ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 10, nº 6 (1993), p. 593-604
<http://www.numdam.org/item?id=AIHPC_1993__10_6_593_0>

Ann. Inst. Henri Poincaré,

Vol. 10, n° 6, 1993, p. 593-604. *Analyse non linéaire*

Multiple solutions of a semilinear elliptic equation in $\mathbb{R}^N$

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ABSTRACT. — In this paper, we are concerned with the existence of multiple solutions of

$$-\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u$$

where $1<p,\ q<\frac{N+2}{N-2}$ if $N\geqq 3$, $1<p,\ q<+\infty$ if $N=2$, $\lambda>0$.

We obtain the existence of multiple solutions by using concentrations-compactness method and dual variational principle to establish the corresponding existence of critical points.

Key words : Semilinear elliptic equations, variation, critical point, concentration-compactness.

RÉSUMÉ. — Nous obtenons dans cet article un résultat d'existence et de multiplicité de solutions de

$$-\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u$$

où $1<p,\ q<\frac{N+2}{N-2}$, $N\geqq 3$, $1<p,\ q<+\infty$ si $N=2$, $\lambda>0$.

A.M.S. Classification: 35 B 05, 35 J 60.

Ces résultats sont prouvés à l'aide de la méthode de concentration-compacité et de principes variationnels duaux pour obtenir l'existence des points critiques correspondants.

1. INTRODUCTION

We consider the existence of multiple solutions of the following semilinear elliptic equation

$$(1.1)\qquad \left\{\begin{array}{l} -\Delta u+u=\lambda b(x)|u|^{p-1}u+c(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N \\ u\in H^1(\mathbb{R}^N) \end{array}\right.$$

where $1<p,\, q<\dfrac{N+2}{N-2}$ if $N\geqq 3$, $1<p,\, q<+\infty$ if $N=2$, $\lambda>0$ is a real number, $b(x)$ and $c(x)$ satisfy

$$(1.2)\qquad \left\{\begin{array}{l} b(x)\in C(\mathbb{R}^N), \qquad b(x)\geqq 0 \quad \text{in } \mathbb{R}^N, \\ b(x) \underset{|x|\to\infty}{\longrightarrow} b_\infty>0, \end{array}\right.$$

$$(1.3)\qquad \left\{\begin{array}{l} c(x)\in C(\mathbb{R}^N), \qquad c(x)\geqq 0 \quad \text{in } \mathbb{R}^N, \\ c(x) \underset{|x|\to\infty}{\longrightarrow} 0. \end{array}\right.$$

Existence of nontrivial solutions (positive solutions, for example) concerning (1.1) has been extensively studied even for more general nonlinearity—*see*, for instance, W. Strauss [12], H. Berestycki and P. L. Lions [4], W. Y. Ding and W. M. Ni [5], P. L. Lions [9], [10], A. Bahri and P. L. Lions [2] and the references therein. For the multiplicity of solutions we refer to H. Berestycki and P. L. Lions [4], X. P. Zhu [13] and Y. Y. Li [8].

It is known to some extent that the equation

$$(1.4)\qquad -\Delta u+u=c(x)|u|^{q-1}u \quad \text{in } \mathbb{R}^N$$

may have infinitely many solutions because (1.3) ensures that the corresponding variational functional

$$(1.5)\qquad I^*(u)=\frac{1}{2}\int|\nabla u|^2+u^2-\frac{1}{q+1}\int c(x)|u|^{q+1}$$

satisfies the (PS) (Palais-Smale) condition and the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] may be applied. When λ is small, (1.1) can be taken as a small perturbation of (1.4) and thus it seems reasonable to hope that (1.1) has more and more solutions as λ tends to 0.

As mentioned in P. L. Lions ([9], [10]) that the variational functional corresponding to (1.1) defined by

$$(1.6)\qquad I_\lambda(u)=\frac{1}{2}\int|\nabla u|^2+u^2-\frac{\lambda}{p+1}\int b(x)|u|^{p+1}-\frac{1}{q+1}\int c(x)|u|^{q+1}$$

fails to satisfy the (PS) condition because of the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^N)\subsetneq L^2(\mathbb{R}^N)$.

Such a failure creates difficulties for the application of standard variational techniques. In section 2, arguing as P. L. Lions [10], we show by using the concentration-compactness principle that $I_\lambda(u)$ satisfies $(PS)_c$ condition if c belongs to an interval depending on λ which becomes large as λ tends to 0. In section 3, using a variant of the dual variational principle (dealing with unbounded even functionals) of A. Ambrosetti and P. Rabinowitz [1] we obtain the existence of multiple solutions by establishing the corresponding existence of critical points of $I_\lambda(u)$ with critical values in the interval in which $I_\lambda(u)$ satisfies $(PS)_c$ condition.

We conclude this introduction by remarking that some more general nonlinearities can be considered and similar existence results can be obtained by the arguments in this paper.

2. EXISTENCE OF A POSITIVE SOLUTION

In this section, we are concerned with the existence of a positive solution of (1.1). As preparations and for the discussion of next section, we first give some notations, definitions and auxiliary results.

Define

$$(2.1)\qquad M_\lambda=\{u\in H^1(\mathbb{R}^N)\,|\,u\not\equiv 0,\ I'_\lambda(u)u=0\}$$

$$(2.2)\qquad M_\lambda^\infty=\{u\in H^1(\mathbb{R}^N)\,|\,u\not\equiv 0,\ I_\lambda^{\infty\prime}(u)u=0\}$$

where $I_\lambda(u)$ is defined by (1.6), $I_\lambda^\infty(u)$ is defined by

$$(2.3)\qquad I_\lambda^\infty(u)=\frac{1}{2}\int|\nabla u|^2+u^2-\frac{\lambda}{p+1}\int b_\infty|u|^{p+1}$$

Let

$$(2.4)\qquad I_\lambda=\inf\{I_\lambda(u)\,|\,u\in M_\lambda\}$$

$$(2.5)\qquad I_\lambda^\infty=\inf\{I_\lambda^\infty(u)\,|\,u\in M_\lambda^\infty\}$$

$$(2.6)\quad I^* = \begin{cases} +\infty & \text{if } c(x)\equiv 0 \text{ in } \mathbb{R}^N \\ \inf\{ I^*(u) \mid u\in H^1(\mathbb{R}^N)\setminus\{0\},\ I^{*\prime}(u)u=0\} & \text{if } c(x)\not\equiv 0 \end{cases}$$

$$(2.7)\qquad S = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int |\nabla u|^2+u^2}{\left(\int |u|^{p+1}\right)^{2/(p+1)}}.$$

We have

PROPOSITION 2.1. – *For each* $\lambda>0$, $I_\lambda \leqq I^*$.

Proof. – If $c(x)\equiv 0$, then $I^*=+\infty$, thus $I_\lambda\leqq I^*$. In what follows, we assume $c(x)\not\equiv 0$.

Suppose $u\in H^1(\mathbb{R}^N)$, $u\not\equiv 0$ such that

$$(2.8)\qquad \int |\nabla u|^2+u^2=\int c(x)|u|^{q+1}.$$

Let $v=\bar{\sigma}u$ such that $v\in M_\lambda$, *i. e.*,

$$(2.9)\qquad \int |\nabla u|^2+u^2=\bar{\sigma}^{p-1}\int \lambda b(x)|u|^{p+1}+\bar{\sigma}^{q-1}\int c(x)|u|^{q+1}$$

Comparing (2.8) and (2.9) we deduce that such $\bar{\sigma}$ exists and $\bar{\sigma}\in(0, 1)$.

Letting $h(\sigma)=\frac{\sigma^2}{2}\int |\nabla u|^2+u^2-\frac{\sigma^{q+1}}{q+1}\int c(x)|u|^{q+1}$, we have

$$h'(\sigma)=\sigma\left(\int |\nabla u|^2+u^2-\sigma^{q-1}\int c(x)|u|^{q+1}\right)>0 \qquad \text{for } \sigma\in(0, 1).$$

$$\begin{aligned}(2.10)\quad I_\lambda(v)&=\frac{\bar{\sigma}^2}{2}\int |\nabla u|^2+u^2-\frac{\bar{\sigma}^{p+1}}{p+1}\int \lambda b(x)|u|^{p+1}\\ &\quad-\frac{\bar{\sigma}^{q+1}}{q+1}\int c(x)|u|^{q+1}\\ &<\frac{\bar{\sigma}^2}{2}\int |\nabla u|^2+u^2-\frac{\bar{\sigma}^{q+1}}{q+1}\int c(x)|u|^{q+1}\\ &<\frac{1}{2}\int |\nabla u|^2+u^2-\frac{1}{q+1}\int c(x)|u|^{q+1}=I^*(u).\end{aligned}$$

Thus $I_\lambda\leqq I^*$ and we have proved Proposition 2.1.

PROPOSITION 2.2. – *We have*

$$(2.11)\qquad I_\lambda^\infty=\frac{p-1}{2(p+1)}S^{(p+1)/(p-1)}(\lambda b_\infty)^{-(2/(p-1))}.$$

Proof. — We can easily find that

$$(2.12)\qquad S=\inf\left\{\int|\nabla u|^2+u^2 \,|\, u\in H^1(\mathbb{R}^N), \int|u|^{p+1}=1\right\}$$

which has a positive minimum $\bar{u}\in H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ satisfying

$$(2.13)\qquad -\Delta u+u=S|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N$$

(*see* W. Strauss [12], P. L. Lions ([9], [10]) for examples). By Gidas, Ni and Nirenberg [7] we may assume $\bar{u}$ is radial.

On the other hand, there exists a positive radial function $\tilde{u}\in H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ achieving I_λ^∞ such that $\tilde{u}$ satisfying

$$(2.14)\qquad -\Delta u+u=\lambda b_\infty|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N$$

(*see* also W. Strauss [12], P. L. Lions ([9], [10]) for examples).

Let $\tilde{u}=\left(\dfrac{S}{\lambda b_\infty}\right)^{1/(p-1)}v$, then $v>0$ in $\mathbb{R}^N$ and solves (2.13). By the uniqueness of radial positive solution due to M. K. Kwong [11] we deduce $v\equiv\bar{u}$ and thus

$$I_\lambda^\infty=I_\lambda^\infty(\tilde{u})=\frac{p-1}{2(p+1)}\int|\nabla\tilde{u}|^2+\tilde{u}^2=\frac{p-1}{2(p+1)}S^{(p+1)/(p-1)}(\lambda b_\infty)^{-(2/(p-1))}$$

proving Proposition 2.2.

LEMMA 2.3. — *$I_\lambda(u)$ satisfies* $(PS)_c$ *condition if*

$$(2.15)\qquad c\in(-\infty, I_\lambda^\infty).$$

Proof. — Suppose $\{u_n\}\subset H^1(\mathbb{R}^N)$ such that

$$(2.16)\qquad I_\lambda(u_n)\underset{n}{\to} c\in(-\infty, I_\lambda^\infty)$$

$$(2.17)\qquad I'_\lambda(u_n)\underset{n}{\to} 0 \quad \text{in } H^1(\mathbb{R}^N)$$

It is easy to deduce from (2.16) and (2.17) that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By choosing subsequence if necessary we assume

$$(2.18)\qquad u_0\underset{n}{\to} u_0 \quad \text{weakly in } H^1(\mathbb{R}^N).$$

By the method of concentration-compactness, as in A. Bahri and P. L. Lions [2], P. L. Lions [10], V. Benci and G. Cerami [3] we deduce that there exist a nonnegative integer k, $\{x_n^i\}$ $(1\leq i\leq k)$ in $\mathbb{R}^N$, solutions $\bar{u}_i\in H^1(\mathbb{R}^N)$ $(1\leq i\leq k)$ of (2.14) such that (extracting subsequence if necessary)

$$(2.19)\qquad \left\|u_n-u_0-\sum_{i=1}^{k}\bar{u}_i(x-x_n^i)\right\|\underset{n}{\to} 0$$

$$c = \mathrm{I}_\lambda(u_0) + \sum_{i=1}^{n} \mathrm{I}_\lambda^\infty(\bar{u}_i). \tag{2.20}$$

Since $\mathrm{I}_\lambda^\infty(\bar{u}_i) = \frac{p-1}{2(p+1)} \int |\nabla \bar{u}_i|^2 + \bar{u}_i^2 \geqq 0$ for $i=1, \ldots, k$ if for some i, $\bar{u}_i \not\equiv 0$, then $\mathrm{I}_\lambda^\infty(\bar{u}_i) \geqq \mathrm{I}_\lambda^\infty$ which implies $c \geqq \mathrm{I}_\lambda^\infty$ because $\mathrm{I}_\lambda(u_0) \geqq 0$. Thus $\bar{u}_i \equiv 0$ for $1 \leqq i \leqq k$. Hence u_n converges to u_0 strongly and therefore Lemma 2.3 has been proved.

We are now going to use the preceding result to obtain the existence of a positive solution.

THEOREM 2.4. – *Suppose* $\mathrm{I}_\lambda < \mathrm{I}_\lambda^\infty$. *Then* (1.1) *has a positive solution.*

Proof. – By Ekeland's variational principle [6] and the definition of I_λ, there exists a minimizing sequence $\{u_n\}$ such that $\{u_n\} \subset \mathrm{M}_\lambda$

$$\mathrm{I}_\lambda(u_n) \underset{n}{\rightarrow} \mathrm{I}_\lambda \tag{2.21}$$

$$\mathrm{I}'_{\lambda \mid \mathrm{M}_\lambda}(u_n) \underset{n}{\rightarrow} 0 \quad \text{in } \mathrm{H}^{-1}(\mathbb{R}^\mathrm{N}). \tag{2.22}$$

$$\mathrm{I}'_\lambda(u_n) \underset{n}{\rightarrow} 0 \quad \text{in } \mathrm{H}^{-1}(\mathbb{R}^\mathrm{N}). \tag{2.23}$$

Indeed, from (2.21), $u_n \in \mathrm{M}_\lambda$, using Sobolev inequality we can find C_1, $\mathrm{C}_2 > 0$ such that

$$\mathrm{C}_1 < \int |\nabla u_n|^2 + u_n^2 < \mathrm{C}_2 \quad \text{for all } n = 1, 2, \ldots \tag{2.24}$$

Letting $\mathrm{J}_\lambda(u) = \int |\nabla u|^2 + u^2 - \int \lambda b(x) |u|^{p+1} - \int c(x) |u|^{q+1}$, we have

$$\mathrm{M}_\lambda = \{ u \in \mathrm{H}^1(\mathbb{R}^\mathrm{N}) \setminus \{0\} \mid \mathrm{J}_\lambda(u) = 0 \}. \tag{2.25}$$

Thus

$$\mathrm{I}'_\lambda(u_n) = \mathrm{I}'_{\lambda \mid \mathrm{M}_\lambda}(u_n) - \theta_n \mathrm{J}'_\lambda(u_n) \tag{2.26}$$

for some $\theta_n \in \mathbb{R}$.

Since $u_n \in \mathrm{M}_\lambda$, we have from (2.26)

$$\mathrm{I}'_{\lambda \mid \mathrm{M}_\lambda}(u_n) u_n - \theta_n \mathrm{J}'_\lambda(u_n) u_n = \mathrm{I}'_\lambda(u_n) u_n = 0 \tag{2.27}$$

$$\begin{aligned} \mathrm{J}'_\lambda(u_n) u_n &= 2 \int |\nabla u_n|^2 + u_n^2 - (p+1) \int \lambda b(x) |u_n|^{p+1} \\ &\quad - (q+1) \int c(x) |u|^{q+1} \\ &= -(p-1) \int \lambda b(x) |u_n|^{p+1} - (q-1) \int c(x) |u_n|^{q+1}. \end{aligned} \tag{2.28}$$

Thus from (2.24), (2.28) and $u_n \in M_\lambda$ we have

$$-C_3 < J'_\lambda(u_n)\,u_n < -C_4 \tag{2.29}$$

for some constants C_3, $C_4 > 0$ independent of n.

From $I'_{\lambda\,|\,M_\lambda}(u_n) \underset{n}{\to} 0$, we obtain by (2.27) and (2.29) that $\theta_n \underset{n}{\to} 0$ which combined with (2.26) deduces $I'_\lambda(u_n) \underset{n}{\to}$ in $H^{-1}(\mathbb{R}^N)$. Thus (2.23) holds.

Following Lemma 2.3, we can assume (by choosing subsequence if necessary)

$$u_n \underset{n}{\to} u_0 \quad \text{strongly in } H^1(\mathbb{R}^N).$$

By Sobolev inequality, we have $I_\lambda > 0$. Thus u_0 is a nontrivial solution of (1.1). Letting $u_0 = u_0^+ + u_0^-$, where $u_0^+ = \max\{u_0, 0\}$, $u_0^- = u_0 - u_0^+$, we have $I_\lambda(u_0) = I_\lambda(u_0^+) + I_\lambda(u_0^-)$. Since $I'_\lambda(u_0^\pm)\,u_0^\pm = 0$, *i.e.*, $u_0^\pm \in M_\lambda$ if $u_0^\pm \not\equiv 0$ we have $I_\lambda(u_0^\pm) \geqq I_\lambda$ if $u_0^\pm \not\equiv 0$. Therefore $u_0^+ \equiv 0$ or $u_0^- \equiv 0$. Without loss of generality, assume $u_0^- \equiv 0$. Thus $u_0 \geqq 0$ in $\mathbb{R}^N$. It follows from standard regularity method and maximum principle that $u_0 \in C^2(\mathbb{R}^N)$, $u_0 > 0$ in $\mathbb{R}^N$. Thus, we conclude the proof of Theorem 2.4.

COROLLARY 2.5. — *Suppose* (1.2) *holds,* $c(x)$ *satisfies*

$$\begin{cases} c(x) \in C(\mathbb{R}^N), & c(x) \geqq 0 \quad in\ \mathbb{R}^N, \\ c(x) \underset{|x| \to \infty}{\longrightarrow} 0, & c(x) \not\equiv 0 \quad in\ \mathbb{R}^N. \end{cases} \tag{2.30}$$

Then (1.1) *has a positive solution provided*

$$\lambda \in \left(0, \left[\frac{p-1}{2(p+1)\,I^*}\right]^{(p-1)/2} S^{(p+1)/2}\, b_\infty^{-1}\right). \tag{2.31}$$

Proof. — From (2.31) we have

$$I^* < \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda\, b_\infty)^{-(2/(p-1))} = I_\lambda^\infty \tag{2.32}$$

which combined with Proposition 2.1 implies

$$I_\lambda < I_\lambda^\infty. \tag{2.33}$$

Thus, by Theorem 2.4 we know (1.1) has a positive solution.

We end this section by a few remarks.

Remark 2.6. — The fact that if $I_\lambda < I_\lambda^\infty$ then I_λ has a minimum has been proved in P. L. Lions ([9], [10]). We reprove this fact for the sake of completeness.

Remark 2.7. — Consider the following equation

$$-\Delta u + u = Q(x)\,|u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \tag{2.35}$$

where $Q(x) \in C(\mathbb{R}^N)$, $Q(x) \geqq 0$ in $\mathbb{R}^N$, $Q(x) \to \bar{Q} > 0$ as $|x| \to \infty$.

(2.35) can be obtained by taking $\lambda=1$, $Q(x)\equiv b(x)$, $c(x)\equiv 0$ in (1.1). From Theorem 2.4 we can deduce the corresponding results concerning the existence of positive solution of (2.35) in section 3 of W. Y. Ding and W. M. Ni [5] [for the case $Q(x)\to\bar{Q}$ as $|x|\to\infty$]. Corollary 2.5 gives a type of precise condition under which $I_\lambda < I_\lambda^\infty$.

Suppose $Q(x)=\lambda b(x)+c(x)$, where $b(x)$ satisfies (1.2) and

$$(2.36)\qquad (b_\infty - b(x))\log(1+|x|)\to+\infty \quad \text{as } |x|\to\infty$$

$c(x)$ satisfies (2.30) with $\operatorname{supp} c(x)$ bounded.

Corollary 2.5 ensures the existence of positive solution if λ is properly small. It should be pointed out that in this case $Q(x)$ does not satisfy the condition proposed by A. Bahri and P. L. Lions in [2].

3. EXISTENCE OF MULTIPLE SOLUTIONS

First of all, let us state a variant of the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] dealing with unbounded even functionals.

Let E be a Banach space, B_r be the ball in E centered at 0 with radius r, ∂B_r be the boundary of B_r. $A\subset E$ is called symmetric if $u\in A$ implies $-u\in A$. Let

$$(3.1)\qquad \Sigma=\{A\,|\,A\subset E\setminus\{0\},\ A \text{ is closed and symmetric}\}$$

For $A\subset\Sigma$, $\nu(A)$ denotes the genus of A. We set for $f\in C^1(E,\mathbb{R})$

$$(3.2)\qquad E_+=\{u\in E\,|\,f(u)\geqq 0\}$$

$$(3.3)\qquad H=\{h\,|\,h\in C(E,E),\ h \text{ is odd homeomorphism } h(B_1)\subset E_+\}$$

$$(3.4)\qquad \Gamma_n=\{A\subset\Sigma\,|\,A \text{ is compact, } \nu(A\cap h(\partial B_1))\geqq n \text{ for any } h\in H\}$$

Replacing (PS) by $(PS)_c$ condition, we have the following lemma proved exactly as in [1].

LEMMA 3.1. – *Suppose* $f\in C^1(E,\mathbb{R})$ *satisfies*

(C1) $f(0)=0$ *and there exist* ρ, $\alpha>0$ *such that* $f(u)>0$ *for any* $u\in B_\rho\setminus\{0\}$, $f(u)\geqq\alpha$ *for all* $u\in\partial B_\rho$;

(C2) *for any finite dimensional subspace* $E^n\subset E$, $E^n\cap E_+$ *is bounded;*

(C3) $f(u)=f(-u)$.

Set

$$(3.5)\qquad b_n=\inf_{A\in\Gamma_n}\sup\{f(u)\,|\,u\in A\},\qquad n=1,2,\ldots$$

Then

(i) $\Gamma_n\neq 0$ *for* $n=1, 2, \ldots$, $b_n\geqq\alpha$;

(ii) b_n *is a critical level if f satisfies* $(PS)_c$ *condition for* $c=b_n$.

Furthermore, if $b=b_n=\ldots=b_{n+m}$, *then* $\nu(K_b)\geqq m+1$, *where*

$$K_b=\{u\in E \mid f(u)=b, f'(u)=0\}.$$

In what follows, we always take $E=H^1(\mathbb{R}^N)$ and use the same notations Σ, B_r, ∂B_r and $\nu(A)$. Let

$$E_\lambda=\{u\in H^1(\mathbb{R}^N) \mid I_\lambda(u)\geqq 0\} \tag{3.6}$$

$$E_*=\{u\in H^1(\mathbb{R}^N) \mid I^*(u)\geqq 0\} \tag{3.7}$$

$$H_\lambda=\{h\in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)),\ h \text{ is odd homeomorphism}, h(B_1)\subset E_\lambda\} \tag{3.8}$$

$$H_*=\{h\in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)),\ h \text{ is odd homeophormism}, h(B_1)\subset E_*\} \tag{3.9}$$

Obviously $E_\lambda\subset E_*$, $H_\lambda\subset H_*$.

PROPOSITION 3.2. – *If* $b(x)$ *satisfies* (1.2), $c(x)$ *satisfies*

$$\left\{\begin{array}{l} c(x)\in C(\mathbb{R}^N), \qquad c(x)\geqq 0 \quad in\ \mathbb{R}^N, \\ \operatorname{meas}\{x\in\mathbb{R}^N \mid c(x)=0\}=0, \\ c(x)\to 0 \quad \text{as} \quad |x|\to\infty \end{array}\right. \tag{3.10}$$

Then $I_\lambda(u)$ *and* $I^*(u)$ *satisfy* (C1), (C2) *and* (C3) *in the previous lemma.*

Proof. – The verification of (C1) and (C3) is trivial. We only show that (C2) holds for $I_\lambda(u)$ [resp. $I^*(u)$]. We argue by way of contradiction. Suppose there exists a m dimensional subspace $E^m\subset H^1(\mathbb{R}^N)$, a sequence $\{u_n\}\subset E^m\cap E_\lambda$ (resp. $\{u_n\}\subset E_*\cap E^m$) such that $\|u_n\|\underset{n}{\to}+\infty$. Let $e_1, e_2, \ldots, e_m$ be the basis of E_m. Then

$$u_n=t_1^n e_1+\ldots+t_m^n e_m \tag{3.13}$$

for some $t_n=(t_1^n, \ldots, t_m^n)\in\mathbb{R}^m$.

Set $|t_n|=\max\limits_{1\leqq i\leqq m}|t_i^n|$, we have $|t_n|\underset{n}{\to}+\infty$.

$$\int|\nabla u_n|^2+u_n^2=0(|t_n|^2) \tag{3.14}$$

$$\int b(x)|u_n|^{p+1}\geqq 0 \tag{3.15}$$

$$\int c(x)|u_n|^{q+1}\geqq C_5|t_n|^{q+1} \qquad \text{for} \quad n \text{ large enough} \tag{3.16}$$

where $C_5>0$ is some constant.

(3.14), (3.15) and (3.16) deduce $I_\lambda(u_n)<0$ for n larger enough [resp. $I^*(u_n)<0$ for n large enough], which contradicts $u_n\in E_\lambda$ (resp. $u_n\in E_*$).

Define

(3.17) $\Gamma_\lambda^n=\{A\subset\Sigma \mid A \text{ is compact and } \nu(A\cap h(\partial B_1))\geqq n \text{ for any } h\in H_\lambda\}, \quad n=1, 2, \ldots,$

(3.18) $\Gamma_*^n=\{A\subset\Sigma \mid A \text{ is compact and } \nu(A\cap h(\partial B_1))\geqq n \text{ for any } h\in H_*\}, \quad n=1, 2, \ldots,$

(3.19) $c_\lambda^n=\inf_{A\in\Gamma_\lambda^n} \max\{I_\lambda(u)\mid u\in A\}, \quad n=1, 2, \ldots,$

(3.20) $c_*^n=\inf_{A\in\Gamma_*^n} \max\{I_*(u)\mid u\in A\}, \quad n=1, 2, \ldots,$

By the definitions we have

(3.21) $\Gamma_\lambda^n\supset\Gamma_*^n \quad \text{for} \quad n=1, 2, \ldots$

Suppose (3.10) holds then by Proposition 3.2 and Lemma 3.1, $\Gamma_*^n\neq\varnothing$ for each $n=1, 2, \ldots$, and consequently $c_*^n<+\infty$.

Let

$$\lambda_k=\left[\frac{p-1}{2(p+1)c_*^k}\right]^{(p-1)/2} S^{(p+1)/2} b_\infty^{-1}, \qquad k=1, 2, \ldots$$

We have

THEOREM 3.3. – *Suppose* (1.2) *and* (3.10) *hold. Then for each* $n=1, 2, \ldots$, (1.1) *has n pair of solutions* $\{-u_i u_i\}$, $i=1, \ldots, n$ *if* $\lambda\in(0, \lambda_n)$.

Proof. – By the definition of c_λ^n, c_*^n, $n=1, 2, \ldots$ we have

$$\begin{aligned} c_\lambda^n &= \inf_{A\in\Gamma_\lambda^n} \max\{I_\lambda(u)\mid u\in A\} \\ &\leqq \inf_{A\in\Gamma_*^n} \max\{I_\lambda(u)\mid u\in A\} \\ &\leqq \inf_{A\in\Gamma_*^n} \max\{I^*(u)\mid u\in A\} \\ &= c_*^n. \end{aligned}$$

Thus

(3.23) $c_\lambda^n\leqq c_*^n \quad \text{for} \quad n=1, 2, \ldots$

Next we claim that for each c_λ^k, $k=1, \ldots, n$, $I_\lambda(u)$ satisfies $(PS)_c$ condition.

Indeed, $\lambda<\lambda_n$ implies

$$\lambda<\left[\frac{p-1}{2(p+1)c_*^n}\right]^{(p-1)/2} S^{(p+1)/2} b_\infty^{-1}.$$

Thus

$$c_*^n < \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))} = I_\lambda^\infty$$

which combining with (3.23) deduces

(3.24) $$c_\lambda^n < I_\lambda^\infty.$$

On the other hand, obviously we have

(3.25) $$c_\lambda^1 \leqq \ldots \leqq c_\lambda^n.$$

Thus, by Lemma 2.3, $I_\lambda(u)$ satisfies $(PS)_c$ condition for c_λ^k, $k=1, 2, \ldots, n$. Following Lemma 3.1, $I_\lambda(u)$ has at least n different critical points $u_i \in H^1(\mathbb{R}^N)$ $(1 \leqq i \leqq n)$ such that $I_\lambda(u_i) = c_\lambda^i$ $(1 \leqq i \leqq n)$. Since $I_\lambda(u)$ is a even functional $-u_i$ is critical point either $(1 \leqq i \leqq n)$, $\{-u_i, u_i\}$ are the solutions we are looking for. Hence we have proved Theorem 3.3.

ACKNOWLEDGEMENTS

This work was completed while the author was visiting CEREMADE, University of Paris-Daupine. He would like to express his gratitude to P. L. Lions for helpful suggestions and comments. He would like also to thank K. C. Wang Fund for financial support of his work.

REFERENCES

[1] A. AMBROSETTI and P. RABINOWITZ, Dual Variational Methods in Critical Point Theory and Applications, *J. Funct. Anal.*, Vol. **14**, 1973, pp. 327-381.

[2] A. BAHRI and P. L. LIONS, *On the Existence of a Positive Solution of Semilinear Elliptic Equations in Unbounded Domains*, preprint.

[3] V. BENCI and G. CERAMI, Positive Solutions of Semilinear Elliptic Problems in Exterior Domains, *Arch. Rat. Mech. Anal.*, Vol. **99**, 1987. pp. 283-300.

[4] H. BERESTYCKI and P. L. LIONS, Nonlinear Scalar Field Equations, I and II, *Arch. Rat. Mech. Anal.*, Vol. **82**, 1983, pp. 313-376.

[5] W. Y. DING and W. M. NI, On the Existence of Positive Entire Solutions of a Semilinear Elliptic Equation, *Arch. Rat. Mech. Anal.*, Vol. **91**, 1986, pp. 288-308.

[6] I. EKELAND, Nonconvex Minimization Problems, *Bull. Amer. Math. Soc.*, Vol. **1**, 1979, pp. 443-474.

[7] B. GIDAS, W. M. NI and L. NIRENBERG, Symmetry of Positive Solutions of Nonlinear Elliptic Equations in $\mathbb{R}^n$, *Advances in Math.*, *Supplementary Studies*, Vol. **7**, 1981, pp. 369-402.

[8] Y. Y. LI, On Second Order Nonlinear Elliptic Equations, *Dissertation*, New York Univ., 1988.

[9] P. L. LIONS, *The Concentration-Compactness Principle in the Calculus of Variations. The Locally Compact Case, I and II*, Vol. **1**, 1984, pp. 109-145 and 223-283.

[10] P. L. LIONS, On Positive Solution of Semilinear Elliptic Equation in Unbounded Domains, In *Nonlinear Diffusion Equations and Their Equilibrium States*, Springer, New York, 1988.

[11] M. K. KWONG, Uniqueness of Positive Solution of $\Delta u - u + u^p = 0$, *Arch. Rat. Mech. Anal.*, Vol. **105**, 1977, pp. 169- .

[12] W. STRAUSS, Existence of Solitary Waves in Higher Dimensions, *Comm. Math. Phys.*, Vol. **55**, 1977, pp. 109-162.

[13] X. P. ZHU, Multiplie Entire Solutions of Semilinear Elliptic Equations, *Nonlinear Anal.*, Vol. **12**, 1988, pp. 1297-1316.

(*Manuscript received April* 15, 1992;
revised October 20, 1992.)