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A characterization of families of function sets described by constraints on the gradient

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ABSTRACT. – One of the results proved in the paper deals with a characterization of the families $\{K(\Omega) : \Omega$ bounded open set$\}$ of subsets of $W_{loc}^{1,p}(\mathbb{R}^n)$ $(p \in [1, +\infty])$ whose elements can be described as $K(\Omega) = \{u \in W_{loc}^{1,p}(\mathbb{R}^n) : Du(x) \in C$ for a.e. x in $\Omega\}$, C being a closed convex subset of $\mathbb{R}^n$.

Characterizations similar to the above mentioned but for families of subsets of $BV_{loc}(\mathbb{R}^n)$ and $L_{loc}^p(\mathbb{R}^n)$ are also proved.

These results are deduced by a general characterization theorem for families of subsets of a Hausdorff locally convex topological vector subspace of $L_{loc}^1(\mathbb{R}^n)$.

An application to the problem of the homogenization of the elastic-plastic torsion of a cylindrical bar is given.

Key words: Differential inclusions, representation of families of sets.

RÉSUMÉ. – Un des résultats démontrés dans notre travail concerne une caractérisation des familles $\{K(\Omega) : \Omega$ ensemble ouvert borné$\}$ des sous-ensembles de $W_{loc}^{1,p}(\mathbb{R}^n)$ $(p \in [1, +\infty])$ dont les éléments peuvent être

décrits comme $K(\Omega) = \{u \in W^{1,p}_{loc}(\mathbb{R}^n) : Du(x) \in C$ pour presque chaque x en $\Omega\}$, C étant un sous-ensemble convexe fermé de $\mathbb{R}^n$.

On prouve aussi des caractérisations similaires pour des familles de sous-ensembles de $BV_{loc}(\mathbb{R}^n)$ et de $L^p_{loc}(\mathbb{R}^n)$.

Nos résultats sont dérivés par un théorème général de caractérisation pour des familles de sous-ensembles d'un sous-espace vectoriel topologique de Hausdorff localement convexe de $L^1_{loc}(\mathbb{R}^n)$.

On donne une application pour le problème d'homogénéisation de la torsion élastoplastique d'une barre cylindrique.

0. INTRODUCTION

Some problems in Calculus of Variations, for example the problem of the elastic-plastic torsion of a cylindrical bar (*cf.* [38], [39], [46], [53], [10] ÷ [12]) lead to the study of variational inequalities or to the minimization of integral functionals defined on sets of functions described by constraints on the gradient. Such sets are of the following type $\{u \in W^{1,p}_{loc}(\mathbb{R}^n) : |Du(x)| \leq \varphi(x)$ for a.e. x in $\Omega\}$, φ being a nonnegative function, Ω an open set and p in $[1, +\infty]$.

The study of the homogenization of the elastic-plastic torsion of a cylindrical bar (*cf.* [2], [5], [15] ÷ [21], [26], [30], [40]) analyzes, for every regular bounded open set Ω, the asymptotic behaviour of sequences of minimum problems for variational functionals defined on convex function sets of the type $\{u \in W^{1,p}_{loc}(\mathbb{R}^n) : |Du(x)| \leq \varphi(hx)$ for a.e. x in Ω, $h \in \mathbb{N}\}$, φ being a nonnegative function also 1-periodic in each variable.

According to a conjecture due to A. Bensoussan, J. L. Lions and G. Papanicolau (*cf.* [5]), such asymptotic behaviour is again described by a minimum problem for a variational functional defined on a set of the type

$$K(\Omega) = \{u \in W^{1,p}_{loc}(\mathbb{R}^n) : Du(x) \in C \quad \text{for a.e.} \quad x \text{ in } \Omega\}, \quad (0.1)$$

C being a suitable convex and closed subset of $\mathbb{R}^n$.

In this context some abstract characterizations of the families $\{K(\Omega) : \Omega$ bounded open set$\}$ of subsets of $W^{1,p}_{loc}(\mathbb{R}^n)$ whose elements can be described as in (0.1) at least for every regular bounded open set Ω seem to be interesting.

We remark that characterization results similar to the ones proved in the present paper, but relative to families of subsets of Sobolev spaces of the type $\{\{u \in W_{loc}^{1,p}(\mathbb{R}^n) : u(x) \geq \psi(x) \text{ for a.e. } x \text{ in } \Omega\} : \Omega \text{ bounded open set}\}$ have been proved in [2] and [3].

The characterization problem considered in the present paper can also be framed in the context of variational convergence theory (*cf.* [2] and [32] for a general exposition on the subject), in particular in the part of the theory that deals with integral representation results in Calculus of Variations (*cf.* [2], [7] ÷ [9], [13], [14], [22], [24], [27], [28], [32], [36], [39], [43], [51]). Indeed our results can also be read as furnishing characterizations of the functionals which can be described by means of variational integrals of the type $\int_\Omega \mathbb{I}_{\{x \in \Omega : Du(x) \in C\}}$ where for every subset E of $\mathbb{R}^n$ we have denoted by $\mathbb{I}_E$ the function defined by $\mathbb{I}_E(x) = 0$ if $x \in E$ and $\mathbb{I}_E(x) = +\infty$ if $x \in \mathbb{R}^n \backslash E$.

Nevertheless the consideration of extended real valued integrands seems to require techniques at least partially different from those utilized in the integral representation theory in Calculus of Variations.

It is straight away verified that, for every bounded open set Ω and every p in $[1, +\infty]$, sets of the type in (0.1) verify the following simple linearity properties

$$u \in K(\Omega), \quad c \in \mathbb{R} \Rightarrow u + c \in K(\Omega); \tag{0.2}$$

$$u \in K(\Omega), \quad x_0 \in \mathbb{R}^n \Rightarrow u(x_0 + \cdot) \in K(\Omega - x_0); \tag{0.3}$$

$$u \in K(\Omega), \quad t > 0 \Rightarrow \frac{1}{t} u(t\cdot) \in K\left(\frac{1}{t}\Omega\right); \tag{0.4}$$

$$K(\Omega) \text{ convex.} \tag{0.5}$$

Moreover, again for every bounded open set Ω, it is clear that the following locality property holds

$$u \in K(\Omega), \quad v \in W_{loc}^{1,p}(\mathbb{R}^n) : v = u \quad \text{a.e. on } \Omega \Rightarrow v \in K(\Omega) \tag{0.6}$$

and, by the closedness of C, it follows that

$$K(\Omega) \text{ is } W^{1,p}(\Omega)\text{-closed in } W_{loc}^{1,p}(\mathbb{R}^n). \tag{0.7}$$

About the dependence of K on the open set, it obviously results that

$$K(\Omega_2) \subseteq K(\Omega_1) \text{ for every couple of bounded open sets } \Omega_1, \Omega_2$$
$$\text{with } \Omega_1 \subseteq \Omega_2; \tag{0.8}$$

$$K\left(\Omega_1\right) \cap K\left(\Omega_2\right) \subseteq K\left(\Omega_1 \cup \Omega_2\right)$$

for every couple of bounded open sets Ω_1, Ω_2;   (0.9)

$$K\left(\Omega_1\right) \cap W_{loc}^{1,\infty}\left(\mathbb{R}^n\right) \subseteq K\left(\Omega_2\right)$$

for every couple of bounded open sets Ω_1, Ω_2

with $\Omega_1 \subseteq \Omega_2$ and meas $\left(\Omega_2 \backslash \Omega_1\right) = 0$.   (0.10)

On the other side, it is clear that, if $z \in C$, then the linear function u_z, defined by $u_z\left(x\right) = \langle z, x \rangle$ for every x in $\mathbb{R}^n$, verifies

$$u_z \in K\left(\Omega\right) \text{ for every bounded open set } \Omega,   (0.11)$$

hence the set C turns out to be given by

$$C = \left\{z \in \mathbb{R}^n : u_z \in K\left(\Omega\right) \text{ for every bounded open set } \Omega\right\}.   (0.12)$$

Then in order to characterize, at least for p in $[1, +\infty[$, the families of subsets of $W_{loc}^{1,p}\left(\mathbb{R}^n\right)$ than can be expressed as in (0.1), we start with a family $\left\{K\left(\Omega\right) : \Omega \text{ bounded open set}\right\}$ of subsets of $W_{loc}^{1,p}\left(\mathbb{R}^n\right)$ verifying, for every bounded open set Ω, conditions $(0.2) \div (0.10)$.

Later we define the set C by the formula (0.12) by using the above introduced family $\left\{K\left(\Omega\right) : \Omega \text{ bounded open set}\right\}$ and prove that C is closed and convex.

Finally we are able to prove that

$$K\left(\Omega\right) = \left\{u \in W_{loc}^{1,p}\left(\mathbb{R}^n\right) : Du\left(x\right) \in C \text{ for a.e. } x \text{ in } \Omega\right\}$$

for every bounded open set with Lipschitz boundary.   (0.13)

The same characterization results holds also when $p = +\infty$ provided that condition (0.7) is replaced by the following

$$K\left(\Omega\right) \text{ is weak}^* - W^{1,\infty}\left(\Omega\right) \text{ closed in } W_{loc}^{1,\infty}\left(\mathbb{R}^n\right).   (0.14)$$

Moreover we are able to obtain results of the same kind also for families of subsets of function spaces wider that $W_{loc}^{1,p}\left(\mathbb{R}^n\right)$.

For example we can prove that if $\left\{K\left(\Omega\right) : \Omega \text{ bounded open set}\right\}$ is a family of subsets of $BV_{loc}\left(\mathbb{R}^n\right)$ verifying conditions $(0.2) \div (0.5)$, $(0.8) \div (0.10)$ and the following ones that replace (0.6) and (0.7)

$$u \in K\left(\Omega\right), v \in BV_{loc}\left(\mathbb{R}^n\right) : v = u \quad \text{a.e. on } \Omega \Rightarrow v \in K\left(\Omega\right),   (0.15)$$

$$K\left(\Omega\right) \text{ is weak}^{*} - BV\left(\Omega\right) \text{ closed in } BV_{loc}\left(\mathbb{R}^{n}\right), \qquad (0.16)$$

then

$$K\left(\Omega\right) = \left\{ u \in BV_{loc}\left(\mathbb{R}^{n}\right) \ : \ \frac{1}{\text{meas}\left(A\right)} \int_{A} Du \in C \right.$$

$$\left. \text{for every open subset } A \text{ of } \Omega \right\}$$

for every bounded open set Ω with Lipschitz boundary, (0.17)

the set C being again given by (0.12).

Conversely, if C is a closed convex subset of $\mathbb{R}^{n}$ and if for every bounded open set Ω $K\left(\Omega\right)$ is given by (0.17), then the family $\{K\left(\Omega\right) \ : \ \Omega \text{ bounded open set}\}$ verifies conditions $(0.2) \div (0.5)$, $(0.8) \div (0.10)$, (0.15) and (0.16).

Furthermore, if $\{K\left(\Omega\right) \ : \ \Omega \text{ bounded open set}\}$ is a family of subsets of $L_{loc}^{p}\left(\mathbb{R}^{n}\right)$, with $p \in [1, +\infty]$, verifying conditions $(0.2) \div (0.5)$, $(0.8) \div (0.10)$ and the following ones that replace (0.6) and (0.7).

$$u \in K\left(\Omega\right), \ v \in L_{loc}^{p}\left(\mathbb{R}^{n}\right) \ : \ v = u \quad \text{a.e. on } \Omega \Rightarrow v \in K\left(\Omega\right), \qquad (0.18)$$

$$K\left(\Omega\right) \text{ is } L^{p}\left(\Omega\right) \text{ closed in } L_{loc}^{p}\left(\mathbb{R}^{n}\right) \text{ if } p \in [1, +\infty[$$

$$\text{and weak}^{*} - L^{\infty}\left(\Omega\right) \text{ closed in } L_{loc}^{\infty}\left(\mathbb{R}^{n}\right) \text{ if } p = +\infty, \qquad (0.19)$$

then

$$K\left(\Omega\right) = \left\{ u \in L_{loc}^{p}\left(\mathbb{R}^{n}\right) \ : \ -\int_{\Omega} uD\varphi \in C \right.$$

$$\left. \text{for every } \varphi \text{ in } C_{0}^{1}\left(\Omega\right) \text{ with } \varphi \geq 0, \int_{\Omega} \varphi = 1 \right\}$$

for every bounded open set Ω with Lipschitz boundary, (0.20)

the set C being again given by (0.12).

Conversely, if C is a closed convex subset of $\mathbb{R}^{n}$ and if for every bounded open set Ω $K\left(\Omega\right)$ is given by (0.20), then the family $\{K\left(\Omega\right) \ : \ \Omega \text{ bounded open set}\}$ verifies conditions $(0.2) \div (0.5)$, $(0.8) \div (0.10)$, (0.18) and (0.19).

Eventually, to characterize the families $\{K\left(\Omega\right) \ : \ \Omega \text{ bounded open set}\}$ as in (0.1) with C not only closed and convex but also with nonempty

interior it seems more natural to drop condition (0.10) and use a condition of topological richness, *i.e.*

$$K\left(\Omega_0\right) \cap C^1\left(\mathbb{R}^n\right) \text{ has interior points in the } C^1\left(\overline{\Omega}_0\right) \text{ topology}$$

$$\text{for some bounded open set } \Omega_0. \tag{0.21}$$

The characterization results for families of subsets of $W_{loc}^{1,\,p}\left(\mathbb{R}^n\right)$ are contained in Theorem 10.3, those for $BV_{loc}\left(\mathbb{R}^n\right)$ in Theorem 10.2 and those for $L_{loc}^{p}\left(\mathbb{R}^n\right)$ in Theorem 10.1. In Theorem 10.4 a characterization result for families in $C^1\left(\mathbb{R}^n\right)$ is also proved.

The above results are deduced as particular cases by general characterization theorems for families of subsets of a Hausdorff locally convex topological vector subspace of $L_{loc}^{1}\left(\mathbb{R}^n\right)$ (theorems 1.1 and 1.2).

An application of some of the above results to the problem of the homogenization of the elastic-plastic torsion of a cylindrical bar is also given in section 11.

In conclusion we observe that in the present paper only the case in which the convex set C is a fixed one is treated. Much more delicate seems to be the study of the characterization of the families of function sets in which the constraint C can depend on the x variable.

1. MAIN RESULTS

In the present section we describe the main results of this paper and give a brief account of the leading ideas of their proofs.

Let us first fix some notations.

For every function u on $\mathbb{R}^n$, y in $\mathbb{R}^n$ and $t > 0$ we denote by $T\left[y\right]u$ the function defined by

$$\left(T\left[y\right]u\right)\left(x\right) = u\left(y + x\right), \qquad x \in \mathbb{R}^n \tag{1.1}$$

and by $O_t u$ the one given by

$$\left(O_t u\right)\left(x\right) = \frac{1}{t}\,u\left(tx\right), \qquad x \in \mathbb{R}^n. \tag{1.2}$$

We also set $W_{loc}^{1,\,\infty} = W_{loc}^{1,\,\infty}\left(\mathbb{R}^n\right)$, $C^1 = C^1\left(\mathbb{R}^n\right)$ and, for every bounded open set Ω, denote by $W^{1,\,\infty}\left(\Omega\right)$ the strong topology of $W^{1,\,\infty}\left(\Omega\right)$ and by $C^1\left(\Omega\right)$ the strong one of $C^1\left(\overline{\Omega}\right)$.

Finally for every Lebesgue measurable subset E of $\mathbb{R}^n$ we denote by $|E|$ its measure.

Let U be a vector subspace of $L^1_{loc}(\mathbb{R}^n)$ verifying

$$u \in U, \ c \in \mathbb{R} \Rightarrow u + c \in U; \tag{1.3}$$

$$u \in U, \ y \in \mathbb{R}^n \Rightarrow T[y]u \in U; \tag{1.4}$$

$$u \in U, \ t > 0 \Rightarrow O_t u \in U; \tag{1.5}$$

$$W^{1,\infty}_{loc} \subseteq U \tag{1.6}$$

and let us consider, for every bounded open set Ω, a topology $\tau(\Omega)$ on U satisfying

$(U, \tau(\Omega))$ is a Hausdforff locally convex topological vector space; (1.7)

$$\left.\begin{array}{l} i) \ \tau(\Omega) \text{ is less fine than } W^{1,\infty}(\Omega) \text{ on } W^{1,\infty}_{loc} \\ ii) \ \tau(\Omega) \text{ is finer than weak-}L^1(\Omega); \end{array}\right\} \tag{1.8}$$

$$\Omega_1 \subseteq \Omega_2 \Rightarrow \tau(\Omega_1) \text{ less fine than } \tau(\Omega_2). \tag{1.9}$$

Moreover we also assume that

for every $x \in \mathbb{R}^n$, $v \in U$ and every bounded open set Ω we have
  i) the function $y \in \mathbb{R}^n \mapsto T[y]v \in U$ is continuous from $\mathbb{R}^n$ endowed with the usual topology to U endowed with the $\tau(\Omega)$ one,
  ii) the function $u \in U \mapsto T[x]u \in U$ is continuous from U endowed with the $\tau(\Omega)$ topology to U endowed with the $\tau(\Omega\text{-}x)$ one, (1.10)
that

for every $u \in U$ and every bounded open set Ω star-shaped with

respect to 0 the function $t \in \,]0, +\infty[\, \mapsto O_t u \in U$

is continuous in $\tau(\Omega)$ (1.11)

and that, if for every $u \in U$ and $\varepsilon > 0$ u_ε denotes the regularization of u,

for every $u \in U$ and every bounded open set Ω

$$u_\varepsilon \to u \text{ in } \tau(\Omega) \text{ as } \varepsilon \to 0^+. \tag{1.12}$$

For every bounded open set Ω we consider a subset $K\,(U,\,\Omega)$ of U satisfying the following conditions

$$u \in K\,(U,\,\Omega), \quad c \in \mathbb{R} \Rightarrow u + c \in K\,(U,\,\Omega); \tag{1.13}$$

$$u \in K\,(U,\,\Omega), \quad y \in \mathbb{R}^n \Rightarrow T[y]\,u \in K\,(U,\,\Omega - y); \tag{1.14}$$

$$u \in K\,(U,\,\Omega), \quad t > 0 \Rightarrow O_t\,u \in K\left(U,\,\frac{1}{t}\,\Omega\right); \tag{1.15}$$

$$u \in K\,(U,\,\Omega), \quad v \in U\,:\,v = u \quad \text{a.e. in } \Omega \Rightarrow v \in K\,(U,\,\Omega); \tag{1.16}$$

$$K\,(U,\,\Omega) \text{ convex}; \tag{1.17}$$

$$K\,(U,\,\Omega) \text{ is } \tau\,(\Omega)\text{-closed in } U \tag{1.18}$$

and

$$\Omega_1 \subseteq \Omega_2 \Rightarrow K\,(U,\,\Omega_2) \subseteq K\,(U,\,\Omega_1); \tag{1.19}$$

$$K\,(U,\,\Omega_1) \cap K\,(U,\,\Omega_2) \subseteq K\,(U,\,\Omega_1 \cup \Omega_2)$$
for every couple of bounded open sets $\Omega_1,\,\Omega_2;$ $\tag{1.20}$

$$K\,(U,\,\Omega_1) \cap W_{loc}^{1,\,\infty} \subseteq K\,(U,\,\Omega_2)$$
for every couple of bounded open sets $\Omega_1,\,\Omega_2$
with $\Omega_1 \subseteq \Omega_2$ and $|\Omega_2 \backslash \Omega_1| = 0.$ $\tag{1.21}$

Furthermore let us defined the set C by

$$C = \{z \in \mathbb{R}^n\,:\,u_z \in K\,(U,\,\Omega) \text{ for every bounded open set } \Omega\} \tag{1.22}$$

and, for every bounded open set Ω, the set $K_C\,(U,\,\Omega)$ by

$$K_C\,(U,\,\Omega) = \left\{u \in U\,:\,-\int_\Omega u D\varphi \in C \right.$$
$$\left. \text{for every } \varphi \in C_0^1\,(\Omega) \text{ with } \varphi \geq 0,\,\int_\Omega \varphi = 1\right\}. \tag{1.23}$$

The following characterization result holds.

THEOREM 1.1. – *Let U be a vector subspace of L^1_{loc} verifying $(1.3) \div (1.6)$ and let, for every bounded open set Ω, $\tau(\Omega)$ be a topology on U satisfying $(1.7) \div (1.12)$.*

For every bounded open set Ω let $K(U, \Omega)$ be a subset of U verifying $(1.13) \div (1.21)$, let C be defined by (1.22) and $K_C(U, \Omega)$ by (1.23).

Then C is closed, convex and

$$K(U, \Omega) = K_C(U, \Omega) \; for \; every \; bounded \; open \; set \; \Omega \; with \; Lipschitz$$

$$boundary \; such \; that \; (U, \tau(\Omega)) \; is \; sequentially \; complete. \tag{1.24}$$

Conversely, given a closed convex subset C of $\mathbb{R}^n$ it turns out that conditions $(1.13) \div (1.21)$ are satisfied by the family $\{K(U, \Omega) : \Omega \; bounded \; open \; set\}$ whose elements are defined for every bounded open set Ω by $K(U, \Omega) = K_C(U, \Omega)$.

A different version of Theorem 1.1 is given by the following result in which conditions (1.6), (1.8) (i) and (1.21) are replaced respectively by

$$C^1 \subseteq U; \tag{1.25}$$

for every bounded open set Ω $\tau(\Omega)$ is less fine than $C^1(\Omega)$ on C^1 (1.26)

and

$$K(U, \Omega_0) \cap C^1 \; has \; nonempty \; interior \; in \; the \; C^1(\Omega_0)$$

$$topology \; for \; some \; bounded \; open \; set \; \Omega_0. \tag{1.27}$$

THEOREM 1.2. – *Let U be a vector subspace of L^1_{loc} verifying $(1.3) \div (1.5)$, (1.25) and let, for every bounded open set Ω, $\tau(\Omega)$ be a topology on U satisfying (1.7), (1.8) (ii), $(1.9) \div (1.12)$ and (1.26).*

For every bounded open set Ω let $K(U, \Omega)$ be a subset of U verifying $(1.13) \div (1.20)$ and (1.27), let C be defined by (1.22) and $K_C(U, \Omega)$ by (1.23).

Then C is closed, convex, has nonempty interior and

$$K(U, \Omega) = K_C(U, \Omega)$$

$$for \; every \; bounded \; open \; set \; \Omega \; with \; Lipschitz$$

$$boundary \; such \; that \; (U, \tau(\Omega)) \; is \; sequentially \; complete. \tag{1.28}$$

Conversely, given a closed convex subset C of $\mathbb{R}^n$ with nonempty interior it turns out that conditions (1.13) ÷ (1.20) *and* (1.27) *are satisfied by the family* $\{K\,(U,\,\Omega)\;:\;\Omega\text{ bounded open set}\}$ *whose elements are defined for every bounded open set* Ω *by* $K\,(U,\,\Omega) = K_C\,(U,\,\Omega)$.

Theorems 1.1 and 1.2 will be deduced in some steps according to the following plan.

In sections 2 and 3 we fix some notations, recall some definitions and the related properties and prove some preliminary results.

In sections 4 and 5, given a closed convex subset C of $\mathbb{R}^n$, we study the properties of the sets in (1.23).

In section 6 we prove a general result yielding sufficient conditions under which $K\,(U,\,\Omega) = \bigcap\limits_{\overline{A} \subset \Omega} K\,(U,\,A)$ (this result plays the same crucial role of the inner regularity ones in the integral representation theory (*cf.* [14], [35]).

Section 7 is devoted to the proof of the properties of the set C in (1.22).

In section 8 we prove representation results for $K\,(U,\,\Omega) \cap C^1$ first under the assumptions of Theorem 1.1 and then under those of Theorem 1.2. The main ideas to achieve these results are a "blow up" argument and an approximation procedure of continuously differentiable functions by piecewise affine functions.

In section 9 theorems 1.1 and 1.2 are proved by using an approximation procedure and a representation result for the regularization of a function.

In section 10 we specialize theorems 1.1 and 1.2 to the case of the most common function spaces in mathematical analysis and in section 11 we give an application of the results obtained.

2. GENERAL NOTATIONS

Let Ω be a bounded open set. We denote by $W^{1,\,\infty}\,(\Omega)$ the set of the functions in $L^\infty\,(\Omega)$ having distributional partial derivatives in $L^\infty\,(\Omega)$.

$W^{1,\,\infty}\,(\Omega)$ is naturally endowed with the topology induced by the norm $\|u\|_{W^{1,\,\infty}\,(\Omega)} = \|u\|_{L^\infty\,(\Omega)} + \|\,|Du|\,\|_{L^\infty\,(\Omega)}$.

Let us now observe that the map $u \in W^{1,\,\infty}\,(\Omega) \to (u,\,D_1 u,\,\ldots,\,D_n u) \in (L^\infty\,(\Omega))^{n+1}$ allows us to identify $W^{1,\,\infty}\,(\Omega)$ with a closed subspace of $(L^\infty\,(\Omega))^{n+1}$, hence, being $(L^\infty\,(\Omega))^{n+1}$ the dual space of $(L^1\,(\Omega))^{n+1}$, we define the weak*-$W^{1,\,\infty}\,(\Omega)$ topology $(w^*\text{-}W^{1,\,\infty}\,(\Omega))$ as the natural relative topology generated by the $(w^*\text{-}L^\infty\,(\Omega))^{n+1}$ one.

In particular, given a generalized sequence $\{u_\lambda\}_{\lambda \in \Lambda}$ and u in $W^{1,\infty}(\Omega)$ it turns out that $\{u_\lambda\}_{\lambda \in \Lambda}$ converges to u in $w^*\text{-}W^{1,\infty}(\Omega)$ if and only if

$$\lim_\lambda \left[\int_\Omega u_\lambda \psi_0 + \sum_{i=1}^n \int_\Omega D_i u_\lambda \, \psi_i \right] = \int_\Omega u\psi_0 + \sum_{i=1}^n \int_\Omega D_i u_\lambda \, \psi_i$$

$$\text{for every } (\psi_0, \psi_1, \ldots, \psi_n) \in (L^1(\Omega))^{n+1}.$$

For every Borel subset B of $\mathbb{R}^n$ we denote by $\mathcal{M}(B)$ the set of the regular, countably additive set functions defined on the Borel subset of B and with finite total variation on B. For every $\mu \in \mathcal{M}(B)$ we denote by $|\mu|$ the total variation of μ.

Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\mu \in \mathcal{M}(\Omega)$, then, being $|\mu|$ a regular measure (*cf.* [37] III Lemma 4.7 and III Lemma 5.12), by Riesz representation theorem (*cf.* [37] IV Theorem 6.3) we have

$$|\mu|(\Omega) = \sup\{|\mu|(K) : K \text{ compact}, K \subseteq \Omega\}$$

$$= \sup\left\{ \sup\left\{ \int_K \varphi \, d\mu : \varphi \in C^0(K), \|\varphi\|_{C^0(K)} \le 1 \right\}\right.$$

$$\left. : K \text{ compact}, K \subseteq \Omega \right\}. \tag{2.1}$$

Let us now observe that

$$\sup\left\{ \sup\left\{ \int_K \varphi \, d\mu : \varphi \in C^0(K), \|\varphi\|_{C^0(K)} \le 1 \right\}\right.$$

$$\left. : K \text{ compact}, K \subseteq \Omega \right\}$$

$$= \sup\left\{ \int_\Omega \varphi \, d\mu : \varphi \in C_0^0(\Omega), \|\varphi\|_{C^0(\overline{\Omega})} \le 1 \right\}, \tag{2.2}$$

hence by (2.1) and (2.2) we obtain

$$|\mu|(\Omega) = \sup\left\{ \int_\Omega \varphi \, d\mu : \varphi \in C_0^0(\Omega), \|\varphi\|_{C^0(\overline{\Omega})} \le 1 \right\}$$

$$\text{for every bounded open set } \Omega, \mu \text{ in } \mathcal{M}(\Omega). \tag{2.3}$$

For every bounded open set Ω we denote by $C_0(\Omega)$ the space of the functions u in $C^0(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$ endowed with the $C^0(\overline{\Omega})$ topology.

We recall the following result (*cf.* Theorem 6.19 in [49]).

PROPOSITION 2.1. – *Let Ω be a bounded open set, then the spaces $(C_0(\Omega))^*$ and $\mathcal{M}(\Omega)$ are isomorphic.*

By $BV(\Omega)$ we denote the set of the functions in $L^1(\Omega)$ having distributional partial derivatives in $\mathcal{M}(\Omega)$, $BV(\Omega)$ is naturally endowed with the topology induced by the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |Du|$ where $\int_\Omega |Du|$ denotes the total variation of the vector measure Du on Ω and is given by

$$\int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} g \, : \, g \in (C_0^1(\Omega))^n, \, |g(x)| \leq 1 \right.$$
$$\left. \text{for every } x \text{ in } \Omega \right\}. \tag{2.4}$$

The map $u \in BV(\Omega) \rightarrow \left(\int_{(\bullet)} u \, dx, \, D_1 u, \ldots, D_n u \right) \in (\mathcal{M}(\Omega))^{n+1}$ allows us to identify $BV(\Omega)$ with a subspace of $(\mathcal{M}(\Omega))^{n+1}$, hence, being by Proposition 2.1 $(\mathcal{M}(\Omega))^{n+1}$ the dual space of $(C_0(\Omega))^{n+1}$, we define the weak*-$BV(\Omega)$ topology $(w^*\text{-}BV(\Omega))$ as the natural relative topology of $BV(\Omega)$ generated by the $(w^*\text{-}\mathcal{M}(\Omega))^{n+1}$ one.

In particular, given a generalized sequence $\{u_\lambda\}_{\lambda \in \Lambda}$ and u in $BV(\Omega)$ it turns out that $\{u_\lambda\}_{\lambda \in \Lambda}$ converges to u in $w^*\text{-}BV(\Omega)$ if and only if

$$\lim_\lambda \left[\int_\Omega u_\lambda \, \psi_0 + \sum_{i=1}^n \int_\Omega D_i u_\lambda \, \psi_i \right] = \int_\Omega \psi_0 u + \sum_{i=1}^n \int_\Omega \psi_i D_i u$$
$$\text{for every } (\psi_0, \, \psi_1, \ldots, \, \psi_n) \in (C_0(\Omega))^{n+1}.$$

For a wide exposition about BV functions we refer to [41], here we only recall that, for every bounded open set Ω, $BV(\Omega)$ compactly embeds in $L^1_{loc}(\Omega)$, that $BV(\Omega)$ endowed with its $w^*\text{-}BV(\Omega)$ topology is sequentially complete and that each function in $BV(\Omega)$ can be extended to a function in $BV(\mathbb{R}^n)$ provided Ω has Lipschitz boundary.

By BV_{loc} we denote the set of functions on $\mathbb{R}^n$ that belong to $BV(A)$ for every bounded open set A of $\mathbb{R}^n$.

Analogously for every $p \in [1, +\infty]$ we set $W_{loc}^{1,p} = W_{loc}^{1,p}(\mathbb{R}^n)$ and $L_{loc}^p = L_{loc}^p(\mathbb{R}^n)$.

For every open set Ω and p in $[1, +\infty]$ we denote by $L^p(\Omega)$, $w - L^p(\Omega)$, $W^{1,p}(\Omega)$ respectively the strong and the weak topology of $L^p(\Omega)$, the strong topology of $W^{1,p}(\Omega)$ and by $C^0(\Omega)$ the usual strong topology on $C^0(\overline{\Omega})$.

Given an open set Ω and a function u on Ω we denote by $\underset{\Omega}{\text{osc}}\ u$ the oscillation of u on Ω, i.e. the value $\underset{x, y \in \Omega}{\sup}\ |u(x) - u(y)|$.

Given a topological space (U, τ) and a subset X of U, by the symbol $(U, \tau) - cl(X)$ we mean the closure of X in U i.e. the set of the points u in U for which there exists a generalized sequence $\{u_\lambda\}_{\lambda \in \Lambda} \subseteq X$ such that $\{u_\lambda\}_{\lambda \in \Lambda}$ converges to u.

Given a vector space V and a subset S of V we denote by $\text{conv}(S)$ the convex envelope of S, i.e. the set of the finite convex combinations of points of S.

Moreover, given a convex subset C of $\mathbb{R}^n$ with $0 \in C$, we denote by $\Sigma(C)$ the vector space generated by C, that is the set of the finite linear combinations of the elements of C; obviously $\Sigma(C)$ turns out to be the smallest vector subspace of $\mathbb{R}^n$ containing C.

Given a subset E of $\mathbb{R}^n$, we denote by χ_E the characteristic function of E defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

Let us recall that a polyhedron is a finite intersection of half spaces.

A function u defined on $\mathbb{R}^n$ is said to be piecewise affine on $\mathbb{R}^n$ if it is continuous and can be expressed as

$$u(x) = \sum_{j=1}^m \left(u_{z_j}(x) + s_j\right) \chi_{P_j}(x) \qquad x \in \mathbb{R}^n$$

where $z_1, \ldots, z_m \in \mathbb{R}^n$, $s_1, \ldots, s_m \in \mathbb{R}$ and $P_1, \ldots, P_m$ are pairwise disjoint polyhedrons of $\mathbb{R}^n$ with nonempty interiors such that $\bigcup_{j=1}^m P_j = \mathbb{R}^n$.

We now recall the concept of integral of a function taking values in a Hausdorff locally convex topological vector space (cf. [48]).

Let (U, τ) be a Hausdorff locally convex topological vector space and let $\{p_a\}_{a \in \mathcal{A}}$ be a family of seminorms generating the topology τ of U.

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and let f be a function from E to U.

DEFINITION 2.2. – *The function f is said to be τ-integrable on E if there exists $u \in U$ such that for every $\eta > 0$ and $a \in \mathcal{A}$ there exists a partition*

$\Delta_{\eta, a} = \{B_{\eta, a, j}\}_{j=1,\ldots, m}$ of E *into measurable sets such that*

$$sup \left\{ p_a \left(\sum_{j=1}^{m} f(y_j)|B_{\eta, a, j}| - u \right) \right.$$

$$\left. : y_j \in B_{\eta, a, j}, \ j \in \{1, \ldots, m\} \right\} < \eta. \tag{2.5}$$

The vector u is the value of the integral of f on E and is denoted by $\int_E f(y) \, dy$.

By Definition 2.2 it is obvious that if f is τ-integrable on E and if σ is another topology on U less fine than τ, then f is also σ-integrable on E.

We recall the following result about integrals of vector valued functions (*cf.* Corollary 5.2 in [48]).

THEOREM 2.3. – *Let $f : E \to U$ be τ-integrable on E and let $L \in U^*$.* *Then $\langle L, f \rangle$ is Lebesgue summable on E and* $\int_E \langle L, f(y) \rangle \, dy = \left\langle L, \int_E f(y) \, dy \right\rangle$.

For sake of completeness we now briefly recall the following integrability condition.

PROPOSITION 2.4. – *Let E be a Lebesgue measurable subset of $\mathbb{R}^n$, let (U, τ) be a sequentially complete Hausdorff locally convex topological vector space and let $f : E \to U$ be continuous with compact support.*

Then f is τ-integrable on E.

Proof. – Let us first observe that, being f continuous and $\mathrm{spt}(f)$ compact, f is uniformly continuous in the following way

$$\text{for every } \eta > 0, \ a \in \mathcal{A} \text{ there exists } \delta > 0$$

$$\text{such that } |x - y| < \delta \Rightarrow p_a(f(x) - f(y)) < \eta. \tag{2.6}$$

For every $h \in \mathbb{N}$ let $\mathcal{R}^h = \{Q_j^h\}_{j \in \mathbb{N}}$ be a partition of $\mathbb{R}^n$ into half open cubes with sidelenght $\dfrac{1}{h}$ and let, for every $j \in \mathbb{N}$, $E_j^h = E \cap Q_j^h$.

Since f has compact support it is not restrictive to assume that E is bounded, hence for every $h \in \mathbb{N}$ there exists $m_h \in \mathbb{N}$ such that $E_j^h \neq \varnothing$ if and only if $j \in \{1, \ldots, m_h\}$.

For every $j \in \{1, \ldots, m_h\}$ let $y_j^h \in E_j^h$ and define $u_h = \sum_{j=1}^{m_h} f(y_j^h)|E_j^h|$.

We prove that $\{u_h\}_h$ is a Cauchy sequence.

To this aim let $\eta > 0$, $a \in \mathcal{A}$ and let δ be given by (2.6). Let $\nu \in \mathbb{N}$ be such that $\dfrac{1}{\nu} < \dfrac{\delta}{2\sqrt{n}}$ then for every h, $k > \nu$ it results

$$
\begin{aligned}
p_a(u_h - u_k) &= p_a\left(\sum_{i=1}^{m_h} f(y_i^h)|E_i^h| - \sum_{j=1}^{m_k} f(y_j^k)|E_j^k|\right) \\
&= p_a\left(\sum_{i=1}^{m_h} f(y_i^h) \sum_{j=1}^{m_k} |E_i^h \cap E_j^k| \right. \\
&\qquad\qquad \left. - \sum_{j=1}^{m_k} f(y_j^k) \sum_{i=1}^{m_h} |E_j^k \cap E_i^h|\right) \\
&\leq \sum_{i=1}^{m_h} \sum_{j=1}^{m_h} |E_i^h \cap E_j^k| \, p_a(f(y_i^h) - f(y_j^k)). \quad (2.7)
\end{aligned}
$$

Let us now observe that if $E_i^h \cap E_j^k \neq \varnothing$ then $|y_i^h - y_j^k| < \delta$, hence by (2.6) we deduce

$$
p_a(u_h - u_k) \leq \sum_{i=1}^{m_h} \sum_{j=1}^{m_k} |E_i^h \cap E_j^k| \, \eta = |E|\eta \quad \text{for every } h, \ k > \nu, \quad (2.8)
$$

hence $\{u_h\}_h$ is a Cauchy sequence.

Since (U, τ) is sequentially complete there exists $u \in U$ such that $u_h \to u$ and obviously $u = \displaystyle\int_E f(y)\, dy$. ∎

3. PRELIMINARY RESULTS

For every $r > 0$ we set

$$
B_r = \{x \in \mathbb{R}^n \ : \ |x| < r\}. \quad (3.1)
$$

Let α be a symmetric mollifier, i.e. a nonnegative function in $C^\infty(\mathbb{R}^n)$ such that $\mathrm{spt}\,(\alpha) \subseteq B_1$, $\displaystyle\int_{\mathbb{R}^n} \alpha = 1$ and $\alpha(-x) = \alpha(x)$ for every x in $\mathbb{R}^n$.

For every $\varepsilon > 0$ let us set $\alpha_\varepsilon(x) = \varepsilon^{-n}\,\alpha\,(x/\varepsilon)$, $x \in \mathbb{R}^n$, and define, for every u in L^1_{loc} the regularization u_ε of u by

$$u_\varepsilon(x) = (\alpha_\varepsilon \star u)(x) = \int_{\mathbb{R}^n} \alpha_\varepsilon(x-y)\,u(y)\,dy \qquad x \in \mathbb{R}^n. \qquad (3.2)$$

The next result yields a description of the regularization of a function in terms of the integral of its translated.

PROPOSITION 3.1. – *Let Ω be an open subset of $\mathbb{R}^n$ and let α be the mollifier appearing in (3.2).*

Then for every $u \in L^1_{loc}$, $\varepsilon > 0$, the function $\alpha\,(\cdot)\,T\,[\varepsilon\cdot]\,u$ is $L^1\,(\Omega)$-integrable on $\mathbb{R}^n$ and

$$\left(\int_{\mathbb{R}^n} \alpha(y)\,T\,[\varepsilon y]\,u\,dy\right)(x) = u_\varepsilon(x) \quad \text{for a.e. } x \text{ in } \Omega. \qquad (3.3)$$

Proof. – Let $\varepsilon > 0$, $u \in L^1_{loc}$.

Let us observe that the space L^1_{loc} endowed with the $L^1(\Omega)$ topology is a Banach space, then by Proposition 2.4 applied with $(U,\,\tau)$ equal to $(L^1_{loc},\,L^1(\Omega))$ the integral $\int_{\mathbb{R}^n} \alpha(y)\,T\,[\varepsilon y]\,u\,dy$ exists and is in L^1_{loc}.

For every v^* in $L^\infty(\Omega)$ the functional $v \in L^1_{loc} \to \int_\Omega v^*(x)\,v(x)\,dx$ is linear and continuous on $(L^1_{loc},\,L^1(\Omega))$, hence by Theorem 2.3 applied with $(U,\,\tau)$ equal to $(L^1_{loc},\,L^1(\Omega))$ we deduce

$$\int_\Omega v^*(x)\left(\int_{\mathbb{R}^n} \alpha(y)\,T\,[\varepsilon y]\,u\,dy\right)(x)\,dx$$

$$= \int_{\mathbb{R}^n}\int_\Omega v^*(x)\,\alpha(y)\,(T\,[\varepsilon y]\,u)(x)\,dx\,dy$$

$$= \int_{\mathbb{R}^n} \alpha(y)\int_\Omega v^*(x)\,u(x+\varepsilon y)\,dx\,dy$$

$$= \int_\Omega v^*(x)\int_{\mathbb{R}^n} \alpha(y)\,u(x+\varepsilon y)\,dy\,dx$$

$$= \int_\Omega v^*(x)\,u_\varepsilon(x)\,dx \quad \text{for every } v^* \in L^\infty(\Omega). \qquad (3.4)$$

By (3.4) equality (3.3) soon follows. ∎

About the left hand side of (3.3) the following result holds.

PROPOSITION 3.2. – *Let $(U,\,\tau)$ be a Hausdorff locally convex topological vector subspace of L^1_{loc} verifying (1.4).*

Let α be the mollifier appearing in (3.2), then if for every $u \in U$, $\varepsilon > 0$ the function $\alpha\,(\cdot)\,T\,[\varepsilon\cdot]\,u$ is τ-integrable on $\mathbb{R}^n$ it results

$$\int_{\mathbb{R}^n} \alpha\,(y)\,T\,[\varepsilon y]\,u\,dy \in (U,\,\tau)\text{-}cl\,(conv\,(\{T\,[\varepsilon y]\,u,\,y \in B_1\}))$$

for every $u \in U$, $\varepsilon > 0$. (3.5)

Proof. – Let $u \in U$, $\varepsilon > 0$. Let us denote by H_u^ε the right hand side of (3.5).

It is well known that, due to separation theorems for closed convex sets, there exists a family $\mathcal{F}_u^\varepsilon = \{(v_\sigma^*,\,a_\sigma),\,\sigma \in S\} \subseteq U^* \times \mathbb{R}$ such that

$$v \in H_u^\varepsilon \Leftrightarrow \langle v_\sigma^*,\,v \rangle + a_\sigma \geq 0 \quad \text{for every } \sigma \in S. \tag{3.6}$$

Let $y \in B_1$, then obviously by (3.6) we deduce

$$\langle v_\sigma^*,\,T\,[\varepsilon y]\,u \rangle + a_\sigma \geq 0 \quad \text{for every } \sigma \in S, \tag{3.7}$$

therefore by multiplying both sides of (3.7) by $\alpha\,(y)$ and then integrating over $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} \alpha\,(y)\,\langle v_\sigma^*,\,T\,[\varepsilon y]\,u \rangle\,dy + a_\sigma \geq 0 \quad \text{for every } \sigma \in S. \tag{3.8}$$

By (3.8) and Theorem 2.3 we infer

$$- a_\sigma \leq \int_{\mathbb{R}^n} \alpha\,(y)\,\langle v_\sigma^*,\,T\,[\varepsilon y]\,u \rangle\,dy = \left\langle v_\sigma^*,\,\int_{\mathbb{R}^n} \alpha\,(y)\,T\,[\varepsilon y]\,u\,dy \right\rangle$$

for every $\sigma \in S$, (3.9)

hence by (3.9) and (3.6) the thesis follows. ∎

For every bounded open set Ω of $\mathbb{R}^n$ and $\varepsilon > 0$ let us define the sets Ω_ε^- and Ω_ε^+ as

$$\left.\begin{array}{l} \Omega_\varepsilon^- = \{x \in \Omega \,:\, \text{dist}\,(x,\,\partial\Omega) > \varepsilon\}, \\ \Omega_\varepsilon^+ = \{x \in \mathbb{R}^n \,:\, \text{dist}\,(x,\,\Omega) < \varepsilon\}. \end{array}\right\} \tag{3.10}$$

The following result holds.

PROPOSITION 3.3. – *Let Ω be an open set of $\mathbb{R}^n$, C be a convex set of $\mathbb{R}^n$ and let ψ be a function in $(L_{loc}^1)^n$ such that*

$$\psi\,(x) \in C \quad \text{for a.e. } x \text{ in } \Omega. \tag{3.11}$$

Then for every $\varepsilon > 0$ *it results*

$$\psi_\varepsilon(x) \in \overline{C} \quad \text{for every } x \text{ in } \Omega_\varepsilon^-. \tag{3.12}$$

Proof. – Since C is a convex set so is also $\overline{C}$, hence there exist two families $\{\beta_\sigma\}_{\sigma \in S} \subseteq \mathbb{R}^n$ and $\{\gamma_\sigma\}_{\sigma \in S} \subseteq \mathbb{R}$ such that

$$\overline{C} = \{z \in \mathbb{R}^n : \langle \beta_\sigma, z \rangle + \gamma_\sigma \geq 0 \quad \text{for every } \sigma \in S\}. \tag{3.13}$$

By (3.11) and (3.13) it follows that

$$\langle \beta_\sigma, \psi(y) \rangle + \gamma_\sigma \geq 0 \quad \text{for every } \sigma \in S \text{ and a.e. } y \text{ in } \Omega, \tag{3.14}$$

therefore by (3.14) it soon follows that

$$\left\langle \beta_\sigma, \int_{\mathbb{R}^n} \alpha_\varepsilon(x - y) \, \psi(y) \, dy \right\rangle + \gamma_\sigma$$
$$= \int_{\mathbb{R}^n} \alpha_\varepsilon(x - y) \left(\langle \beta_\sigma, \psi(y) \rangle + \gamma_\sigma \right) dy \geq 0$$
$$\text{for every } \sigma \in S \text{ and every } x \text{ in } \Omega_\varepsilon^-. \tag{3.15}$$

By (3.15) and (3.13) (3.12) soon follows. ∎

Let us now prove the following result.

PROPOSITION 3.4. – *Let G be a convex subset of $\mathbb{R}^m$ such that $\partial G \neq \varnothing$ and $B_r \subseteq \overset{\circ}{G}$ for some $r > 0$.*

Let $t \in]0, 1[$, then

$$dist(tG, \partial G) \geq r(1 - t). \tag{3.16}$$

Proof. – Let $y \in G$, then (see for example [37] page 413)

$$ty + (1 - t) x \in \overset{\circ}{G} \quad \text{for every } x \text{ in } B_r, \tag{3.17}$$

hence, by (3.17), it soon follows that

$$ty + B_{r(1-t)} \subseteq \overset{\circ}{G}. \tag{3.18}$$

In conclusion by (3.18) it follows that the distance of every point ty of tG from ∂G is greater than or equal to $r(1 - t)$, hence (3.16) follows. ∎

For every $n \times n$ matrix M we denote by M^{-1}, if it exists, the inverse of M and by M^T the transpose of M.

Given two bounded open sets Ω_1, Ω_2, we say that $\Omega_1 \Subset \Omega_2$ if $\overline{\Omega}_1 \subseteq \Omega_2$.

We now select a particular class of star-shaped open sets.

DEFINITION 3.5. – *We say that an open set Ω is strongly star-shaped if it is star-shaped with respect to some point x_0 in Ω and if for every x in $\overline{\Omega}$ the half open line segment joining x_0 to x, and not containing x, is contained in Ω.*

Let Ω be a strongly star-shaped bounded open set and let x_0 be given by Definition 3.5, then it is obvious that for every $t > 0$ the open set $x_0 + t(\Omega - x_0)$ is still strongly star-shaped and that, if $t > 1$, $\overline{\Omega} \subseteq x_0 + t(\Omega - x_0)$.

This implies

$$x_0 + s(\Omega - x_0) \Subset \Omega \Subset x_0 + t(\Omega - x_0)$$

$$\text{for every } s, \ t \in \mathbb{R} \text{ such that } 0 \le s < 1 < t. \tag{3.19}$$

Let Ω be an open set of $\mathbb{R}^n$; we say that Ω has Lipschitz boundary if $\partial\Omega$ is locally the graph of a Lipschitz continuous function.

The following result holds for open sets with Lipschitz boundary (*see* [39], [24]).

PROPOSITION 3.6. – *Let Ω be a bounded open set with Lipschitz boundary, then there exists a finite open covering $\{\tilde{\Omega}_j\}_{j=1,\ldots,s}$ of $\overline{\Omega}$ such that for every $j = 1, \ldots, s$ $\tilde{\Omega}_j \cap \Omega$ is strongly star-shaped with Lipschitz boundary.*

4. SOME AUXILIARY RESULTS

In the present section, given a vector subspace U of L^1_{loc} and a subset C of $\mathbb{R}^n$, we study, for every bounded open set Ω, some properties of the sets in (1.23).

To this aim it is useful to consider first some particular cases.

We need to prove some lemmas.

LEMMA 4.1. – *Let Ω be a bounded open set and let C be a closed and convex subset of $\mathbb{R}^n$.*

Let $u \in W^{1,1}_{loc}$, then

$$-\int_\Omega uD\varphi \in C \quad \text{for every} \quad \varphi \in C^1_0(\Omega) \quad \text{with} \quad \varphi \ge 0,$$

$$\int_\Omega \varphi = 1 \Leftrightarrow Du(x) \in C \quad \text{for a.e. } x \quad \text{in} \quad \Omega. \tag{4.1}$$

Proof. – Let u verify the left hand side of (4.1), then we have

$$\int_\Omega \varphi Du \in C \quad \text{for every} \quad \varphi \in C_0^1(\Omega) \quad \text{with} \quad \varphi \geq 0, \quad \int_\Omega \varphi = 1. \quad (4.2)$$

By (4.2) we easily deduce that if x is a Lebesgue point for Du then $Du(x) \in C$, hence one implication follows.

Let us now prove the opposite implication.

Since C is closed and convex there exist two families $\{a_\sigma\}_{\sigma \in S} \subseteq \mathbb{R}^n$ and $\{b_\sigma\}_{\sigma \in S} \subseteq \mathbb{R}$ such that

$$z \in C \Leftrightarrow \langle a_\sigma, z \rangle + b_\sigma \geq 0 \quad \text{for every } \sigma \in S. \quad (4.3)$$

Let us assume that $Du(x) \in C$ for a.e. x in Ω and let $\varphi \in C_0^1(\Omega)$ be such that $\varphi \geq 0$ and $\int_\Omega \varphi = 1$, then by (4.3) we get

$$\varphi(x)\langle a_\sigma, Du(x) \rangle + \varphi(x)b_\sigma \geq 0 \quad \text{for a.e. } x \text{ in } \Omega \text{ and every } \sigma \in S. \quad (4.4)$$

By integrating both sides of (4.4) over Ω and by applying the divergence theorem we obtain

$$\left\langle a_\sigma, -\int_\Omega uD\varphi \right\rangle + b_\sigma = \left\langle a_\sigma, \int_\Omega \varphi Du \right\rangle + b_\sigma \geq 0$$
$$\text{for every} \quad \sigma \in S, \quad\quad\quad\quad\quad\quad\quad (4.5)$$

hence by (4.5) and (4.3) the implication follows. ∎

LEMMA 4.2. – *Let Ω be a bounded open set and let C be a closed and convex subset of $\mathbb{R}^n$.*

Let $u \in BV_{loc}$, then

$$\left.\begin{array}{c} -\int_\Omega uD\varphi \in C \quad for~every~\varphi \in C_0^1(\Omega) \quad with \quad \varphi \geq 0, \\[2mm] \int_\Omega \varphi = 1 \quad \Leftrightarrow \quad \dfrac{1}{|A|}\int_A Du \in C \quad for~every \\[2mm] open~subset~A~of~\Omega. \end{array}\right\} \quad (4.6)$$

Proof. – Let u verify the left hand side of (4.6); let A be an open subset of Ω and let $\{\varphi_h\}_h \subseteq C_0^1(A)$ be such that $0 \leq \varphi_h \leq \varphi_{h+1}$ and $\varphi_h(x) \to \chi_A(x)$ for every x in A.

Since $u \in BV_{loc}$ we have

$$\int_\Omega \varphi_h \, Du = -\int_\Omega u D\varphi_h \quad \text{for every} \quad h \in \mathbb{N},\qquad (4.7)$$

therefore by (4.7) and the closedness of C we deduce

$$\frac{1}{|A|} \int_A Du = \lim_h \frac{1}{\displaystyle\int_\Omega \varphi_h} \int_\Omega \varphi_h \, Du$$

$$= \lim_h -\int_\Omega u \, D\left(\frac{1}{\displaystyle\int_\Omega \varphi_h} \varphi_h\right) \in C. \qquad (4.8)$$

By (4.8) we get the direct implication in (4.6).

In order to prove the reverse one let u verify the right hand side of (4.6), then by using the closedness of C, it is easy to prove that

$$\frac{1}{|E|} \int_E Du \in C \quad \text{for every Borel subset } E \text{ of } \Omega. \qquad (4.9)$$

Let φ be in $C_0^1\,(\Omega)$ be such that $\varphi \geq 0$, $\displaystyle\int_\Omega \varphi = 1$.

For every $\nu \in \mathbb{N}$ let $\mathcal{R}^\nu$ be a partition of $\mathbb{R}^n$ into half open cubes with sidelenght $\dfrac{1}{\nu}$ and let Q_j^ν be such cubes.

Let $S_\nu = \{j \in \mathbb{N} \; : \; |Q_j^\nu \cap \Omega| > 0\}$ and set, for every $j \in S_\nu$,
$\varphi_j^\nu = \dfrac{1}{|Q_j^\nu \cap \Omega|} \displaystyle\int_{Q_j^\nu \cap \Omega} \varphi.$

Since by (4.9), for every $j \in S_\nu$, $\dfrac{1}{|Q_j^\nu \cap \Omega|} \displaystyle\int_{Q_j^\nu \cap \Omega} Du \in C$, by the convexity of C we have

$$\frac{1}{\displaystyle\int_\Omega \varphi} \int_\Omega \sum_{j \in S_\nu} \varphi_j^\nu \, \chi_{Q_j^\nu \cap \Omega} \, Du$$

$$= \sum_{j \in S_\nu} \frac{\displaystyle\int_{Q_j^\nu \cap \Omega} \varphi}{\displaystyle\int_\Omega \varphi} \frac{1}{|Q_j^\nu \cap \Omega|} \int_{Q_j^\nu \cap \Omega} Du \in C$$

$$\text{for every } \nu \in \mathbb{N}. \qquad (4.10)$$

By (4.10), the closedness of C, once observed that $\displaystyle\sum_{j\in S_\nu} \varphi_j^\nu \, \chi_{Q_j^\nu \cap \Omega} \rightharpoonup \varphi$ uniformly on Ω as $\nu \to \infty$, we deduce that

$$\frac{1}{\displaystyle\int_\Omega \varphi} \int_\Omega \varphi \, Du \in C \quad \text{for every} \quad \varphi \in C_0^1(\Omega) \quad \text{with} \quad \varphi \ge 0. \qquad (4.11)$$

By (4.11) the reverse implication in (4.6) holds and the thesis follows. ∎

Now let us define for every bounded open set Ω and $p \in [1, +\infty]$

$$K_C^1(\Omega) = \left\{ u \in C^1 \ : \ Du(x) \in C \quad \text{for every } x \text{ in } \Omega \right\}, \qquad (4.12)$$

$$K_C^{1,p}(\Omega) = \left\{ u \in W_{loc}^{1,p} \ : \ Du(x) \in C \quad \text{for a.e. } x \text{ in } \Omega \right\} \qquad (4.13)$$

and

$$K_C(\Omega) = \left\{ u \in BV_{loc} \ : \ \frac{1}{|A|} \int_A Du \in C \right.$$
$$\left. \text{for every open subset } A \text{ of } \Omega \right\}. \qquad (4.14)$$

Then by the previous lemmas the following result holds.

PROPOSITION 4.3. – *Let* C *be a closed convex subset of* $\mathbb{R}^n$. *For every bounded open set* Ω *let* $K_C(C^1, \Omega)$, $K_C(W_{loc}^{1,p}, \Omega)$, $p \in [1, +\infty]$, *and* $K_C(BV_{loc}, \Omega)$ *be defined by* (1.23) *respectively with* $U = C^1$, $U = W_{loc}^{1,p}$ $p \in [1, +\infty]$, *and* $U = BV_{loc}$; *moreover let* $K_C^1(\Omega)$, $K_C^{1,p}(\Omega)$, $p \in [1, +\infty]$, *and* $K_C(\Omega)$ *be defined by* (4.12), (4.13) *and* (3.14).

Then

$$K_C(C^1, \Omega) = K_C^1(\Omega) \quad \text{for every bounded open set } \Omega, \qquad (4.15)$$

$$K_C(W_{loc}^{1,p}, \Omega) = K_C^{1,p}(\Omega) \quad \text{for every bounded open set } \Omega, \qquad (4.16)$$

$$K_C(BV_{loc}, \Omega) = K_C(\Omega) \quad \text{for every bounded open set } \Omega. \qquad (4.17)$$

Proof. – The proof immediately follows by Lemma 4.1 and Lemma 4.2. ∎

We can now study the properties of $K_C(U, \cdot)$.

PROPOSITION 4.4. – *Let U be a vector subspace of L^1_{loc} verifying $(1.3) \div (1.5)$, let C be a subset of $\mathbb{R}^n$ and let, for every bounded open set Ω, $K_C(U, \Omega)$ be given by (1.23).*
Then for every bounded open set Ω it results

$$u \in K_C(U, \Omega), \ c \in \mathbb{R} \Rightarrow u + c \in K_C(U, \Omega); \tag{4.18}$$

$$u \in K_C(U, \Omega), \ y \in \mathbb{R}^n \Rightarrow T[y] u \in K_C(U, \Omega - y); \tag{4.19}$$

$$u \in K_C(U, \Omega), \ t > 0 \Rightarrow O_t u \in K_C\left(U, \frac{1}{t}\Omega\right); \tag{4.20}$$

$$u \in K_C(U, \Omega), \ v \in U \ : \ v = u \, a.e. \ in \ \Omega \Rightarrow v \in K_C(U, \Omega); \tag{4.21}$$

$$\Omega_1 \subseteq \Omega_2 \Rightarrow K_C(U, \Omega_2) \subseteq K_C(U, \Omega_1). \tag{4.22}$$

Moreover for every bounded open set Ω we have

$$C \ convex \Rightarrow K_C(U, \Omega) \ convex; \tag{4.23}$$

$$C \ closed \Rightarrow K_C(U, \Omega) \ w - L^1(\Omega) \ closed \ in \ U \tag{4.24}$$

and

$$C \ convex \Rightarrow K_C(U, \Omega_1) \cap K_C(U, \Omega_2) \subseteq K_C(U, \Omega_1 \cup \Omega_2) \tag{4.25}$$

for every couple of bounded open sets Ω_1, Ω_2;

$$C \ closed \ and \ convex \Rightarrow K_C(U, \Omega_1) \cap W^{1,\infty}_{loc} \subseteq K_C(U, \Omega_2) \tag{4.26}$$

for every couple of bounded open sets Ω_1, Ω_2 with $\Omega_1 \subseteq \Omega_2$ and $|\Omega_2 \backslash \Omega_1| = 0$.

Proof. – The proof of $(4.18) \div (4.20)$ easily follows by $(1.3) \div (1.5)$, moreover $(4.21) \div (4.23)$ are straight away verified.

In order to prove (4.24) let $u \in U$ and let $\{u_\lambda\}_{\lambda \in \Lambda} \subseteq K_C(U, \Omega)$ be such that $\{u_\lambda\}_{\lambda \in \Lambda}$ converges to u in the $w - L^1(\Omega)$ topology, then for every $\varphi \in C_0^1(\Omega)$ with $\varphi \geq 0$, $\int_\Omega \varphi = 1$, by the closedness of C, it follows that

$$- \int_\Omega u D\varphi = \lim_\lambda - \int_\Omega u_\lambda \, D\varphi \in C. \tag{4.27}$$

By (4.27) the $w - L^1(\Omega)$ closedness of $K_C(U, \Omega)$ in U follows.

We now prove (4.25).

Let Ω_1 and Ω_2 be two bounded open sets, let $u \in K_C(U, \Omega_1) \cap K_C(U, \Omega_2)$ and let $\varphi \in C_0^1(\Omega_1 \cup \Omega_2)$ with $\varphi \geq 0$, $\int_{\Omega_1 \cup \Omega_2} \varphi = 1$.

By using the finite partition of unity lemma (cf. [37] XIV, Lemma 2.4) let $\varphi_1 \in C_0^1(\Omega_1)$, $\varphi_2 \in C_0^1(\Omega_2)$ be such that

$$\varphi_1 \geq 0, \ \varphi_2 \geq 0, \ \varphi_1 + \varphi_2 = 1 \quad \text{in spt}(\varphi). \tag{4.28}$$

By (4.28) we get

$$\varphi = \varphi_1 \varphi + \varphi_2 \varphi \tag{4.29}$$

and we have

$$\int_{\Omega_1 \cup \Omega_2} u D\varphi = \int_{\Omega_1} u D(\varphi_1 \varphi) + \int_{\Omega_2} u D(\varphi_2 \varphi). \tag{4.30}$$

By (4.30) we deduce

$$- \int_{\Omega_1 \cup \Omega_2} u D\varphi = - \int_{\Omega_1} \varphi_1 \varphi \int_{\Omega_1} u D \left(\frac{1}{\int_{\Omega_1} \varphi_1 \varphi} \varphi_1 \varphi \right)$$

$$- \int_{\Omega_2} \varphi_2 \varphi \int_{\Omega_2} u D \left(\frac{1}{\int_{\Omega_2} \varphi_2 \varphi} \varphi_2 \varphi \right). \tag{4.31}$$

Let us now set $t_1 = \int_{\Omega_1} \varphi_1 \varphi$ and $t_2 = \int_{\Omega_2} \varphi_2 \varphi$, then it is clear that

$$\frac{1}{t_i} \varphi_i \varphi \geq 0, \ \int_{\Omega_i} \frac{1}{t_i} \varphi_i \varphi = 1 \quad \text{for } i = 1, 2 \tag{4.32}$$

and that

$$t_1 + t_2 = \int_{\Omega_1 \cup \Omega_2} (\varphi_1 + \varphi_2) \varphi = 1, \tag{4.33}$$

therefore, since $-\displaystyle\int_{\Omega_i} uD\left(\dfrac{1}{t_i}\,\varphi_i\varphi\right) \in C$ for $i = 1, 2$, by $(4.31) \div (4.33)$ and the convexity of C we deduce that

$$\left.\begin{array}{c} -\displaystyle\int_{\Omega_1\cup\Omega_2} uD\varphi \in C \\[2mm] \text{for every } \varphi \in C_0^1\,(\Omega_1 \cup \Omega_2) \text{ with } \varphi \geq 0, \\[2mm] \displaystyle\int_{\Omega_1\cup\Omega_2} \varphi = 1. \end{array}\right\} \qquad (4.34)$$

By (4.34) inclusion (4.25) follows.

In conclusion we prove (4.26).

Let Ω_1, Ω_2 be two bounded open sets with $\Omega_1 \subseteq \Omega_2$ and $|\Omega_2\backslash\Omega_1| = 0$ and let $u \in K_C\,(U,\,\Omega_1) \cap W_{loc}^{1,\,\infty}$.

Then by Lemma 4.1 we have

$$Du\,(x) \in C \quad \text{for a.e. } x \text{ in } \Omega_1. \qquad (4.35)$$

Since $|\Omega_2\backslash\Omega_1| = 0$, by (4.35) we get

$$Du\,(x) \in C \quad \text{for a.e. } x \text{ in } \Omega_2, \qquad (4.36)$$

hence, by (4.36) and again Lemma 4.1, we obtain that $u \in K_C\,(U,\,\Omega_2)$, therefore (4.26) holds. ∎

5. A REPRESENTATION RESULT

Let C be a closed convex subset of $\mathbb{R}^n$ and let, for every bounded open set Ω, $K_C^1\,(\Omega)$ be defined by (4.12).

In this section, given a vector subspace U of L_{loc}^1 and, for every bounded open set Ω, a topology $\tau\,(\Omega)$ on U, we describe the sets $\displaystyle\bigcap_{\varepsilon>0} (U,\,\tau\,(\Omega)) - cl\,(K_C^1\,(\Omega_\varepsilon^-) \cap U)$.

We assume that

for every bounded open set Ω $\tau\,(\Omega)$ is finer than $w - L^1\,(\Omega)$ \qquad (5.1)

and that, if for $u \in U$ and $\varepsilon > 0$ u_ε denotes the regularization of u defined by (3.2),

$$\left.\begin{array}{l} i)\ u_\varepsilon \in U \quad \text{for every } u \in U,\ \varepsilon > 0, \\[2mm] ii)\ u_\varepsilon \to u \text{ in } \tau\,(\Omega) \text{ as } \varepsilon \to 0^+ \\[2mm] \quad \text{for every } u \in U,\ \Omega \text{ bounded open set.} \end{array}\right\} \qquad (5.2)$$

The following result holds.

PROPOSITION 5.1. – *Let U be a vector subspace of L^1_{loc} and let, for every bounded open set Ω, $\tau\,(\Omega)$ be a topology on U verifying (5.1) and (5.2).*

Let C be a closed convex subset of $\mathbb{R}^n$ and let, for every bounded open set Ω, $K^1_C\,(\Omega)$ be defined by (4.12) and $K_C\,(U,\,\Omega)$ by (1.23). Then

$$\bigcap_{\varepsilon>0} (U,\,\tau\,(\Omega)) - cl\,(K^1_C\,(\Omega^-_\varepsilon)\cap U) = K_C\,(U,\,\Omega)$$

$$for\ every\ bounded\ open\ set\ \Omega. \tag{5.3}$$

Proof. – Since C is closed and convex there exist two families $\{a_\sigma\}_{\sigma\in S}\subseteq\mathbb{R}^n$ and $\{b_\sigma\}_{\sigma\in S}\subseteq\mathbb{R}$ such that

$$C = \{z\in\mathbb{R}^n \ : \ \langle a_\sigma,\,z\rangle + b_\sigma \geq 0\ \text{for every}\ \sigma\in S\}. \tag{5.4}$$

Let Ω be a bounded open set and let $u\in\bigcap_{\varepsilon>0}(U,\tau(\Omega))-cl(K^1_C(\Omega^-_\varepsilon)\cap U)$, then for every $\varepsilon>0$ there exists a generalized sequence $\{u^\varepsilon_\lambda\}_{\lambda\in\Lambda}\subseteq C^1\cap U$ such that $u^\varepsilon_\lambda\to u$ in $\tau\,(\Omega)$ and

$$Du^\varepsilon_\lambda\,(x)\in C\quad\text{for every x in}\ \Omega^-_\varepsilon,\,\lambda\in\Lambda. \tag{5.5}$$

Let $\varphi\in C^1_0\,(\Omega)$ be such that $\varphi\geq 0$, $\int_\Omega\varphi=1$ and let $\varepsilon>0$ be such that $\varphi\in C^1_0(\Omega^-_\varepsilon)$, then by (5.4) and (5.5) we soon obtain

$$\left\langle a_\sigma,\,\int_{\Omega^-_\varepsilon}\varphi Du^\varepsilon_\lambda\right\rangle + b_\sigma\int_{\Omega^-_\varepsilon}\varphi\geq 0\quad\text{for every}\quad\lambda\in\Lambda,\,\sigma\in S. \tag{5.6}$$

Since $\varphi\in C^1_0\,(\Omega^-_\varepsilon)$ and $u^\varepsilon_\lambda\to u$ in $\tau\,(\Omega)$, by (5.1) and (5.6) we deduce

$$0\leq\lim_\lambda-\left\langle a_\sigma,\,\int_{\Omega^-_\varepsilon}u^\varepsilon_\lambda\,D\varphi\right\rangle + b_\sigma$$

$$\leq-\left\langle a_\sigma,\,\int_\Omega uD\varphi\right\rangle + b_\sigma\quad\text{for every}\ \sigma\in S, \tag{5.7}$$

therefore by (5.7) and (5.4) we have

$$\bigcap_{\varepsilon>0}(U,\,\tau\,(\Omega)) - cl\,(K^1_C\,(\Omega^-_\varepsilon)\cap U)\subseteq K_C\,(U,\,\Omega). \tag{5.8}$$

In order to prove the reverse inclusion in (5.8) let u be in $K_C\,(U,\,\Omega)$ and let, for every $\varepsilon>0$, u_ε be te regularization of u given by (3.2).

Let $\varepsilon > 0$, then for every $x \in \Omega_\varepsilon^-$ the function $\alpha_\varepsilon (x - \cdot)$ is in $C_0^1 (\Omega)$ with $\alpha_\varepsilon (x - \cdot) \geq 0$, $\int_\Omega \alpha_\varepsilon (x - y) \, dy = 1$; therefore we have

$$Du_\varepsilon (x) = \int_\Omega (D\alpha_\varepsilon) (x - y) u (y) \, dy$$

$$= - \int_\Omega D (\alpha_\varepsilon (x - \cdot)) (y) u (y) \, dy \in C$$

$$\text{for every } \varepsilon > 0, \ x \in \Omega_\varepsilon^-, \tag{5.9}$$

that is by i) of (5.2)

$$u_\varepsilon \in K_C^1 (\Omega_\varepsilon^-) \cap U \quad \text{for every } \varepsilon > 0. \tag{5.10}$$

Let us now observe that

$$\varepsilon_1 < \varepsilon_2 \Rightarrow K_C^1 (\Omega_{\varepsilon_1}^-) \subseteq K_C^1 (\Omega_{\varepsilon_2}^-) \tag{5.11}$$

hence, for fixed $\varepsilon_0 > 0$, by (5.11) we get

$$u_\varepsilon \in K_C^1 (\Omega_{\varepsilon_0}^-) \cap U \quad \text{for every } \varepsilon < \varepsilon_0. \tag{5.12}$$

By (5.12) and ii) of (5.2) we soon deduce

$$K_C (U, \Omega) \subseteq \bigcap_{\varepsilon > 0} (U, \tau (\Omega)) - cl (K_C^1 (\Omega_\varepsilon^-) \cap U). \tag{5.13}$$

By (5.8) and (5.13) equality (5.3) follows. ■

6. AN INNER REGULARITY RESULT

Let us briefly recall the concept of inner regular envelope of an increasing set function (cf. [35]).

Let F be an extended real valued function defined on the set of all bounded open sets of $\mathbb{R}^n$, we say that F is increasing if

$$\Omega_1 \subseteq \Omega_2 \Rightarrow F (\Omega_1) \leq F (\Omega_2). \tag{6.1}$$

Given an increasing function defined for every bounded open set of $\mathbb{R}^n$, we define the inner regular envelope F_- of F on Ω as

$$F_- (\Omega) = \sup_{A \Subset \Omega} F (A) \tag{6.2}$$

Obviously by (6.1) it follows that

$$F_- (\Omega) = \sup_{\varepsilon > 0} F (\Omega_\varepsilon^-). \qquad (6.3)$$

Let us now consider a vector subspace V of L^1_{loc} and let, for every bounded open set Ω, $H (\Omega)$ be a subset of V verifying the following monotonicity assumption

$$\Omega_1 \subseteq \Omega_2 \Rightarrow H (\Omega_2) \subseteq H (\Omega_1), \qquad (6.4)$$

then by (6.4) for every $u \in V$ the function

$$\mathbb{I} (\Omega, u) = \begin{cases} 0 & \text{if} \quad u \in H (\Omega) \\ +\infty & \text{if} \quad u \notin H (\Omega) \end{cases} \qquad (6.5)$$

is increasing.

Therefore, given a bounded open set Ω, we define the inner regular envelope $H_- (\Omega)$ of $H (\cdot)$ on the open set Ω as the domain of $\mathbb{I}_- (\Omega, \cdot)$, i.e. by (6.4) and (6.3)

$$H_- (\Omega) = \bigcap_{A \Subset \Omega} H (A) = \bigcap_{\varepsilon > 0} H (\Omega_\varepsilon^-). \qquad (6.6)$$

In the present section, given a family $\{H (\Omega) : \Omega \text{ bounded open set}\}$ of subsets of V we give sufficient conditions in order to prove an identity result between $H_- (\Omega)$ and $H (\Omega)$ at least for every bounded open set Ω with Lipschitz boundary.

We assume that V is a vector subspace of L^1_{loc} verifying (1.4), (1.5) and that, for every bounded open set Ω, a topology $\sigma (\Omega)$ on V is given such that

$$T [-y] O_t T [y] u \to u \quad \text{in} \quad \sigma (\Omega) \quad \text{as} \quad t \to 1^-$$
for every $u \in V$ and every strongly star-shaped
bounded open set Ω star-shaped with respect to y. $\qquad (6.7)$

For every bounded open set Ω let $H (\Omega)$ be a subset of V such that

$$u \in H (\Omega), \ y \in \mathbb{R}^n \Rightarrow T [y] u \in H (\Omega - y), \qquad (6.8)$$

$$u \in H (\Omega), \ t > 0 \Rightarrow O_t u \in H \left(\frac{1}{t} \Omega \right), \qquad (6.9)$$

$$H (\Omega) \quad \text{is} \quad \sigma (\Omega)\text{-closed in } V \qquad (6.10)$$

and

$$H (\Omega_1) \cap H (\Omega_2) \subseteq H (\Omega_1 \cup \Omega_2)$$
for every couple of bounded open sets Ω_1, Ω_2. $\qquad (6.11)$

The following result holds.

PROPOSITION 6.1. – *Let V be a vector subspace of L^1_{loc} verifying (1.4), (1.5).*

For every bounded open set Ω let $\sigma(\Omega)$ be a topology on V satisfying (6.7) and let $H(\Omega)$ be a subset of V verifying (6.8) ÷ (6.11) and (6.4). Then

$$H(\Omega) = H_-(\Omega) \tag{6.12}$$

for every bounded open set Ω with Lipschitz boundary.

Proof. – Let Ω be a bounded open set, then by (6.4) it soon follows that

$$H(\Omega) \subseteq \bigcap_{\varepsilon > 0} H(\Omega_\varepsilon^-). \tag{6.13}$$

In order to prove the reverse inclusion in (6.13) let us first assume that Ω is strongly star-shaped, let $x_0 \in \Omega$ be such that Ω is star-shaped with respect to x_0.

Let $\varepsilon > 0$, since Ω is strongly star-shaped, by (3.19) there exists $t_\varepsilon \in]0, 1[$ such that

$$x_0 + t(\Omega - x_0) \subseteq \Omega_\varepsilon^- \quad \text{for every } t \in]0, t_\varepsilon[, \tag{6.14}$$

hence by (6.14) and (6.4) it turns out that

$$H(\Omega_\varepsilon^-) \subseteq H(x_0 + t(\Omega - x_0)) \quad \text{for every } t \in]0, t_\varepsilon[. \tag{6.15}$$

Let $u \in \bigcap_{\varepsilon > 0} H(\Omega_\varepsilon^-)$, then, once observed that $t_\varepsilon \to 1^-$ as $\varepsilon \to 0^+$, by (6.15) it results that

$$u \in H(x_0 + t(\Omega - x_0)) \quad \text{for every } t \in]0, 1[, \tag{6.16}$$

therefore by (6.8) and (6.9) we deduce that

$$T[-x_0] O_t T[x_0] u \in H(\Omega). \tag{6.17}$$

By (6.17), (6.7) and (6.10) we infer, as $t \to 1^-$, that

$$\bigcap_{\varepsilon > 0} H(\Omega_\varepsilon^-) \subseteq H(\Omega)$$

$$\text{for every strongly star-shaped bounded open set } \Omega. \tag{6.18}$$

In order to prove (6.18) for every bounded open set Ω with Lipschitz boundary let $\{\tilde{\Omega}_j\}_{j=1,\ldots,s}$ be a finite open covering of $\overline{\Omega}$ given by

Proposition 3.6, such that for every $j = 1, \ldots, s$ $\Omega_j = \tilde{\Omega}_j \cap \Omega$ is strongly star-shaped.

Let $u \in \bigcap_{\varepsilon > 0} H(\Omega_\varepsilon^-)$, then by (6.4) it results that

$$u \in \bigcap_{\varepsilon > 0} H((\Omega_j)_\varepsilon^-) \quad \text{for every } j = 1, \ldots, s. \tag{6.19}$$

Since Ω_j is strongly star-shaped, by (6.18) and (6.19) it follows that

$$u \in \bigcap_{j=1}^{s} H(\Omega_j). \tag{6.20}$$

At this point, by (6.20) and (6.11) it follows that

$$\bigcap_{\varepsilon > 0} H(\Omega_\varepsilon^-) \subseteq H(\Omega)$$

for every bounded open set Ω with Lipschitz boundary. (6.21)

By (6.13) and (6.21) equality (6.12) follows. ∎

7. THE CHARACTERIZATION PROBLEM

Let U be a vector subspace of L^1_{loc} and let, for every bounded open set Ω, $K(U, \Omega)$ be a subset of U.

In the present section and in the next ones we propose sufficient conditions on the family $\{K(U, \Omega) : \Omega \text{ bounded open set}\}$ in order to deduce the existence of a closed convex subset C of $\mathbb{R}^n$ such that $K(U, \Omega) = K_C(U, \Omega)$ at least for every regular bounded open set Ω, $K_C(U, \Omega)$ being defined by (1.23).

We assume that U is a vector subspace of L^1_{loc} satisfying $(1.3) \div (1.5)$.

For every bounded open set Ω let $K(U, \Omega)$ be a subset of U satisfying, according to Proposition 4.4, assumptions $(1.13) \div (1.21)$.

For every bounded open set Ω let us define the sets $C(\Omega)$ and C by

$$C(\Omega) = \{z \in \mathbb{R}^n : u_z \in K(U, \Omega)\} \tag{7.1}$$

and by (1.22).

PROPOSITION 7.1. – *Let U be a vector subspace of L^1_{loc} satisfying $(1.3) \div (1.5)$, let, for every bounded open set Ω, $K(U, \Omega)$ be a subset*

of U satisfying $(1.13) \div (1.15)$ and (1.19), $C\,(\Omega)$ be given by (7.1) and let C be defined by (1.22). Then

$$C\,(\Omega) = C \quad for\ every\ bounded\ open\ set\ \Omega. \qquad (7.2)$$

Proof. – We only need to prove that the sets $C\,(\Omega)$ are independent on Ω. To this aim let A, B be bounded open sets and let us prove that

$$C\,(A) \subseteq C\,(B). \qquad (7.3)$$

We can obviously assume that $C\,(A) \neq \varnothing$.

Let $z \in C\,(A)$ then $u_z \in K\,(U,\,A)$ and, if $x_0 \in A$, by (1.14) it results that $T\,[x_0]\,u_z \in K\,(U,\,A - x_0)$.

Since $0 \in A - x_0$ let $t > 0$ be such that

$$\frac{A - x_0}{t} \supseteq B, \qquad (7.4)$$

then by (1.15), (1.19) and (7.4) it turns out that

$$O_t\,T\,[x_0]\,u_z \in K\left(U,\,\frac{A - x_0}{t}\right) \subseteq K\,(U,\,B). \qquad (7.5)$$

At this point we only have to observe that

$$O_t\,T\,[x_0]\,u_z\,(x) = u_z\,(x) + \frac{1}{t}\,\langle z,\,x_0 \rangle \quad for\ every\ x\ in\ \mathbb{R}^n, \qquad (7.6)$$

hence by (7.6), (1.13) and (7.5) we soon get $u_z \in K\,(U,\,B)$, establishing (7.3).

By (7.3), changing the role of A and B, we obtain

$$C\,(A) = C\,(B)\ \text{for every couple of bounded open sets } A \text{ and } B. \quad \blacksquare\ (7.7)$$

In order to study further properties of the set C we need to assume that

$$K\,(U,\,\Omega) \cap C^1 \text{ is } C^1\,(\Omega) \text{ closed in } C^1 \text{ for every bounded open set } \Omega. \quad (7.8)$$

PROPOSITION 7.2. – *Let U be a vector subspace of L^1_{loc} satisfying $(1.3) \div (1.5)$, let, for every bounded open set Ω, $K\,(U,\,\Omega)$, be a subset of U satisfying $(1.13) \div (1.15)$, (1.19) and let C be defined by (1.22). We have that*

a) *if in addition we assume (7.8) then C is closed;*

b) *if in addition (1.17) holds then C is convex.*

Proof. – Let Ω be a bounded open set. Let $z \in \mathbb{R}^n$ and let $\{z_h\}_h \subseteq C$ be such that

$$z_h \to z, \tag{7.9}$$

then the functions u_{z_h} are in $K\,(U,\,\Omega) \cap C^1$ and by (7.9) it results

$$u_{z_h} \to u_z \quad \text{in } C^1\,(\Omega). \tag{7.10}$$

By (7.10) and (7.8) it follows that $u_z \in K\,(U,\,\Omega)$; by Proposition 7.1 this yields that $z \in C$ and the closedness of C follows.

Finally if (1.17) holds the convexity of C immediately follows. ∎

Remark 7.3. – We observe that, in general, conditions of geometric type on the family $\{K\,(U,\,\Omega)\,:\,\Omega$ bounded open set$\}$ change into geometric properties of the set C in (1.22).

For example it is easy to verify that if in addition to $(1.13) \div (1.15)$, (1.17) and (1.19) we assume the following condition

$$\left.\begin{array}{l} \text{there exists a bounded open set } \Omega \text{ and } z_0 \in \mathbb{R}^n \text{ such that} \\ u \in K\,(U,\,\Omega) \Rightarrow u\,(R^{-1}\cdot) \in K\,(U,\,R\Omega) \\ \text{for every orthogonal transformation } R \text{ with } R\,(z_0) = z_0 \end{array}\right\} \tag{7.11}$$

then the set C is a ball with centre in z_0 and radius $r = \sup\{|z - z_0|\,:\,u_z \in K\,(U,\,\Omega)\}$.

In fact, by Proposition 7.2, the set C turns out to be convex, moreover, by (7.11), for every $z \in C$ it results that $Rz \in C$; this implies that C is the above described ball.

In the same way if we assume that

$$\left.\begin{array}{l} \text{there exists a bounded open set } \Omega \text{ such that} \\ u \in K\,(U,\,\Omega),\, t > 0 \Rightarrow tu \in K\,(U,\,\Omega) \end{array}\right\} \tag{7.12}$$

then C is a cone.

Let us now prove the following result.

LEMMA 7.4. – *Let U be a vector subspace of L^1_{loc} satisfying $(1.3) \div (1.5)$, let $\{K\,(U,\,\Omega)\,:\,\Omega$ bounded open set$\}$ be a family of subset of U verifying $(1.13) \div (1.15)$, (1.19), (7.8) and let C be defined by (1.22). Let Ω be a bounded open set and let u be in $K\,(U,\,\Omega) \cap C^1$, then*

$$Du\,(x_0) \in C \quad \text{for every } x_0 \in \Omega. \tag{7.13}$$

Proof. – Let Ω be a bounded open set, u in $K(U, \Omega) \cap C^1$ and x_0 in Ω, then, by virtue of $(1.13) \div (1.15)$, for every $t > 0$ the function

$$T[-x_0] O_t T[x_0] u - \frac{u(x_0)}{t} = \frac{u(x_0 + t(\cdot - x_0)) - u(x_0)}{t} \qquad (7.14)$$

is in $K\left(U, x_0 + \dfrac{\Omega - x_0}{t}\right) \cap C^1$.

Let us now observe that $0 \in \Omega - x_0$, hence for every positive t sufficiently small, it follows that

$$x_0 + \frac{\Omega - x_0}{t} \supseteq \Omega. \qquad (7.15)$$

By (7.15) and (1.19) there exists $t_0 > 0$ such that

$$T[-x_0] O_t T[x_0] u \in K(U, \Omega) \cap C^1$$
$$\text{for every } t \in]0, t_0[. \qquad (7.16)$$

Let us recall now that u is differentiable in x_0, hence

$$\sup_{x \in \Omega} \left| \frac{u(x_0 + t(x - x_0)) - u(x_0)}{t} - \langle Du(x_0), x - x_0 \rangle \right| \to 0$$
$$\text{as } t \to 0^+; \qquad (7.17)$$

moreover it is easy to verify that by the continuity of Du it results

$$D(T[-x_0] O_t T[x_0] u) \to Du(x_0) \quad \text{uniformly on } \Omega \text{ as } t \to 0^+. \qquad (7.18)$$

By (7.17) and (7.18) we deduce that

$$T[-x_0] O_t T[x_0] u - \frac{u(x_0)}{t} \to \langle Du(x_0), \cdot - x_0 \rangle \quad \text{in} \quad C^1(\Omega)$$
$$\text{as } t \to 0^+, \qquad (7.19)$$

therefore by (7.19), (7.16) and (7.8) we infer that

$$\langle Du(x_0), \cdot - x_0 \rangle \in K(U, \Omega) \cap C^1. \qquad (7.20)$$

By (7.20) and (1.13) we get that

$$Du(x_0) \in C(\Omega), \qquad (7.21)$$

hence by (7.21) and Proposition 7.1 the thesis follows. ∎

8. THE CASE OF CONTINUOUSLY
DIFFERENTIABLE FUNCTIONS

In the present section, given U and a family $\{K(U, \Omega) : \Omega \text{ bounded open}$
set$\}$ of subsets of U as in section 7, we characterize the sets $K(U, \Omega) \cap C^1$.

For every subset C of $\mathbb{R}^n$ and every bounded open set Ω let $K_C^1(\Omega)$ and
$K_C^{1, \infty}(\Omega)$ be defined by (4.12) and (4.13).

PROPOSITION 8.1. – *Let U be a vector subspace of L_{loc}^1 verifying*
(1.3) ÷ (1.5); *for every bounded open set Ω let $K(U, \Omega)$ be a subset of U*
verifying (1.13) ÷ (1.15), (1.19), (7.8) *and let C be given by* (1.22). *Then*

$$K(U, \Omega) \cap C^1 \subseteq K_C^1(\Omega) \quad \text{for every bounded open set } \Omega. \qquad (8.1)$$

Proof. – The proof immediately follows by Lemma 7.4. ∎

We now prove, by various steps, the reverse inclusion in (8.1).

To do this we need to assume that

$$K(U, \Omega) \cap W_{loc}^{1, \infty} \text{ is } W^{1, \infty}(\Omega) \text{ closed in } W_{loc}^{1, \infty}$$

$$\text{for every bounded open set } \Omega. \qquad (8.2)$$

LEMMA 8.2. – *Let U be a vector subspace of L_{loc}^1 satisfying* (1.3) ÷ (1.6);
let $\{K(U, \Omega) : \Omega \text{ bounded open set}\}$ be a family of subsets of U verifying
(1.13) ÷ (1.16) *and* (1.19) ÷ (1.21).

Let C be given by (1.22), *then for every bounded open set Ω its results*

$$\left. \begin{array}{l} u \text{ piecewise affine function on } \mathbb{R}^n, \\ u \in K_C^{1, \infty}(\Omega) \Rightarrow u \in K(U, \Omega). \end{array} \right\} \qquad (8.3)$$

Proof. – Let Ω be a bounded open set and let $u = \sum_{j=1}^{m} \left(u_{z_j} + s_j \right) \chi_{P_j}$
be a piecewise affine function on $\mathbb{R}^n$.

Since $u \in K_C^{1, \infty}(\Omega)$, it turns out that

$$z_j \in C \quad \text{for every } j = 1, \dots, m, \qquad (8.4)$$

C being given by (1.22).

By (8.4) and (1.13) we soon get that

$$u_{z_j} + s_j \in K(U, \Omega \cap \overset{\circ}{P}_j) \quad \text{for every } j = 1, \dots, m, \qquad (8.5)$$

hence by (8.5), (1.6) and (1.16) we obtain

$$\bar{u} \in K\left(U, \Omega \cap \overset{\circ}{P}_j\right) \quad \text{for every } j = 1, \ldots, m. \tag{8.6}$$

At this point by (8.6) and (1.20) we deduce that

$$u \in K\left(U, \Omega \cap \bigcup_{j=1}^{m} \overset{\circ}{P}_j\right) \tag{8.7}$$

and finally by (8.7) and (1.21), once recalled that $\left|\Omega \backslash \left(\Omega \cap \bigcup_{j=1}^{m} \overset{\circ}{P}_j\right)\right| = 0$,

the thesis follows. ■

Let U verify $(1.3) \div (1.5)$ and let $\{K\,(U,\,\Omega) : \Omega$ bounded open set$\}$ be a family of subsets of U verifying $(1.13) \div (1.15)$, (1.17), (1.19) and let C be defined by (1.22).

Then by Proposition 7.2 C turns out to be convex.

Let us observe that it is not restrictive to assume that, if $C \neq \varnothing$, $0 \in C$; in fact, if this is not the case, taken z_0 in C, it is sufficient to consider the sets $K'\,(U,\,\Omega) = K\,(U,\,\Omega) - u_{z_0}$ and $C' = C - z_0$.

Let $\Sigma\,(C)$ be the vector space generated by C and let $\nu\,(\leq n)$ be its dimension, then it is soon observed that C possesses interior points in the topology of $\Sigma\,(C)$ and that, by using the same argument as before, it is not restrictive to assume that

$$0 \in \overset{\circ}{C} \quad \text{the interior being taken in the topology of } \Sigma\,(C). \tag{8.8}$$

If $\nu < n$ let us denote by 0_ν, respectively by $0_{n-\nu}$, the origin of $\mathbb{R}^\nu$, respectively of $\mathbb{R}^{n-\nu}$.

Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation if $\nu = n$ and an orthogonal linear transformation such that

$$R\,(\Sigma\,(C)) = \mathbb{R}^\nu \times \{0_{n-\nu}\} \tag{8.9}$$

if $\nu < n$.

For every function u in $W_{loc}^{1,\infty}$ let us define the function u' as

$$u'\,(y) = u\,(R^{-1}\,y) \quad y \in \mathbb{R}^n, \tag{8.10}$$

then since $R^T = R^{-1}$ we have that

$$D_y u'\,(y) = D_x u\,(R^{-1}y) \cdot R^{-1} = D_x u\,(R^{-1}y) \cdot R^T$$
$$= (R \cdot D_x u\,(R^{-1}y)^T)^T, \quad y \in \mathbb{R}^n. \tag{8.11}$$

Let $\overline{x} \in \mathbb{R}^n$, let B be an open ball of $\mathbb{R}^n$ centered at $\overline{x}$, then for every u in $K_C^{1,\infty}(B)$ by (8.9), (8.11) and the convexity of B, it follows that u' effectively depends only on $(y_1, \ldots, y_\nu)$ when $(y_1, \ldots, y_n)$ varies in RB, RB obviously being an open ball centered at $\overline{y} = R\overline{x}$.

Hence we can define the function $\hat{u}$ as

$$\hat{u}(y_1, \ldots, y_\nu) = \begin{cases} u(y_1, \ldots, y_n) & \text{if} \quad \nu = n \\ u'(y_1, \ldots, y_\nu, \overline{y}_{\nu+1}, \ldots, \overline{y}_n) & \text{if} \quad \nu < n \end{cases}$$
$$(y_1, \ldots, y_\nu) \in \mathbb{R}^\nu. \qquad (8.12)$$

By (8.12), (8.11) and (8.9) it soon follows that

$$D\hat{u}(y_1, \ldots, y_\nu) \in Pr_\nu(RC)$$
$$\text{for every} \quad (y_1, \ldots, y_\nu) \in Pr_\nu(RB), \qquad (8.13)$$

where Pr_ν is the projection function from $\mathbb{R}^n$ to $\mathbb{R}^\nu$ defined by

$$Pr_\nu(y_1, \ldots, y_n) = (y_1, \ldots, y_\nu) \quad (y_1, \ldots, y_n) \in \mathbb{R}^n; \qquad (8.14)$$

moreover by (8.8) and (8.9) it follows that

$$0_\nu \in (Pr_\nu(RC))^0. \qquad (8.15)$$

LEMMA 8.3. – *Let U be a vector subspace of L_{loc}^1 satisfying $(1.3) \div (1.6)$ and let $\{K(U, \Omega) : \Omega$ bounded open set$\}$ be a family of subsets of U verifying $(1.13) \div (1.17)$, $(1.19) \div (1.21)$ and (8.2).*

Let C be given by (1.22), then

$$K_C^1(\Omega) \subseteq \bigcap_{\varepsilon > 0} K(U, \Omega_\varepsilon^-) \quad \text{for every bounded open set } \Omega. \qquad (8.16)$$

Proof. – Let us first prove the following stronger inclusion

$$K_C^1(B) \subseteq K(U, B) \quad \text{for every open ball } B \text{ of } \mathbb{R}^n. \qquad (8.17)$$

To this aim let B be an open ball of $\mathbb{R}^n$, let u be in $K_C^1(B)$ and let $\hat{u}$ be the function deduced from u as in (8.12), then obviously (8.13) holds, moreover we can assume (8.15).

Let Ω' be a bounded open set with Lipschitz boundary such that $\Omega' \ni B$, let $t \in]0, 1[$ and let $\{\hat{u}_h\}_h$ be a sequence of piecewise affine functions such that

$$\left. \begin{array}{ll} \hat{u}_h \to t\hat{u} & \text{uniformly on } \overline{Pr_\nu \left(R\Omega' \right)}, \\ D\hat{u}_h \to tD\hat{u} & \text{in } L^\infty_{loc} \left(Pr_\nu \left(R\Omega' \right) \right) \end{array} \right\} \tag{8.18}$$

(see for example Proposition 2.1 at page 309 in [39]).

By (8.13), (8.15) and Proposition 3.4, applied with $m = \nu$ and $G = Pr_\nu \left(RC \right)$ (the convention that $\text{dist} \left(z, \varnothing \right) = +\infty$ for every $z \in \mathbb{R}^n$ is adopted), we get the existence of $\delta > 0$ such that

$$\text{dist} \left(tD\hat{u} \left(y_1, \ldots, y_\nu \right), \partial Pr_\nu \left(RC \right) \right) > \delta$$

$$\text{for every } \left(y_1, \ldots, y_\nu \right) \in Pr_\nu \left(RB \right), \tag{8.19}$$

hence by (8.18) and (8.19) we soon obtain that

$$D\hat{u}_h \left(y_1, \ldots, y_\nu \right) \in Pr_\nu \left(RC \right)$$

$$\text{for a.e. } \left(y_1, \ldots, y_\nu \right) \in Pr_\nu \left(RB \right), \ h \in \mathbb{N} \text{ sufficiently large.} \tag{8.20}$$

Let us define now the functions u'_h and u_h as

$$\left. \begin{array}{l} u'_h \left(y_1, \ldots, y_n \right) = \hat{u}_h \left(y_1, \ldots, y_\nu \right) \\ \text{for every } \left(y_1, \ldots, y_\nu \right) \in \mathbb{R}^n \\ u_h \left(x_1, \ldots, x_n \right) = u'_h \left(R \left(x_1, \ldots, x_n \right) \right) \\ \text{for every } \left(x_1, \ldots, x_n \right) \in \mathbb{R}^n, \end{array} \right\} \tag{8.21}$$

then, by (8.21) and (8.18), the functions u_h are piecewise affine and

$$u_h \to tu \quad \text{in} \quad W^{1, \infty} \left(B \right); \tag{8.22}$$

moreover, by (8.20) we have

$$Du_h \left(x \right) \in C \quad \text{for a.e. } x \in B, \ h \in \mathbb{N} \text{ sufficiently large.} \tag{8.23}$$

At this point, by Lemma 8.2 we get that

$$u_h \in K \left(U, B \right) \quad \text{for every } h \in \mathbb{N} \text{ sufficiently large,} \tag{8.24}$$

hence, by (8.22), (8.24) and (8.2) we soon obtain that

$$tu \in K \left(U, B \right) \quad \text{for every } t \in]0, 1[. \tag{8.25}$$

Finally, by (8.25) and again (8.2) we deduce (8.17) as $t \to 1^-$.

By (8.17) the thesis easily follows, in fact let Ω be a bounded open set, let $\varepsilon > 0$ and let $B^1, \ldots, B^k$ be a finite covering of $\overline{\Omega_\varepsilon^-}$ made up of open balls such that $B^j \subseteq \Omega$ for every $j \in \{1, \ldots, k\}$.

By (8.17) we get

$$K_C^1 (B^j) \subseteq K (U, B^j) \quad \text{for every} \quad j \in \{1, \ldots, k\}, \qquad (8.26)$$

hence by (8.26), (1.20) and (1.19) we deduce

$$K_C^1 (\Omega) \subseteq \bigcap_{j=1}^k K_C^1 (B^j) \subseteq \bigcap_{j=1}^k K (U, B^j)$$

$$\subseteq K \left(U, \bigcup_{j=1}^k B^j \right) \subseteq K (U, \Omega_\varepsilon^-). \qquad (8.27)$$

Since (8.27) holds for every $\varepsilon > 0$, inclusion (8.16) follows by (8.27). ∎

We can now prove the main result of this section.

THEOREM 8.4. – *Let U be a vector subspace of L_{loc}^1 satisfying* (1.3) ÷ (1.6).

For every bounded open set ᵕ *let $K (U, \Omega)$ be a subset of U verifying* (1.13) ÷ (1.17), (1.19) ÷ (1.21), (8.2) *and let $K_C^1 (\Omega)$ be defined by* (4.12) *with C given by* (1.22).

Then

$$K (U, \Omega) \cap C^1 = K_C^1 (\Omega)$$

for every bounded open set Ω with Lipschitz boundary.   (8.28)

Proof. – The proof follows by Proposition 8.1, Lemma 8.3 and Proposition 6.1 applied with $H (\Omega) = K (U, \Omega) \cap C^1$, $V = C^1$ and $\sigma (\Omega)$ equal to the $C^1 (\Omega)$ topology, once observed that, with these choices, (6.7) trivially holds. ∎

Let us observe that assumption (1.21) has been utilized directly just to prove one step of the proof of Theorem 8.4, i.e. the one performed in Lemma 8.2.

In the rest of the present section our goal will consist in replacing condition (1.21) with assumption (1.27) suggested mainly by the existing studies in literature on gradient constrained problems (*cf.* [2], [5], [19] ÷ [21], [26], [30], [39]) and prove again the thesis of Theorem 8.4.

A first consequence of (1.27) is proved in the following result.

PROPOSITION 8.5. – *Let U be a vector subspace of L_{loc}^1 verifying* (1.3) ÷ (1.5); *for every bounded open set Ω let $K (U, \Omega)$ be a subset*

of U verifying (1.13) $\div$ (1.15), (1.19), (7.8), (1.27) *and let C be given by* (1.22). *Then*

$$C^0 \neq \varnothing. \tag{8.29}$$

Proof. – Let Ω_0 be given by (1.27) and let x_0 be in Ω_0.

Let us denote by L_{x_0} the operator defined by

$$L_{x_0} \; : \; u \in C^1\left(\overline{\Omega}_0\right) \mapsto Du\left(x_0\right) \in \mathbb{R}^n, \tag{8.30}$$

then it is clear that L_{x_0} is a linear, continuous operator onto $\mathbb{R}^n$, therefore by (1.27) and the open mapping theorem we obtain that

$$\left(L_{x_0}\left(K\left(U,\,\Omega\right) \cap C^1\right)\right)^0 \neq \varnothing. \tag{8.31}$$

By Lemma 7.4 we have

$$\left(L_{x_0}\left(K\left(U,\,\Omega\right) \cap C^1\right)\right) \subseteq C, \tag{8.32}$$

hence by (8.31) and (8.32) the thesis follows. ∎

We now study the properties of the interiors in the $C^1\left(\Omega\right)$ topology of the elements $K\left(U,\,\Omega\right) \cap C^1$ when Ω is a bounded open set.

To do this we need to assume (1.25).

LEMMA 8.6. – *Let U be a vector subspace of L^1_{loc} verifying* (1.3), (1.4), (1.25) *and let, for every bounded open set Ω, $K\left(U,\,\Omega\right)$ be a subset of U such that* (1.13), (1.14), (1.16), (1.19), (1.20) *and* (1.27) *hold.*

Then there exist $z_0 \in \mathbb{R}^n$ and $\varepsilon_0 > 0$ such that

$$\left.\begin{array}{c} \Omega \text{ bounded open set,} \\ u \in C^1 \; : \; \|u - u_{z_0}\|_{C^1\,(\Omega)} < \varepsilon_0 \Rightarrow u \in K\left(U,\,\Omega\right) \cap C^1. \end{array}\right\} \tag{8.33}$$

Proof. – By (1.27) let u_0 be in the interior in the $C^1\left(\Omega_0\right)$ topology of $K\left(U,\,\Omega_0\right) \cap C^1$.

Let $x_0 \in \Omega_0$. For every $h \in \mathbb{N}$ let B_h be the open ball centered at x_0 and with radius $\dfrac{1}{h}\,\text{dist}\left(x_0,\,\partial\Omega_0\right)$ and let $\{u_h\}_h$ be a sequence of functions such that

$$u_h \in C^1, \; u_h \text{ affine in } B_h, \; u_h \to u_0 \text{ in } C^1\left(\Omega_0\right). \tag{8.34}$$

By virtue of (8.34), for h large enough, the functions u_h will be in the interior, with respect to the $C^1\left(\Omega_0\right)$ topology, of $K\left(U,\,\Omega_0\right) \cap C^1$. Let us denote by $\tilde{u}_0$ one of such functions, say $\tilde{u}_0 = u_{h_0}$, and set $B_0 = B_{h_0}$, then

$$\left.\begin{array}{c} \tilde{u}_0 \text{ is in the interior in the } C^1\,(\Omega_0) \text{ topology of } K\,(U,\,\Omega_0) \cap C^1, \\ \tilde{u}_0 \text{ affine in } B_0. \end{array}\right\} \quad (8.35)$$

By virtue of (8.35) let $z_0 = D\tilde{u}_0\,(x_0)$, let $\varepsilon_0 > 0$ be such that

$$u \in C^1 \ : \ \|u - \tilde{u}_0\|_{C^1\,(\Omega_0)} < 3\varepsilon_0 \Rightarrow u \in K\,(U,\,\Omega_0) \cap C^1 \qquad (8.36)$$

and set $\delta = \operatorname{diam}\,(B_0)$.

Let us first prove that

$$\left.\begin{array}{c} B \text{ open ball centered at } x_0, \ \operatorname{diam}\,(B) < \delta, \\ v \in C^1 \ : \ \|v - u_{z_0}\|_{C^1\,(B)} < 3\varepsilon_0 \Rightarrow v \in K\,(U,\,B) \cap C^1. \end{array}\right\} \quad (8.37)$$

To do this let B and v be as in (8.37), then since B has C^1 boundary there exists a function w such that

$$\left.\begin{array}{c} w \in C^1, \ w = v + \tilde{u}_0\,(x_0) - u_{z_0}\,(x_0) \text{ in } B \\ \text{and } \|w - \tilde{u}_0\|_{C^1\,(\Omega_0)} < 3\varepsilon_0. \end{array}\right\} \quad (8.38)$$

By (8.38) and (8.36) we get

$$w \in K\,(U,\,\Omega_0) \cap C^1, \qquad (8.39)$$

hence by (8.39), (1.19), (8.38), (1.16) and (1.13) we obtain

$$v \in K\,(U,\,B) \cap C^1, \qquad (8.40)$$

that is (8.37).

Let us now prove that

$$\left.\begin{array}{c} B \text{ open ball, } \operatorname{diam}\,(B) < \delta, \\ v \in C^1 \ : \ \|v - u_{z_0}\|_{C^1\,(B)} < 3\varepsilon_0 \quad \Rightarrow \quad v \in K\,(U,\,B) \cap C^1. \end{array}\right\} \quad (8.41)$$

Let B, v be as in (8.41) and let $y_0 \in B$, then it is clear that

$$\|T\,[y_0 - x_0]\,v - T\,[y_0 - x_0]\,u_{z_0}\|_{C^1\,(x_0 - y_0 + B)}$$
$$= \|T\,[y_0 - x_0]\,v - \langle z_0,\,y_0 - x_0\rangle - u_{z_0}\|_{C^1\,(x_0 - y_0 + B)} < 3\varepsilon_0, \quad (8.42)$$

that $x_0 \in x_0 - y_0 + B$ and that diam $(x_0 - y_0 + B) < \delta$, hence by (8.37) we obtain

$$T\left[y_0 - x_0\right] v - \langle z_0,\, y_0 - x_0 \rangle \in K\left(U,\, x_0 - y_0 + B\right) \cap C^1. \qquad (8.43)$$

By (8.43), (1.13) and (1.14), (8.41) soon follows.

We now prove (8.33).

To this aim let Ω, u be as in (8.33) and let $\{B_s\}_{s=1,2\ldots m}$ be a finite covering of $\overline{\Omega}$ made up by open balls such that

$$\text{diam}\,(B) < \delta,\; \underset{B_s}{\text{osc}}\;u < \varepsilon_0,\; \underset{B_s}{\text{osc}}\;u_{z_0} < \varepsilon_0$$
$$\text{for every } s \in \{1, 2, \ldots, m\}, \qquad (8.44)$$

then by (8.44) we have that

$$\|u - u_{z_0}\|_{C^1\,(B_s)} < 3\varepsilon_0 \quad \text{for every } s \in \{1, 2, \ldots, m\}, \qquad (8.45)$$

hence by (8.45) and (8.41) we obtain

$$u \in K\left(U,\, B_s\right) \cap C^1 \quad \text{for every } s \in \{1, 2, \ldots, m\}. \qquad (8.46)$$

At this point by (8.46), (1.20) and (1.19) we get

$$u \in \bigcap_{s=1}^{m} K\left(U,\, B_s\right) \cap C^1 \subseteq K\left(U,\, \bigcap_{s=1}^{m} B_s\right) \cap C^1$$
$$\subseteq K\left(U,\, \Omega\right) \cap C^1, \qquad (8.47)$$

hence by (8.47) the thesis follows. ■

We can now prove the analogous of Theorem 8.4.

THEOREM 8.7. – *Let U be a vector subspace of L^1_{loc} satisfying* (1.3) ÷ (1.5) *and* (1.25). *For every bounded open set Ω let $K\left(U,\, \Omega\right)$ be a subset of U verifying* (1.13) ÷ (1.17), (1.19), (1.20), (7.8), (1.27) *and let $K^1_C\left(\Omega\right)$ be defined by* (4.12) *with C given by* (1.22).

Then

$$K\left(U,\, \Omega\right) \cap C^1 = K^1_C\left(\Omega\right) \quad \text{for every bounded open set } \Omega. \qquad (8.48)$$

Proof. – Let us prove that

$$K_C^1(\Omega) \subseteq K(U, \Omega) \cap C^1. \tag{8.49}$$

To this aim let $u \in K_C^1(\Omega)$, let z_0 and ε_0 be given by Lemma 8.6 and let $\tau \in [0, 1[$.

Let $t \in [\tau, 1[$ and let $Q_1^t, \ldots, Q_{m_t}^t$ be open cubes centered respectively at the points $x_1^t, \ldots, x_{m_t}^t \in \Omega$ with $\overline{\Omega} \subseteq \bigcup\limits_{j=1}^{m_t} Q_j^t$ and such that, by setting for every $j \in \{1, \ldots, m_t\}$ $z_j^t = \tau Du(x_j^t) - \tau z_0$,

$$\frac{1}{1-t} \| \tau (u - u_{z_0}) - \tau (u(x_j^t) - u_{z_0}(x_j^t))$$
$$+ u_{z_j^t}(x_j^t) - u_{z_j^t} \|_{C^1(Q_j^t)} < \varepsilon_0$$
$$\text{for every } j \in \{1, \ldots, m_t\}. \tag{8.50}$$

Let us now observe that obviously $z_0 \in C$ and that, since $Du(x) \in C$ for every x in Ω and C is convex, we have $z_0 + \frac{1}{t} z_j^t = \frac{\tau}{t} Du(x_j^t) + \left(1 - \frac{\tau}{t}\right) z_0 \in C$ for every $j \in \{1, \ldots, m_t\}$, hence by (1.22) and (1.13) we obtain

$$u_{z_0} + \frac{1}{t} u_{z_j^t} \in K(U, Q_j^t) \quad \text{for every } j \in \{1, \ldots, m_t\}. \tag{8.51}$$

At this point by (8.50), Lemma 8.6, (8.51) and (1.17) we infer

$$u_{z_0} + \tau (u - u_{z_0}) + \tau (u(x_j^t) - u_{z_0}(x_j^t)) + u_{z_j^t}(x_j^t)$$

$$= t \left(u_{z_0} + \frac{1}{t} u_{z_j^t} \right)$$

$$+ (1-t) \left(u_{z_0} + \frac{\tau (u - u_{z_0}) + \tau (u(x_j^t) - u_{z_0}(x_j^t)) + u_{z_j^t}(x_j^t)}{1-t} \right)$$

$$\in K(U, Q_j^t) \quad \text{for every } j \in \{1, \ldots, m_t\}, \tag{8.52}$$

therefore by (8.52), (1.13), (1.20) and (1.19) we obtain

$$u_{z_0} + \tau (u - u_{z_0}) \in \bigcap\limits_{j=1}^{m_t} K(U, Q_j^t) \subseteq K\left(U, \bigcup\limits_{j=1}^{m_t} Q_j^t\right) \subseteq K(U, \Omega)$$
$$\text{for every } \tau \in [0, 1[. \tag{8.53}$$

As $\tau \to 1^-$, by (8.53) and (7.8) we get (8.49).

Finally by (8.49) and Proposition 8.1 the thesis follows. ∎

9. THE GENERAL CASE

In the present section we prove a complete characterization of the sets $K\,(U,\,\Omega)$.

LEMMA 9.1. – *Let* U *be a vector subspace of* L_{loc}^1 *satisfying* (1.4), (1.25) *and let, for every bounded open set* Ω, $\tau\,(\Omega)$ *be a topology on* U *verifying* (1.7), (1.9), (1.10) (i) *and* (1.12).

Let $\{K\,(U,\,\Omega)\ :\ \Omega$ *bounded open set*$\}$ *be a family of subsets of* U *verifying* (1.14), (1.16) $\div$ (1.19). *Then*

$$K\,(U,\,\Omega) \subseteq \bigcap_{\varepsilon>0} (U;\,\tau\,(\Omega)) - cl\,(K\,(U,\,\Omega_\varepsilon^-) \cap C^1) \qquad (9.1)$$

for every bounded open set Ω *such that* $(U;\,\tau\,(\Omega))$ *is sequentially complete.*

Proof. – Let Ω be a bounded open set such that $(U;\,\tau\,(\Omega))$ is sequentially complete and let u be in $K\,(U,\,\Omega)$.

Let $\varepsilon > 0$ and $y \in B_\varepsilon$. By (1.14) and (1.19) it follows that

$$T\,[y]\,u \in K\,(U,\,\Omega - y) \subseteq K\,(U,\,\Omega_\varepsilon^-), \qquad (9.2)$$

hence by (9.2) and (1.17) we have

$$\mathrm{conv}\,(\{T\,[y]\,u\ :\ y \in B_\varepsilon\}) \subseteq K\,(U,\,\Omega_\varepsilon^-). \qquad (9.3)$$

Let us now observe that, by (1.9) and (1.18), $K\,(U,\,\Omega_\varepsilon^-)$ is also $\tau\,(\Omega)$-closed in U, hence by (9.3) we deduce that

$$(U;\,\tau\,(\Omega)) - cl\,(\mathrm{conv}\,(\{T\,[y]\,u\ :\ y \in B_\varepsilon\})) \subseteq K\,(U,\,\Omega_\varepsilon^-). \qquad (9.4)$$

By (1.10) (i) the function $\alpha\,(\cdot)\,T\,[\varepsilon\cdot]\,u$ is continuous with compact support from $\mathbb{R}^n$ to U, therefore, by Proposition 2.4, Proposition 3.2, both applied with $\tau = \tau\,(\Omega)$, and by (9.4) we obtain that the integral $\displaystyle\int_{\mathbb{R}^n} \alpha\,(y)\,T\,[\varepsilon y]\,u\,dy$ exists and that

$$\int_{\mathbb{R}^n} \alpha\,(y)\,T\,[\varepsilon y]\,u\,dy \in K\,(U,\,\Omega_\varepsilon^-). \qquad (9.5)$$

We now observe that by (9.5), Proposition 3.1, (1.25) and (1.16) we obtain

$$u_\varepsilon \in K\,(U,\,\Omega_\varepsilon^-) \cap C^1 \quad \text{for every} \quad \varepsilon > 0, \qquad (9.6)$$

hence by virtue of (9.6) and (1.12), once observed that from (1.19) it follows that $\varepsilon_1 < \varepsilon_2$ implies $K\,(U,\,\Omega_{\varepsilon_1}^-) \subseteq K\,(U,\,\Omega_{\varepsilon_2}^-)$, we have

$$u \in (U;\,\tau\,(\Omega)) - cl\,(K\,(U,\,\Omega_\varepsilon^-) \cap C^1) \quad \text{for every } \varepsilon > 0. \qquad (9.7)$$

By (9.7) inclusion (9.1) follows. ∎

We now prove the reverse inclusion to (9.1).

LEMMA 9.2. – *Let U be a vector subspace of L_{loc}^1 and let, for every bounded open set Ω, $\tau\,(\Omega)$ be a topology on U satisfying* (1.9).

Let $\{K\,(U,\,\Omega)\ :\ \Omega$ bounded open set$\}$ be a family of subsets of U verifying (1.18). *Then*

$$\bigcap_{\varepsilon > 0} (U;\,\tau\,(\Omega)) - cl\,(K\,(U,\,\Omega_\varepsilon^-) \cap C^1) \subseteq \bigcap_{\varepsilon > 0} K\,(U,\,\Omega_\varepsilon^-)$$

for every bounded open set Ω. (9.8)

Proof. – The proof follows easily once observed that, by (1.9) and (1.18), for every $\varepsilon > 0$, $K\,(U,\,\Omega_\varepsilon^-)$ is also $\tau\,(\Omega)$-closed in U. ∎

We can now prove Theorem 1.1.

Proof of Theorem 1.1. – By (1.18), (1.8) (i) condition (7.8) holds, hence by Proposition 7.2 the set C in (1.22) turns out to be closed; moreover by (1.17) it is also convex.

Let Ω be a bounded open set with Lipschitz boundary such that $(U;\,\tau\,(\Omega))$ is sequentially complete.

We observe that by (1.10) (ii) and (1.11) condition (6.7) holds, hence by virtue of (1.10) (i), Lemma 9.1, Lemma 9.2 and Proposition 6.1 applied with $V = U$, $\sigma\,(\Omega) = \tau\,(\Omega)$, and $H\,(\Omega) = K\,(U,\,\Omega)$ we get

$$K\,(U,\,\Omega) = \bigcap_{\varepsilon > 0} K\,(U,\,\Omega_\varepsilon^-)$$

$$= \bigcap_{\varepsilon > 0} (U;\,\tau\,(\Omega)) - cl\,(K\,(U,\,\Omega_\varepsilon^-) \cap C^1). \qquad (9.9)$$

Let us now observe that for every $\varepsilon > 0$ we can find an open set with Lipschitz boundary Ω_ε' such that $\Omega_\varepsilon^- \Subset \Omega_\varepsilon' \Subset \Omega_{\varepsilon/2}^-$, therefore by (1.19) we deduce

$$\bigcap_{\varepsilon > 0} (U; \tau(\Omega)) - cl(K(U, \Omega_\varepsilon^-) \cap C^1)$$
$$= \bigcap_{\varepsilon > 0} (U; \tau(\Omega)) - cl(K(U, \Omega_\varepsilon') \cap C^1). \qquad (9.10)$$

At this point we observe that, by (1.6) and (1.8) (i), the assumptions of Theorem 8.4 are satisfied, hence it turns out that

$$K(U, \Omega_\varepsilon') \cap C^1 = K_C^1(\Omega_\varepsilon') \quad \text{for every } \varepsilon > 0, \qquad (9.11)$$

the sets $K_C^1(\cdot)$ being defined by (4.12); therefore by $(9.9) \div (9.11)$ we get

$$K(U, \Omega) = \bigcap_{\varepsilon > 0} (U; \tau(\Omega)) - cl(K(U, \Omega_\varepsilon') \cap C^1)$$
$$= \bigcap_{\varepsilon > 0} (U; \tau(\Omega)) - cl(K_C^1(\Omega_\varepsilon'))$$
$$= \bigcap_{\varepsilon > 0} (U; \tau(\Omega)) - cl(K_C^1(\Omega_\varepsilon^-)). \qquad (9.12)$$

By (9.12) and Proposition 5.1, once recalled that by (1.6) $K_C^1(\Omega_\varepsilon^-)$ is contained in U for every $\varepsilon > 0$, equality (1.24) follows.

Finally the last part of the thesis follows by Proposition 4.4 and by (1.8) (ii). ∎

By using the results of the last part of section 8 we can now prove Theorem 1.2.

Proof of Theorem 1.2. – We observe that by (1.18) and (1.26) condition (7.8) holds, hence by Proposition 7.2 the set C in (1.22) turns out to be closed and convex, moreover by Proposition 8.5 it also results that C has nonempty interior.

Let Ω be a bounded open set with Lipschitz boundary such that $(U, \tau(\Omega))$ is sequentially complete, then by virtue of (1.10) (i), Lemma 9.1, Lemma 9.2, (1.10) (ii), (1.11) and Proposition 6.1 applied with $V = U$, $\sigma(\Omega) = \tau(\Omega)$ and $H(\Omega) = K(U, \Omega)$ we get

$$K(U, \Omega) = \bigcap_{\varepsilon > 0} K(U, \Omega_\varepsilon^-) = \bigcap_{\varepsilon > 0} (U, \tau(\Omega)) - cl(K(U, \Omega_\varepsilon^-) \cap C^1). \quad (9.13)$$

Let $K_C^1(\Omega)$ be defined by (4.12), then by (9.13) and Theorem 8.7 we obtain

$$K(U, \Omega) = \bigcap_{\varepsilon > 0} (U, \tau(\Omega)) - cl\left(K_C^1(\Omega_\varepsilon^-)\right), \qquad (9.14)$$

hence by (9.14) and Proposition 5.1, once recalled that by (1.25) $K_C^1(\Omega_\varepsilon^-)$ is contained in U for every $\varepsilon > 0$, equality (1.28) follows.

Finally the last part of the thesis follows by Proposition 4.4 and by (1.8) (ii) once observed that if $\overset{\circ}{C} \neq \varnothing$ then obviously $K(U, \Omega) \cap C^1$ has nonempty interior in the $C^1(\Omega)$ topology for every bounded open set Ω. ∎

10. SOME PARTICULAR CASES

In this section, by using the results of section 5, we specialize Theorem 1.1 and Theorem 1.2 to the case of the most common function spaces.

THEOREM 10.1. – *Let $p \in [1, +\infty]$ and let $\{K(L_{loc}^P, \Omega) : \Omega$ bounded open set$\}$ be family of subsets of L_{loc}^p verifying $(1.13) \div (1.20)$ with $U = L_{loc}^p$ and $\tau(\Omega)$ equal to the $L^p(\Omega)$ topology if $p \in [1, +\infty[$ and to the $w^*\text{-}L^\infty(\Omega)$ one if $p = +\infty$.*

Let C be defined by (1.22) with $U = L_{loc}^p$.

Then C is closed, convex and

a) if in addition (1.21) too is satisfied, it results that

$$K(L_{loc}^p, \Omega) = \left\{ u \in L_{loc}^p \; : \; -\int_\Omega uD\varphi \in C \quad for\; every\; \varphi \in C_0^1(\Omega) \right.$$
$$\left. with\; \varphi \geq 0, \int_\Omega \varphi = 1 \right\}$$

for every bounded open set Ω with Lipschitz boundary; (10.1)

b) if in addition (1.27) is satisfied, it results that C has also interior points and that (10.1) holds.

Conversely, given a closed convex subset C of $\mathbb{R}^n$ it turns out that conditions $(1.13) \div (1.21)$ with $U = L_{loc}^p$ and $\tau(\Omega)$ equal to the $L^p(\Omega)$ topology if $p \in [1, +\infty[$ and to the $w^*\text{-}L^\infty(\Omega)$ one if $p = +\infty$ are satisfied by $K(U, \Omega) = K_C^p(\Omega)$ and that if in addition C has also interior points then (1.27) too holds.

Proof. – The proof follows by Theorem 1.1 and Theorem 1.2 applied with $U = L_{loc}^p$ and $\tau(\Omega)$ equal to the $L^p(\Omega)$ topology if $p \in [1, +\infty[$

and to the w^*-L^∞ (Ω) one if $p = +\infty$, once observed that, for every bounded open set Ω, $(L^p_{loc}, L^p(\Omega))$ if $p \in [1, +\infty[$ and $(L^\infty_{loc}, w^*$-$L^\infty(\Omega))$ if $p = +\infty$ are sequentially complete Hausdorff locally convex topological vector spaces. ∎

In the case of BV_{loc} functions the following result holds.

THEOREM 10.2. – *Let* $\{K(BV_{loc}, \Omega) : \Omega$ *bounded open set*} *be a family of subsets of* BV_{loc} *verifying* $(1.13) \div (1.20)$ *with* $U = BV_{loc}$ *and* $\tau(\Omega)$ *equal to the* w^*-$BV(\Omega)$ *topology.*

Let C *be defined by* (1.22) *with* $U = BV_{loc}$ *and* $K_C(\Omega)$ *by* (4.14).

Then C *is closed, convex and*

a) *if in addition* (1.21) *too is satisfied, it results that*

$$K(BV_{loc}, \Omega) = K_C(\Omega)$$

for every bounded open set Ω *with Lipschitz boundary;* (10.2)

b) *if in addition* (1.27) *is satisfied, it results that* C *has also interior points and that* (10.2) *holds.*

Conversely, given a closed convex subset C *of* $\mathbb{R}^n$ *it turns out that conditions* $(1.13) \div (1.21)$ *with* $U = BV_{loc}$ *and* $\tau(\Omega)$ *equal to the* w^*-$BV(\Omega)$ *topology are satisfied by* $K(U, \Omega) = K_C(\Omega)$ *and that if in addition* C *has also interior points then* (1.27) *too holds.*

Proof. – Let us first observe that, for every bounded open set Ω with Lipschitz boundary $(BV_{loc}, w^*$-$BV(\Omega))$ is a sequentially complete Hausdorff locally convex topological vector space.

Therefore the proof follows by Theorem 1.1 and Theorem 1.2 applied with $U = BV_{loc}$ and $\tau(\Omega)$ equal to the w^*-$BV(\Omega)$ topology and the Proposition 4.3. ∎

We now treat the case of $W^{1,p}_{loc}$ functions.

THEOREM 10.3. – *Let* $p \in [1, +\infty]$ *and let* $\{K(W^{1,p}_{loc}, \Omega) : \Omega$ *bounded open set*} *be a family of subsets of* $W^{1,p}_{loc}$ *verifying* $(1.13) \div (1.20)$ *with* $U = W^{1,p}_{loc}$ *and* $\tau(\Omega)$ *equal to the* $W^{1,p}(\Omega)$ *topology if* $p \in [1, +\infty[$ *and to the* w^*-$W^{1,\infty}(\Omega)$ *one if* $p = +\infty$.

Let C *be defined by* (1.22) *with* $U = W^{1,p}_{loc}$ *and* $K^{1,p}_C(\Omega)$ *by* (4.13).

Then C *is closed, convex and*

a) *if in addition* (1.21) *too is satisfied, it results that*

$$K(W^{1,p}_{loc}, \Omega) = K^{1,p}_C(\Omega)$$

for every bounded open set Ω *with Lipschitz boundary;* (10.3)

b) *if in addition* (1.27) *is satisfied, it results that* C *has also interior points and that* (10.3) *holds.*

Conversely, given a closed convex subset C *of* $\mathbb{R}^n$ *it turns out that conditions* (1.13) ÷ (1.21) *with* $U = W_{loc}^{1,p}$ *and* $\tau(\Omega)$ *equal to the* $W^{1,p}(\Omega)$ *topology if* $p \in [1, +\infty[$ *and to the* $w^*\text{-}W^{1,\infty}(\Omega)$ *one if* $p = +\infty$ *are satisfied by* $K(U, \Omega) = K_C^{1,p}(\Omega)$ *and that if in addition* C *has also interior points then* (1.27) *too holds.*

Proof. – Let us first observe that for every bounded open set Ω with Lipschitz boundary the spaces $(W_{loc}^{1,p}, W^{1,p}(\Omega))$, $p \in [1, +\infty[$, and $(W_{loc}^{1,\infty}, w^*\text{-}W^{1,\infty}(\Omega))$ are sequentially complete Hausdorff locally convex topological vector spaces.

Hence the thesis follows by Theorem 1.1 and Theorem 1.2 applied with $U = W_{loc}^{1,p}$ and $\tau(\Omega)$ equal to the $w\text{-}W^{1,p}(\Omega)$ topology if $p \in [1, +\infty[$ and to the $w^*\text{-}W^{1,\infty}(\Omega)$ one if $p = +\infty$, and by Proposition 4.3. ∎

In the case of C^1 functions the following result holds.

THEOREM 10.4. – *Let* $\{K(C^1, \Omega) : \Omega$ *bounded open set*$\}$ *be a family of subsets of* C^1 *verifying* (1.13) ÷ (1.20) *and* (1.27) *with* $U = C^1$ *and* $\tau(\Omega)$ *equal to the* $C^1(\Omega)$ *topology.*

Let C *be defined by* (1.22) *with* $U = C^1$ *and* K_C^1 *by* (4.12).

Then C *is closed, convex, has nonempty interior and*

$$K(C^1, \Omega) = K_C^1(\Omega) \quad for \; every \; bounded \; open \; set \; \Omega. \qquad (10.4)$$

Conversely, given a closed convex subset C *of* $\mathbb{R}^n$ *with nonempty interior it turns out that conditions* (1.13) ÷ (1.20) *and* (1.27) *with* $U = C^1$ *and* $\tau(\Omega)$ *equal to the* $C^1(\Omega)$ *topology are satisfied by* $K(U, \Omega) = K_C^1(\Omega)$.

Proof. – The proof follows by Theorem 8.7 and by Proposition 4.4 applied with $U = C^1$ and $\tau(\Omega)$ equal to the $C^1(\Omega)$ topology, and by Proposition 4.3. ∎

11. AN APPLICATION

In the present section we apply some of the results obtained in this paper to the problem of the homogenization of the elastic-plastic torsion of a cylindrical bar.

As already mentioned in the introduction, in the theory of homogenization of the elastic-plastic torsion of a cylindrical bar it is interesting to study

the "convergence" of sets of the type

$$K_h^0\,(\Omega) = \{u \in W_{loc}^{1,\,\infty} \; : \; u = 0 \quad \text{on} \quad \partial\Omega,$$

$$|Du\,(x)| \le \varphi\,(hx) \quad \text{for a.e. } x \text{ in } \Omega\}, \qquad (11.1)$$

h being an integer number, Ω a bounded open set and φ a function verifying

$$\varphi \; : \; \mathbb{R}^n \to [0,\,+\infty[,\; \varphi \text{ bounded and 1-periodic in each variable } x_i, \quad (11.2)$$

to a set of the type

$$K_\infty^0\,(\Omega) = \{u \in W_{loc}^{1,\,\infty} \; : \; u = 0 \quad \text{on} \quad \partial\Omega,$$

$$Du\,(x) \in C_\infty \quad \text{for a.e. } x \text{ in } \Omega\}, \qquad (11.3)$$

C_∞ being described by the formula

$$C_\infty = \{z \in \mathbb{R}^n \; : \; \text{there exists } w \in W_{loc}^{1,\,\infty}$$

$$\text{with } w \text{ 1-periodic in each variable } x_i$$

$$\text{and } |z + Dw\,(x)| \le \varphi\,(x) \quad \text{for a.e. } x \text{ in }]0,\,1[^n\}. \quad (11.4)$$

This study is just the one we want to carry out in this section by using Theorem 10.3 together with Γ-convergence theory (*cf.* [32], [34]).

Let us briefly recall the definition and the main properties of Γ-convergence. We refer to [34] and [2] for complete references.

Let $(U,\,\tau)$ be a topological space satisfying the first countability axiom and let F_h, $h \in \mathbb{N}$, F' and F'' be functionals from U to $[-\infty + \infty]$.

Let $u \in U$, we say that

$$F'\,(u) = \Gamma^-\,(\tau)\,\liminf_{\substack{h \to \infty \\ v \to u}} F_h(v) \qquad (11.5)$$

if for every $\{u_h\}_h \subseteq U$ such that $u_h \overset{\tau}{\to} u$ it results

$$F'\,(u) \le \liminf_{h \to \infty} F_h(u_h) \qquad (11.6)$$

and if there exists $\{v_h\}_h \subseteq U$ such that $v_h \overset{\tau}{\to} u$ and

$$F'\,(u) \ge \liminf_{h \to \infty} F_h(v_h); \qquad (11.7)$$

we say that

$$F''(u) = \Gamma^-(\tau) \limsup_{\substack{h \to \infty \\ v \to u}} F_h(v) \tag{11.8}$$

if (11.6) and (11.7) hold with the operator "$\liminf_{h\to\infty}$" replaced by "$\limsup_{h\to\infty}$".

When $F'(u) = F''(u)$ we say that there exists the limit $\Gamma^-(\tau) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v)$.

We recall that the functional F' and F'' in (11.7) and (11.8) are τ-lower semicontinuous on U and that, if in addition (U, τ) verifies the second countability axiom, it results

there exists $\{F_{h_k}\}_k \subseteq \{F_h\}_h$ such that the limit

$$\Gamma^-(\tau) \lim_{\substack{h \to \infty \\ v \to u}} F_{h_k}(v) \text{ exists for every } u \in U. \tag{11.9}$$

For every subset S of $W_{loc}^{1,\infty}$ we set

$$\mathbb{I}_S : u \in W_{loc}^{1,\infty} \mapsto \begin{cases} 0 & \text{if } u \in S \\ +\infty & \text{if } u \notin S \end{cases} \tag{11.10}$$

and prove the following result.

THEOREM 11.1. – *Let φ be as in (11.2), let C_∞ be given by (11.4) and, for every bounded open set Ω and $h \in \mathbb{N}$, $K_h^0(\Omega)$ by (11.1) and $K_\infty^0(\Omega)$ by (11.3).*

Assume that

$$(C_\infty)^\circ \neq \varnothing, \tag{11.11}$$

then

$$\mathbb{I}_{K_\infty^0(\Omega)}(u) = \Gamma^-(C^0(\Omega)) \lim_{\substack{h \to \infty \\ v \to u}} \mathbb{I}_{K_h^0(\Omega)}(v)$$

for every bounded open set Ω, u in $W_{loc}^{1,\infty}$. \tag{11.12}

We remark that the convergence result in Theorem 11.1 turns out to be equivalent to the Kuratowski notion of convergence for sequences of sets (*cf.* [45]).

Moreover we observe that a sufficient condition on φ to get (11.11) is the following one proposed in [20]

there exist $m > 0$, $\theta \in \left[0, \dfrac{1}{2}\right[$ such that

$$m < \varphi(x) \text{ for a.e. } x \text{ in }]0, 1[^n \setminus \left]\dfrac{1}{2} - \theta, \dfrac{1}{2} + \theta\right[^n. \qquad (11.13)$$

Let φ be a function as in (11.2), for every bounded open set Ω and $h \in \mathbb{N}$ let us define the set

$$K_h(\Omega) = \{u \in W_{loc}^{1,\infty} : |Du(x)| \leq \varphi(hx) \quad \text{for a.e. } x \text{ in } \Omega\} \qquad (11.14)$$

and its characteristic function $\mathbb{I}_h(\Omega, \cdot) = \mathbb{I}_{K_h(\Omega)}(\cdot)$, moreover let us define the following limits

$$F'(\Omega, u) = \Gamma^-(C^0(\Omega)) \liminf_{\substack{h \to \infty \\ v \to u}} \mathbb{I}_h(\Omega, v) \quad u \in W_{loc}^{1,\infty}, \qquad (11.15)$$

$$F''(\Omega, u) = \Gamma^-(C^0(\Omega)) \limsup_{\substack{h \to \infty \\ v \to u}} \mathbb{I}_h(\Omega, v) \quad u \in W_{loc}^{1,\infty}. \qquad (11.16)$$

We are going to prove that for every u in $W_{loc}^{1,\infty}$ the inner regular envelopes (see Section 6 for the definition) $F'_-(\cdot, u)$ and $F''_-(\cdot, u)$ of the functionals in (11.15) and (11.16) agree for every bounded open set and that they are equal to the characteristic function $\mathbb{I}_\infty(\Omega, \cdot)$ of a set of the type

$$K_\infty(\Omega) = \{u \in W_{loc}^{1,\infty} : Du(x) \in C_\infty \quad \text{for a.e. } x \text{ in } \Omega\}, \qquad (11.17)$$

C_∞ being an explicitly described closed convex subset of $\mathbb{R}^n$.

By virtue of this we will prove that for every bounded open set Ω the $\Gamma^-(C^0(\Omega))$ limit of the sequence $\{\mathbb{I}_{K_h^0}(\Omega)\}_h$ exists and is equal to $\mathbb{I}_{K_\infty^0}(\Omega)$, $K_\infty^0(\Omega)$ being given by (11.3).

From this result the "convergence" one for the sets in (11.1) to the one in (11.3) will follow.

For every $h \in \mathbb{N}$ let us set $\varphi_h(x) = \varphi(hx)$, $x \in \mathbb{R}^n$.

Let us prove some properties of F'' in (11.16).

LEMMA 11.2. – *Let φ be as in* (11.2) *and let F'' be given by* (11.16). *Then*

$$F''_-(\Omega - x_0, T[x_0] u) = F''_-(\Omega, u)$$

for every bounded open set Ω, u in $W_{loc}^{1,\infty}$ and x_0 in $\mathbb{R}^n$. (11.18)

Proof. – Let Ω, u, x_0 be as in (11.18), let us prove that

$$F''_- (\Omega - x_0, T [x_0] u) \leq F''_- (\Omega, u). \qquad (11.19)$$

To do this we can assume that $F''_- (\Omega, u) = 0$. Let A, B be open sets with $B \Subset A \Subset \Omega$ and let $\{u_h\}_h \subseteq W^{1,\infty}_{loc}$ be such that $u_h \to u$ in $C^0 (A)$ and $|Du_h| \leq \varphi_h$ a.e. in A.

Let $\{m_h\}_h \subseteq \mathbb{Z}^n$ be such that $\lim_h m_h/h = x_0$; for every $h \in \mathbb{N}$ we define the functions $v_h = u_h \left(\cdot + \dfrac{m_h}{h} \right)$, then obviously $v_h \to T [x_0] u$ in $C^0 (B - x_0)$ and

$$|Dv_h (x)| = \left| Du_h \left(x + \frac{m_h}{h} \right) \right| \leq \varphi (hx + m_h) = \varphi_h (x)$$

$$\text{for a.e. } x \text{ in } B - x_0 \text{ and every } h \in \mathbb{N}. \qquad (11.20)$$

By (11.20) we have that $v_h \in K_h (B - x_0)$ for every $h \in \mathbb{N}$, that is

$$F'' (B - x_0, T [x_0] u) = 0$$

$$\text{for every open set } B \text{ with } B \Subset \Omega, \qquad (11.21)$$

hence by (11.21) inequality (11.19) soon follows.

Finally, by simmetry, (11.18) follows by (11.19). ∎

LEMMA 11.3. – *Let φ be as in (11.2) and let F'' be given by (11.16). Then*

$$F'_- \left(\frac{1}{t} \Omega, O_t u \right) = F'_- (\Omega, u)$$

for every bounded open set Ω, u in $W^{1,\infty}_{loc}$ and $t > 0$. (11.22)

Proof. – Let Ω, u, t be as in (11.22), let us prove that

$$F'_- \left(\frac{1}{t} \Omega, O_t u \right) \leq F'_- (\Omega, u). \qquad (11.23)$$

Obviously we can assume that $F'_- (\Omega, u) = 0$. Let A, B be open sets with $B \Subset A \Subset \Omega$, let $\{u_h\}_h \subseteq W^{1,\infty}_{loc}$ and let $\{k_h\}_h$ be an increasing sequence of integer numbers such that $u_h \to u$ in $C^0 (A)$ and $|Du_{k_h}| \leq \varphi_{k_h}$ a.e. in A.

Let $\{t_h\}_h \subseteq \mathbb{N}$ be such that $\lim_h t_h/k_h = t$; for every $h \in \mathbb{N}$ we define the functions $w_h = \dfrac{k_h}{t_h} u_h \left(\dfrac{t_h}{k_h} \cdot \right)$, then obviously $w_h \to O_t u$ in $C^0 \left(\dfrac{1}{t} B \right)$ and

$$|Dw_h(x)| = \left|Du_h\left(\frac{t_h}{k_h}x\right)\right| \le \varphi(t_h x)$$

for a.e. x in $\frac{1}{t}B$ and every $h \in \mathbb{N}$. (11.24)

By (11.24) we have that $w_h \in K_{t_h}\left(\frac{1}{t}B\right)$ for every $h \in \mathbb{N}$, from which we obtain

$$F'\left(\frac{1}{t}B, O_t u\right) = 0 \quad \text{for every open set } B \text{ with } B \Subset \Omega, \qquad (11.25)$$

hence by (11.25) inequality (11.26) soon follows, once observed that

$$\sup_{B \Subset \Omega} F'\left(\frac{1}{t}B, O_t u\right) = \sup_{E \Subset \frac{1}{t}\Omega} F'(E, O_t u).$$

Finally, by simmetry, (11.22) follows by (11.23). ∎

We can now prove the representation result for F'_- and F''_-.

For every bounded open set Ω let us set

$$K_\infty(\Omega) = \{u \in W^{1,\infty}_{loc} : Du(x) \in C_\infty \quad \text{for a.e. } x \text{ in } \Omega\}. \qquad (11.26)$$

PROPOSITION 11.4. – *Let φ be as in* (11.2), *let F', F'' be given by* (11.15) *and* (11.16), *C_∞ by* (11.4) *and, for every bounded open set Ω, $K_\infty(\Omega)$ by* (11.26).

Assume that (11.11) *holds, then*

$$F'_-(\Omega, u) = F''_-(\Omega, u) = \mathbb{I}_{K_\infty(\Omega)}(u)$$

for every bounded open set Ω, u in $W^{1,\infty}_{loc}$. (11.27)

Proof. – Let us preliminarly observe that by (11.9), since we are going to describe explicitly the inner regular envelopes of the functionals in (11.15) and (11.16) for every bounded open set Ω, it is not restrictive to assume that

$$F'_-(\Omega, u) = F''_-(\Omega, u)$$

for every bounded open set Ω, u in $W^{1,\infty}_{loc}$. (11.28)

By (11.28) it is clear that for every bounded open set Ω there exists a subset $K\left(\Omega\right)$ of $W_{loc}^{1,\,\infty}$ such that

$$F_{-}'\left(\Omega,\,u\right) = F_{-}''\left(\Omega,\,u\right) = \mathbb{I}_{K\left(\Omega\right)}\left(u\right)$$

$$\text{for every bounded open set } \Omega,\,u \text{ in } W_{loc}^{1,\,\infty}. \tag{11.29}$$

By (11.29) the family $\{K\left(\Omega\right) \,:\, \Omega \text{ bounded open set}\}$ satisfies all the assumptions of Theorem 10.3 with $p = +\infty$. In fact (1.13), (1.16), (1.19) trivially hold, (1.14) and (1.15) respectively follow by Lemma 11.2 and Lemma 11.3, (1.17) follows by the convexity of $F_{-}''\left(\Omega,\cdot\right)$, (1.18) by the $C^{0}\left(\Omega\right)$ lower semicontinuity of $F_{-}''\left(\Omega,\cdot\right)$ for every bounded open set Ω; moreover (1.20) comes for example by (11.11) and Proposition 2.5 in [26] and (1.27) by Proposition 3.5 again in [26], both applied with $f \equiv 0$.

By Theorem 10.3 we deduce the existence of a closed convex subset C of $\mathbb{R}^{n}$ with nonempty interior such that

$$K\left(\Omega\right) = \{u \in W_{loc}^{1,\,\infty} : Du\left(x\right) \in C \text{ for a.e. } x \text{ in } \Omega\}$$

$$\text{for every bounded open set } \Omega \text{ with Lipschitz boundary,} \tag{11.30}$$

moreover, once recalled that

$$F_{-}''\left(\Omega,\,u\right)$$

$$= \sup\{F_{-}''\left(A,\,u\right) \,:\, A \text{ open set with Lipschitz boundary, } A \Subset \Omega\}$$

$$\text{for every bounded open set } \Omega,\,u \text{ in } W_{loc}^{1,\,\infty}, \tag{11.31}$$

by (11.29), (11.31), and (11.30) we obtain

$$K\left(\Omega\right) = \bigcap_{\substack{A \Subset \Omega \\ A \text{ with Lipschitz boundary}}} K\left(A\right)$$

$$= \{u \in W_{loc}^{1,\,\infty} \,:\, Du\left(x\right) \in C \quad \text{for a.e. } x \text{ in } \Omega\}$$

$$\text{for every bounded open set } \Omega, \tag{11.32}$$

hence in order to complete the proof we only have to verify that

$$C = C_{\infty}. \tag{11.33}$$

Equality (11.33) follows by (11.29) and Lemma 4.1 and Lemma 4.3 in [26], both applied with $f \equiv 0$. ∎

We can prove Theorem 11.1.

Proof of Theorem 11.1. – Let Ω, u be as in (11.12).

If $u = 0$ on $\partial\Omega$ the proof follows by Theorem 11.3 and Proposition 2.4 in [26] applied to $f \equiv 0$.

Otherwise the thesis follows once observed that $I_{K_\infty^0 (\Omega)}(u) = +\infty$ and that for every $\{u_h\}_h \subseteq W_{loc}^{1,\infty}$ such that $u_h \to u$ in $C^0(\Omega)$ it must result that $u_h \notin K_h(\Omega)$ definitively in h. ∎

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