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GUY KATRIEL

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Mountain pass theorems and global homeomorphism theorems

by

Guy KATRIEL

Department of Mathematics, University of Haifa
Haifa 31999, Israel

ABSTRACT. — We show that mountain-pass theorems can be used to derive global homeomorphism theorems. Two new mountain-pass theorems are proved, generalizing the "smooth" mountain-pass theorem, one applying in locally compact topological spaces, using Hofer's concept of mountain-pass point, and another applying in complete metric spaces, using a generalized notion of critical point similar to the one introduced by Ioffe and Schwartzman. These are used to prove global homeomorphism theorems for certain topological and metric spaces, generalizing known global homeomorphism theorems for mappings between Banach spaces.

Key words : Mountain-pass theorems, global homeomorphism theorems.

RÉSUMÉ. — On montre que des théorèmes du col peuvent être utilisés pour dériver des théorèmes d'homéomorphisme global. On prouve deux nouveaux théorèmes du col qui généralisent le théorème du col « lisse », l'un s'appliquant à des espaces topologiques localement compacts, avec emploi du concept du point de col de Hofer, et l'autre s'appliquant à des espaces métriques complets, en utilisant un concept généralisé de point critique ressemblant à celui introduit par Ioffe et Schwartzman. Ils sont

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utilisés pour prouver des théorèmes d'homéomorphisme global pour certains espaces topologiques et métriques généralisant des théorèmes d'homéomorphisme global connus concernant des applications entre espaces de Banach.

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1. INTRODUCTION

Mountain-pass theorems and global homeomorphism theorems are among the important tools for dealing with nonlinear problems in analysis. The main object of this work is to show that mountain-pass theorems can be used to prove global homeomorphism theorems. The technique presented here can be used to prove new global homeomorphism theorems as well as give elegant proofs of known theorems. The idea is extremely simple and will be demonstrated presently. Let us first remind ourselves of the simplest mountain-pass theorem for finite-dimensional spaces.

THEOREM 1.1. – *Let* X *be a finite-dimensional Euclidean space,* $f: X \to R$ *be a* C^1 *function with* $f(x) \to \infty$ *as* $\|x\| \to \infty$. *If* f *has two strict local minima* x_0 *and* x_1, *then it has a third critical point* x_2 *with* $f(x_2) > \max\{f(x_0), f(x_1)\}$.

We now use theorem 1.1 to prove the following global homeomorphism theorem of Hadamard:

THEOREM 1.2. – *Let* X, Y *be finite dimensional Euclidean spaces,* $F: X \to Y$ *be a* C^1 *mapping satisfying*:

(1) $F'(x)$ *is invertible for all* $x \in X$.

(2) $\|F(x)\| \to \infty$ *as* $\|x\| \to \infty$.

Then F *is a diffeomorphism of* X *onto* Y.

Proof. – By (1) and the inverse function theorem, F is an open mapping (takes open sets to open sets). Thus F(X) is open in Y. Condition (2) easily implies that F(X) is closed in Y (here the finite-dimensionality is used, that is the fact that closed bounded sets are compact). Hence since Y is connected, F(X) = Y. In order to show that F is a diffeomorphism it remains to show that F is one-to-one. Suppose by way of contradiction that $F(x_0) = F(x_1) = y$. We define a function $f: X \to R$ by $f(x) = \frac{1}{2}\|F(x) - y\|^2$. f is C^1 and: $f'(x) = F'^*(x)(F(x) - y)$. By (2), $f(x) \to +\infty$ as $\|x\| \to \infty$. Clearly x_0 and x_1 are (global) minima of f, and since by the inverse function theorem $F(x) \neq F(x_i)$ for x in a neighborhood of $x_i (i=0, 1)$, we get that x_0, x_1 are strict local minima. Therefore we conclude from theorem 1.1 that there exists a third critical point x_2 with $f(x_2)>0$. So we have $\|F(x_2)-y\|>0$, so $F(x_2)-y \neq 0$. But since x_2 is a critical point of f we have: $F'^*(x_2)(F(x_2)-y)=0$. But this contradicts the invertability of $F'(x_2)$ (assumption 1). ■

Our aim is to use the simple method of this proof in more general settings. In section 3 we prove a global homeomorphism theorem valid in certain topological spaces. For this we will need to prove a topological mountain-pass theorem, which we do in section 2. This theorem is, I believe, of interest in itself, since it shows that an "analytic" theorem (theorem 1.1), usually proved by "analytic" means (deformation along gradient curves), is in fact, a consequence of a general-topological theorem. In section 4 we compare the results of section 3 with known global homeomorphism theorems based on the monodromy argument. The global homeomorphism theorem of section 3 does not, however, apply to spaces which are not locally compact. The same thing happens in the smooth case, where theorem 1.1 above is not true for infinite-dimensional spaces, and we need an extra assumption, the Palais-Smale condition, for it to be true. The Palais-Smale condition, however, is a metric condition, and makes no sense in a topological space. Therefore in section 5 we define a new concept of critical point on a metric space, formulate a Palais-Smale condition for functions on metric spaces, and prove a mountain pass theorem in metric spaces. This theorem is applied in section 6 to generalize Banach space global homeomorphism theorems to certain metric spaces, and also to nonsmooth mappings on Banach spaces. In Section 7 we use a recent mountain pass theorem of Schechter to prove another type of global homeomorphism theorem, proved first by Hadamard.

2. A TOPOLOGICAL MOUNTAIN-PASS THEOREM

In this section we will prove a mountain-pass theorem which is valid in a large class of topological spaces. This may initially seem impossible, since the standard mountain-pass theorems like theorem 1.1 contain the concept of "critical point", and in order to formulate this concept we need a differentiable structure. However, certain critical points can be characterized topologically, like local minima and maxima, and also mountain-pass points, a concept introduced by Hofer for functions on Banach spaces, but which makes sense in any topological space.

DEFINITION 2.1. – *Let* X *be a topological space,* $f: X \to R$ *a function.* $x \in X$ *is called a global mountain-pass* (MP) *point of* f *if for every neighborhood* N *of* x *the set* $\{y \mid f(y)<f(x)\} \cap N$ *is disconnected.* x *is called a*

local MP point *of f if there is a neighborhood* M *of x such that x is a global MP point for $f|_M$*.

Hofer [5] proved in the Banach space case that the critical point ensured by the mountain-pass theorem is in fact either a local minimum point or a global MP point. Note that if x is a differentiable manifold and f is C^1 then any local MP point is automatically a critical point of f. This follows at once from the following "linearization lemma":

LEMMA 2.1. – *If X is a Banach space, $U \subset X$ is open, $f: U \to R$ is C^1, $x \in U$ and $f'(x) \neq 0$, then there is an open ball B in X with center at 0, a diffeomorphism $H: B \to H(B) \subset U$ with $H(0) = x$, and a linear functional l on X such that $f(H(w)) = l(w) + f(x)$ for $w \in B$. Moreover, we may choose $l = f'(x)$, and H so that $H'(0) = I$.*

Proof. – Let $l = f'(x)$, and let Z be the subspace of X annihilated by l. Choose $v \notin Z$, and let π be the projection of X onto Z defined by: $\pi(h) = h - [l(h)/l(v)]\, v$. Define $F: U \to X$ by:

$$F(y) = \pi(y - x) + [(f(y) - f(x))/l(v)]\, v.$$

The derivative of F at x is given by: $F'(x)(h) = \pi(h) + (l(h)/l(v))\, v = h$, that is $F'(x) = I$, so by the inverse function theorem F is invertible in a neighborhood of $0 \in X$, that is, there is a diffeomorphism H from an open ball B with center 0, with $H(0) = x$ and $F(H(w)) = w$ for $w \in B$, that is:

$$\pi(H(w) - x) + [(f(H(w)) - f(x))/l(v)]\, v = w$$

for $w \in B$. Applying l to both sides we get: $f(H(w)) - f(x) = l(w)$ for $w \in B$, which is what we wanted. ■

This lemma shows that near a regular point (*i.e.* a non-critical point) a smooth function looks topologically like a nontrivial linear functional, and since a linear functional has no MP points, we get that a regular point cannot be a MP point, so a MP point must be a critical point.

Now we will see that the concept of MP point not only allows us to gain a better understanding of what the critical point ensured by the mountain-pass theorem is like, as in the smooth case, but also to formulate a mountain-pass theorem in a context where the notion of critical-point is nonexistent. All topological spaces will be assumed to be regular.

DEFINITION 2.2. – *A topological space X will be called compactly connected if for each pair $x_0, x_1 \in X$ there is a compact connected set $K \in X$ with $x_0, x_1 \in K$.*

DEFINITION 2.3. – *A function $f: X \to R$ will be said to be increasing at infinity if for every $x \in X$ there is a compact set $K \subset X$ such that $f(z) > f(x)$ for every $z \notin K$.*

We note here that a topological space admiting a continuous function f which is increasing at infinity must be locally compact, since the sets

$\{x \mid f(x) < c\}$ $(-\infty < c < \sup_X f)$ form an open covering of the space by pre-compact sets.

LEMMA 2.2. – *Let* X *be compactly connected, locally connected and locally compact and* C *an open and connected subset of* X. *Then* C *is compactly connected.*

Proof. – Let $x_0 \in C$ and let B be the union of all compact connected sets contained in C and containing x_0. We must show $B = C$. Since C is connected it suffices to show that B is open and closed in C. For $x \in B$, let N_0 be a compact neighborhood of x (X is locally compact). Let O_1 and O_2 be nonintersecting open sets containing x and C^c respectively (we use here the regularity of X). Let $N_1 = N_0 \cap \bar{O}_1$. Then N_1 is a compact neighborhood of x contained in C. By local connectedness of X there is a connected neighborhood $N \subset N_1$ of x, so $\bar{N}$ is a compact connected neighborhood of x contained in C. Since $x \in B$ there is a compact connected set $K \subset C$ containing x_0 and x. Since $K \cup \bar{N}$ is compact, connected, and contained in C, we see $N \subset B$, so B is open. To see that B is closed in C, let $x \in \bar{B} \cap C$. As previously, let $\bar{N}$ be a compact connected neighborhood of x contained in C. Since $x \in \bar{B}$, there is some $y \in B \cap N$. Let $K \subset C$ be a compact connected set containing x_0 and y. Then $K \cup \bar{N}$ is a compact connected set containing x_0 and x and contained in C, so $x \in B$, hence B is closed. ■

THEOREM 2.1. – *Let* X *be a locally connected, compactly connected topological space,* $f: X \to R$ *continuous and increasing at infinity. Suppose* $x_0, x_1 \in X$, $S \subset X$ *separates* x_0 *and* x_1 *(that is,* x_0 *and* x_1 *lie in different components of* $X - S$*), and:*

$$\max\{f(x_0), f(x_1)\} < \inf_{x \in S} f(x) = p \tag{1}$$

Then there is a point x_2 *which is either a local minimum or a global MP point of* f, *with:* $f(x_2) > \max\{f(x_0), f(x_1)\}$.

Proof. – Let Γ be the set of all connected compact subsets of X containing both x_0 and x_1. Since X is compactly connected, Γ is nonempty. We define $\Phi : \Gamma \to R$ by $(A \in \Gamma)$:

$$\Phi(A) = \max_{x \in A} f(x).$$

Let $c = \inf\{\Phi(A) \mid A \in \Gamma\}$. Since S separates x_0 and x_1, every $A \in \Gamma$ intersects S, so $c \geqq p$. We will now show that the infimum c is attained, that is, there is $B \in \Gamma$ with $\Phi(B) = c$. To do this, we choose a sequence $\{A_n\} \subset \Gamma$ such that: $\Phi(A_1) \geqq \Phi(A_2) \geqq \Phi(A_3) \geqq \ldots$ and $\Phi(A_n) \to c$. By the assumption that f is increasing at infinity there is a compact set K such that $f(z) > \Phi(A_1)$ for $z \notin K$. It follows that $A_n \subset K$ for all n. This implies that the sets $B_k = \overline{\bigcup \{ A_n \mid n \geqq k \}}$ are contained in K, and they are also connected since each A_n is connected and $x_0 \in A_n$ for all n. So B_k form a descending sequence of closed connected subsets of a compact set, hence by an elementary topological result (*see* [3], theorem 4.A.8) $B = \bigcap \{ B_k \mid k \geqq 1 \}$ is compact and connected, and contains x_0 and x_1. So $B \in \Gamma$ and it is easy to see that $\Phi(B) = c$.

We will now show that $f^{-1}(c)$ contains either a local minimum or a global MP point of f. Suppose by way of contradiction that this is not the case. Let C be the connected component of $X - f^{-1}(c)$ containing x_0. C is open because X is locally connected. We now show that:

$$\bar{C} \subset C \cup f^{-1}(c) \tag{2}$$

Suppose $x \in \bar{C}$ and $x \notin f^{-1}(c)$. Then the set $C \cup \{ x \}$ is connected (since it lies between C and $\bar{C}$) and contained in $X - f^{-1}(c)$, but since C is maximal among such sets we must have $C \cup \{ x \} = C$ so $x \in C$. We shall show that $B \subset \bar{C}$, so that together with 2 we get: $B \subset C \cup f^{-1}(c)$. But this implies that $x_1 \in C$ (since $x_1 \in f^{-1}(c)$ is impossible by 1 and the fact that $c \geqq p$). However, since C is open and connected and X is compactly connected and locally compact (since it admits a function increasing at infinity), C is compactly connected by lemma 2.2. So there is a compact connected set $K \subset C$ containing x_0 and x_1, and since $f(x) < c$ for $x \in C$, we get: $\Phi(K) < c$ — in contradiction to the definition of c. So we only have to show that $B \subset \bar{C}$, or: $B \cap \bar{C} = B$. Clearly $B \cap \bar{C}$ is relatively closed in B. Since B is connected it suffices to show that $B \cap \bar{C}$ is relatively open in B. So let $x \in B \cap \bar{C}$. We must construct a neighborhood N of x such that $N \cap B \subset B \cap \bar{C}$. If $x \in C$, we can take $N = C$, because C is open. Otherwise, by 2, $x \in f^{-1}(c)$. In this case we use the assumption that x is not a global MP point to find a neighborhood N of x such that $M = \{ y \mid f(y) < c \} \cap N$ is connected. Since $x \in \bar{C}$, there exists $u \in C \cap N$. Since $u \in C$, $f(u) < c$, so $u \in M$. So we see that the two connected sets C and M intersect, and since C is a maximal connected set in $X - f^{-1}(c)$:

$$M \subset C \tag{3}$$

Suppose now $w \in f^{-1}(c) \cap N$. By assumption, w is not a local minimum point, so: $w \in \bar{M}$. So:

$$f^{-1}(c) \cap N \subset \bar{M} \tag{4}$$

Now using 3 and 4:

$$N \cap B \subset N \cap \{ y \mid f(y) \leqq c \} = M \cup (N \cap f^{-1}(c)) \subset C \cup \bar{M} \subset \bar{C}$$

So $N \cap B \subset B \cap \bar{C}$, which is what we wanted. ■

The following corollary is easier to apply:

COROLLARY 2.1. — *Let* X *be a locally connected and compactly connected topological space,* f: X → R *continuous and increasing at infinity. If* x_0 *and*

x_1 *are strict local minima of* f *then there exists* $x_2 \in X$ *different from* x_0, x_1 *such that* x_2 *is either a local minimum or a global MP point of* f, *and* $f(x) > \max\{f(x_0), f(x_1)\}$.

Proof. – Without loss of generality we assume $f(x_0) \geqq f(x_1)$. As remarked before, X must be locally compact. Let N be a compact neighborhood of x_0 such that $f(x) > f(x_0)$ for $x \in N - \{x_0\}$ (x_0 is a strict local minimum). ∂N is compact and does not contain x_0. Let the minimum of f on ∂N be p. Then $p > f(x_0)$, $x_1 \notin N$ by the choice of N and the assumption that $f(x_0) \geqq f(x_1)$. So taking $S = \partial N$, S separates x_0 and x_1, and we can apply theorem 2.1 to conclude the existence of the desired point. ■

Note that corollary 2.1 together with the fact that a MP point of a smooth function is a critical point, implies theorem 1.1, since a function on R^n satisfying $f(x) \to +\infty$ as $\|x\| \to \infty$ is increasing at infinity. I would like to draw attention to the elementary general-topological nature of the proof of theorem 2.1 in contrast with the "classical" proof of theorem 1.1 which uses deformation along gradient curves and thus requires in an essential way a differentiable structure.

3. APPLICATION OF THE TOPOLOGICAL MOUNTAIN-PASS THEOREM TO GLOBAL HOMEOMORPHISM THEOREMS

A mapping $F : X \to Y$ (X, Y topological spaces) is called a local homeomorphism at x_0 if there is a neighborhood U of x_0 such that F(U) is a neighborhood of $F(x_0)$ and $F : U \to F(U)$ is a homeomorphism. It is called a local homeomorphism if it is a local homeomorphism at each point of X. A global homeomorphism is a homeomorphism of X onto Y. It is important to know under what additional conditions a local homeomorphism is a global homeomorphism. $F : X \to Y$ is called proper if for every compact set $K \subset Y$, $F^{-1}(K)$ is compact.

The following lemma is not hard to prove:

LEMMA 3.1. – *Let* X, Y *be topological spaces,* Y *connected and locally compact. Let* $F : X \to Y$ *be a local homeomorphism and a proper mapping. Then the cardinality of* $F^{-1}(y)$ *is finite and constant for each* $y \in Y$.

We are now ready to prove the main theorem of this section:

THEOREM 3.1. – *Let* X, Y *be topological spaces,* Y *connected and* X *locally connected and compactly connected. Then at least one of the following holds*:

(I) *Every continuous* $f : Y \to R$ *which is increasing at infinity has either an infinite number of local minima or a local MP point* (*or both*).

(II) *Every proper local homeomorphism* $F: X \to Y$ *is a global homeomorphism.*

Proof. — As we remarked before, the existence of a continuous function on Y which is increasing at infinity implies that Y is locally compact, so if Y is not locally compact (I) is satisfied in a trivial way. So we may assume from now on that Y is locally compact. Let us assume that (II) is not satisfied and prove (I) holds. Let $f: Y \to R$ be continuous and increasing at infinity. f has at least one local minimum: a global minimum whose existence is assured by the fact that f is increasing at infinity. Suppose it has only a finite number of local minima. Let y_0 be a local minimum point at which the value of f is maximal. Since there are only a finite number of local minima, y_0 is a strict local minimum. Since (II) is not satisfied there exists a proper local homeomorphism $F: X \to Y$ which is not a global homeomorphism. Since F is a local homeomorphism it is open. If $\operatorname{card}(F^{-1}(y))$ were equal to 1 for all $y \in Y$, F would be a global homeomorphism. Hence $\operatorname{card}(F^{-1}(y)) \neq 1$ for some $y \in Y$. By lemma 3.1 $\operatorname{card}(F^{-1}(y)) = \operatorname{card}(F^{-1}(y_0))$ for every $y \in Y$, so we see that $\operatorname{card}(F^{-1}(y_0)) \geqq 2$. Let x_0 and x_1 be such that $F(x_0) = F(x_1) = y_0$ $(x_0 \neq x_1)$. Define $g: X \to R$ by $g = f \circ F$. Since f is increasing at infinity and F is proper, g is increasing at infinity. Since F is a local homeomorphism and y_0 is a strict local minimum of f, x_0 and x_1 are strict local minima of g. We can thus apply corollary 2.1 to conclude that there is a point $x \in X$ which is either a local minimum or a global MP point of g, with $g(x) > \max\{g(x_0), g(x_1)\} = f(y_0)$. So $f(F(x)) > f(y_0)$. But since F is a local homeomorphism, $y_1 = F(x)$ is a local minimum or a local MP point of f (local MP points are invariant under local homeomorphisms, though global MP points are not). But since y_0 was chosen to attain the maximal value among local minima, y_1 must be a MP point. So we have proved (I). ■

Theorem 3.1 can be applied in two ways. Suppose we are given spaces X, Y satisfying the assumptions of the theorem, and suppose we can find a continuous function f on Y which is increasing at infinity and has only a finite number of local minima and no MP point. Then we know that (II) holds, that is, every proper local homeomorphism is a global homeomorphism. On the other hand, if we can find a single proper local homeomorphism $F: X \to Y$ which is not a global homeomorphism, then (I) holds. We begin with the first kind of application, and make the following definition:

DEFINITION 3.1. — *A continuous function* $f: Y \to R$ *which is increasing at infinity and which has only a finite number of local minima and no local MP point is called a simple function. A topological space* Y *which admits at least one simple function is called a simple space.*

So we have:

THEOREM 3.2. — *If* X *is locally connected and compactly connected and* Y *is simple and connected, then any proper local homeomorphism:* $F: X \to Y$ *is a global homeomorphism.*

Note that if f is a continuous function which is increasing at infinity, and if f has no local minima or MP points except for the global minimum, then f is a simple function. This allows us to give examples of simple spaces:

(1) $Y = R^n$. Take $f(x) = \|x\|$.

(2) $Y = S^n = \{x \in R^{n+1} \mid \|x\| = 1\}$ for $n > 1$. Take $f(x) = x^1$ (the first component). For $n = 1$ this does not work since in this case the maximum point of f is also a local MP point.

(3) $Y = B^n = \{x \in R^n \mid \|x\| \leqq 1\}$. Again take $f(x) = x^1$.

Applying 3.2 for $X = Y = R^n$ we get that every proper local homeomorphism from R^n to itself is a global homeomorphism. This generalizes 1.2. Applying 3.2 for $X = Y = S^n$ ($n > 1$), we get that every local homeomorphism from S^n to itself is a global homeomorphism (properness is immediate since S^n is compact). Both of these results are not new. Their usual proof depends, however, on a monodromy-type argument and is based on the simple-connectedness of the spaces. In the case of S^n, it is much easier to construct a simple function than to show simple-connectedness. In the next section we shall make a comparison of theorem 3.2 and the monodromy theorem.

We now turn to the second kind of application of theorem 3.1, which is proving the existence of "critical" points.

THEOREM 3.3. — *Let* W *be a compactly connected, locally connected topological space. Let* $X = S^1 \times W$. *Then every continuous function on* X *which is increasing at infinity has either an infinite number of local minima or a MP point (or both).*

Proof. — X inherits the compact connectedness and local connectedness of W. We want to prove (I) of theorem 3.1, so by that theorem it is sufficient to display a proper local homeomorphism $F: X \to X$ which is not a global homeomorphism. Representing points of S^1 by $(\cos(t), \sin(t))$, such a mapping is given by:

$$F(\cos(t), \sin(t), w) = (\cos(2t), \sin(2t), w). \quad \blacksquare$$

As an example we can take $X = T^n = S^1 \times S^1 \times \ldots \times S^1$ (n factors, $n \geqq 2$). Since $X = S^1 \times T^{n-1}$ theorem 3.3 applies and we conclude that every continuous function on T^n has either an infinite number of local minima or a local MP point or both (since X is compact any function is increasing at infinity).

As another application we take $X = S^n$, $Y = P^n$ (n-dimensional projective space, and $F : X \to Y$ the standard projection identifying antipodal points. F is a proper local homeomorphism but not a global homeomorphism. We conclude from theorem 3.1 that every continuous function on P^n has an infinite number of local minima or a local MP point.

4. COMPARISON WITH THE MONODROMY ARGUMENT

A map $F : X \to Y$ is called a covering map if each $y \in Y$ has an open neighborhood U such that $F^{-1}(U)$ is a disjoint union of open sets, each one of which is mapped homeomorphically onto U by F.

The following theorem will be called the monodromy theorem (the proof follows from [12], theorem 2.3.9):

THEOREM 4.1. — *If* X *is path-connected,* Y *is connected and simply-connected, and* F *is a covering map, then* F *is a global homeomorphism.*

In order to apply theorem 4.1 to our question of giving criteria for local homeomorphisms to be global, we must give conditions under which a local homeomorphism is a covering map. Various such conditions have been given by Browder [2]. For example, he proved that if X is normal, Y is connected and has a countable base of neighborhoods at every point, then every proper local homeomorphism from X to Y is a covering map. Together with the monodromy theorem we get, for example, that every proper local homeomorphism from a Banach space X to a Banach space Y is a global homeomorphism. This famous theorem of Banach and Mazur cannot be derived from theorem 3.2 unless Y is finite dimensional, since an infinite-dimensional Banach space is not simple because it is not locally compact. Therefore we see that the monodromy argument works in situations where our argument fails.

On the other hand, while the monodromy theorem requires the space X to be path-connected, this assumption is not made in theorem 3.2. In fact there exist spaces which are compactly connected and locally connected but not path connected (example: a countable set with the co-finite topology: finite sets are closed), so that theorem 3.2 can be applied to such spaces, but not any theorem which is based on the monodromy theorem.

The major difference, however, between the monodromy theorem and theorem 3.2 is that Y is assumed to be simply-connected in the first and simple in the second. The relation between these two properties is not clear at present. It seems reasonable that at least for "nice" spaces (manifolds, for example) the two properties coincide. In fact I have not found any examples of simple spaces which are not simply-connected or vice versa. We leave these as open problems.

5. CRITICAL POINTS IN METRIC SPACES

In this section we prove a mountain-pass theorem which generalizes the Banach space mountain-pass theorem to continuous functions on metric spaces. We will use a definition of "critical point" of a continuous function on a metric space, which reduces to the regular definition in the smooth case, and which is inspired by, but different from, a definition given by Ioffe and Schwartzman [7] in the context of Banach spaces.

DEFINITION 5.1. – *Let* (X, d) *be a metric space,* $x \in X$, *and* f *a real function defined in a neighborhood of* x. *Given* $\delta > 0$, x *is said to be a* δ-*regular point of* f *if there is a neighborhood* U *of* x, *a constant* $\alpha > 0$, *and a continuous mapping* $\Psi : U \times [0, \alpha] \to X$ *such that for all* $(u, t) \in U \times [0, \alpha]$:

(1) $d(\Psi(u, t), u) \leqq t$.

(2) $f(u) - f(\Psi(u, t)) \geqq \delta t$

Ψ *is called a* δ-*regularity mapping for* f *at* x. x *will be called a regular point of* f *if it is* δ-*regular for some* $\delta > 0$. *Otherwise it is called a critical point of* f. *We define the regularity constant of* f *at* x *to be*:

$$\delta(f, x) = \sup \{ \delta \mid f \text{ is } \delta\text{-regular at } x \}.$$

If x *is a critical point of* f *we set* $\delta(f, x) = 0$. *We say that* f *satisfies the generalized Palais-Smale* (*PS*) *condition if: Any sequence* $\{x_n\} \subset X$ *with* $\{f(x_n)\}$ *bounded and* $\delta(f, x_n) \to 0$ *has a convergent subsequence. It is said to satisfy the weaker condition* $(PS)_c$ (*where* $c \in R$) *if the above is true under the additional assumption that* $f(x_n) \to c$.

LEMMA 5.1. – *If* X *is a Banach space and* $f : X \to R$ *is* C^1, *then* $\delta(f, x) = \| f'(x) \|$. *In particular, the generalized notion of critical point coincides with the usual one in this case.*

Proof. – First we show that f is δ-regular at x for every $0 < \delta < \| f'(x) \|$. Choose $v \in X$ with $\| v \| < 1$ and $f'(x)(v) > \delta$. Let $\Psi(u, t) = u - tv$. Condition (1) in the definition of regularity mapping is obviously satisfied. By strict differentiability of f at x we have:

$$\limsup_{u \to x,\, t \to 0+} \frac{f(u) - f(\Psi(u, t))}{t} > \delta$$

So condition (2) also holds. So f is δ-regular at x. We now show that if $\delta > \| f'(x) \|$ then f is not δ-regular at x. Suppose it were. Then we would have $\Psi : U \times [0, \alpha] \to X$ with:

(1) $\|\Psi(u, t) - u\| \leqq t$

(2) $f(u) - f(\Psi(u, t)) \geqq \delta t$.

Combining these two inequalities we get:

$$f(u) - f(\Psi(u, t)) \geqq \delta \|u - \Psi(u, t)\|$$

But differentiability of f implies that there is a neighborhood V of x such that for $u, v \in V$ we have:

$$|f(u) - f(v)| < \delta \|u - v\|$$

contradicting the previous inequality when (u, t) is close enough to $(x, 0)$. ■

LEMMA 5.2. — *$\delta(f, x)$ is lower-semicontinuous as a function of x. In particular, if $x_n \to x$ and $\delta(f, x_n) \to 0$ then x is a critical point of f.*

Proof. — We only have to note that a δ-regularity mapping for f at x is a δ-regularity mapping for any point in a neighborhood of x. ■

We are now prepared to state and prove a mountain-pass theorem based on this notion of critical point. The proof of the theorem is an adaptation of the proof of the smooth mountain path theorem given in [8].

THEOREM 5.1. — *Let f be a continuous function on a path-connected complete metric space X. Suppose $x_0, x_1 \in X$, Γ is the set of continuous curves $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ and the function $\Upsilon : \Gamma \to R$ is defined by:*

$$\Upsilon(\gamma) = \max_{t \in [0, 1]} f(\gamma(t)).$$

Let $c = \inf\{\Upsilon(\gamma) \mid \gamma \in \Gamma\}$ and $c_1 = \max\{f(x_0), f(x_1)\}$. If $c > c_1$ and f satisfies $(PS)_c$ then there is a critical point x_2 of f with $f(x_2) = c$.

Proof. — We make Γ into a metric space by defining:

$$\rho(\gamma_1, \gamma_2) = \max_{t \in [0, 1]} d(\gamma_1(t), \gamma_2(t))$$

It is well-known that this makes Γ a complete metric space, and the function Υ continuous. We shall construct, for every $\varepsilon > 0$, a point $v \in X$ such that $c - \varepsilon \leqq f(v) \leqq c + \varepsilon$ and $\delta(f, v) \leqq \varepsilon^{1/2}$. By the $(PS)_c$ condition and lemma 5.2, this will imply the existence of a critical point w with $f(w) = c$, which is what we want.

We may assume $0 < \varepsilon < c - c_1$. Choose $\gamma_0 \in \Gamma$ with $\Upsilon(\gamma_0) \leqq c + \varepsilon$. According to Ekeland's variational principle [4], there is $\gamma_1 \in \Gamma$ such that $\Upsilon(\gamma_1) \leqq \Upsilon(\gamma_0)$, $\rho(\gamma_0, \gamma_1) \leqq \varepsilon^{1/2}$, and

$$\Upsilon(\gamma) > \Upsilon(\gamma_1) - \varepsilon^{1/2} \rho(\gamma, \gamma_1) \tag{5}$$

for any $\gamma \in \Gamma$, $\gamma \neq \gamma_1$. We shall show that for some $s \in [0, 1]$ we have: $c - \varepsilon \leqq f(\gamma_1(s))$ and $\delta(f, \gamma_1(s)) \leqq \varepsilon^{1/2}$. If not, then for each

$s\in S=\{t\in[0, 1]\,|\,c-\varepsilon\leqq f(\gamma_1(t))\}$ there exists $r(s)>0$, $\alpha(s)>0$ and a regularity map $\Psi_s: B(\gamma_1(s), r(s))\times[0, \alpha(s)]\to X$ with:

(1) $d(\Psi_s(u, t), u)\leqq t$

(2) $f(u)-f(\Psi_s(u, t))\geqq\varepsilon^{1/2}\, t$

By continuity, for each $s\in S$ there is an interval $I(s)$, relatively open in $[0, 1]$, containing s such that $\gamma_1(I(s))\subset B(\gamma_1(s), r(s)/2)$. Since S is compact, we may choose a finite subcovering $I(s_1), I(s_2), \ldots, I(s_k)$ of S. We define for $1\leqq i\leqq k$:

$$\mu_i(t)=\begin{cases}\dfrac{d(t, I(s_i)^c)}{\sum_{j=1}^{k} d(t, I(s_j)^c)} & \text{if } t\in\bigcup\{I(s_i)\,|\,1\leqq i\leqq k\}\\ 0 & \text{otherwise.}\end{cases}$$

Let $\phi:[0, 1]\to[0, 1]$ be a continuous function such that:

$$\phi(t)=1 \text{ when } c\leqq f(\gamma_1(t))$$
$$\phi(t)=0 \text{ when } f(\gamma_1(t))\leqq c-\varepsilon$$

Let:

$$\eta=\min\left(\frac{1}{2}\min\{r(s_i)\,|\,1\leqq i\leqq k\}, \min\{\alpha(s_i)\,|\,1\leqq i\leqq k\}\right)$$

We now define $y_i:[0, 1]\to X$, $1\leqq i\leqq k+1$, by: $y_1=\gamma_1$. For $1\leqq i\leqq k$:

$$y_{i+1}(t)=\begin{cases}\Psi_{s_i}(y_i(t), \eta\phi(t)\mu_i(t)) & \text{if } \phi(t)\mu_i(t)\neq 0\\ y_i(t) & \text{if } \phi(t)\mu_i(t)=0\end{cases}$$

We show, by induction, that each y_i is defined, and that:

$$\rho(y_i, \gamma_1)<r(s_i)/2 \tag{6}$$

This is certainly true for $i=1$. Assuming it is true for i, suppose $t\in[0, 1]$ is such that $\phi(t)\mu_i(t)\neq 0$. Then $t\in I(s_i)$, so $\gamma_1(t)\in B(\gamma_1(s_i), r(s_i)/2)$. From this and from equation 6 we get that for each $t\in[0, 1]$:

$$d(y_i(t), \gamma_1(s_i))\leqq d(y_i(t), \gamma_1(t))+d(\gamma_1(t), \gamma_1(s_i))<r(s_i)/2+r(s_i)/2=r(s_i)$$

so:

$$y_i(t)\in B(\gamma_1(s_i), r(s_i))$$

also:

$$\eta\phi(t)\mu_i(t)\leqq\eta\leqq\alpha(s_i)$$

and these last two inequalities show that y_{i+1} is well-defined. Continuity of y_{i+1} is easy to verify. By property (1) of regularity mappings:

$$d(y_{i+1}(t), y_i(t))\leqq d(\Psi_{s_i}(y_i(t), \eta\phi(t)\mu_i(t)), y_i(t))\leqq\eta\phi(t)\mu_i(t)\leqq\eta$$

implying:

$$d(y_{i+1}(t), \gamma_1(t))\leqq\sum_{j=1}^{i} d(y_j(t), y_{j+1}(t))\leqq\eta\leqq r(s_i)/2 \tag{7}$$

$\left(\text{we use here the fact that } \sum_{i=1}^{k} \mu_i(t) \leqq 1\right)$ so 6 is true for $i+1$. By the assumption $\varepsilon < c - c_1$ we have:

$$f(\gamma_1(0)) = f(x_0) \leqq c_1 < c - \varepsilon$$
$$f(\gamma_1(1)) = f(x_1) \leqq c_1 < c - \varepsilon$$

So $\phi(0) = \phi(1) = 0$. Thus $y_i(0) = x_0$, $y_i(1) = x_1$ for each $1 \leqq i \leqq k+1$, so $y_i \in \Gamma$. We now set $\gamma_2 = y_{k+1}$. By property (2) of the regularity mappings, for $1 \leqq i \leqq k$:

$$f(y_{i+1}(t)) - f(y_i(t)) \leqq f(\Psi_{s_i}(y_i(t), \eta\phi(t)\mu_i(t))) - f(y_i(t)) \leqq -\varepsilon^{1/2}\eta\phi(t)\mu_i(t)$$

adding these inequalities, we get, for all $t \in [0, 1]$:

$$f(\gamma_2(t)) - f(\gamma_1(t)) \leqq -\varepsilon^{1/2}\eta\phi(t)$$

If t_0 is such that $f(\gamma_2(t_0)) = \Upsilon(\gamma_2)$ we get $f(\gamma_1(t_0)) \geqq f(\gamma_2(t_0)) \geqq c$, so $\phi(t_0) = 1$. Therefore $f(\gamma_2(t_0)) - f(\gamma_1(t_0)) \leqq -\varepsilon^{1/2}\eta$, so:

$$\Upsilon(\gamma_2) + \varepsilon^{1/2}\eta \leqq f(\gamma_1(t_0)) \leqq \Upsilon(\gamma_1) \tag{8}$$

In particular $\gamma_1 \neq \gamma_2$. From equation 7:

$$\rho(\gamma_1, \gamma_2) \leqq \eta$$

From this and 8:

$$\Upsilon(\gamma_2) + \varepsilon^{1/2}\rho(\gamma_1, \gamma_2) \leqq \Upsilon(\gamma_1)$$

which contradicts the choice of γ_1 (equation 5). ■

6. GLOBAL HOMEOMORPHISM THEOREMS IN METRIC SPACES

We now apply the mountain-pass theorem of the previous section to prove criteria for global homeomorphism theorems in metric spaces, using the same idea as before of associating a function to the mapping. We are able to do this in a nonsmooth context, using the surjection constant introduced by Ioffe [6]. Given a mapping $F: X \to Y$ (metric spaces) we set for $x \in X$, $t > 0$:

$$\mathrm{Sur}(F, x)(t) = \sup\{r \geqq 0 \mid B(F(x), r) \subset F(B(x, t))\}$$
$$\mathrm{sur}(F, x) = \liminf_{t \to 0+} t^{-1}\,\mathrm{Sur}(F, x)(t)$$

$\mathrm{sur}(F, x)$ is called the surjection constant of F at x.

If X, Y are Banach spaces and F is a local diffeomorphism at x, we have, by Ljusternik's theorem:

$$\mathrm{sur}(F, x) = \frac{1}{\|[F'(x)]^{-1}\|}$$

LEMMA 6.1. — *If* $F: X \to Y$ *is a local homeomorphism,* $g: Y \to R$, $f = g \circ F$, $x \in X$ *and* $\operatorname{sur}(F, x') \geqq c > 0$ *for* x' *in a neighborhood of* x. *Then:* $\delta(f, x) \geqq \delta(g, F(x))\, c$.

Proof. — Let B be a ball around x in which F is invertible and such that $\operatorname{sur}(F, x') \geqq c$ for $x' \in B$, and let $G: F(B) \to B$ be the inverse of $F|_B$. Let $0 < \delta < \delta(g, F(x))$, and let $\Psi: B' \times [0, \alpha] \to Y$ be a δ-regularity mapping for g at $F(x)$, where B' around $F(x)$ and $\alpha > 0$ are chosen so small that $\Psi(B' \times [0, \alpha]) \subset F(B)$. We define $\Psi_1: G(B') \times [0, \alpha/c] \to X$ by: $\Psi_1(u, t) = G(\Psi(F(u), ct))$. We have:

$$d(\Psi_1(u, t), u) = d(G(\Psi(F(u), ct)), u)$$
$$\leqq (1/c)\, d(\Psi(F(u), ct), F(u)) \leqq (1/c)\, ct = t$$

and:

$$f(u) - f(\Psi_1(u, t)) = f(u) - f(G(\Psi(F(u), ct)))$$
$$= g(F(u)) - g(\Psi(F(u), ct)) \geqq \delta ct$$

These last two inequalities show that Ψ_1 is a δc regularity mapping for f at x. ■

DEFINITION 6.1. — *A metric space* Y *will be called nice if it admits, for each* $y \in Y$, *a continuous* PS *function* $g_y: Y \to R$, *satisfying*:

(A) *y is a unique global minimum of* g_y

(B) *The only critical points that* g_y *has are* y *and (possibly) global maximum points, and these are isolated.*

(C) *There is a* $\beta > 0$ *such that* $\delta(g_y, u) > \beta$ *for* $u \neq y$ *in a neighborhood of* y.

LEMMA 6.2. — *A Banach space is a nice space.*

Proof. — Let Y be Banach space. For each $y \in Y$, take $g_y(z) = \| z - y \|$. (A) is obvious. To prove (B) and (C) it suffices to show that $\delta(g_y, x) \geqq 1$ for each $x \neq y$. Without loss of generality we take $y = 0$. Fix $x \neq 0$. To show that g_0 is 1-regular at x, we construct a regularity map: $\Psi(u, t) = u - tu/\| u \|$. We have:

$$\| \Psi(u, t) - u \| = t$$
$$g_0(u) - g_0(\Psi(u, t)) = \left(1 - \left|1 - \frac{t}{\| u \|}\right|\right) \| u \| = t$$

(the last equality is true for $t < \| u \|$). Which shows that Ψ is a 1-regularity mapping for g_0. ■

LEMMA 6.3. — *If* X *is the unit sphere in a Hilbert space* H *then* X *is a nice space.*

Proof. — Fix $y \in X$ and let M be the tangent hyperplane to X at y. We define the Riemann projection F from $X - \{-y\}$ to H by taking, for each

$x \in X - \{y\}$, $F(x)$ to be the point at which the line through $-y$ and x intersects M. It is not hard to prove that F is a homeomorphism and $\mathrm{sur}(F, x) \geqq 1$ for every $x \in X - \{-y\}$. Defining $h: X - \{-y\} \to R$ by $h(x) = \| F(x) - y \|$, we see by lemma 6.1 that $\delta(h, x) \geqq 1$ for $x \neq y$. Defining $g_y(x) = 1 - e^{-h(x)}$ for $x \neq -y$ and $g_y(-y) = 1$, we get that g_y is continuous. We have $\delta(g_y, x) \geqq e^{-h(x)}$, which easily implies that g_y is PS and also (B) and (C) of the definition of nice space. (A) is obviously satisfied. ■

It should also be possible to prove that spheres in more general Banach spaces are nice spaces, though there are some technical difficulties.

LEMMA 6.4. – *If $f: X \to R$ is continuous and bounded from below, then there is a sequence $\{x_n\} \subset X$ with $f(x_n) \to \inf_X f$ and $\delta(f, x_n) \to 0$.*

Proof. – Given $\varepsilon > 0$, there exists, by Ekeland's variational principle, a point $v \in X$ with $f(v) \leqq \varepsilon + \inf_X f$ and

$$f(u) > f(v) - \varepsilon d(u, v) \tag{9}$$

for all $u \neq v$. This implies that f is not ε-regular at v. For if Ψ were a regularity mapping at v, we would have, combining (1) and (2) in the definition of regularity map:

$$f(v) - f(\Psi(v, t)) \geqq \varepsilon \| v - \Psi(v, t) \|$$

contradicting 9 (take $u = \Psi(v, t)$). ■

The following lemma appears in [1] for smooth functions:

LEMMA 6.5. – *If f is continuous,* (PS), *and bounded from below, $x_0 \in X$, then*:

$$\lim_{d(x, x_0) \to \infty} f(x) = +\infty.$$

Proof. – Assume on the contrary $d(x_n, x_0) \to \infty$ while $f(x_n) \leqq \inf_X f + \varepsilon/2$ for all n. Then using the strong form of Ekeland's Variational Principle, we can find for each n u_n such that:

$$d(u_n, x_n) \leqq d(x_n, x_0)/2 \tag{10}$$

and for all $u \neq u_n$:

$$f(u_n) < f(u) + 2 \frac{\varepsilon}{d(x_n, x_0)} d(u, u_n) \tag{11}$$

From 10 we get that $u_n \to \infty$. From 11, we get, as in the proof of the previous lemma, that: $\delta(f, u_n) \to 0$. These two facts together contradict the assumption that f is (PS). ■

A set S in a topological space will be called discrete if each point $x \in S$ has a neighborhood N such that $S \cap N = \{x\}$.

THEOREM 6.1. – *Let* X, Y *be complete path-connected complete metric spaces, such that* X *remains path-connected after the removal of any discrete* *set, and* Y *is nice. Then if* $F: X \to Y$ *is a local homeomorphism satisfying, for some (hence any)* $y_0 \in y$:

$$\inf\{\operatorname{sur}(F, x) \mid d(F(x), y_0) < k\} > 0$$

for all $k > 0$, *then* F *is a global homeomorphism.*

Proof. – Let $F: X \to Y$ be a local homeomorphism satisfying the assumption. Fixing $y \in Y$, we shall show that $F^{-1}(y)$ consists of precisely one point. Let g_y be a function satisfying (A)-(C). Let $f = g_y \circ F$. By Lemma 6.4, there exists a sequence $\{x_n\} \subset X$ with

$$f(x_n) \to \inf_X f \tag{12}$$

$$\delta(f, x_n) \to 0 \tag{13}$$

Since by lemma 6.5 $g_y(w) \to \infty$ as $w \to \infty$, we have by 12 that $F(x_n)$ is bounded, so suppose $d(F(x_n), y_0) < k$ for all n. By our assumption there is some $d > 0$ such that $\operatorname{sur}(F, x) > d$ if $F(x) \in B(y_0, k)$. By lemma 6.1 this implies $\delta(f, x_n) \geqq \delta(g_y, F(x_n))\, d$ for each n, so by 13 we get: $\delta(g_y, F(x_n)) \to 0$, which implies, since g_y is (PS) that $\{F(x_n)\}$ has a convergent subsequence. Without loss of generality, we assume that $\{F(x_n)\}$ itself converges to a point $z \in Y$. By lemma 5.2, z is a critical point of g_y. By property (B), either $z = y$ or z is a strict local maximum of g_y. If the second possibility were the case, then we would have, for n large enough:

$$g_y(z) > g_y(F(x_n)) = f(x_n) \geqq \inf_X f = m$$

but on the other hand $f(x_n) \to m$, that is $g_y(F(x_n)) \to m$, so $g_y(z) = m$-contradiction. Therefore $z = y$, so by (C) either $F(x_n) = y$ for n sufficiently large, or $\delta(g_y, F(x_n)) > \beta > 0$ for all n large enough, contradicting the fact that $\delta(g_y, F(x_n)) \to 0$. Thus we have $F(x_n) = y$ for n large enough, showing that $F^{-1}(y)$ is nonempty.

To show that $F^{-1}(y)$ contains no more than one point, we assume by way of contradiction $F(x_0) = F(x_1) = y$ $(x_0 \neq x_1)$. We let c, c_1 be as in theorem 5.1. Since y is a unique global minimum of g_y and F is a local homeomorphism, x_0 and x_1 are strict local minima of f, which implies $c > c_1$. We now show that f satisfies $(PS)_c$. Suppose $f(x_n) \to c$ and $\delta(f, x_n) \to 0$. As before, this implies $\delta(g_y, F(x_n)) \to 0$, so $F(x_n) \to w$. We have $g_y(w) = c$. So $w \neq y$. By lemma 5.2 w is a critical point so by (B) it is a global maximum point, so $c = \max_Y g_y$. Therefore the set $g_y^{-1}(c)$ consists of global maxima, so by the second part of (B), it is discrete, so $f^{-1}(c) = F^{-1}(g_y^{-1}(c))$ is discrete since F is a local homeomorphism. So by our assumption on X, $X - f^{-1}(c)$ is path-connected. So there exists a path $\gamma: [0, 1] \to X - f^{-1}(c)$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Since $f(x_0) \leqq c_1 < c$ we must have $f(\gamma(t)) < c$ for all $t \in [0, 1]$, so (in the notation of theorem 5.1) $\Upsilon(\gamma) < c$, contradicting the definition of c. Therefore we have shown there can exist no sequence x_n as above, so $(PS)_c$ is satisfied in a trivial way. Thus we see that f satisfies the conditions of theorem 5.1, which implies the existence of a critical point x of f with $f(x)=c$. By lemma 6.1 and the fact that $\operatorname{sur}(F, x)>0$, we get that $u=F(x)$ is a critical point of g_y with $g_y(u)=c$. But this, as we have shown in the argument above, is impossible, and we have arrived at the desired contradiction. Therefore $F^{-1}(y)$ consists of exactly one point for each $y \in Y$, and F is open since it is a local homeomorphism, so F is a global homeomorphism. ■

As an immediate consequence we get the following theorem, which generalizes a theorem of Plastock ([9], theorem 2.1), both in the sense of applying not only in Banach spaces, and in the sense that it applies to non-smooth mappings in the Banach space case:

THEOREM 6.2. — *Let* X *be a path-connected metric-space and such that it is not path-disconnected when a discrete set is removed. Let* Y *be a nice metric space. Let* F *be a local homeomorphism satisfying*:

(*a*) $F(x) \to \infty$ *as* $x \to \infty$, *that is, for some (hence any)* $x_0 \in X$, $y_0 \in Y : d(F(x), y_0) \to +\infty$ *as* $d(x, x_0) \to +\infty$.

(*b*) *For any ball* $B \subset X : \inf\{\operatorname{sur}(F, x) \mid x \in B\} > 0$

then F *is a global homeomorphism.*

Proof. — We show that (*a*), (*b*) imply the assumption of theorem 6.1. Given $k>0$, there exists, by assumption (*a*), some $r>0$ such that $d(F(x), y_0)>k$ for $d(x, x_0) \geqq r$. We have, using (*b*):

$$\inf\{\operatorname{sur}(F, x) \mid d(F(x), y_0)<k\} \geqq \inf\{\operatorname{sur}(F, x) \mid d(x, x_0)<r\}>0$$

which is what we wanted. ■

As another consequence, we get a generalization of [11], theorem 1.22:

THEOREM 6.3. — X *is a path-connected metric space which is not path-disconnected by the removal of a discrete set, and* Y *a nice metric space. Let* $F : X \to Y$ *be a local homeomorphism satisfying*:

$$\inf\{\operatorname{sur}(F, x) \mid x \in X\}>0$$

Then F *is a global homeomorphism.*

Proof. — Immediate. ■

7. SCHECHTER'S MOUNTAIN PASS THEOREM AND HADAMARD'S GLOBAL HOMEOMORPHISM THEOREM

We now move to another type of global homeomorphism theorem proved by Plastock ([9], theorem 3.2), which he attributes to Hadamard, in which the assumption is that an integral involving the derivative of the mapping diverges.

THEOREM 7.1. — *Let* X, Y *be Banach spaces,* $F: X \to Y$ *a* C^1 *mapping with* $F'(x)$ *invertible for all* $x \in X$. *If:*

$$\int_0^\infty \inf_{\|x\| \leqq s} \frac{1}{\|[F'(x)]^{-1}\|} ds = \infty$$

then F *is a diffeomorphism of* X *onto* Y.

Happily, a new mountain pass theorem proved by Schechter ([10], theorem 2.1), together with our technique and a theorem of Ioffe, implies this theorem immediately. Actually, we need a non-smooth generalization of Schechter's theorem using Clarke's subgradient, but to obtain this generalization all we have to change in Schechter's proof is the construction of a Lipschitz pseudo-gradient field for a non-smooth function, as shown by Zaslavskii ([13], lemma 1).

THEOREM 7.2. — *Let* X *be a Banach space,* $f: X \to R$ *a locally Lipschitz function. For each* $x \in X$ *we define:*

$$\chi(x) = \operatorname{dist}(\partial f(x), 0)$$

Where $\partial f(x)$ *denotes Clarke's subgradient. Let* $e \in X$ *be an element such that* $e \neq 0$, $r > 0$, $\rho \in R$, *and suppose that:*

$$f(0) \leqq \rho, \qquad f(e) \leqq \rho$$
$$\|e\| \geqq r$$
$$f(u) \geqq \rho \qquad \text{for} \quad \|u\| = r$$

Then for every positive non-increasing function $\psi(t)$ *in* $(0, \infty)$ *such that:*

$$\int_1^\infty \psi(t)\, dt = \infty$$

there is a sequence $\{u_k\} \subset X$ *such that:*

$$f(u_k) \to b \geqq \rho$$
$$\frac{\chi(u_k)}{\psi(\|u_k\|)} \to 0$$

The following was proved by Ioffe [6]:

THEOREM 7.3. — *Let* $F: X \to Y$ *be continuous,* m *a positive lower-semicontinuous function on* $[0, \infty)$ *such that, for all* $x \in X$:

$$\operatorname{sur}(F, x) \geqq m(\|x\|)$$

Then, for every $r > 0$:

$$\operatorname{Sur}(F, 0)(r) \geqq \int_0^r m(t)\, dt$$

Proof of theorem 7.1. — Defining:

$$m(t) = \inf_{\|x\| \leq t} \frac{1}{\|[F'(x)]^{-1}\|}$$

We have that $m(t)$ is continuous since F is C^1, and $\mathrm{sur}(F, x) \geqq m(\|x\|)$, so the divergence of the integral of $m(t)$ and theorem 7.3 imply that F is surjective.

To show F is one-to-one, we assume $F(x_0) = F(x_1) = y$ $(x_0 \neq x_1)$. Define: $f(x) = \frac{1}{2}\|F(x) - y\|^2$. f is clearly locally Lipschitz, and:

$$\partial f(x) = F'^*(x)(\partial \|\,.\,\|(F(x) - y))$$

We have: $f(x_0) = f(x_1) = 0$, and since F is a local homeomorphism there is some $r > 0$ such that $f(x) \geqq \rho > 0$ for x satisfying $\|x - x_0\| = r$. Thus from theorem 7.2, there is a sequence $\{u_k\}$ with $f(u_k) \to b \geqq \rho$ and:

$$\frac{\chi(u_k)}{m(\|u_k\|)} \to 0 \tag{14}$$

Since we have: $\|x^*\| = 1$ for $x^* \in \partial \|.\|(x)$, $x \neq 0$, we get, for x such that $F(x) \neq y$:

$$\chi(x) \geqq \inf_{\|v\| = 1} F'^*(x) v = \mathrm{sur}(F, x) \geqq m(\|x\|)$$

This contradicts 14 for k large enough, since $F(u_k) \neq y$ for k large enough. ■

We should mention that Ioffe ([6], theorem 2) extended theorem 7.1 to nonsmooth mappings using Plastock's method. This, and an extension of theorem 7.1 to some metric spaces, could be achieved by our method, provided Schechter's theorem can be generalized to metric spaces using the generalized notion of critical point. We shall not pursue this here.

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