Annales de l'I. H. P., section C

Yiming Long

The minimal period problem of classical hamiltonian systems with even potentials

Annales de l'I. H. P., section C, tome 10, nº 6 (1993), p. 605-626
<http://www.numdam.org/item?id=AIHPC_1993__10_6_605_0>

Ann. Inst. Henri Poincaré,

Vol. 10, n° 6, 1993, p. 605-626. *Analyse non linéaire*

The minimal period problem of classical Hamiltonian systems with even potentials

by

Yiming LONG (*)

Nankai Institute of Mathematics, Nankai University
Tianjin 300071, P.R. China

ABSTRACT. — In this paper, we study the existence of periodic solutions with prescribed minimal period for even superquadratic autonomous second order Hamiltonian systems defined on $\mathbf{R}^n$ with no convexity assumptions. We use a direct variational approach for this problem on a $W^{1,2}$ space of functions invariant under the action of a transformation group isomorphic to the Klein Fourgroup $\mathbf{V}_4 = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ to find symmetric periodic solutions, and prove a new iteration inequality on the Morse index by iterating such functions properly. Using these tools and the Mountain-pass theorem, we show that for every $T > 0$ the abobe mentioned system possesses a T-periodic solution $x(t)$ with minimal period T or T/3, and this solution is even about $t=0$, T/2 and odd about $t=T/4$, 3 T/4.

Key words : V_4-symmetry, direct variational method, Morse index, iteration inequality, minimal period, even potential, even solution, superquadratic condition, non-convexity, second order Hamiltonian systems.

RÉSUMÉ. — Dans cet article, on étudie l'existence de solutions périodiques avec la période minimale prescrit pour les systèmes hamiltoniens pairs autonomes d'ordre secondaire à croissance super-quadratique, définis

A.M.S. Classification: 58 F 05, 58 E 05, 34 C 25.

(*) Partially supported by NNSF and YTF of the Edu, Comm. of China.

dans $\mathbf{R}^n$ sans hypothèse de convexité. Pour trouver des solutions périodiques symétriques, on utilise une approche directe variationnelle pour ce problème dans $W^{1,2}$, espace de fonctions invariantes sous l'action d'un groupe de transformation, qui est isomorphe avec le Quatre-groupe de Klein $\mathbf{V}_4=\mathbf{Z}_2\oplus\mathbf{Z}_2$, et prouve les nouvelles inégalités d'itération sur les indices de Morse pour l'itération propre de telles fonctions. En utilisant ces outils et le théorème de Col de Montagne, on montre que pour chaque $T>0$ le système ci-dessus possède une solution T-période $x(t)$ avec la période minimale T ou T/3, et que cette solution est paire sur $t=0$, T/2 et impaire sur $t=T/4$, 3 T/4.

1. INTRODUCTION AND MAIN RESULTS

We consider the existence of non-constant periodic solutions with prescribed minimal period for the following autonomous second order Hamiltonian systems,

$$\ddot{x}+V'(x)=0, \qquad \forall x\in\mathbf{R}^n, \tag{1.1}$$

where n is a positive integer. $V:\mathbf{R}^n\to\mathbf{R}$ is a function, and V' denotes its gradient. In his pioneering work [21] of 1978, P. Rabinowitz proved that if the potential function V is non-negative and superquadratic at both the infinity and the origin, then the system (1.1) possesses a non-constant periodic solution with any prescribed period $T>0$. Because a T/k-periodic function is also a T-periodic function for every $k\in\mathbf{N}$, Rabinowitz conjectured that (1.1) or the first order Hamiltonian system

$$\dot{z}=JH'(z) \qquad \text{for} \quad z\in R^{2n} \tag{1.2}$$

possesses a non-constant solution with any prescribed minimal period under his conditions. Since then, a large amount of contributions on this minimal period problem have been made by many mathematicians. Among all these results, a significant progress was made by Ekeland and Hofer in 1985 (*cf.* [10]). They gave an affirmative answer to Rabinowitz' conjecture for strictly convex Hamiltonian systems (1.2). Their proof is based upon the dual action principle for convex Hamiltonians, Ekeland index theory and Hofer's topological characterization of mountain-pass points. Their work was extended to the case of system (1.1) when V is strictly convex by Coti Zelati, Ekeland, and P. L. Lions (*cf.* Theorem IV.5.3 [9]). Generalizations of their results under different or weaker convexity assumptions can be found in [9], [14], [15], [16]. Most of these results deal with convex Hamiltonian functions. As far as the author knows, there are only three papers ([12], [13], [20]) dealing with the Hamiltonian functions with no convexity assumptions. In [12] and [13], by an *a priori* estimation method Girardi and Matzeu obtained T-periodic solutions of (1.2) with a lower bound on the minimal period by assuming Rabinowitz' conditions hold globally on $\mathbf{R}^{2n}$ and additional assumptions that $H(z)$ and $H'(z)$ are sufficiently close to functions $|z|^\beta$ and $|z|^{\beta-1}$ with $\beta>2$, and also obtained T-minimal periodic solutions of (1.2) under further assumption that H is homogeneous of degree β or a pinching condition holds. In the recent paper [20] of the author, by using the natural $\mathbf{Z}_2$-symmetry possessed by the system (1.1) and a Morse index theory method, under precisely Rabinowitz' superquadratic condition, it was proved that for every $T>0$ there exists an even T-periodic solution of (1.1) with minimal period not smaller than $T/(n+2)$. The key observation made in [20] is that certain Morse indices do increase by iterating an even periodic solution without any particular assumptions on V'', and that this phenomenon can be used to get lower bounds for the minimal period of this solution.

In this paper, we further develop the ideas used in [20], and study the minimal period problem of (1.1) when the potential function V is even. In thid case we observe that the usual direct variational formulation of the system (1.1) possesses a natural $\mathbf{V}_4$-symmetry, where $\mathbf{V}_4=\mathbf{Z}_2\oplus\mathbf{Z}_2$ is the Klein Fourgroup, and that this symmetry can be used to reach the following purposes:

1° To eliminate the subspace $\mathbf{R}^n$ from $L^2(S_T, \mathbf{R}^n)$ so that the mountain-pass theorem can be applied to get a T-periodic solution x of (1.1) with its symmetric Morse index defined in this paper not larger than 1, and its derivative $\dot{x}$ is anti-symmetric in the sense of the Definition 2.1 below.

2° To eliminate the possibility that this x is a $2m$-th iteration of some $\dfrac{T}{2m}$-periodic function for every natural integer m.

3° To show the symmetric Morse index does increase by iterating this solution.

Our argument depends on the mentioned $\mathbf{V}_4$-symmetry of the problem inherited from the natural $\mathbf{Z}_2$-symmetry of the system (1.1) and the evenness as of V, but does not depend on any particular property of the second derivative V'' of the potential function V (for example, convexity type property). To realize the point 1°, we work on a $W^{1,2}$-space SE_T of $\mathbf{V}_4$-symmetric T-periodic functions, which are even about the time $t=0$ and odd about $t=T/4$. By using the Mountain-Pass theorem, we then get a non-constant T-periodic $\mathbf{V}_4$-symmetric solution x of (1.1) with its symmetric Morse index defined on SE_T being not larger than 1. The symmetry possessed by x automatically realizes the point 2°. To prove 3°, we noticed that the derivative $\dot{x}$ of this solution we found is anti-symmetric, and is not in our working space SE_T. We then constructed a sequence of symmetric functions from this $\dot{x}$ to show that an iteration inequality of the symmetric Morse index holds, and that can be used to reduce the minimal period of x to not smaller than T/3. Then combining with 2° we conclude that this solution x must possess its minimal period T or T/3.

The main results we obtained in this paper are the following theorems. In the text of this paper, we denote by $a . b$ and $|a|$ the usual inner product and norm in $\mathbf{R}^n$ respectively.

THEOREM 1.1. – *Suppose* V *satisfies the following conditions.*

(V1) $V \in C^2(\mathbf{R}^n, \mathbf{R})$.

(V2) *There exists constants* $\mu > 2$ *and* $r_0 > 0$ *such that*

$$0 < \mu V(x) \leqq V'(x) \cdot x, \qquad \forall |x| \geqq r_0.$$

(V3) $V(x) \geqq V(0) = 0$, $\forall x \in \mathbf{R}^n$.

(V4) $V(x) = o(|x|^2)$, *at* $x = 0$.

(V5) V is even, *i.e*, $V(-x) = V(x)$, $\forall x \in \mathbf{R}^n$.

Then, for every $T > 0$, *the system* (1.1) *possesses a non-constant* T*-periodic solution with minimal period* T *or* T/3, *and which is even about* $t = 0$, T/2, *and odd about* $t = T/4$, 3T/4.

Next we consider the potential functions which are quadratic at the origin, *i.e.* satisfying the following condition (V6) at the origin.

(V6) *There exists constants* $\omega > 0$ *and* $r_1 > 0$ *such that*

$$V(x) \leqq \frac{\omega}{2} |x|^2, \qquad \forall |x| \leqq r_1.$$

A similar result is also true.

THEOREM 1.2. – *Suppose* V *satisfies conditions* (V1)-(V3), (V5) *and* (V6). *Then, for every positive* $T < \frac{1}{\sqrt{\omega}}$, *the conclusion of Theorem* 1.1 *holds.*

This paper is organized as follows. In section 2, we describe the mentioned $\mathbf{V}_4$-symmetric $W^{1,2}$-approach for Hamiltonian systems. In section 3, we establish the new iteration inequalities of Morse indices for linear second order Hamiltonian systems without convexity type assumption. Finally in section 4, we estimate the order of the isotropy subgroup of periodic symmetric solutions of (1.1) in terms of their Morse indices, study the minimal period problem for the system (1.1), and prove the above mentioned theorems.

2. A VARIATIONAL APPROACH ON A V_4-SYMMETRIC FUNCTION SPACE

In his pioneering work [21], Rabinowitz introduced the following variational formulation for the system (1.1),

$$\psi(x)=\int_0^{\mathrm{T}}\left(\frac{1}{2}|\dot{x}|^2-\mathrm{V}(x)\right)dt, \qquad \forall x\in \mathrm{W}^{1,2}(\mathrm{S_T}, \mathbf{R}^n), \tag{2.1}$$

where $\mathrm{T}>0$ and $\mathrm{S_T}=\mathbf{R}/(\mathrm{T}\,\mathbf{Z})$, and proved the existence of T-periodic solutions of (1.1) via the saddle point theorem. When the potential function V is even, this problem possesses a $\mathbf{V}_4$-symmetry as explained below. In this section we describe a variational formulation for this problem on $\mathrm{W}^{1,2}$-spaces of the $\mathbf{V}_4$-invariant functions.

For $\mathrm{T}>0$, we define the mentioned $\mathbf{V}_4$-action for any T-periodic measurable function $x:\mathrm{S_T}\to\mathbf{R}^n$ with $\mathbf{V}_4=\{\delta_0,\delta_1,\delta_2,\delta_3\}$ by

$$\begin{aligned}&\delta_0 x(t)=x(t), &&\delta_1 x(t)=x(-t),\\ &\delta_2 x(t)=-x\left(t-\frac{\mathrm{T}}{2}\right), &&\delta_3 x(t)=-x\left(\frac{\mathrm{T}}{2}-t\right),\end{aligned} \qquad a.e.$$

They are commutative and satisfy $\delta_1^2=\delta_2^2=\delta_3^2=\delta_0=\mathrm{id}$ and $\delta_1\delta_2=\delta_3$. We define another transformation group by $\hat{\mathbf{V}}_4=\{\delta_0,-\delta_1,\delta_2,-\delta_3\}$. It possesses similar properties as the first one. Note that both these groups are isomorphic to the Klein Fourgroup $\mathbf{Z}_2\oplus\mathbf{Z}_2$.

DEFINITION 2.1. – For $\mathrm{T}>0$, a T-periodic measurable function $x:\mathrm{S_T}\to\mathbf{R}^n$ is symmetric. if it satisfies

$$\delta x=x, \qquad \forall\delta\in\mathbf{V}_4.$$

a T-periodic measurable function $x:\mathrm{S_T}\to\mathbf{R}^n$ is anti-symmetric, if it satisfies

$$\delta x=x, \qquad \forall\delta\in\hat{\mathbf{V}}_4.$$

Note that for $\mathrm{T}>0$, a T-periodic function is symmetric (anti-symmetric) if and only if it is even (odd) about $t=0$ and $\mathrm{T}/2$, and is odd (even) about $t=\mathrm{T}/4$ and $3\mathrm{T}/4$.

Let $\mathrm{E_T}=\mathrm{W}^{1,2}(\mathrm{S_T},\mathbf{R}^n)$ with the usual norm

$$\|x\|_{\mathrm{T}}=\left(\int_0^{\mathrm{T}}(|\dot{x}|^2+|x|^2)\,dt\right)^{1/2}, \qquad \forall x\in\mathrm{E_T}.$$

Then $\mathrm{E_T}$ is a Hilbert space. We denote by $(\cdot,\cdot)_{\mathrm{T}}$ the corresponding inner product in $\mathrm{E_T}$.

Define

$$\mathrm{SE_T}=\{x\in\mathrm{E_T}\,|\,\delta x=x, \forall\delta\in\mathbf{V}_4\}.$$

SE_T is a closed subspace of E_T. Note that for given $T>0$, and $x \in E_T$, if $x(t)$ is odd (or even) about $t=t_0$, it is also odd (or even) about $t=T/2+t_0$.

LEMMA 2.2. – *For* $T>0$, *let* $x \in SE_T$. *Then*

1° *x is symmetric and satisfies* $x(0)=-x(T/2)$, $x(T/4)=x(3T/4)=0$ *and* $[x]_T=[x]_{T/2}=0$, *where* $[x]_T=\int_0^T x(t)\,dt$. *Its derivative* $\dot{x}$ *is anti-symmetric.*

2° *If* $x(0)\neq 0$, *then x is not* $T/(2m)$*-periodic for any* $m \in \mathbf{N}$, *where* $\mathbf{N}$ *is the set of all positive integers.*

3° *If* $x \not\equiv 0$, *then it can not be viewed as a symmetric* $2mT$*-periodic function for any* $m \in \mathbf{N}$.

4° *If* (V1) *holds and* $V'(0)=0$, *and if this x is a non-constant solution of the system* (1.1), *then* $x(0)\neq 0$.

5° *On* SE_T, the norm $\|x\|_T$ *is equivalent to the* L^2*-norm of the derivative* $\dot{x}$, i.e. $\left(\int_0^T |\dot{x}(t)|^2\,dt\right)^{1/2}$.

Proof. – 1° follows from the definition; 2° follows from the fact $x(0)=-x(T/2)\neq 0$. Let y be the function x viewed as a $2mT$-periodic function. Since x is even about $t=0$ and $t=T/2$, y is even about $t=mT/2$. So if y is symmetric, then it must be odd about $t=mT/2$, therefore $y\equiv 0$ and so is x. This proves 3°. In the case of 4°, x is smooth and $\dot{x}(0)=0$. So by the uniqueness theorem for initial value problems of (1.1) we obtain $x(0)\neq 0$. Since $x(T/4)=0$, we obtain for every $t \in [0, T]$

$$|x(t)| \leqq \int_{T/4}^{t} |\dot{x}(s)|\,ds \leqq \sqrt{T}\,\|\dot{x}\|_{L^2}. \tag{2.2}$$

This implies the equivalence between the norms claimed in 5°. The proof is complete. ■

PROPOSITION 2.3. – *Suppose* V *satisfies the condition* (V1). *Then for every* $T>0$ *we have*

1° $\psi \in C^2(E_T, \mathbf{R})$, i.e. ψ *is continuously* 2*-times Fréchet differentiable on* E_T.

2° *There holds*

$$(\psi'(x), y)_T=\int_0^T (\dot{x}.\dot{y}-V'(x).y)\,dt, \qquad \forall\, x, y \in E_T. \tag{2.3}$$

3° *There holds*

$$(\psi''(x)y, z)_T=\int_0^T (\dot{y}.\dot{z}-V''(x)y.z)\,dt, \qquad \forall\, x, y, z \in E_T. \tag{2.4}$$

4° *If in addition,* (V5) *holds, then* ψ *is* $\mathbf{V}_4$*-invariant,* i.e.

$$\psi(\delta x)=\psi(x), \qquad \forall\, x \in E_T \text{ and } \delta \in \mathbf{V}_4.$$

5° *All the above conclusions still hold, if we substitute* E_T *by* SE_T.

Proof. — 1°-3° are well-known.

4° We only need to prove the invariance under δ_1 and δ_2, since δ_3 is a composition of them. For $x \in E_T$, we have

$$\begin{aligned}\psi(\delta_1 x) &= \int_0^T \left\{\frac{1}{2}|-\dot{x}(-t)|^2 - V(x(-t))\right\} dt \\ &= -\int_0^{-T} \left\{\frac{1}{2}|\dot{x}(t)|^2 - V(x(t))\right\} dt \\ &= \psi(x),\end{aligned}$$

and by the condition (V5) we obtain

$$\begin{aligned}\psi(\delta_2 x) &= \int_0^T \left\{\frac{1}{2}\left|-\dot{x}\left(t-\frac{T}{2}\right)\right|^2 - V\left(-x\left(t-\frac{T}{2}\right)\right)\right\} dt \\ &= \int_0^T \left\{\frac{1}{2}\left|\dot{x}\left(t-\frac{T}{2}\right)\right|^2 - V\left(x\left(t-\frac{T}{2}\right)\right)\right\} dt \\ &= \int_{-T/2}^{T/2} \left\{\frac{1}{2}\left|\dot{x}(t)\right|^2 - V(x(t))\right\} dt \\ &= \psi(x).\end{aligned}$$

Note that the δ_1-symmetry is naturally possessed by ψ without using (V5) as we have noticed in [20]. Thus 4° holds, and then 5° follows. ■

It is well-known that critical points of ψ on E_T corresponds to $C^2(S_T, \mathbf{R}^n)$-solutions of (1.1).

PROPOSITION 2.4. — *Suppose* V *satisfies* (V1) *and* (V5). *Then the following holds*

1° *If* $x \in SE_T$ *is a critical point of* ψ *on* SE_T, *then it is a symmetric* $C^3(S_T, \mathbf{R}^n)$-*solution of* (1.1).

2° *Conversely, if* $x \in C^3(S_T, \mathbf{R}^n)$ *is a solution of* (1.1), *and is symmetric, then* $x \in SE_T$, *and it is a critical point of* ψ *on* SE_T.

Proof. — 1° Suppose $x \in SE_T$ is a critical point of ψ on SE_T. By (2.3) there holds

$$\int_0^T (\dot{x} \cdot \dot{y} - V'(x) \cdot y)\, dt = 0, \qquad \forall y \in SE_T. \tag{2.5}$$

Since $V \in C^2$, we have $w \equiv V'(x) \in W^{1,2}(S_T, \mathbf{R}^n)$, and so it is in $C(S_T, \mathbf{R}^n)$. By (V5), it is symmetric, since so is x. Therefore $[w]_T = [w]_{T/2} = 0$. The linear system

$$\left\{\begin{array}{l}\dot{q} = p \\ \dot{p} = -w\end{array}\right. \tag{2.6}$$

possesses a unique solution $(Q, P) \in C^2(\mathbf{R}, \mathbf{R}^n) \times C^1(\mathbf{R}, \mathbf{R}^n)$ satisfying $P(0)=0$ and $Q(T/4)=0$. Since $[w]_T=0$, P is T-periodic. Since w is symmetric, P is anti-symmetric. So we have $[P]_T=0$ and $P(T)=P(0)=0$. Thus Q is T-periodic and symmetric. So $Q \in SE_T$. From (2.6) we obtain that for every $y \in SE_T$ there holds

$$\int_0^T (\dot{Q} \cdot \dot{y} - V'(x) \cdot y)\, dt = \int_0^T (\dot{Q} - P) \cdot \dot{y}\, dt + P(t) \cdot y(t)\,|_0^T = 0.$$

Combining with (2.5) it yields

$$\int_0^T (\dot{x} - \dot{Q}) \cdot \dot{y}\, dt = 0, \qquad \forall\, y \in SE_T.$$

Letting $y = x - Q$, by the fact $x(T/4) = Q(T/4) = 0$ we obtain

$$|x(t) - Q(t)| \leqq \int_{T/4}^{t} |\dot{x}(s) - \dot{Q}(s)|\, ds \leqq \sqrt{T}\, \|\dot{x} - \dot{Q}\|_{L^2} = 0, \qquad \forall\, t \in [0, T].$$

Thus $x = Q \in C^2(S_T, \mathbf{R}^n)$ and is a solution of (1.1) by (2.6). Then by (V1) and the system (1.1), x is C^3.

2° is clear and the proof is complete. ■

Remark 2.5. – The proof of 1° uses an idea of Rabinowitz given in [24].

DEFINITION 2.6. – Given a C^1 real functional f defined on a real Hilbert space E. A sequence $\{u_k\} \subset E$ is said to be a (PS)-sequence, if $\{|f(u_k)|\}$ is bounded and $f'(u_k) \to 0$ as $k \to \infty$. The functional f is said to satisfy the Palais-Smale condition (PS) on E, if every (PS) sequence $\{u_k\} \subset E$ possesses a subsequence convergent in E.

PROPOSITION 2.7. – *Suppose* V *satisfies* (V1) *and* (V2). *Then* ψ *satisfies* (PS) *on* E_T. *Suppose* V *further satisfies* (V5). *Then* ψ *satisfies* (PS) *on* SE_T.

Proof. – It is well-known that ψ satisfies the (PS) condition on E_T. For a proof we refer to [21], [23]. When (V5) holds, if $\{u_k\}$ is a (PS) sequence in SE_T, it is also a (PS) sequence in E_T. Therefore it possesses a subsequence which converges to some element $u \in E_T$. Since SE_T is a closed subspace of E_T, we obtain $u \in SE_T$, and the proof is complete. ■

3. A MORSE INDEX THEORY AND ITS ITERATION INEQUALITY

In section 4, we shall find a critical point x_0 of ψ on SE_T. By Proposition 2.4, x_0 is a symmetric $C^2(S_T, \mathbf{R}^n)$-solution of (1.1). Let

$A(t)=V''(x_0(t))$. Since V is even, so is V'' on $\mathbf{R}^n$. Therefore $A(t)$ is continuous, T-periodic, and is even about $t=0$. By Proposition 2.3, $\psi''(x_0)$ defines the following bilinear form on SE_T

$$\phi_T(x, y)=\int_0^T (\dot{x}.\dot{y}-A(t)x.y)\,dt, \qquad \forall x, y \in SE_T. \tag{3.1}$$

Note that ϕ_T is also defined on E_T. The Morse index of ψ at the critical point x_0 in SE_T is defined to be the Morse index of the quadratic form $\phi_T(x, x)$ in SE_T. The main goal in this section is to establish iteration inequalities for such a Morse index theory. Note that ϕ_T corresponds to the following linear second order Hamiltonian system,

$$\ddot{x}+A(t)x=0, \qquad \forall x \in \mathbf{R}^n. \tag{3.2}$$

Let $\mathscr{L}_S(\mathbf{R}^n)$ denote the space of symmetric $n \times n$ matrices on the field $\mathbf{R}$. If the above mentioned x_0 has minimal period T/k for some integer $k \geqq 1$, we shall prove in section 4 that $A(t)=V''(x_0(t))$ is $T/(2k)$-periodic and is even about $t=0$ and $t=T/(4k)$. Enlarging these numbers by k times, in this section for given $T>0$, we always suppose the following condition holds,

(AS) $A \in C(S_{T/2}, \mathscr{L}_S(\mathbf{R}^n))$, and it is even about $t=0$ and $T/4$.

DEFINITION 3.1. – We say that x and $y \in SE_T$ are ϕ_T-orthogonal and write $x \oplus_T y$, if $\phi_T(x, y)=0$. Two subspaces F and G of SE_T are ϕ_T-orthogonal, if $x \oplus_T y=0$ for all $x \in F$ and all $y \in G$. We write $F \oplus_T G$.

PROPOSITION 3.2. – *Suppose the condition* (AS) *holds.*

1° SE_T *possesses a* ϕ_T-*orthogonal decomposition*

$$SE_T=SE_T^+ \oplus_T SE_T^0 \oplus_T SE_T^-$$

such that ϕ_T *is positive, null, and negative definite on* SE_T^+, SE_T^0, *and* SE_T^- *respectively.*

2° $SE_T^0=\ker \phi_T$, *in* SE_T, *and* $\dim SE_T^0<+\infty$.

3° $\dim SE_T^- <+\infty$.

Proof. – Define an operator $A_T : SE_T \to SE_T$ by

$$(A_T x, y)_T=\int_0^T (A(t)x \cdot y+x \cdot y)\,dt, \qquad \forall x, y \in SE_T. \tag{3.3}$$

Since the quadratic functional $\int_0^T (A(t)x \cdot y+x \cdot y)\,dt$ is weakly continuous and uniformally Fréchet differentiable on SE_T, its gradient A_T is compact by a theorem of Tsitlanadze (*cf.* [18]). Then A_T is a linear compact self-adjoint operator on SE_T. Therefore by the spectral theory for such operators in a Hilbert space, SE_T possesses a basis $\{e_m \mid m \in \mathbf{N}\}$ and corresponding eigenvalues $\{\lambda_m\}$, such that $\lambda_m \to 0$ in $\mathbf{R}$, and

$$\begin{cases} (e_i, e_j)_T = \delta_{ij}, & \forall\, i, j \in \mathbf{N}, \\ (A_T e_m, x)_T = \lambda_m (e_m, x)_T, & \forall\, m \in \mathbf{N}, \quad x \in SE_T, \end{cases} \tag{3.4}$$

and for any $x \in SE_T$, there exists $\{\alpha_m\} \subset \mathbf{R}$, such that $x = \sum_{m \geq 1} \alpha_m e_m$ in L^2. Thus we have

$$\begin{aligned} \phi_T(x, x) &= (x, x)_T - (A_T x, x)_T \\ &= \sum \alpha_m^2 - (\sum \alpha_m \lambda_m e_m, \sum \alpha_m e_m)_T \\ &= \sum (1 - \lambda_m) \alpha_m^2. \end{aligned}$$

Let

$$\begin{aligned} SE_T^+ &= \{\sum \alpha_m e_m \mid \alpha_m = 0 \text{ if } 1 - \lambda_m \leq 0\}, \\ SE_T^0 &= \{\sum \alpha_m e_m \mid \alpha_m = 0 \text{ if } 1 - \lambda_m \neq 0\}, \\ SE_T^- &= \{\sum \alpha_m e_m \mid \alpha_m = 0 \text{ if } 1 - \lambda_m \geq 0\}. \end{aligned}$$

Notice that $1 - \lambda_m \to 1$ in $\mathbf{R}$, the proof if complete. ■

DEFINITION 3.3. – Define

$$si_T = \dim SE_T^-, \qquad s\nu_T = \dim SE_T^0.$$

si_T and $s\nu_T$ are called the symmetric Morse index and the symmetric nullity of ϕ_T on SE_T respectively,

Let E_T^0 be the kernel of ϕ_T on E_T, *i.e.* the set of all T-periodic solutions of (3.2).

If x is a critical point of ψ in SE_T with $x \neq 0$, then x is a symmetric solution of (1.1) by Proposition 2.4. Therefore $\dot{x}$ is an anti-symmetric solution of the linear system (3.2) with $A(t) = V''(x(t))$ and satisfies $\dot{x}(0) = 0$. Since $x \neq 0$, we have $\dot{x} \notin SE_T^0$. Therefore we define the space of such anti-symmetric solutions of (3.2) by

$$AE_T^0 = \{y \in E_T^0 \mid y \text{ is anti-symmetric}\}.$$

DEFINITION 3.4. – Define

$$a\nu_T = \dim AE_T^0.$$

$a\nu_T$ is called the anti-symmetric nullity of ϕ_T in E_T.

Let $y = \dot{x}$ and $z = (y, x)$. Then the system (3.2) is equivalent to the following first order Hamiltonian system,

$$\dot{z} = JB(t) z, \qquad z \in \mathbf{R}^{2n}, \tag{3.5}$$

where $B(t)$ is defined to be $\begin{pmatrix} I & 0 \\ 0 & A(t) \end{pmatrix}$, and J is the standard symplectic matrix. Denote by $M(t)$ the fundamental solution of (3.5), *i.e.* it satisfies

$$\begin{cases} \dot{M}(t) = JB(t)M(t), & \forall t \in \mathbf{R}, \\ M(0) = I. \end{cases}$$

The following result is well-known.

PROPOSITION 3.5. – *Suppose the condition* (AS) *holds. Then*

$$av_T \leqq \dim \ker (M(T) - I) \leqq 2n. \tag{3.6}$$

In order to further study these indices, we define the following maps for given $x : [0, T] \to \mathbf{R}^n$ and $y : \mathbf{R} \to \mathbf{R}^n$. By Lemma 2.2, we only need to consider odd iterations of functions in E_T. Fix an odd integer $k \geqq 3$. Define

$$r_- x(t) = \begin{cases} x(t), & 0 \leqq t \leqq \frac{T}{2}, \\ 0, & \frac{T}{2} < t \leqq T \end{cases} \qquad r_+ x(t) = \begin{cases} 0, & 0 \leqq t \leqq \frac{T}{2}, \\ x(t), & \frac{T}{2} < t \leqq T. \end{cases}$$

$$px(t) = \begin{cases} x\left(t - \frac{(k-1)}{2} T\right), & \frac{(k-1)}{2} T \leqq t \leqq \frac{(k+1)}{2} T \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_- y(t) = y\left(t + \frac{T}{2}\right) \quad \text{and} \quad \eta_+ y(t) = y\left(t - \frac{T}{2}\right), \qquad \forall t \in \mathbf{R}.$$

Then it is clear that $r_\pm : E_T \to E_T$, $p : E_T \to E_{kT}$, and $\eta_\pm : E_{kT} \to E_{kT}$.

The next lemma collects special properties of elements in AE_T^0.

LEMMA 3.6. – *Suppose the condition* (AS) *holds. Let* $x \in AE_T^0 \setminus \{0\}$. *Then x is an anti-symmetric solution of* (3.2), *satisfies* $\dot{x}(0) = -\dot{x}(T/2) \neq 0$, *and* $r_\pm x \in E_T$.

Proof. – By the definition of $x \in AE_T^0$, $x(0) = x(T/2) = 0$. If $\dot{x}(0) = 0$, then by the uniqueness of the initial value problem of (3.2), $x \equiv 0$. This contradicts the assumption. Then $\dot{x}(T/2) = -\dot{x}(0) \neq 0$. Other claims are clear. ■

The following iteration inequality on the symmetric Morse indices is the main result in this section.

THEOREM 3.7. – *Suppose the condition* (AS) *holds. Then*

$$si_{kT} \geqq \left(\frac{k-1}{2}\right) av_T + si_T, \qquad \forall k \in 2\mathbf{N} - 1. \tag{3.7}$$

Proof. – Supposse $av_T \geqq 1$ and fix an odd integer $k \geqq 3$. Other cases follows from the proof immediately. We carry out the proof in several steps.

Step 1. – For $1 \leqq i \leqq (k-1)/2$, we define

$$\mathrm{N}_i = \begin{cases} \{(\eta_-^{k-i} pr_-) - (\eta_-^{i-1} pr_-) + (\eta_+^{i-1} \mathrm{pr}_+) - (\eta_+^{k-i} pr_+)\} \mathrm{AE}_\mathrm{T}^0, \\ \qquad \text{if } i \text{ is odd}, \\ \{(\eta_+^{k-i+1} pr_-) - (\eta_+^{i} pr_-) + (\eta_-^{i} pr_+) \\ \qquad - (\eta_-^{k-i+1} pr_+)\} \mathrm{AE}_\mathrm{T}^0, \\ \qquad \text{if } i \text{ is even}, \end{cases} \tag{3.8}$$

$$\mathrm{N} = \bigoplus_{i=1}^{(k-1)/2} \mathrm{N}_i, \tag{3.9}$$

and

$$\mathrm{M} = \left\{ id + \sum_{i=1}^{(k-1)/2} (\eta_-^{2i} + \eta_+^{2i}) \right\} p\, \mathrm{SE}_\mathrm{T}^-. \tag{3.10}$$

where $\eta_\pm^2 = \eta_\pm \circ \eta_\pm$, and $\oplus$ means the direct sum. Note that M is simply the space SE_T viewed as a subspace of $\mathrm{SE}_{k\mathrm{T}}$ (*cf.* Figures 1 and 2 in Appendix for an illustration of functions in N_i's and M). By the definition and Lemma 3.6, all these spaces are subspaces of $\mathrm{SE}_{k\mathrm{T}}$. Since T-periodic symmetric and anti-symmetric functions are determined by their values on $[0, \mathrm{T}/4]$, from the definitions (3.8) and (3.10) we obtain

$$\dim \mathrm{M} = si_\mathrm{T}, \qquad \text{and} \qquad \dim \mathrm{N}_i = a\nu_\mathrm{T} \quad \text{for } 1 \leqq i \leqq \frac{k-1}{2}. \tag{3.11}$$

We claim that for $1 \leqq i \leqq (k-1)/2$,

$$\alpha \oplus_{k\mathrm{T}} \beta, \qquad \forall \alpha, \beta \in \mathrm{N}_i. \tag{3.12}$$

We only prove the case when i is odd. The other case can be proved similarly. In fact, since i is odd, for any $\alpha_j \in \mathrm{N}_i$, $j = 1, 2$, by the first formula in (3.8), there exist $u_j \in \mathrm{AE}_\mathrm{T}^0$, $j = 1, 2$, such that

$$\begin{aligned} \alpha_j &= \{(\eta_-^{k-i} pr_-) - (\eta_-^{i-1} pr_-) + (\eta_+^{i-1} pr_+) - (\eta_+^{k-i} pr_+)\} u_j \\ &\equiv \alpha_{j,1}^- - \alpha_{j,2}^- + \alpha_{j,1}^+ - \alpha_{j,2}^+. \end{aligned} \tag{3.13}$$

Therefore by (3.8)-(3.10), (AS), and the evenness of the integrand, we pick the first and the third terms in the integration and obtain

$$\begin{aligned} \phi_{k\mathrm{T}}(\alpha_1, \alpha_2) &= 2 \int_{(i-1)\mathrm{T}/2}^{i\mathrm{T}/2} \{\dot\alpha_{1,1}^- \cdot \dot\alpha_{2,1}^- - \mathrm{A}(t)(\alpha_{1,1}^-) \cdot (\alpha_{2,1}^-)\}\, dt \\ &\quad + 2 \int_{(k+i-1)\mathrm{T}/2}^{(k+i)\mathrm{T}/2} \{\dot\alpha_{1,1}^+ \cdot \dot\alpha_{2,1}^+ - \mathrm{A}(t)(\alpha_{1,1}^+) \cdot (\alpha_{2,1}^+)\}\, dt \\ &= 2 \int_0^{\mathrm{T}/2} \{(r_- \dot u_1) \cdot (r_- \dot u_2) - \mathrm{A}(t)(r_- u_1) \cdot (r_- u_2)\}\, dt \end{aligned}$$

$$+2\int_{T/2}^{T}\{(r_+\dot{u}_1)\cdot(r_+\dot{u}_2)-A(t)(r_+u_1)\cdot(r_+u_2)\}\,dt$$
$$=2\int_0^T\{\dot{u}_1\cdot\dot{u}_2-A(t)(u_1)\cdot(u_2)\}\,dt$$
$$=0. \tag{3.14}$$

Here we have used the fact $u_1, u_2 \in AE_T^0$. So $\alpha_1 \oplus_{kT} \alpha_2$ and (3.12) is proved.

By the definition (3.8), when $i \neq j$, functions in N_i and N_j have disjoint supports, therefore we have

$$N_i \oplus_{kT} N_j, \quad \text{if } i \neq j, \qquad \text{and} \qquad 1 \leqq i, j \leqq \frac{k-1}{2}. \tag{3.15}$$

We claim that

$$N_i \oplus_{kT} M, \qquad \text{for} \quad 1 \leqq i \leqq \frac{k-1}{2}. \tag{3.16}$$

We only prove the case when i is odd. The other case can be proved similarly. In fact, since i is odd, for any $\alpha \in N_i$, by the first formula in (3.8), there exist $u \in AE_T^0$ such that

$$\alpha=(\eta_-^{k-i}pr_- - \eta_-^{i-1}pr_- + \eta_+^{i-1}pr_+ - \eta_+^{k-i}pr_+)u$$
$$\equiv \alpha_1^- - \alpha_2^- + \alpha_1^+ - \alpha_2^+. \tag{3.17}$$

For any $\beta \in M$, by the definition of M, there exists $v \in SE_T^-$ such that viewing v as a function in SE_{kT} gives β. That is

$$\beta=(\mathrm{id}+\sum_{i=1}^{(k-1)/2}(\eta_-^{2i}+\eta_+^{2i}))pv. \tag{3.18}$$

Therefore by (3.8)-(3.10), (AS), and the evenness of the integrand, similar to (3.14) we obtain

$$\phi_{kT}(\alpha, \beta)=2\int_{(i-1)T/2}^{iT/2}\{\dot{\alpha}_1^-\cdot\dot{\beta}-A(t)(\alpha_1^-)\cdot\beta\}\,dt$$
$$+2\int_{(k+i-1)T/2}^{(k+i)T/2}\{\dot{\alpha}_1^+\cdot\dot{\beta}-A(t)(\alpha_1^+)\cdot\beta\}\,dt$$
$$=2\int_0^{T/2}\{(r_-\dot{u})\cdot\dot{v}-A(t)(r_-u)\cdot v\}\,dt$$
$$+2\int_{T/2}^{T}\{(r_+\dot{u})\cdot\dot{v}-A(t)(r_+u)\cdot v\}\,dt$$
$$=2\int_0^T\{\dot{u}\cdot\dot{v}-A(t)u\cdot v\}\,dt$$
$$=0. \tag{3.19}$$

So $\alpha \oplus_{kT} \beta$. This proves the claim (3.16).

Thus these subspaces N_i's and M are all mutually ϕ_{kT}-orthogonal.

Since N_i and N_j, $i \neq j$, contain functions with disjoint supports, they are linearly independent. Since all the functions in N are identically zero on the set $[(k-1)T/4, (k+1)T/4] \cup [(3k-1)T/4, (3k+1)T/4]$, but all the non-trivial functions in M are not identically zero on any non-empty subinterval, M and N are linearly independent. Therefore from (3.11), we obtain

$$\dim (M \oplus N) = \left(\frac{k-1}{2}\right) a \nu_T + s i_T. \tag{3.20}$$

Step 2. – We claim that

$$\begin{cases} \phi_{kT}(x, x) < 0, & \forall x \in M \setminus \{0\}, \\ \phi_{kT}(x, x) = 0, & \forall x \in N, \\ \phi_{kT}(x, x) \leqq 0, & \forall x \in M \oplus N, \end{cases} \tag{3.21}$$

and

$$\phi_{kT}(x, x) = 0 \quad \text{and} \quad x \in M \oplus N \text{ imply that } x \in N. \tag{3.22}$$

In fact, for any $\beta \in M \setminus \{0\}$, let $v \in SE_T^-$ such that viewing v as a function in SE_{kT} gives β, *i.e.* (3.18) holds. Then

$$\phi_{kT}(\beta, \beta) = k \phi_T(v, v) < 0, \qquad \forall \beta \in M \setminus \{0\}. \tag{3.23}$$

For $1 \leqq i \leqq (k-1)/2$, from (3.14) and a similar derivation when i is even, we obtain

$$\phi_{kT}(\alpha, \alpha) = 0, \qquad \forall \alpha \in N_i. \tag{3.24}$$

By the ϕ_{kT}-orthogonalities we just proved among these subspaces, we obtain the claims (3.21) and (3.22).

Step 3. – We claim that

$$N \cap SE_{kT}^0 = \{0\}. \tag{3.25}$$

In order to prove (3.25), we prove the following claim first:

The derivative of every $\alpha \in N \setminus \{0\}$

is discontinuous somewhere in $[0, kT]$. (3.26)

In fact, by definition, α must have the form

$$\alpha = \sum_{i=1}^{k-1} \alpha_i, \quad \text{for some } \alpha_i \in N_i, \qquad 1 \leqq i \leqq \frac{k-1}{2}. \tag{3.27}$$

Let j be the smallest subscript in $\{1, \ldots, (k-1)/2\}$ such that $\alpha_j \neq 0$. We only prove the case when j is odd. The other case can be proved similarly. By the definition (3.8) there exists $u \in AE_T^0$ such that

$$\alpha_j = \{(\eta_-^{k-i} pr_-) - (\eta_-^{i-1} pr_-) + (\eta_+^{i-1} pr_+) - (\eta_+^{k-i} pr_+)\} u. \tag{3.28}$$

By Lemma 3.6, we have $\dot{u}(0)\neq 0$. Therefore the first and the fourth terms in the right hand side of (3.28) are not C^1 at $t=(i-1)T/2$ and $t=(2k-i+1)T/2$ respectively, and therefore so is α_j. Since j is the smallest subscript with $\alpha_j\not\equiv 0$, by definition all the other $\dot{\alpha}_i$'s are identically equal to zero near these two times. Therefore $\dot{\alpha}$ is discontinuous at these two times. This proves the claim (3.26).

Now we prove the following claim which is stronger than (3.25), since $SE^0_{kT}\subset E^0_{kT}$.

$$N\cap E^0_{kT}=\{0\}. \tag{3.29}$$

In fact, let $x\in N\cap E^0_{kT}$. Then by the definition of E^0_{kT}, x is a $C^2(S_{kT}, \mathbf{R}^n)$-solution of (3.2). Therefore $\dot{x}$ must be continuous everywhere. By (3.26), this implies $x\equiv 0$. Thus (3.29) and therefore (3.25) is true.

Here we give another proof of (3.29) using the following property of functions in N in stead of (3.26). From the definition (3.8), every $x\in N$ satisfies

$$x(t)=0, \qquad \forall\, t\in\left[\frac{(k-1)T}{4}, \frac{(k+1)T}{4}\right]\cup\left[\frac{(3k-1)T}{4}, \frac{(3k+1)T}{4}\right]. \tag{3.30}$$

If $x\in N\cap E^0_{kT}$, then it is a solution of (3.2). Then by the uniqueness theorem of the initial value problem of (3.2), we must have $x=0$ on $\mathbf{R}$. This proves (3.29) and (3.25).

Step 4. – Let $D:SE_{kT}\to SE_{kT}$ be the linear operator associated to the bilinear form $\phi_{kT}(x, y)$, *i.e.*

$$\phi_{kT}(x, y)=(Dx, y)_{kT}, \qquad \forall\, x, y\in SE_{kT}.$$

Then D is linear, continuous, self-adjoint, and is actually the gradient of the quadratic functional $\phi_{kT}(x, x)$ on SE_{kT}. Therefore when $|h|$ is sufficiently small, $F_h=\mathrm{id}+hD$: $SE_{kT}\to SE_{kT}$ is a linear homeomorphism. Define

$$(M\oplus N)_h=F_h(M\oplus N),$$
$$S=\{x\in SE_{kT}\,|\,\|v\|_{kT}=1\},$$

and

$$f_h(x)=\frac{F_h(x)}{\|F_h(x)\|_{kT}}, \qquad \forall\, x\in SE_{kT}\setminus\{0\}.$$

Because $f_h(\lambda x)=f_h(x)$ holds for all $\lambda>0$, it is easy to see that

$$\begin{aligned} S\cap(M\oplus N)_h &=\{f_h(x)\,|\,x\in(M\oplus N)\setminus\{0\}\}\\ &=\{f_h(x)\,|\,x\in S\cap(M\oplus N)\}. \end{aligned} \tag{3.31}$$

By elementary calculations we find that for every $x\in S\cap(M\oplus N)$,

$$\left\{\frac{d}{dh}f_h(x)\right\}\bigg|_{h=0}=Dx-\phi_{kT}(x, x)\,x.$$

So

$$\left\{\frac{d}{dh}\phi_{kT}(f_h(x), f_h(x))\right\}\Bigg|_{h=0} = 2\left(\mathrm{D}\, f_h(x), \frac{d}{dh} f_h(x)\right)_{kT}\Bigg|_{h=0}$$
$$= 2\,\|\mathrm{D}\, x\|_{kT}^2 - \phi_{kT}^2(x, x). \tag{3.32}$$

If $x \in \mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N})$ and $\phi_{kT}(x, x) = 0$, (3.31) yields $x \in \mathrm{S} \cap \mathrm{N}$. By (3.25), $\mathrm{N} \cap \mathrm{SE}_{kT}^0 = \{0\}$, so $x \notin \mathrm{SE}_{kT}^0 = \ker \mathrm{D}$. This implies that $\mathrm{D}x \neq 0$. Therefore (3.32) yields

$$\left\{\frac{d}{dh}\phi_{kT}(f_h(x), f_h(x))\right\}\Bigg|_{h=0} = \|\mathrm{D}x\|_{kT}^2 > 0$$
$$\text{if} \quad x \in \mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N}) \quad \text{and} \quad \phi_{kT}(x, x) = 0.$$

Since there also holds $\phi_{kT}(x, x) \leqq 0$ for any $x \in \mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N})$, by the compactness of $\mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N})$, there exists a constant $\varepsilon > 0$ such that

$$\phi_{kT}(f_h(x), f_h(x)) < 0, \qquad \forall\, x \in \mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N}) \quad \text{and} \quad -\varepsilon < h < 0.$$

Thus from (3.31), we obtain

$$\phi_{kT}(x, x) < 0, \qquad \forall\, x \in \mathrm{S} \cap (\mathrm{M} \oplus \mathrm{N})_h, \quad -\varepsilon < h < 0.$$

Therefore ϕ_{kT} is negative definite on $(\mathrm{M} \oplus \mathrm{N})_h$, and from (3.20) we get

$$si_{kT} \geqq \dim (\mathrm{M} \oplus \mathrm{N})_h = \dim (\mathrm{M} \oplus \mathrm{N}) = \left(\frac{k-1}{2}\right) a\, \nu_T + si_T.$$

The proof is complete. ■

Remarks 3.8. – 1° For first order linear Hamiltonian systems (3.5) with positive definite coefficients $\mathrm{B}(t)$, the following iteration inequality was first proved by Ekeland in 1984 in terms of his index theory (*cf.* Theorem I.5.1 [9] and [7], [8]).

$$i_{kT} \geqq k i_T + (k-1)\, \nu_T. \tag{3.33}$$

Similar iteration inequalities on various Morse indices for general linear second order Hamiltonian systems (3.2) without convexity type assumptions on the coefficients were proved in [20] by the author.

2° The proof of Theorem 3.7 uses ideas of Ekeland (*cf.* the proof of Theorem I.5.1 [9]) and the author (*cf.* the proof of Theorem 3.10 [20]). As mentioned in the section 1, here special efforts are made in order to construct functions with the required symmetry. Our arguments depend on the T/2-periodicity, continuity and symmetry of $\mathrm{A}(t)$ given by the condition (AS), but not on its positivity.

4. THE EXISTENCE OF SOLUTIONS WITH PRESCRIBED MINIMAL PERIOD

In this section, we prove theorems 1.1 and 1.2.

DEFINITION 4.1. — Given $T>0$, for every non-constant T-periodic solution x of the system (1.1), $O(x)$ is defined to be the order of the isotropy subgroup of x for the S^1-action a_θ on T-periodic functions, where $a_\theta x(t)=x(t+\theta T)$. In another words, $O(x)$ is the greatest positive integer k such that x is T/k-periodic.

By Proposition 2.4, every T-periodic solution x of (1.1) which is even about $t=0$, odd about $t=T/4$ corresponds to a critical point of the functional ψ on SE_T defined in (2.1) with the potential function $V=V(x)$ given in (1.1). In the discussion of the section 3, let $A(t)=V''(x(t))$. The functional ϕ_T defined by (3.1) is precisely the quadratic form of the second Fréchet differential of ψ on SE_T. We denote the corresponding symmetric and anti-symmetric Morse indices defined in the section 3 of ψ at x by $si_T(x)$ and $a\nu_T(x)$, etc. respectively. Our following theorem estimates $O(x)$ in terms of $si_T(x)$.

THEOREM 4.2. — *Suppose that the conditions* (V1) *and* (V5) *hold. For* $T>0$, *and every non-constant* $C^3(S_T, \mathbf{R}^n)$*-solution* x *of* (1.1) *which is even about* $t=0$ *and odd about* $t=T/4$, *there holds*

$$O(x)\leqq 2(si_T(x))+1. \tag{4.1}$$

Proof. — Let $k=O(x)$. Since x is a non-constant T/k-periodic solution of (1.1), and it is even about $t=0$ and odd about $t=T/4$, we have $x(0)=-x(T/2)\neq 0$. Thus k is odd by Lemma 2.2. Then we must have $k=4m+1$ or $k=4m+3$ for some integer $m\geqq 0$. Therefore $T/4$ can be rewritten as one of the following forms:

$$\frac{T}{4}=\frac{T(4m+1)}{4k}=m\frac{T}{k}+\frac{T}{4k},$$
$$\frac{T}{4}=\frac{T(4m+3)}{4k}=m\frac{T}{k}+\frac{3T}{4k}.$$

Since x is odd about $T/4$ and T/k-periodic, the abobe equalities show that it must be odd about $T/(4k)$. Then we have $y=\dot{x}$ is a non-trivial T/k-periodic solution of the lineat system (3.2) with $A(t)=V''(x(t))$, and y is odd about $t=0$, even about $t=T/(4k)$. Therefore $y\in AE^0_{T/k}$. This shows $a\nu_{T/k}(x)\geqq 1$. From the symmetry of x and the evenness of V, $A(t)$ is T/k-periodic and even about all integer multiples of $T/(4k)$. Therefore it is $T/(2k)$-periodic and is even about times $t=0$ and $T/(4k)$. So the condition (AS) holds. From Theorem 3.7 we obtain

$$si_T(x) = si_{k \cdot T/k}(x) \geqq \frac{k-1}{2}.$$

This yields (4.1) and completes the proof. ■

Remark 4.3. – For T-periodic solutions of the strictly convex Hamiltonian systems (1.2), a similar estimate,

$$O(x) \leqq i_T(x) + 1,$$

was first proved in 1985 by Ekeland and Hofer in terms of Ekeland index theory (*cf.* Theorem III.6 [11]). For the system (1.1) under only the condition (V1), a similar estimate,

$$O(x) \leqq si_T(x) + 1 - \sigma_T^+(x),$$

for even T-periodic solutions in terms of the symmetric Morse index defined there was proved by the author in [20]. There are also other similar estimates established in [20] in terms of various Morse indices.

For given $T > 0$, in order to find T-periodic solutions of (1.1), we use the following well-known Mountain-pass theorem of Ambrosetti and Rabinowitz.

THEOREM 4.4. – *Let* E *be a real Hilbert space suppose* $f \in C^2(E, \mathbf{R})$, *satisfies the* (PS) *condition, and the following conditions.*

(F1) *There exist* ρ *and* $\alpha > 0$ *such that* $f(u) \geqq \alpha$, *for all* $u \in \partial B_\rho(0)$.

(F2) *There exist* $R > \rho$ *and* $e \in E$ *with* $\|e\| \geqq R$ *such that* $f(e) \leqq 0$.

Then 1° *f possesses a critical value* $c \geqq \alpha$, *which is given by*

$$c = \inf_{h \in \Gamma} \max_{u \in h([0,1])} f(u),$$

where $\Gamma = \{ h \in C([0, 1], E) \mid h(0) = 0,\ h(1) = e \}$.

2° *There exists an element* $u_0 \in \mathscr{K}_c \equiv \{ u \in E \mid f'(u) = 0, f(u) = c \}$ *such that the negative Morse index* $i(u_0)$ *of f at* u_0 *satisfies*

$$i(u_0) \leqq 1. \tag{4.2}$$

Remark 4.5. – The proof of this theorem can be found in [4], [11], [17], [19],]23], [25], [26]. Combining theorems 4.2 and 4.4 together, we obtain the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. – Given $T > 0$, in Theorem 4.4, let $E = SE_T$, and $f = \psi$ defined by (2.1) on SE_T for $V = V(x)$. Propositions 2.3 and 2.7 show that ψ is C^2 and satisfies the (PS) condition. Note that by Lemma 2.2, on SE_T the E_T norm $\|x\|_T$ is equivalent to the norm $\|\dot{x}\|_{L^2}$. By the condition (V4), for any $\varepsilon > 0$ small, there is a constant $\rho > 0$ such that

$$0 \leqq V(x) \leqq \varepsilon |x|^2, \qquad \forall x \in B_\rho(0).$$

Then for $x \in SE_T$ with $\|x\|_T$ small, by (2.2) we obtain

$$\begin{aligned}\psi(x) &\geqq \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 - \varepsilon |x(t)|^2\right) dt \\ &\geqq \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 - \varepsilon T^2 |\dot{x}(t)|^2\right) dt \\ &= \frac{1}{2}(1 - 2\varepsilon T^2) \|\dot{x}\|^2_{L^2(0,T)}.\end{aligned}$$

Therefore if we choose $\varepsilon > 0$ to be small enough, the condition (F1) holds. It is standard to show that under assumptions (V1) and (V2), the condition (F2) holds. Since the proof of Rabinowitz given in the section 6 of his book [23] works here with only minor notational modifications, we omit the verification of this condition here.

So we get a critical point $x \in SE_T$ of ψ with $\psi(x) > 0$ and for this x the inequality (4.2) holds, that is

$$si_T(x) \leqq 1, \tag{4.3}$$

Since $\psi(x) > 0$, x is not a constant function. By Proposition 2.4, x is a non-constant T-periodic symmetric classical solution of (1.1). By Theorem 4.2 and (4.3) we get

$$O(x) \leqq 2(si_T(x)) + 1 \leqq 3. \tag{4.4}$$

By Lemma 2.2, $O(x)$ is odd. Thus $O(x) = 3$ or $O(x) = 1$. The proof is complete. ■

Proof of Theorem 1.2. – For $x \in SE_T$ with $\|x\|_T$ being sufficiently small, by the Sobolev imbedding Theorem and Lemma 2.2, we have $\|x\|_C \leqq r_1$ for r_1 defined in (V6). So by (V6) and (2.2), we have for such x,

$$\begin{aligned}\psi(x) &\geqq \int_0^T \left(\frac{1}{2}|\dot{x}|^2 - \frac{\omega}{2}|x|^2\right) dt \\ &\geqq \int_0^T \left(\frac{1}{2}|\dot{x}|^2 - \frac{\omega}{2}T^2|\dot{x}|^2\right) dt \\ &\geqq \frac{1}{2}(1 - T^2\omega) \|\dot{x}\|^2_{L^2(0,T)}.\end{aligned}$$

Thus when $0 < T < \frac{1}{\sqrt{\omega}}$, the condition (F1) holds for the functional ψ resticted to SE_T. Now the remaining part of the proof can be carried out as that of Theorem 1.1, and therefore is omitted. ■

ACKNOWLEDGEMENTS

The author would like to express his sincere thanks to Professors J. Moser and E. Zehnder for their invitation to visit E.T.H., their encouragements, and for their valuable comments on this work, especially for Edi Zehnder who carefully read the manuscript of this paper and for his valuable remarks on it. The author also sincerely thank the hospitality of the Forschungsinstitut für Mathematik, E.T.H.-Zürich, where this paper was written.

APPENDIX

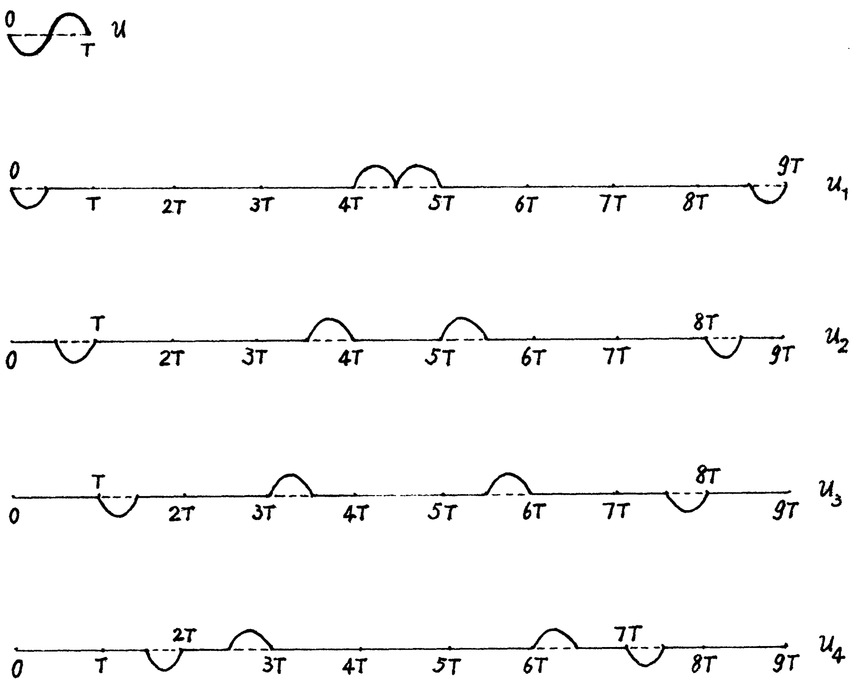

FIG. 1. Functions in N_i defined by (3.8) for the case of $k=9$, where $u \in AE_T^0$.

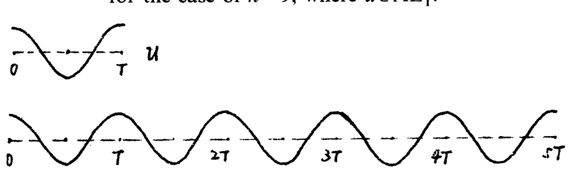

FIG. 2. Functions in M defined by (3.10) for the case of $k=5$, where $u \in SE_T^-$.

REFERENCES

[1] H. AMANN and E. ZEHNDER, Nontrivial Solutions for a Class of Nonresonance Problems and Applications to Nonlinear Defferential Equations, *Ann. Scuola Norm. Super. Pisa*, Vol. 7, 1980, pp. 539-603.

[2] A. AMBROSETTI and V. COTI ZELATI, Solutions with Minimal Period for Hamiltonian Systems in a Potential Well, *Ann. Inst. Henri-Poincaré, Anal. non linéaire*, Vol. **4**, 1987, pp. 275-296.

[3] A. AMBROSETTI and G. MANCINI, Solutions of Minimal Period for a Class of Convex Hamiltonian Systems, *Math. Ann.*, Vol. **255**, 1981, pp. 405-421.

[4] A. AMBROSETTI and P. RABINOWITZ, Dual Variational Methods in Critical Point Theory, *J. Funct. Anal.*, Vol. **14**, 1973, pp. 343-387.

[5] F. CLARKE and I. EKELAND, Hamiltonian Trajectories Having Prescribed Minimal Period, *Comm. Pure Appl. Math.*, Vol. **33**, 1980, pp. 103-116.

[6] S. DENG, Minimal Period Solutions of a Class of Hamiltonian Equation Systems, *Acta Math. Sinica*, Vol. **27**, 1984, pp. 664-675.

[7] I. EKELAND, Une théorie de Morse pour les systèmes hamiltoniens convexes, *Ann. I.H.P. Anal. non linéaire*, Vol. **1**, 1984, pp. 19-78.

[8] I. EKELAND, An Index Theory for Periodic Solutions of Convex Hamiltonian Systems, *Proc. Symp. in Pure Math.*, Vol. **45**, 1986, pp. 395-423.

[9] I. EKELAND, *Convexity Methods in Hamiltonian Mechanics*, Springer, Berlin, 1990.

[10] I. EKELAND and H. HOFER, Periodic Solutions with Prescribed Period for Convex Autonomous Hamiltonian Systems, *Inven. Math.*, Vol. **81**, 1985, pp. 155-188.

[11] N. GHOUSSOUB, *Location, Multiplicity and Morse Indices of Min-Max Critical Points*, Preprint.

[12] M. GIRARDI and M. MATZEU, Some Results on Solutions of Minimal Period to Superquadratic Hamiltonian Equations, *Nonlinear Anal. T.M.A.*, Vol. **7**, 1983, pp. 475-482.

[13] M. GIRARDI and M. MATZEU, Solutions of Minimal Period for a Class of Nonconvex Hamiltonian Systems and Applications to the Fixed Energy Problem, *Nonlinear Anal. T.M.A.*, Vol. **10**, 1986, pp. 371-382.

[14] M. GIRARDI and M. MATZEU, Periodic Solutions of Convex Autonomous Hamiltonian Systems with a Quadratic Growth at the Origin and Superquadratic at Infinity, *Ann. Mat. pura ed appl.*, Vol. **147**, 1987, pp. 21-72.

[15] M. GIRARDI and M. MATZEU, Dual Morse Index Estimates for Periodic Solutions of Hamiltonian Systems in Some Nonconvex Superquadratic Case, *Nonlinear Anal. T.M.A.*, Vol. **17**, 1991, pp. 481-497.

[16] M. GIRARDI and M. MATZEU, *Essential Critical Points of Linking Type and Solutions of Minimal Period to Superquadratic Hamiltonian Systems*, Preprint, 1991.

[17] H. HOFER, A Geometric Description of the Neighbourhood of a Critical Point Given by the Mountain-Pass Theorem, *J. London Math. Soc.*, Vol. **31**, 1985, pp. 556-570.

[18] M. A. KRASNOSEL'SKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1963.

[19] A. LAZER and S. SOLIMINI, Nontrivial Solutions of Operator Equations and Morse Indices of Critical Points of Min-Max Type, *Nonlinear Anal. T.M.A.*, Vol. **12**, 1988, pp. 761-775.

[20] Y. LONG, *The Minimal Period Problem of Periodic Solutions for Autonomous Superquadratic Second Order Hamiltonian System*, Preprint of Nankai, Inst. Math. Nankai Univ., April 1991. Revised version, Preprint of F.I. Math. E.T.H.-Zürich, Nov. 1991, May 1992, *J. Diff. Equa.* (to appear).

[21] P. RABINOWITZ, Periodic Solutions of Hamiltonian Systems, *Comm. Pure Appl. Math.*, Vol. **31**, 1978, pp. 157-184.

[22] P. Rabinowitz, Periodic Solutions of Hamiltonian Systems: a Survey, *S.I.A.M. J. Math. Anal.*, Vol. **13**, 1982, pp. 343-352.
[23] P. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, *C.B.M.S. Reg. Conf. Ser. in Math.*, No. 65, Amer. Math. Soc., 1986.
[24] P. Rabinowitz, On the Existence of Periodic Solutions for a class of Symmetric Hamiltonian Systems, *Nonlinear Anal. T.M.A.*, Vol. **11**, 1987, pp. 599-611.
[25] S. Solimini, Morse Index Estimates in Min-Max Theorems, *Manus. Math.*, Vol. **63**, 1989, pp. 421-453.
[26] G. Tian, On the Mountain-Pass Lemma, *Kexue Tongbao*, *Vol.* **29**, 1984, pp. 1151-1154.
[27] S. Zhang, *Doctoral Thesis*, Nankai University, 1991.

(*Manuscript received June* 22, 1992;
revised March 25, 1993.)