ANNALES DE L'I. H. P., SECTION C

VLADIMIR ŠVERÁK

On Tartar's conjecture

Annales de l'I. H. P., section C, tome 10, nº 4 (1993), p. 405-412
<http://www.numdam.org/item?id=AIHPC_1993__10_4_405_0>

Ann. Inst. Henri Poincaré,

Vol. 10, n° 4, 1993, p. 405-412. *Analyse non linéaire*

On Tartar's conjecture

by

Vladimír ŠVERÁK

Department of Mathematics,
Heriot-Watt University, Edinburgh EH14 4AS, UK.
(On leave from MFF UK, Sokolovská 83, Praha)

ABSTRACT. – We prove that the only probability measures supported at connected subsets of 2×2 matrices without rank-one connections and commuting with the determinant are Dirac masses. We also prove some regularity results for fully nonlinear 2×2 elliptic systems of the first order.

Key words : Young measures, compactness, regularity.

RÉSUMÉ. – Soit K un sous-ensemble connexe de matrices deux par deux sans connexion de rang un et soit ν une mesure de probabilité concentrée sur K qui commute avec le déterminant. On démontre que ν est une masse de Dirac. On démontre aussi quelques résultats de régularité pour des systèmes elliptiques deux par deux du premier ordre.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^2$ be open an bounded. For functions $v : \Omega \to \mathbf{R}^2$ we consider nonlinear systems given by $\mathrm{D}\, v(x) \in \mathrm{K}$, where K is a submanifold of the

Classification A.M.S. : 35 B.

set $M^{2\times 2}$ of all 2×2 matrices. We shall be interested in regularity of solutions of these systems and also in the following question: if $v_j:\Omega\to\mathbf{R}^2$ is a sequence of functions such that $|Dv_j|\leqq c$ and $\mathrm{dist}(Dv_j(.), K)\to 0$ in L^p, what can be said about compactness of the sequence Dv_j in L^p? Since for every A, $B\in M^{2\times 2}$ with rank $(A-B)=1$ we can construct a sequence of piecewise linear functions whose gradients oscillate between A and B, a necessary condition to get some positive results is that rank $(A-B)\geqq 2$ for any two distinct matrices A, $B\in K$. Tartar's conjecture (*see* [14]) in our special situation says that this condition should be also sufficient for the compactness of the sequences above. Here we prove that this holds true under the additional assumption that K is connected. (Without additional assumptions the conjecture fails. For a counterexample with K consisting of four matrices see [7]. Counterexamples in higher dimensions can be found in [2].) We also give a simple proof of the fact that if K is connected, $\mathrm{rank}(A-B)\geqq 2$ for each A, $B\in K$ distinct, and the system $Dv(x)\in K$ is elliptic (*i.e.* planes tangent to K do not contain rank-one directions), then the solutions which are Lipschitzian belong to $C^{1,\alpha}$ for some $\alpha>0$. If, moreover, K is smooth, then the solutions are smooth. *A priori* estimates for the $C^{1,\alpha}$-norm of twice differentiable solutions of the systems considered here are well-known. (*See*, for example, [8], Chapter 12.) I am not aware of any previous regularity results for Lipschitzian solutions, with the exception of the Monge-Ampère equation, which, of course, can be considered as a first-order elliptic system. In general, if K is two dimensional and is contained in symmetric matrices, then the equation $Dv(x)\in K$ can be viewed as a fully nonlinear scalar equation of the second order for the potential of the vector field v. *A priori* estimates for solutions of such equations in arbitrary dimensions have been obtained in [5]. *See* also [8], Chapter 17.

2. PRELIMINARIES

Throughout this paper Ω denotes a nonempty, bounded, open subset of $\mathbf{R}^2$. The Lebesgue spaces L^p, the Sobolev spaces $W^{k,p}$ and the spaces $C^{k,\alpha}$ of Hölder continuous functions are defined in the usual way.

Let us briefly recall basic facts concerning Young measures. (We refer the reader to [1] or [14] for more details.) Let $z_j:\Omega\to\mathbf{R}^n$ be a sequence of functions bounded in $L^\infty(\Omega)$. It is possible to prove that there exists a subsequence z_μ of z_j such that for any continuous function $f:\mathbf{R}^n\to\mathbf{R}$ the sequence $f\circ z_\mu$ converges weakly* in $L^\infty(\Omega)$ to some function h_f. Moreover, it is also possible to prove that there is a subset S of Ω of measure zero and a family $\{\nu_x,\ x\in\Omega\setminus S\}$ of probability measures on $\mathbf{R}^n$ such that for each continuous $f: \mathbf{R}^n \to \mathbf{R}$ we have $h_f(x) = \int_{\mathbf{R}^n} f(\lambda)\, d\nu_x(\lambda)$ for almost every $x \in \Omega$. We shall use the notation $\int_{\mathbf{R}^n} f(\lambda)\, d\nu_x(\lambda) = \langle \nu_x, f \rangle$. If almost all of the measures ν_x are Dirac masses, then the sequence z_μ is compact in $L^r(\Omega)$ for any $r < \infty$ and *vice versa*. The measures ν_x are called Young measures.

We shall use the following lemma.

LEMMA 1. — *Let* K *be a connected topological space and let* $g: \mathrm{K} \times \mathrm{K} \to \mathbf{R}$ *be a continuous function such that* $g(x, y) = g(y, x) \neq 0$ *for every* $x, y \in \mathrm{K}$, $x \neq y$ *and* $g(x, x) = 0$ *for every* $x \in \mathrm{K}$. *Then either* $g(x, y) \geqq 0$ *for every* $x, y \in \mathrm{K}$ *or* $g(x, y) \leqq 0$ *for every* $x, y \in \mathrm{K}$.

Proof. — We notice that if g changes sign on $\mathrm{K} \times \mathrm{K}$, then there exists $y \in \mathrm{K}$ such that $g(\cdot, y)$ changes sign on K. Indeed, supposing this is not the case, we consider the sets $\mathrm{K}^+ = \{ y \in \mathrm{K}, g(\cdot, y) \geqq 0 \text{ on } \mathrm{K} \}$ and $\mathrm{K}^- = \{ y \in \mathrm{K}, g(\cdot, y) \leqq 0 \text{ on } \mathrm{K} \}$. These sets are clearly closed and $\mathrm{K}^+ \cap \mathrm{K}^- = \varnothing$. Since K is connected, we cannot have $\mathrm{K}^+ \cup \mathrm{K}^- = \mathrm{K}$. Therefore the lemma will be proved if we show that under our assumptions the function $g(\cdot, y)$ does not change sign for any $y \in \mathrm{K}$. Suppose this is not true and let $y_0 \in \mathrm{K}$ be such that $g(\cdot, y_0)$ changes sign. Let $\mathrm{K}_+ = \{ x \in \mathrm{K}, g(x, y_0) \geqq 0 \}$ and $\mathrm{K}_- = \{ x \in \mathrm{K}, g(x, y_0) \leqq 0 \}$. We claim that K_+ and K_- are connected. To see this, suppose that $\mathrm{K}_+ = \mathrm{U} \cup \mathrm{V}$, where U, V are nonempty disjoint closed subsets of K_+. We can suppose $y_0 \notin \mathrm{V}$. We now consider the sets $\tilde{\mathrm{U}} = \mathrm{K}_- \cup \mathrm{U}$ and $\tilde{\mathrm{V}} = \mathrm{V}$. These are closed sets covering K, *i.e.* $\tilde{\mathrm{U}} \cup \tilde{\mathrm{V}} = \mathrm{K}$. We have

$$\tilde{\mathrm{U}} \cap \tilde{\mathrm{V}} = (\mathrm{K}_- \cap \mathrm{V}) \cup (\mathrm{U} \cap \mathrm{V}) \subset (\mathrm{K}_- \cap \mathrm{K}_+) \setminus \{ y_0 \}.$$

Since g does not vanish outside the diagonal, the last set is empty. Since K is connected and $\tilde{\mathrm{U}}$ is nonempty, the set $\mathrm{V} = \tilde{\mathrm{V}}$ must be empty. This shows that K_+ is connected. The proof for K_- is the same. Let $x_+ \in \mathrm{K}_+ \setminus \{ y_0 \}$ and $x_- \in \mathrm{K}_- \setminus \{ y_0 \}$. The function $g(x_+, .)$ is positive at y_0 and does not vanish on the connected set K_- containing y_0. Therefore it is positive on K_- and in particular $g(x_+, x_-) > 0$. On the other hand, the function $g(x_-, \cdot)$ is negative at y_0 and does not vanish on the connected set K_+ containing y_0 and therefore $g(x_-, x_+) = g(x_+, x_-) < 0$, a contradiction. The proof is finished.

3. COMPACTNESS

LEMMA 2. — *Let* K *be a connected subset of* $M^{2\times 2}$ *and suppose that* $\operatorname{rank}(X-Y)\geqq 2$ *for every two distinct matrices* $X, Y\in K$. *Then either* $\det(X-Y)\geqq 0$ *for all* $X, Y\in K$ *or* $\det(X-Y)\leqq 0$ *for all* $X, Y\in K$.

Proof. — This is an obvious consequence of Lemma 1.

LEMMA 3. — *Let* K *be a bounded Borel measurable subset of* $M^{2\times 2}$ *such that* $\operatorname{rank}(X-Y)\geqq 2$ *for any two distinct* $X, Y\in K$ *and suppose that* $\det(X-Y)$ *does not change sign on* $K\times K$. *Let* ν *be a probability measure on* $M^{2\times 2}$ *carried by* K (i.e. $\nu(M^{2\times 2}\setminus K)=0$) *and satisfying* $\langle \nu, \det\rangle=\det\langle \nu, \text{identity}\rangle$. *Then* ν *is a Dirac mass,* i.e. $\nu=\delta_A$ *for some* $A\in K$.

Proof. — Let $A=\langle \nu, \text{identity}\rangle$ be the centre of mass of ν. Let b be the symmetric bilinear form on $M^{2\times 2}$ determined by $\det X=\frac{1}{2}b(X, X)$. We can write

$$\begin{aligned}
&\int_{M^{2\times 2}} d\nu(X)\int_{M^{2\times 2}} d\nu(Y)\det(X-Y)\\
&=\int_{M^{2\times 2}} d\nu(X)\int_{M^{2\times 2}} d\nu(Y)(\det X+\det Y-b(X, Y))\\
&=\int_{M^{2\times 2}} d\nu(X)(\det X+\det A-b(X, A))\\
&=\det A+\det A-b(A, A)=0.
\end{aligned}$$

Since $\det(X-Y)$ does not change sign and vanishes only at the diagonal of $K\times K$, we see that the measure $\nu\otimes\nu$ is supported at the diagonal of $K\times K$ and therefore it must be a Dirac mass. The proof is finished.

THEOREM 1. — *Let* $U^{(j)}=\begin{pmatrix} u_1^{(j)} & u_2^{(j)}\\ v_1^{(j)} & v_2^{(j)}\end{pmatrix}$ *be a uniformly bounded sequence of matrix-valued functions on* Ω *and suppose that the sequences* $\operatorname{curl} u^{(j)}$ *and* $\operatorname{curl} v^{(j)}$ *are compact in* $H^{-1}(\Omega)$. *Let* K *be a closed connected subset of* $M^{2\times 2}$ *such that* $\operatorname{rank}(X-Y)\geqq 2$ *for any two distinct* $X, Y\in K$ *and suppose that* $\operatorname{dist}(U^{(j)}(x), K)\to 0$ *for* a.e. $x\in\Omega$. *Then the sequence* $U^{(j)}$ *is compact in* $L^p(\Omega)$ *for every* $1\leqq p<\infty$.

Proof. — Following L. Tartar [14] we consider a family of Young measures ν_x associated to a subsequence of the sequence $U^{(j)}$ we and use the div-curl lemma (*see* [14]) to infer that $\langle \nu_x, \det\rangle=\det\langle \nu_x, \text{identity}\rangle$ for almost every $x\in\Omega$. Our assumptions clearly imply that ν_x is supported on a bounded subset of K for a. e. $x \in \Omega$. From Lemma 2 and Lemma 3 we see that ν_x is a Dirac mass for almost every $x \in \Omega$. The proof is finished.

4. RANK-ONE CONNECTIONS IN SETS OF GRADIENTS

The results of Section 3 can be used to generalize some results of [2], Section 5.

THEOREM 2. — *Let $u: \Omega \to \mathbf{R}^2$ be a Lipschitzian function which coincides with an affine function* A *at the boundary of* Ω *and suppose that* Du *is continuous in* Ω. *If u is not affine, then there exist $x, y \in \Omega$ such that* $\operatorname{rank}(\mathrm{D}u(x) - \mathrm{D}u(y)) = 1$.

Proof. — Let us first assume that Ω is connected and $\mathrm{A}=0$. Let $\mathrm{K} = \{\mathrm{D}u(x), x \in \Omega\}$ and let ν be the probability measure on $\mathrm{M}^{2\times 2}$ given by

$$\langle \nu, f \rangle = \frac{1}{\operatorname{meas} \Omega} \int_\Omega f(\mathrm{D}u(x))\, dx$$

for every continuous function $f: \mathrm{M}^{2\times 2} \to \mathbf{R}$. Under our assumptions the set K is bounded and connected. The measure ν is carried by K. We claim that $\langle \nu, \det \rangle = \det \langle \nu, \text{identity} \rangle$. For this it is enough to prove that under our assumptions we have $\int_\Omega \mathrm{D}u(x)\, dx = 0$ and $\int_\Omega \det \mathrm{D}u(x)\, dx = 0$. This is well known if u is Lipschitzian and compactly supported in Ω. (*See*, for example, [11].) The general case can be brought to this case by extending u by 0 outside Ω and integrating over a sufficiently large ball in which Ω is compactly contained. We can now apply Lemma 2 and Lemma 3 and we see that if Du is not constant, then there must be rank-one connections in K. The proof in the case when Ω is connected and $\mathrm{A}=0$ is finished. The general case follows easily, since clearly $u = \mathrm{A}$ on the boundary of every connected component of Ω and since we can replace u by $u - \mathrm{A}$, if necessary.

Remarks. — 1. For any open set $\Omega \subset \mathbf{R}^2$ it is possible to construct a Lipschitzian function $u: \Omega \to \mathbf{R}^2$ vanishing at the boundary of Ω and a bounded countable set $\mathrm{S} \subset \mathrm{M}^{2\times 2}$ such that there are no rank-one connections in the closure K of S, $0 \notin \mathrm{K}$, and $\mathrm{D}u \in \mathrm{S}$ a. e. in Ω. *See* [13].

2. For examples showing that Theorem 2 fails in higher dimensions (except, perhaps, for mappings from $\Omega \subset \mathbf{R}^2$ to $\mathbf{R}^3$) *see* [2].

5. REGULARITY

THEOREM 3. — *Let* K *be a bounded subset of* $M^{2\times 2}$ *and suppose that there is* $\lambda>0$ *such that either* $\det(X-Y)\geqq\lambda|X-Y|^2$ *for each* $X, Y\in K$ *or* $\det(X-Y)\leqq-\lambda|X-Y|^2$ *for each* $X, Y\in K$. *Let* $v:\Omega\to\mathbf{R}^2$ *be a Lipschitzian function satisfying* $Dv(x)\in K$ *for almost every* $x\in K$. *Then there is* $p>2$ *such that* v *belongs to* $W^{2,p}_{loc}(\Omega)$. *In particular, the gradient* Dv *of* v *is Hölder continuous.*

Proof. — We will consider only the case $\det(X-Y)\geqq\lambda|X-Y|^2$. For the proof in the case $\det(X-Y)\leqq-\lambda|X-Y|^2$ it is enough to replace det by $-\det$ in the formulae below. Let $a\in\mathbf{R}^2$ and for $h>0$ let $v_h(x)=(v(x+ha)-v(x))/h$. (We can extend v by zero outside Ω, for example.) Let η be a smooth nonnegative function compactly supported in Ω. Let $b\in\mathbf{R}^2$. For sufficienly small h we have

$$
\begin{aligned}
0&=\int_\Omega \det D(\eta(v_h-b))\,dx\\
&\geqq\int_\Omega(-\eta|Dv_h||D\eta||v_h-b|+\eta^2\det Dv_h)\,dx\\
&\geqq\int_\Omega(-\eta|Dv_h||D\eta||v_h-b|+\lambda\eta^2|Dv_h|^2)\,dx\\
&\geqq-\frac{1}{2\lambda}\int_\Omega|D\eta|^2|v_h-b|^2\,dx+\frac{\lambda}{2}\int_\Omega\eta^2|Dv_h|^2\,dx.
\end{aligned}
$$

We see that the L^2-norm of Dv_h on compact subsets of Ω is estimated by the L^2-norm of v_h. We can now use the well-known Nirenberg's Lemma to infer that $Dv\in W^{1,2}_{loc}(\Omega)$. It is well-known that if there exists $C>0$ such that

$$\int_\Omega\eta^2|Dv_h|^2\,dx\leqq C\int_\Omega|D\eta|^2|v_h-b|^2\,dx$$

for every η as above and every $b\in\mathbf{R}^2$, or in another words, if v_h satisfies the Caccioppoli's inequality, then there exists a $p>2$ such that the L^p-norm of Dv_h on every set $\tilde{\Omega}$ compactly contained in Ω is bounded by $C_1\|v_h\|_{L^2(\Omega)}$, where C_1 depends only on C, p, $\tilde{\Omega}$ and Ω. (For a proof of this which is based on the technique of reverse Hölder inequalities see [6].) Using Nirenberg's Lemma again, we see that Dv is bounded in $W^{1,p}_{loc}(\Omega)$. The Hölder continuity of Dv follows from the Sobolev Imbedding Theorem. The proof is finished.

COROLLARY. — *Let* K *be a closed connected smooth submanifold of* $M^{2\times 2}$ *such that* $\operatorname{rank}(X-Y)\geqq 2$ *for any two distinct* $X, Y\in K$. *Suppose moreover* *that* K *is "elliptic", or in other words, that for any* $X \in K$ *the tangent space to* K *passing through* X *does not contain rank-one directions. Then every Lipschitz function* $v: \Omega \to \mathbf{R}^2$ *satisfying* $Dv(x) \in K$ *for* a. e. $x \in K$ *is smooth.*

Proof. — We notice that the ellipticity condition together with Lemma 1 implies that for each bounded subset K_1 of K there exists $\lambda > 0$ such that either

$$\det(X-Y) \geqq \lambda |X-Y|^2 \quad \text{for every} \quad X, Y \in K_1$$

or

$$\det(X-Y) \leqq -\lambda |X-Y|^2 \quad \text{for every} \quad X, Y \in K_1.$$

We can use Theorem 2 to infer that Dv is Hölder continuous and that v belongs to the space $W^{2,2}_{\text{loc}}(\Omega)$. Since $Dv(x) \in K$ in Ω, the derivatives $\frac{\partial}{\partial x_i} Dv(x)$ belong to the tangent space of K at $Dv(x)$ for a.e. $x \in \Omega$. Since Dv is Hölder continuous and K is elliptic, we see that $\frac{\partial}{\partial x_i} v(x)$ can be viewed as solutions of a certain linear first order elliptic system with Hölder continuous coefficients. Therefore $D^2 v$ is Hölder continuous. (*See*, for example, [11].) Applying the usual procedure of improving regularity we see that v must be smooth. The proof is finished.

6. EXAMPLES

Classical examples of K's which are elliptic in the above sense are

$$K_0 = \{ X \in M^{2 \times 2}, \text{ X is symmetric and Trace } X = 0 \}$$

and

$$K_1 = \{ X \in M^{2 \times 2}, \text{ X is symmetric, positive definite, and } \det X = 1 \}.$$

Clearly K_0 can be viewed as the tangent space to K_1 at the unit matrix.

The following examples arise in connection with problems concerning invariant "wells" which appear in the theory of microstructures. (*See*, for example, [3], [4], [9], and [10] for motivation). Let $A_1, \ldots, A_m \in M^{2 \times 2}$ with $\det A_k > 0$ for each $k = 1, \ldots, m$ and let

$$K_w = SO(2) \cdot A_1 \cup \ldots \cup SO(2) \cdot A_m.$$

It is easy to check that if K_w does not contain rank-one connections (*i. e.* rank $(X-Y) \geqq 2$ for any two distinct $X, Y \in K_w$), then there exists $\nu > 0$ such that $\det(X-Y) \geqq \nu |X-Y|^2$ for each $X, Y \in K_w$. We see that in this case Lemma 3 and Theorem 3 can be applied to K_w. This shows, for example, that if K_w does not contain rank-one connections, then the deformations $\varphi: \Omega \to \mathbf{R}^2$ satisfying $D\varphi \in K_w$ a.e. in Ω belong to $C^{1, \alpha}_{\text{loc.}}(\Omega)$

for some $\alpha>0$. Using this it is not difficult to see that if K_w does not contain rank-one connections, then $D\varphi \in K_w$ a.e. in Ω implies that in fact $D\varphi$ is locally constant in Ω.

We can also consider continuous families of invariant wells. A simple example is the following: let $\mu:[0, 1]\to \mathbf{R}$ and $\lambda:[0, 1]\to \mathbf{R}$ be smooth strictly positive functions with $\mu'(t)>0$ and $\lambda'(t)>0$ for all $t\in[0, 1]$ and let $K_c = \bigcup_{t\in[0,1]} SO(2)\cdot\begin{pmatrix}\lambda(t) & 0\\ 0 & \mu(t)\end{pmatrix}$. It is easy to check that K_c satisfies the assumptions of Theorem 1 and Theorem 3.

ACKNOWLEDGEMENTS

I thank John Ball for several useful suggestions. I also thank the referee for useful remarks.

REFERENCES

[1] J. M. BALL, A Version of the Fundamental Theorem for Young Measures, in *Partial Differential Equations and Continuum Models of Phase Transitions*, M. RASCLE, D. SERRE and M. SLEMROD Eds., pp. 107-215, Springer-Verlag.

[2] J. M. BALL, Sets of Gradients with No Rank-One Connections, *J. Math. pures et appl.*, Vol. **69**, 1990, pp. 241-159.

[3] J. M. BALL and R. D. JAMES, Fine Phase Mixtures as Minimizers of Energy, *Arch. Rat. Mech. Anal.*, Vol. **100**, 1987, pp. 13-52.

[4] J. M. BALL and R. D. JAMES, *Proposed Experimental Tests of a Theory of Fine Microstructures and the Two-Well Problem*, preprint, 1990.

[5] L. C. EVANS, Classical Solutions of Fully Nonlinear Second Order Elliptic Equations, *Comm. Pure Appl. Math.*, Vol. **25**, 1982, pp. 333-363.

[6] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations*, Princeton University Press, 1983.

[7] N. FIROOZYE, R. D. JAMES and R. KOHN (to appear).

[8] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer, 1983.

[9] D. KINDERLEHRER, Remarks about equilibrium configurations of crystals, in *Symp. Material Instabilities in Continuum Mechanics*, pp. 217-242, J. M. BALL Ed., Heriot-Watt, Oxford University Press, 1988.

[10] J. P. MATOS, *Young Measures and the Absence of Fine Microstructures in the $\alpha-\beta$ Quartz Phase Transition*, preprint, 1991.

[11] Ch. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer, 1966.

[12] V. ŠVERÁK, *On the Problem of Two Wells* (to appear).

[13] V. ŠVERÁK (to appear).

[14] L. TARTAR, The Compensated Compactness Method Applied to Systems of Conservations Laws, in *Systems of Nonlinear Partial Differential Equations*, J. M. BALL Ed., NATO ASI Series, Vol. C111, Reidel, 1982.

(*Manuscript received May* 21, 1991; *revised August* 21, 1991.)