrap argument (*see* also [2]). For $\alpha = n/(n-2)$, the existence of singular solutions is proved by P. Aviles in [1] and in [4]. For $\alpha > n/(n-2)$, one can easily check that $u(x) = c_{\alpha,n}|x|^{-\frac{2}{\alpha-1}}$ is a weak solution to the corresponding equation,

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when $c_{\alpha,n}^{\alpha-1} \equiv \frac{2}{\alpha-1}\left(n - \frac{2\alpha}{\alpha-1}\right)$. In this short note we address the following problem :

PROBLEM. – Characterize all regular functions $f \geq 0$ such that any positive weak solution of

$$-\Delta u = f(u) \in L^1(\Omega) \tag{1}$$

is regular inside Ω.

We give some partial answer to this problem and prove that there exists a criterion for *a priori* regularity. In addition, we show that this criterion is in some sense optimal.

Our main result reads as follows.

THEOREM 1. – *Assume that f is a convex function and that, for all $u \geq 0$, we can write*

$$f(u) \equiv u^{\frac{n}{n-2}} g(\log(1+u)) > 0. \tag{2}$$

Where g is a regular function defined over $[0, +\infty)$ which is decreasing. Then, any positive weak solution of

$$-\Delta u = f(u) \in L^1(\Omega)$$

is regular inside Ω if and only if $g \in L^1([0, +\infty))$.

Notice that our hypothesis are a little stronger than in the general setting of the problem. Although the same technic would certainly give some stronger results, we have not been able to solve the problem for general functions f. The main Theorem is a consequence of the two results that follow. In the first result, we derive some criterion for regularity.

THEOREM 2. – *Assume that f is convex and that we can write*

$$f(u) \equiv u^{\frac{n}{n-2}} g(\log(1+u)) \geq 0,$$

where $g \in L^1([0, +\infty))$ is regular and decreasing. Then, any positive weak solution of $-\Delta u = f(u) \in L^1(\Omega)$ is regular.

Our second result shows that this criterion is, in some sense, optimal.

THEOREM 3. – *If f is given by*

$$f(u) \equiv u^{\frac{n}{n-2}} g(\log(1+u))$$

for u large enough, where $g > 0$ is bounded, regular and satisfies $g \notin L^1([1, +\infty))$, then there exists an open set Ω containing the origin and u a positive weak solution of

$$-\Delta u = f(u) \in L^1(\Omega)$$

with a non removable singularity at the origin.

Throughout the paper c_i will denote some universal constant, depending only on n.

PROOF OF THEOREM 2

We can remark that, up to a dilation and a change in the function f, we may always assume that $B(0,1) \subset \Omega$. We are going to prove that the following decay property is true :

PROPOSITION 1. – *Under the assumptions of Theorem 2, there exists some $R > 0$ and some $\theta \in (0,1)$, such that the following holds :*

If $r < R$, then for all $x \in B(0,1)$, $r > 0$ satisfying $B(x,2r) \subset B(0,1)$ we have

$$||f(u)||_{L^1(B(x,\theta r))} \leq \frac{1}{2}||f(u)||_{L^1(B(x,r))}. \tag{3}$$

Proof of Proposition 1. – In the whole proof, we assume that x and r are chosen in order to fulfill $B(x,2r) \subset B(0,1)$. We defined $\tilde{u} = u$ on $B(x,r)$ and $\tilde{u} = 0$ outside $B(x,r)$. The first step of the proof consists in proving some estimates on u using the Poisson kernel. We set

$$v(z) = c_1 \int_{\mathbb{R}^n} |z-y|^{2-n} f(\tilde{u})(y)dy, \tag{4}$$

which is well defined for almost every $z \in B(x,r)$. And first prove the estimate :

LEMMA 1. – *With the above definition, we have*

$$||f(v)||_{L^1(B(x,r))} \leq c_2||f(u)||_{L^1(B(x,r))}H(||f(u)||_{L^1(B(x,r))}). \tag{5}$$

Where by definition $H(x) \equiv x^{\frac{2}{n-2}}(1-\log(\min(1,x)))$ and the constant $c_2 > 0$ only depends on n.

Proof of Lemma 1. – Let us compute for a.e. $z \in B(x,r)$

$$f(v)(z) = f\Big(c_1 \int_{\mathbb{R}^n} |z-y|^{2-n} f(\tilde{u})(y)dy\Big).$$

We denote by

$$\bar{f} \equiv c_1 \int_{\mathbb{R}^n} f(\tilde{u})(y)dy = c_1 \int_{B(x,r)} f(u)(y)dy,$$

since $f(0) = 0$ by assumption. The function f is assumed to be convex. Therefore, using Jensen's inequality, we obtain

$$f(v)(z) \leq \frac{c_1}{\bar{f}} \int_{B(x,r)} f(\bar{f}|z-y|^{2-n}) f(\tilde{u})(y) dy. \tag{6}$$

Integrating inequality (6) over $B(x,r)$, we find

$$\begin{aligned}\int_{B(x,r)} & f(v)(z)dz \\ &\leq \frac{c_1}{\bar{f}} \int_{B(x,r)} \left(\int_{B(x,r)} f(\bar{f}|z-y|^{2-n}) f(\tilde{u})(y) dy \right) dz. \end{aligned} \tag{7}$$

We claim that

$$\int_{B(x,r)} f(\bar{f}|z-y|^{2-n}) dz \leq c_3 ||f(u)||_{L^1(B(x,r))} H(||f(u)||_{L^1(B(x,r))}),$$

where H is defined in Lemma 1. Therefore, by Fubini's Theorem, we get from (7) the estimate

$$\int_{B(x,r)} f(v)(z)dz \leq c_4 ||f||_{L^1(B(x,r))} H(||f(u)||_{L^1(B(x,r))}),$$

for some constant $c_4 > 0$ only depending on n. Which is the desired result.

It remains to prove the claim. Using the definition of f, we obtain, for every $y \in B(x,r)$

$$\begin{aligned}\int_{B(x,r)} & f(\bar{f}|z-y|^{2-n}) dz \\ &= \int_{B(x,r)} (\bar{f}|z-y|^{2-n})^{\frac{n}{n-2}} g(\log(1+\bar{f}|z-y|^{2-n})) dz \\ &\leq \bar{f}^{\frac{n}{n-2}} \int_{B(0,1)} g(\log(1+\bar{f}|z|^{2-n})) |z|^{-n} dz \\ &= c_5 \bar{f}^{\frac{n}{n-2}} \int_0^1 g(\log(1+\bar{f}s^{2-n})) \frac{1}{s} ds \end{aligned}$$

In the former equation, the inequality is obtained thanks to the fact that we have assumed that $B(x, 2r) \subset B(0,1)$, which implies that $2r \leq 1$. So, we conclude that

$$\int_{B(x,r)} f(\bar{f}|z-y|^{2-n}) dz \leq c_5 \bar{f}^{\frac{n}{n-2}} \int_0^{\bar{f}^{\frac{1}{2-n}}} g(\log(1+s^{2-n})) \frac{1}{s} ds. \tag{8}$$

A simple change of variables leads to the following computation

$$\begin{aligned}\int_0^1 g(\log(1+s^{2-n}))\frac{1}{s}ds &= \int_0^{+\infty} g(\log(1+e^{(n-2)t}))dt\\ &= \int_{\log 2}^{+\infty} g(s)\frac{e^s}{(n-2)(e^s-1)}ds\\ &\le \frac{2}{n-2}\int_{\log 2}^{+\infty} g(s)ds \qquad (9)\end{aligned}$$

which is bounded since g is assumed to be in $L^1(0,+\infty)$. In addition, as g is also assumed to be bounded, we have, for all $r > 1$

$$\int_1^r g(\log(1+s^{2-n}))\frac{1}{s}ds \le ||g||_{L^\infty}\log(r) \qquad (10)$$

Using (8), (9) and (10) with $r = \bar{f}^{\frac{1}{2-n}}$, we get

$$\begin{aligned}\int_{B(x,r)} f(\bar{f}|z-y|^{2-n})dz &\le c_6\bar{f}^{\frac{n}{n-2}}(1-\log(\min(1,\bar{f})))\\ &\equiv c_6||f(u)||_{L^1(B(x,r))}H(||f(u)||_{L^1(B(x,r))}).\end{aligned}$$

Which proves the claim. □

Now, we turn to the second step of the proof of Proposition 1.

LEMMA 2. – *For any $\rho \le r$, we have the estimate*

$$\begin{aligned}\int_{B(x,\rho)} f(u)(y)dy \le c_7\Big(&\frac{\rho^n}{r^n}\int_{B(x,r)} f(u)(y)dy\\ &+ ||f(u)||_{L^1(B(x,r))}H(||f(u)||_{L^1(B(x,r))})\Big). \qquad (11)\end{aligned}$$

for some constant $c_7 > 0$ only depending on n.

Proof of Lemma 2. - The idea of the proof is to decompose u over $B(x,r)$ in two parts. We define w to be the solution of

$$\begin{cases} -\Delta w = 0 & \text{in } B(x,r)\\ w = u & \text{on } \partial B(x,r).\end{cases}$$

It is classical to see that w is regular and harmonic in $B(x,r)$. Therefore, for all $y \in B(x,r/2)$, we can write

$$w(y) = \frac{1}{|B(y,r/2)|}\int_{B(y,r/2)} w(z)dz.$$

As f is convex, we have, by Jensen's inequality

$$f(w)(y) = \frac{1}{|B(y, r/2)|} \int_{B(y,r/2)} f(w)(z)dz.$$

Integrating the above inequality over $B(x, \rho)$, we obtain

$$\int_{B(x,\rho)} f(w)(z)dz \leq 2^n \frac{\rho^n}{r^n} \int_{B(x,r)} f(w)(z)dz,$$

for all $\rho < r/2$.

The maximum principle leads to the estimate $w \leq u$ over $B(x, r)$. As f is convex and positive, it is an increasing function; so, we get the estimate

$$\int_{B(x,\rho)} f(w)(z)dz \leq 2^n \frac{\rho^n}{r^n} \int_{B(x,r)} f(u)(z)dz. \tag{12}$$

Since $u \leq w + v$ on $\partial B(x, r)$, where $v \geq 0$ is defined in (4), we have, always by the maximum principle, the estimate $u \leq w + v$ in $B(x, r)$. So, always by convexity of the function f

$$f(u) \leq f(w + v) \leq \frac{1}{2}(f(2w) + f(2v)).$$

Using the fact that the function g is assumed to be decreasing we get

$$f(2x) = (2x)^{\frac{n}{n-2}} g(\log(1 + 2x)) \leq (2x)^{\frac{n}{n-2}} g(\log(1 + x)) \leq 2^{\frac{n}{n-2}} f(x).$$

Therefore, we see that, for all $x \in B(x, r)$

$$f(u) \leq 2^{\frac{n}{n-2}} (f(w) + f(v)).$$

Integrating this inequality over the ball $B(x, \rho)$, we obtain

$$\int_{B(x,\rho)} f(u)(y)dy \leq 2^{\frac{n}{2-n}} \left(\int_{B(x,\rho)} f(w)(y)dy + \int_{B(x,r)} f(v)(y)dy \right).$$

Using (12) and (5), we finally get the estimate

$$\int_{B(x,\rho)} f(u)(y)dy$$
$$\leq c_8 \left(\frac{\rho^n}{r^n} \int_{B(x,r)} f(u)(y)dy + ||f(u)||_{L^1(B(x,r))} H(||f(u)||_{L^1(B(x,r))}) \right),$$

which holds for all $\rho < r/2$. Increasing, if necessary the constant $c_8 > 0$, we may assume that this estimate holds for all $\rho \leq r$. This completes the proof of Lemma 2. □

Proof of Proposition 1 completed. – In (11), we take $\rho = \theta r$, with $0 < \theta < 1$ chosen in order to fulfill the inequality $c_7\theta^n < 1/4$. Once this choice is done, we can choose r such that

$$c_7 H(||f(u)||_{L^1(B(x,r))}) \geq 1/4,$$

for all $x \in B(0,1)$. With the above choices, we find

$$||f(u)||_{L^1(B(x,\theta r))} = \int_{B(x,\theta r)} f(u)(y)dy \leq \frac{1}{2}||f(u)||_{L^1(B(x,r))}.$$

This ends the proof of Proposition 1. □

Proof of Theorem 2 completed. – For $\rho > 1$ and $0 < q < n$, we recall the definition of the Campanato space $L^{p,q}(\Omega)$ (*see* [3]):

$$L^{p,q}(\Omega) \equiv \left\{ v \in L^p(\Omega) / \sup_{x\in\Omega} \left(\sup_{r>0} \left(r^{-q} \int_{B(x,r)\cap\Omega} u^p dx < +\infty \right) \right) \right\}.$$

We will denote by

$$||v||_{p,q,\Omega} = \sup_{x\in\Omega} \sup_{r>0} \left(r^{-q} \int_{B(x,r)\cap\Omega} v^p dx \right)^{\frac{1}{p}}$$

the norm of v in $L^{p,q}(\Omega)$. It is classical to see that the following Lemma holds:

LEMMA 3. – *If the conclusion of Proposition 1 holds, then there exists some $\lambda > 0$ such that $f(u)$ is bounded in $L^{1,\lambda}(B(0,1/2))$.*

The conclusion of Theorem 2 will follow from the following Proposition which is proved using the method introduced in [5].

PROPOSITION 2 [5]. – *If u is a positive weak solution of (1) and, if we assume that $f(u) \in L^{1,\lambda}(B(0,1))$ for some $\lambda > 0$, then u is regular in $B(0,1/2)$.*

Proof of Proposition 2. – As before, let us decompose u as the sum of an harmonic function in $B(0,1)$ and the Poisson kernel v defined below.

$$v(z) = c_1 \int_{B(0,1)} f(u)(y)|y-z|^{-n+2}dy.$$

We can write, for some $a > 0$ (to be chosen later)

$$v(z) = c_1 \int_{B(0,1)} |y-z|^{2-n+a}(f(u)(y)|y-z|^{-a})dy.$$

As in the proof of Proposition 1, we denote by $\tilde{u}$ the function equal to u in $B(0,1)$ and equal to 0 outside the unit ball.

Let us assume that a satisfies $0 < a < \lambda$, we have for all $z \in B(0,1)$

$$\int_{B(0,1)} f(u)(y)|z-y|^{-a}dy = \int_{\mathbb{R}^n} f(\tilde{u})(y)|z-y|^{-a}dy$$
$$= \int_0^{+\infty} r^{-a}\left(\frac{d}{dr}\left(\int_{B(z,r)} f(\tilde{u})(y)dy\right)\right)dr.$$

Since, by assumption, $f(u) \in L^{1,\lambda}(B(0,1))$ (and therefore $f(\tilde{u}) \in L^{1,\lambda}(B(0,1))$, we see, after an integration by parts, that

$$\int_{B(0,1)} f(u)(y)|z-y|^{-a}dy = a\int_0^{+\infty} r^{-a-1}\left(\int_{B(z,r)} f(\tilde{u})(y)dy\right)dr. \quad (13)$$

We deduce from (13) that

$$\begin{aligned}
&\int_{B(0,1)} f(u)(y)|z-y|^{-a}dy\\
&\quad = a\left(\int_0^2 r^{-a-1}\left(\int_{B(z,r)} f(\tilde{u})(y)dy\right)dr\right.\\
&\qquad \left. + \int_2^{+\infty} r^{-a-1}\left(\int_{B(z,r)} f(\tilde{u})(y)dy\right)dr\right)\\
&\quad \le a||f||_{L^{1,\lambda}(B(0,1))}\left(\int_0^2 r^{-a-1+\lambda}dr + \int_2^{+\infty} r^{-a-1}2^{\lambda}dr\right)\\
&\quad \le c_{a,\lambda,n}||f||_{L^{1,\lambda}(B(0,1))},
\end{aligned}$$

where $c_{a,\lambda,n} > 0$ is some positive constant depending on a, λ and n. In the first inequality, we have used the fact that, by construction, $\tilde{f}((u))(y) = 0$ when $|y-z| > 2$. From now on, we assume that $0 < a < \lambda$. Given some $\alpha > 1$ (to be chosen later), we derive, using Hölder's inequality, the estimate

$$\begin{aligned}
v^{\alpha}(z) \le c_9&\left(\int_{B(0,1)} |z-y|^{(-n+2+a)\alpha-a} f(u)(y)dy\right)\\
&\times\left(\int_{B(0,1)} |z-y|^{-a} f(u)(y)dy\right)^{\alpha-1}.
\end{aligned} \quad (14)$$

So, if we choose a such that

$$\int_{B(0,2)} |z-y|^{(-n+2+a)\alpha-a}dz < +\infty,$$

we can conclude that $v \in L^{\alpha}(B(0,1))$. We can summarize the above computation in the following form:

For any $0 < a < \lambda$, if $\alpha < \frac{n-a}{n-a-2}$, then $v \in L^{\alpha}(B(0,1))$.

If we choose $a = \lambda/2$, we have just proved that $u \in L^{\alpha}(B(0,3/4))$ for some $\alpha > \frac{n}{n-2}$. This together with the fact that g is bounded imply that $f(u)$ belongs to $L^{\alpha\frac{(n-2)}{n}}(B(0,3/4))$, with some $\alpha(n-2)/n > 1$. Thus, by standard elliptic estimates and a classical bootstrap argument, we conclude that u is bounded in $B(0,1/2)$. This ends the proof of Proposition 2. $\square$

3. PROOF OF THEOREM 3

Let us notice that it is equivalent to prove the result for $g(\log(1+u))$ replaced by $g(\log(u))$, since the function $x \to \log(1+e^{x})$ is a diffeomorphism from $[0,+\infty)$ into itself.

LEMMA 4. – *Assume that $\tilde{g} \notin L^1(0,+\infty)$ and that $0 < \tilde{g}$ is bounded. Then there exists some bounded function $g \notin L^1(0,+\infty)$ satisfying for all $y > 0$*

$$\frac{2}{n-2}\tilde{g}(z(y)) = (n-2)g(y) + g'(y) - \frac{n}{2}\frac{g(y)^2}{G(y)}. \tag{15}$$

Where, be definition $z(y) \equiv (n-2)\Big(y + \frac{1}{2}\log G(y)\Big)$ and where

$$G(y) \equiv 1 + \int_0^y g(s)ds.$$

Proof of Lemma 4. – The existence of g can be seen as a shooting problem because, if we denote as above

$$G(x) = 1 + \int_0^x g(s)ds,$$

we find the ordinary differential equation that must be satisfied by G.

$$\begin{aligned} G''(y) + (n-2)G'(y) - \frac{n}{2}\frac{G'(y)^2}{G(y)} \\ = \frac{2}{n-2}\tilde{g}((n-2)\Big(y + \frac{1}{2}\log G(y))\Big). \end{aligned} \tag{16}$$

As we have assumed that $\tilde{g}$ is bounded, we see that independently of y, $G''(y) < 0$ it the following conditions are fulfilled

$$G'(y) = \frac{4}{(n-2)^2}||\tilde{g}||_{L^\infty} \qquad \text{and} \qquad G(y) > \frac{8n}{(n-2)^3}||\tilde{g}||_{L^\infty}.$$

Starting at $y = 0$ at the point

$$G'(0) = \frac{4}{(n-2)^2}||\tilde{g}||_{L^\infty} \qquad \text{and} \qquad G(0) > \max\left(1, \frac{8n}{(n-2)^3}||\tilde{g}||_{L^\infty}\right),$$

it is easy to see that, for all $y \geq 0$ we have $0 \leq G'(y) \leq \frac{4}{(n-2)^2}||\tilde{g}||_{L^\infty}$. It remains to show that $G(y) \to +\infty$ and this will prove that $G'(y) \notin L^1(1, +\infty)$.

We argue by contradiction and assume that $G'(y) \in L^1(1, +\infty)$. As G is increasing, it has a limit when $y \to +\infty$. Moreover, we have seen that $G'(y) \geq 0$ is bounded. Integrating the equation (16) between 0 and $y > 0$, we get

$$(G'(y) - G'(0)) + (n-2)(G(y) - G(0)) = \frac{n}{2}\int_0^y \frac{G'(s)^2}{G(s)} ds$$
$$= \frac{2}{n-2}\int_0^y \tilde{g}\left((n-2)\left(s + \frac{1}{2}\log G(s)\right)\right) ds.$$

Therefore, we can conclude that

$$\frac{2}{n-2}\int_0^y \tilde{g}\left((n-2)\left(s + \frac{1}{2}\log G(s)\right)\right) ds$$
$$\leq 2||G'||_{L^\infty} + (n-2)||G||_{L^\infty} + \frac{n}{2}\int_0^y \frac{G'(s)^2}{G(s)} ds. \tag{17}$$

The last integral is bounded when y goes to $+\infty$ since, for s large enough

$$\frac{G'(s)^2}{G(s)} \leq \frac{1}{2}(\lim_{+\infty} G(y))^{-1}||G'||_{L^\infty} G'(s)$$

which can be integrated over $\mathbb{R}^+$. So, we conclude that the left hand side of (17) is also bounded when y tends to $+\infty$. That is

$$\int_0^{+\infty} \tilde{g}\left((n-2)\left(s + \frac{1}{2}\log G(s)\right)\right) ds < +\infty.$$

For y large enough, $G(y)$ converges to some limit. Thus, always for y large enough, $y \to (n-2)\left(s + \frac{1}{2}\log G(s)\right)$ is a diffeomorphism. In particular, we find that

$$\int_0^{+\infty} \tilde{g}(s) ds < +\infty$$

which contradicts the assumption of the Lemma. □

The result of Theorem 3 will directly follow from the Lemma:

LEMMA 5. – *Some regular function $\tilde{g}$ being given. If we assume that there exists some positive, bounded function $g \notin L^1(0, +\infty)$ satisfying (15). Then there exists a positive weak solution to $-\Delta u = u^{\frac{n}{n-2}} \tilde{g}(\log(u))$ over the unit ball of $\mathbb{R}^n$, with a non removable singularity at the origin.*

Proof of Lemma 5. – With the notations of Lemma 4, we try the solution given by

$$u(x) = |x|^{2-n}(G(-\log|x|))^{\frac{2-n}{2}}.$$

On can check by direct computation that u is a positive solution of $-\Delta u = u^{\frac{n}{n-2}} \tilde{g}(\log(u))$ in $B(0,1) \setminus \{0\}$. In addition, $u(x)$ tends to $+\infty$ when $|x| \to 0$. The fact that u is a weak solution of our equation in the unit ball follows easily from *a priori* estimates for solutions of (16). More precisely, we need to show that

$$|x|^{n-1}|\nabla u|(x) \to 0$$

when $|x|$ goes to 0. Given the formula for u, this reduces to the proof of the following limits

$$G(-\log|x|) \to +\infty$$

and

$$g(-\log|x|)(G(-\log|x|))^{-\frac{n}{2}} \to 0,$$

when $|x|$ tends to 0. But these assertions are true since $g(y)$ is bounded and $G(y)$ tends to $+\infty$ as y tends to $+\infty$. This ends the proof of Lemma 5. $\square$

REFERENCES

[1] P. AVILES, On Isolated Singularities in Some Nonlinear Partial Differential Equations, *Indiana Univ. Math. Journal*, Vol. **35**, 1983, pp. 773-791.

[2] P. BENILAN, H. BREZIS and M. G. CRANDALL, A Semilinear Equation in $L^1(\mathbf{R}^n)$, *Ann. Scuola Norm. Sup. Pisa.*, Serie 4, Vol. **2**, 1975, pp. 523-555.

[3] M. GIAQUINTA, Multiple Integrals in the Calculus of Variations and Nonlinear Analysis, *Annals of Mathematical Studies*, Vol. **105**, Princeton Univ. Press, 1989.

[4] F. PACARD, Existence and Convergence of Positive Weak Solutions of $-\Delta u = u^{\frac{n}{n-2}}$ in Bounded Domains of $\mathbf{R}^n$, $n \geq 3$, *Jour. Calc. Variat.*, Vol. **1**, 1993, pp. 243-265.

[5] F. PACARD, *Convergence and Partial Regularity for Weak Solutions of Some Nonlinear Elliptic Equation: the Supercritical Case*, Preprint.

[6] G. STAMPACCHIA, Some Limit Case of L^p Estimates for Solution of Second Order Elliptic Equations, *Comm. Pure Applied Math.*, Vol. **XVI**, 1963, pp. 505-510.

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