

ANNALES DE L'I. H. P., SECTION C

ANDREA BRAIDES

VALERIA CHIADÓ PIAT

ANNELIESE DEFRANCESCHI

Homogenization of almost periodic monotone operators

Annales de l'I. H. P., section C, tome 9, n° 4 (1992), p. 399-432

http://www.numdam.org/item?id=AIHPC_1992__9_4_399_0

© Gauthier-Villars, 1992, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section C » (<http://www.elsevier.com/locate/anihpc>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Homogenization of almost periodic monotone operators

by

Andrea BRAIDES

Dipartimento Automazione Industriale,
Università degli Studi di Brescia,
Via Valotti 9, I-25060 Brescia, Italy

and

Valeria CHIADÒ PIAT ⁽¹⁾
Anneliese DEFRANCESCHI ⁽²⁾

Sissa, Strada Costiera 11,
I-34014 Trieste, Italy

ABSTRACT. — We determine some sufficient conditions for the G -convergence of sequences of quasi-linear monotone operators, together with an asymptotic formula for the G -limit. We then prove a homogenization theorem for quasiperiodic monotone operators and, eventually, extend this result to general almost periodic monotone operators using an approximation result and a closure lemma.

Key words : Homogenization, G -convergence, almost periodic functions, monotone operators, quasi-linear equations.

RÉSUMÉ. — Nous déterminons quelques conditions suffisantes pour la G -convergence de suites d'opérateurs monotones quasi linéaires, avec une

Classification A.M.S. : 35 B 27, 35 B 40, 35 B 15, 34 C 27, 42 A 75, 73 B 27.

⁽¹⁾ *Permanent address*: Politecnico di Torino, Dip. di Matematica, C.so Duca degli Abruzzi 24, I-10129 Torino, Italy.

⁽²⁾ *Permanent address*: Università degli Studi di Trento, Dip. di Matematica, I-38050 Povo (TN), Italy.

formule asymptotique pour la G-limite. Ensuite nous démontrons un théorème d'homogénéisation pour les opérateurs monotones quasi-périodiques et, enfin, nous étendons ce résultat aux opérateurs monotones presque périodiques en utilisant un résultat d'approximation.

INTRODUCTION

In this paper we consider a class of quasi-linear operators $\mathcal{A} : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$ of the form

$$\mathcal{A}u = -\operatorname{div}(a(x, Du)),$$

where Ω is a bounded open subset of \mathbf{R}^n , $1 < p < +\infty$, $1/p + 1/q = 1$, and the function $a : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies suitable measurability, continuity, and monotonicity assumptions. By M_Ω we denote the set of such functions a . In order to study the behaviour of boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, Du)) = f & \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

under perturbations of the function $a \in M_\Omega$, a notion of G-convergence has been introduced in [16]. Its definition and main properties are recalled in Section 1.

The main purpose of Section 2 is to determine some conditions on a sequence of functions (a_h) in M_Ω which imply G-convergence. It turns out that one of them is simply the strong convergence in $(L^1(\Omega))^n$ of the sequence $(a_h(\cdot, \xi))$, for every $\xi \in \mathbf{R}^n$ (see Theorem 2.1). A necessary and sufficient condition involves the limit behaviour, as h tends to $+\infty$, of the integrals

$$\int_A a_h(x, Dv_h + \xi) dx$$

for every $\xi \in \mathbf{R}^n$ and for every A in a suitable family of open sets, where the functions v_h are the solutions to the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a_h(x, Dv_h + \xi)) = 0 & \text{on } A, \\ v_h \in H_0^{1,p}(A) \end{cases}$$

(see Theorem 2.3). The proof of this result relies on a representation formula for functions $a \in M_\Omega$ given in Theorem 2.2.

Section 3 is concerned with the homogenization of operators defined by functions of the class M_{Ω} ; *i.e.*, the G-convergence of sequences of functions (a_h) of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

where (ε_h) is a sequence of positive real numbers converging to 0. We suppose that the function $a: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies the usual measurability, continuity, and monotonicity assumptions, and a condition of quasiperiodicity with respect to x (*see* Definition 3.1). By using the results obtained in Section 2, we prove that there exists a monotone operator $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that, for every $f \in H^{-1, q}(\Omega)$, the solutions u_h and the momenta $a\left(\frac{x}{\varepsilon_h}, Du_h\right)$ of the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Du_h\right)\right) = f & \text{on } \Omega, \\ u_h \in H_0^1, p(\Omega) \end{cases}$$

converge, as (ε_h) tends to 0, to the solution u and the momentum $b(Du)$ of the homogenized problem

$$\begin{cases} -\operatorname{div}(b(Du)) = f & \text{on } \Omega, \\ u \in H_0^1, p(\Omega). \end{cases}$$

Theorem 3.4 gives also an asymptotic formula for the function b ; its proof generalizes a construction used in [36].

In Section 4 we extend the homogenization result of Section 3 to almost periodic operators (in the sense of Besicovitch; *see* Definition 4.1) defined by functions of the class $M_{\mathbf{R}^n}$ using an approximation result (Lemma 4.4) and a closure lemma (Lemma 4.3).

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (*see* [40], [41], [42], [21]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (*see* [43], [44], and [34]). A further extension to higher order linear elliptic operators can be found in [47] together with an extensive bibliography on this subject. Results for the quasi-linear case are given, among others, in [46], [37], [23], [22] and [16].

For the related problems in homogenization theory under periodicity hypotheses on $a(\cdot, \xi)$, we refer to the books [2], [39], and [1], which contain a wide bibliography on this topic. Homogenization results for quasi-linear operators are obtained in [4], [5], [6], [25], [26], [17], while the almost periodic case for linear equations is studied in [28] and [36]. A corrector result for quasi-linear periodic equations has been obtained in

[18], whereas an analogous theorem for the almost periodic case will appear in [10].

From another point of view, the homogenization of a class of variational integrals of the form

$$F_h(u) = \int_{\Omega} f\left(\frac{x}{\varepsilon_h}, Du\right) dx,$$

which is related to the homogenization of the operators $-\operatorname{div}\left(\partial_{\xi} f\left(\frac{x}{\varepsilon_h}, \xi\right)\right)$, has been studied in [32], [15], and [7], using the techniques of Γ -convergence introduced by E. De Giorgi. Homogenization results for variational integrals under almost periodicity assumptions have been proven in [8], [9], [11].

ACKNOWLEDGEMENTS

We wish to thank Gianni Dal Maso for the many helpful discussions on the subject and his constant interest in this work.

1. NOTATIONS AND PRELIMINARY RESULTS

Let p be a real constant, $1 < p < +\infty$, and let q be its dual exponent, $1/p + 1/q = 1$. The Euclidean norm and the scalar product in \mathbf{R}^n are denoted by $|\cdot|$ and (\cdot, \cdot) , respectively.

DEFINITION 1.1. — Given four constants α, β, c_1 , and c_2 , such that $c_1 > 0, c_2 > 0, 0 \leq \alpha \leq 1 \wedge (p - 1), p \vee 2 \leq \beta < +\infty$, we denote by $\mathbf{M}(\alpha, \beta, c_1, c_2)$ the class of all functions $a: \mathbf{R}^n \rightarrow \mathbf{R}^n$ which fulfill the following conditions:

- (i) $|a(0)| \leq c_1$;
- (ii) a satisfies the following inequalities of equicontinuity and strict monotonicity:

$$|a(\xi_1) - a(\xi_2)| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha \tag{1.1}$$

$$(a(\xi_1) - a(\xi_2), \xi_1 - \xi_2) \geq c_2 (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \tag{1.2}$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

For every open subset \mathcal{O} of \mathbf{R}^n , by $\mathbf{M}_{\mathcal{O}}(\alpha, \beta, c_1, c_2)$ we denote the class of all functions $a: \mathcal{O} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ which satisfy the following conditions:

- (iii) for a. e. $x \in \mathcal{O}, |a(x, \cdot)| \in \mathbf{M}(\alpha, \beta, c_1, c_2)$;
- (iv) for every $\xi \in \mathbf{R}^n, a(\cdot, \xi)$ is Lebesgue measurable.

It is easy to see that (iii) implies that there exist constants $c_3 > 0$, $c_4 > 0$, such that

$$|a(x, \xi)| \leq c_3 (1 + |\xi|)^{p-1}, \quad (1.3)$$

$$|\xi|^p \leq c_4 (1 + (a(x, \xi), \xi)) \quad (1.4)$$

for a.e. $x \in \mathcal{O}$, for every $\xi \in \mathbf{R}^n$. The proof of (1.3) is trivial. As for the proof of (1.4), we just observe that by Young's inequality we have

$$|\xi|^p \leq \frac{p}{\beta} (1 + |\xi|)^{p-\beta} |\xi|^\beta + \frac{\beta-p}{\beta} (1 + |\xi|)^p.$$

In the case $\mathcal{O} = \mathbf{R}^n$, we simply use the notation $M_{\mathbf{R}^n}$ for $M_{\mathbf{R}^n}(\alpha, \beta, c_1, c_2)$.

Let us fix from now on a bounded open subset Ω of \mathbf{R}^n . Given $a \in M_\Omega(\alpha, \beta, c_1, c_2)$, it can be proved that for every $f \in H^{-1, q}(\Omega)$ there exists a unique solution $u \in H_0^{1, p}(\Omega)$ to the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1, p}(\Omega). \end{aligned} \right\} \quad (1.5)$$

For a proof we refer, for instance, to [27], Chapter III, Corollary 1.8, or to [31], Chapter 2, Theorem 2.1. The solution to (1.5) satisfies a Meyers' regularity estimate (see [33]) that will be needed in the sequel in the particular case where Ω is a cube, as stated in the following theorem.

THEOREM 1.2. — *Let Q be a cube in \mathbf{R}^n and let $w \in H_0^{1, p}(Q)$ be the weak solution to the equation*

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dw)) &= 0 \quad \text{on } Q, \\ w &\in H_0^{1, p}(Q). \end{aligned} \right\}$$

Then there exists $\eta > 0$ such that $w \in H^{1, p+\eta}(Q)$ and

$$\|Dw\|_{(L^{p+\eta}(Q))^n} \leq C(Q) \|Dw\|_{(L^p(Q))^n}. \quad (1.6)$$

The constant η depends only on c_1, c_2, n, p , while $C(Q)$ depends in addition on Q . Moreover, a simple rescaling argument shows that we can take

$$C(tQ) = t^{-n/r} C(Q)$$

for all $t > 0$, where $\frac{1}{r} + \frac{1}{q} + \frac{1}{p+\eta} = 1$.

In order to study the behaviour of problem (1.5) under perturbations of the function a we make use of the following notion of G-convergence.

DEFINITION 1.3. — We say that a sequence (a_h) in $M_\Omega(\alpha, \beta, c_1, c_2)$ G-converges to $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ if, for every $f \in H^{-1, q}(\Omega)$ and for every sequence (f_h) converging to f strongly in $H^{-1, q}(\Omega)$, the solutions u_h to

the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega), \end{aligned} \right\} \tag{1.7}$$

satisfy the following conditions:

$$u_h \rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \tag{1.8}$$

$$a_h(x, Du_h) \rightarrow a(x, Du) \quad \text{weakly in } (L^q(\Omega))^n, \tag{1.9}$$

where u is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \tag{1.10}$$

Remark 1.4. — It can be proved that this definition of G-convergence is independent of the boundary condition. More precisely, if $\varphi \in H^{1,p}(\Omega)$, if the sequence $(a_h) \in M_\Omega(\alpha, \beta, c_1, c_2)$ G-converges to $a \in M_\Omega(\alpha, \beta, c_1, c_2)$, (f_h) converges to f strongly in $H^{-1,q}(\Omega)$, and u_h are the solutions in $H^{1,p}(\Omega)$ to the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h - \varphi &\in H_0^{1,p}(\Omega), \end{aligned} \right\}$$

then

$$\left. \begin{aligned} u_h &\rightarrow u \quad \text{weakly in } H^{1,p}(\Omega), \\ a_h(x, Du_h) &\rightarrow a(x, Du) \quad \text{weakly in } (L^q(\Omega))^n, \end{aligned} \right\}$$

where u is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u - \varphi &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

A proof of this fact can be found, for instance, in [16], Theorem 3.8.

The next two theorems concern a localization property and a compactness result for G-convergence. Their proofs can be deduced from Theorem 6.1 and Theorem 4.1 in [16] respectively, by using Theorem 7.9 and Corollaries 7.10-7.12 therein.

Let Ω' be an open subset of Ω . For $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ we denote by a' the function of $M_{\Omega'}(\alpha, \beta, c_1, c_2)$ defined by $a' = a|_{\Omega' \times \mathbb{R}^n}$. Then the following localization property holds.

THEOREM 1.5. — *Let (a_h) be a sequence in $M_\Omega(\alpha, \beta, c_1, c_2)$ which G-converges to a in $M_\Omega(\alpha, \beta, c_1, c_2)$. Then (a'_h) G-converges to a' in $M_{\Omega'}(\alpha, \beta, c_1, c_2)$.*

THEOREM 1.6. — *Let (a_h) be a sequence in $M_\Omega(\alpha, \beta, c_1, c_2)$. Then there exist suitable positive constants c'_1, c'_2 and a subsequence $(a_{\sigma(h)})$ of (a_h) which G-converges to a function a of the class $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$.*

In order to simplify the notation, the classes $M_\Omega(\alpha, \beta, c_1, c_2)$ and $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$ given by Theorem 1.6 will be denoted, from now on, by M_Ω and M'_Ω respectively.

Finally, we recall a lemma of compensated compactness type (see [35], [45]) which will be used in the sequel. For its proof see, for example, Lemma 3.4 in [16].

LEMMA 1.7. — *Let (u_h) be a sequence converging to u weakly in $H^{1,p}(\Omega)$. Let (g_h) be a sequence converging to g weakly in $(L^q(\Omega))^n$ with $(\operatorname{div} g_h)$ converging to $\operatorname{div} g$ strongly in $H^{-1,q}(\Omega)$. Then*

$$\int_{\Omega} (g_h, D u_h) \varphi \, dx \rightarrow \int_{\Omega} (g, D u) \varphi \, dx$$

for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

2. SUFFICIENT CONDITIONS FOR THE G-CONVERGENCE AND REPRESENTATION FORMULA FOR MONOTONE OPERATORS

In this section we investigate some conditions on a sequence of functions (a_h) in M_Ω which imply G-convergence. Furthermore, we give a representation formula for functions $a \in M_\Omega$ showing that $a(x, \xi)$ can be determined by the knowledge of the solutions v to the Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 && \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'), \end{aligned} \right\}$$

where $\xi \in \mathbf{R}^n$ and Ω' is an open subset of Ω . More precisely, we prove that $a(x, \xi)$ can be calculated by a differentiation process of the set function

$$\Omega' \rightarrow \int_{\Omega'} a(x, Dv(x) + \xi) \, dx$$

along a family of open subsets of Ω . Similar results for minima of variational functionals were proved in [21] and [24] for the quadratic case, and in [19] for the general case.

The following theorem shows that the strong convergence $(L^1(\Omega))^n$ of a sequence (a_h) in M_Ω implies the G-convergence.

THEOREM 2.1. — *Let a_h and $a \in M_\Omega$. Assume that $(a_h(\cdot, \xi))$ converges to $a(\cdot, \xi)$ strongly in $(L^1(\Omega))^n$ for every $\xi \in \mathbf{R}^n$. Then, the sequence (a_h) G-converges to a .*

Proof. — Let (f_h) be a sequence in $H^{-1,q}(\Omega)$ converging to f strongly in $H^{-1,q}(\Omega)$. Let u_h be the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

By the definition of G-convergence we have to prove that (1.8), (1.9) and (1.10) are satisfied. Since (f_h) is bounded in $H^{-1,q}(\Omega)$, condition (1.4) implies that (u_h) is bounded in $H_0^{1,p}(\Omega)$, hence $(a_h(\cdot, Du_h(\cdot)))$ is bounded in $(L^q(\Omega))^n$ by (1.3). Therefore, up to a subsequence,

$$\begin{aligned} u_h &\rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \\ a_h(x, Du_h) &\rightarrow g \quad \text{weakly in } (L^q(\Omega))^n, \end{aligned}$$

with $-\operatorname{div}g=f$. We shall show that $g(x)=a(x, Du(x))$ for a.e. $x \in \Omega$, hence u is the unique solution to (1.10). Therefore, the whole sequences (u_h) and $(a_h(\cdot, Du_h(\cdot)))$ converge, and the proof of our assertion is complete. By the strong convergence in $(L^1(\Omega))^n$ of the sequence $(a_h(\cdot, \xi))$ to $a(\cdot, \xi)$ and by the equicontinuity [see Definition 1.1 (iii)], there exists a subsequence, still denoted by (a_h) , such that $(a_h(x, \xi))$ converges to $a(x, \xi)$ for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Furthermore, by taking the equiboundedness condition [Definition 1.1 (iii)] for a_h into account, the dominated convergence theorem implies that

$$a_h(\cdot, \xi) \rightarrow a(\cdot, \xi) \quad \text{strongly in } (L^q(\Omega))^n, \quad \text{for every } \xi \in \mathbf{R}^n.$$

Since $a_h(x, \cdot)$ is monotone for a.e. $x \in \Omega$, we have

$$\int_{\Omega} (a_h(x, Du_h) - a_h(x, \xi), Du_h(x) - \xi) \varphi(x) dx \geq 0$$

for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. Passing to the limit as h tends to $+\infty$ we obtain by means of Lemma 1.7 that

$$\int_{\Omega} (g(x) - a(x, \xi), Du(x) - \xi) \varphi(x) dx \geq 0 \tag{2.1}$$

holds for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$. By a standard density argument, (2.1) implies that

$$(g(x) - a(x, \xi), Du(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$ [remind that $a(x, \xi)$ is continuous with respect to ξ by (iii) in Definition 1.1]. By Minty's lemma (see, for example, [27], Chapter III, Lemma 1.5) it follows that

$$(g(x) - a(x, Du(x)), Du(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Hence, $g(x)=a(x, Du(x))$ for a.e. $x \in \Omega$, which completes the proof. \square

In order to state the representation formula for functions in the class M_Ω , given $a \in M_\Omega$, let us define the function

$$\Psi(\xi, \Omega') = \int_{\Omega'} a(x, Dv(x) + \xi) dx \quad (2.2)$$

for every $\xi \in \mathbf{R}^n$, for every open subset Ω' of Ω , where the function v , depending on ξ and Ω' , is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'). \end{aligned} \right\} \quad (2.3)$$

THEOREM 2.2. — *Let $a \in M_\Omega$. Then there exists a measurable subset N of Ω with $|N| = 0$ such that*

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0^+} \frac{\Psi(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \quad (2.4)$$

for every $\xi \in \mathbf{R}^n$, $x_0 \in \Omega \setminus N$, where $|\cdot|$ denotes the Lebesgue measure, $A_\rho(x_0) = x_0 + \rho A$, and A is any bounded open subset of \mathbf{R}^n .

Proof. — By a standard density argument and by Lebesgue's differentiation theorem (see, for instance, [38], Theorem 8.8) there exists a measurable subset N of Ω with $|N| = 0$ such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{|A_\rho(x_0)|} \int_{A_\rho(x_0)} |a(x, \xi) - a(x_0, \xi)| dx = 0 \quad (2.5)$$

for every $x_0 \in \Omega \setminus N$, for every $\xi \in \mathbf{R}^n$, where $A_\rho(x_0) = x_0 + \rho A$, and A is any bounded open subset of \mathbf{R}^n . Given $x_0 \in \Omega \setminus N$, $\rho > 0$, $\xi \in \mathbf{R}^n$ we denote by v the function depending on ξ and $A_\rho(x_0)$ which is the unique solution to the Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\} \quad (2.6)$$

By performing the change of variables $y = \frac{(x - x_0)}{\rho}$, problem (2.6) becomes

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0 + \rho y, D_y u + \xi)) &= 0 \quad \text{on } A, \\ u(y) &= \frac{1}{\rho} v(x_0 + \rho y) \in H_0^{1,p}(A). \end{aligned} \right\} \quad (2.7)$$

We may suppose that ρ runs through a sequence (ρ_h) which tends to 0^+ as h tends to $+\infty$. Let us set $a_h(y, \xi) = a(x_0 + \rho_h y, \xi)$ for every $y \in A$, $\xi \in \mathbf{R}^n$. By (2.5) we have

$$a_h(\cdot, \xi) \rightarrow a(x_0, \xi) \quad \text{strongly in } (L^1(A))^n,$$

which guarantees by Theorem 2.1 that

$$(a_h) \text{ G-converges to } a \text{ on } A. \tag{2.8}$$

Since $w=0$ is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0, D_y w(y) + \xi)) &= 0 \quad \text{on } A, \\ w &\in H_0^{1,p}(A), \end{aligned} \right\}$$

if u_h denotes the solution to (2.7) corresponding to $\rho = \rho_h$, the G-convergence condition (2.8) and Remark 1.4 imply that (u_h) converges to 0 weakly in $H_0^{1,p}(A)$ and

$$a(x_0 + \rho_h y, D_y u_h(y) + \xi) \rightarrow a(x_0, \xi) \quad \text{weakly in } (L^q(A))^n. \tag{2.9}$$

By (2.9) we have then

$$a(x_0, \xi) = \lim_{h \rightarrow \infty} \frac{1}{|A|} \int_A a(x_0 + \rho_h y, D_y u_h(y) + \xi) dy,$$

which by a change of variables proves (2.4). \square

The aim of the next theorem is to obtain a necessary and sufficient condition for the G-convergence of a sequence $a_h \in M_\Omega$ by means of the convergence of the momenta related to the Dirichlet problems

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)), \end{aligned} \right\}$$

where $\xi \in \mathbf{R}^n$.

THEOREM 2.3. — *Let (a_h) be a sequence in M_Ω . Let Ψ_h be the function associated to a_h by (2.2). For every $\rho > 0$ and $x_0 \in \mathbf{R}^n$, let $A_\rho(x_0) = x_0 + \rho A$, where A is any bounded open subset of \mathbf{R}^n . Let N be a measurable subset of Ω with $|N| = 0$. Then, the following conditions are equivalent:*

(a) *the limit*

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, A_\rho(x_0))$$

exists for every $\xi \in \mathbf{R}^n$ and for every $x_0 \in \Omega \setminus N$;

(b) *there exists a function $a \in M'_\Omega$ such that (a_h) G-converges to a . Moreover, if the previous conditions are satisfied, then*

$$a(x, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, A_\rho(x))}{|A_\rho(x)|} \tag{2.10}$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$.

Proof. — Assume (a). Let $(a_{\tau(h)})$ be a subsequence of (a_h) . By the compactness theorem 1.6 there exist a further subsequence $(a_{\tau(\sigma(h))})$ and $a \in M'_\Omega$ such that

$$(a_{\tau(\sigma(h))}) \text{ G-converges to } a. \tag{2.11}$$

If we show that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \quad (2.12)$$

for a.e. $x_0 \in \Omega$, for every $\xi \in \mathbf{R}^n$, by the existence of the limit in (a) and the Urysohn property of the G-convergence (see [16], Remark 3.7) we may conclude that (b) and (2.10) hold.

Given $\xi \in \mathbf{R}^n$, $x_0 \in \Omega \setminus N$, and $\rho > 0$, we denote by v_h the solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a_{\tau(\sigma(h))}(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\}$$

Since v is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)), \end{aligned} \right\} \quad (2.13)$$

the G-convergence condition (2.11), Theorem 1.5 and Remark 1.4 imply

$$\left. \begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_{\tau(\sigma(h))}(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n. \end{aligned} \right\}$$

Therefore

$$\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0)) \rightarrow \int_{A_\rho(x_0)} a(x, Dv + \xi) dx = \Psi(\xi, A_\rho(x_0)).$$

Now, by the representation Theorem 2.2 we conclude that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|}$$

for a.e. $x_0 \in \Omega$, for every $\xi \in \mathbf{R}^n$, proving (2.12).

Assume (b). Then, Theorem 1.5 and Remark 1.4 guarantee that the solutions v_h to

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)) \end{aligned} \right\}$$

satisfy

$$\left. \begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_h(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n, \end{aligned} \right\}$$

where v is the unique solution to (2.13). Hence, condition (a) follows immediately. \square

Remark 2.4. — Theorem 2.3 provides a simple characterization of G-convergence in the special case of functions a_h in M_Ω satisfying

$$-\operatorname{div}_x(a_h(x, \xi)) = 0 \quad \text{on } \Omega$$

for every $h \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$. In this case (a_h) G-converges to a function $a \in \mathbf{M}_\Omega$ if and only if $(a_h(\cdot, \xi))$ tends to $a(\cdot, \xi)$ weakly in $(L^1(\Omega))^n$, for every $\xi \in \mathbf{R}^n$. According to Theorem 2.3 this condition is clearly sufficient, since in this case

$$\Psi_h(\xi, A_p(x_0)) = \int_{A_p(x_0)} a_h(x, \xi) dx,$$

while its necessity follows easily from the local character of the G-convergence (see Theorem 1.5 and Remark 1.4). In the linear case, the previous characterization was proved in [13]. See also [12] for a similar result in the case $a_h(x, \xi) = \partial_\xi f_h(x, \xi)$ with f_h convex in ξ .

3. HOMOGENIZATION OF QUASIPERIODIC OPERATORS

In this section we give a characterization of the G-limit of a sequence of functions a_h of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

with $a \in \mathbf{M}_{\mathbf{R}^n}$ verifying suitable hypotheses of quasiperiodicity in the first variable (see Definition 3.1). This result will be used in Section 4 to derive the homogenization theorem for general almost periodic operators.

DEFINITION 3.1 (see [30] 3.3). — A continuous function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is *quasiperiodic* if there exist $m_1, \dots, m_n \in \mathbf{N}$ and a continuous function $F: \mathbf{R}^N \rightarrow \mathbf{R}$, where $N = m_1 + \dots + m_n$, $N \geq n$, such that

$$f(x_1, \dots, x_n) = F(\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}),$$

and F periodic of periods $\frac{2\pi}{\lambda_1^1}, \frac{2\pi}{\lambda_1^2}, \dots, \frac{2\pi}{\lambda_1^{m_1}}, \frac{2\pi}{\lambda_2^1}, \frac{2\pi}{\lambda_2^2}, \dots, \frac{2\pi}{\lambda_2^{m_2}}, \dots, \frac{2\pi}{\lambda_n^1}, \frac{2\pi}{\lambda_n^2}, \dots, \frac{2\pi}{\lambda_n^{m_n}}$, with

$\lambda_r^l \in]0, +\infty[$. It is not restrictive to assume that the frequencies $\lambda_r^1, \dots, \lambda_r^{m_r}$ are linearly independent on \mathbf{Z} for every $r = 1, \dots, n$. This will be done constantly in the sequel. Under this assumption, Kronecker's lemma (see Appendix, Section A) guarantees that F is uniquely determined by f .

Given m_1, \dots, m_n as above and given $\lambda \in \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_n}$, $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_r = (\lambda_r^1, \dots, \lambda_r^{m_r})$ for $r = 1, \dots, n$, we denote by $\text{QP}(\lambda)$ the set of all quasiperiodic functions $f: \mathbf{R}^n \rightarrow \mathbf{R}$ with frequencies λ . Furthermore, by $\text{Trig}(\lambda)$ we indicate the set of all trigonometric poly-

mials with frequencies λ ; *i. e.*, finite sums of terms of the form

$$P(x) = \operatorname{Re}(c \exp(i \sum_{l,r} k_r^l \lambda_r^l x_r)), \quad (3.1)$$

where $k \in \mathbf{Z}^N$, $c \in \mathbf{C}$.

Remark 3.2. — Trigonometric polynomials are obviously quasiperiodic functions. Moreover, it can be proved that every function f of $\operatorname{QP}(\lambda)$ is the uniform limit of a sequence of trigonometric polynomials belonging to $\operatorname{Trig}(\lambda)$ (see [30] 3.3).

For every $s > 0$, for every $z \in \mathbf{R}^n$, let $Q_s(z)$ be the cube of side length s and center z . For every $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ we define

$$\int f(x) dx = \limsup_{s \rightarrow \infty} \frac{1}{|Q_s(0)|} \int_{Q_s(0)} f(x) dx.$$

It can be seen easily that for every function f in $\operatorname{QP}(\lambda)$ we have

$$\int f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx$$

uniformly with respect to z . This limit is called the *mean value* of f (on \mathbf{R}^n) (see [30], 2.3).

In the sequel it will be useful to consider the function $j: \mathbf{R}^n \rightarrow \mathbf{R}^N$ defined by

$$j(x) = (\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}).$$

Then, introducing the variables

$$(y_1, \dots, y_N) = (y_1^1, \dots, y_1^{m_1}, \dots, y_n^1, \dots, y_n^{m_n})$$

on \mathbf{R}^N , for every $v \in \mathcal{C}^1(\mathbf{R}^N)$ and $u(x) = v(j(x))$ we get

$$D_r u(x) = \sum_{l=1}^{m_r} \frac{\partial v}{\partial y_r^l}(j(x)) \quad \text{for every } r=1, \dots, n,$$

or briefly, $Du(x) = (\partial v) \circ j(x)$, where $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$.

Let us denote by $\operatorname{Trig}_0(\lambda)$ the set of all trigonometric polynomials in $\operatorname{Trig}(\lambda)$ with mean value 0. By Birkhoff's theorem (see Appendix, Theorem A) it is easy to prove that

$$\|u\|_{1,p} = \left(\sum_{r=1}^n \int |D_r u|^p dx \right)^{1/p}$$

is a norm on $\text{Trig}_0(\lambda)$. Since $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$ is not complete, we study its completion. To this aim let us introduce

$$T = \prod_{i,r} \left[0, \frac{2\pi}{\lambda_r^i} \right] \subset \mathbf{R}^N, \tag{3.2}$$

and let us denote by $\text{Trig}(T)$ the set of all trigonometric polynomials in \mathbf{R}^N with period T . Let us also define the set $\text{Trig}_0(T)$ of all functions v in $\text{Trig}(T)$ with mean value zero. On $\text{Trig}_0(T)$ we consider the norm

$$\|v\|_{\mathscr{W}} = \|\partial v\|_{(L^p(T))^n} = \left(\sum_{r=1}^n \frac{1}{|T|} \int_T |\partial_r v|^p dy \right)^{1/p};$$

in fact, if $\partial v = 0$, then $u = v \circ j$ satisfies $Du = 0$. Hence, u is constant. Since v depends uniquely on u (see Definition 3.1), v is constant too; hence $v = 0$. By Birkhoff's theorem (see Appendix, Theorem A) the linear map

$$\text{Re} \left(c \exp \left(i \sum_{i,r} k_r^i \lambda_r^i x_r \right) \right) \mapsto \text{Re} \left(c \exp \left(i \sum_{i,r} k_r^i \lambda_r^i y_r^i \right) \right)$$

is a bijective isometry between $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$ and $(\text{Trig}_0(T), \|\cdot\|_{\mathscr{W}})$, and will be denoted by J .

Let us denote by \mathscr{W} the completion of $\text{Trig}_0(T)$ with respect to the norm $\|\cdot\|_{\mathscr{W}}$, which we can identify with the completion of $\text{Trig}_0(\lambda)$ with respect to the norm $\|\cdot\|_{1,p}$. Finally, let us remark that the isometry

$$\begin{aligned} \partial: \text{Trig}_0(T) &\rightarrow (L^p(T))^n \\ u &\mapsto (\partial_1 u, \partial_2 u, \dots, \partial_n u) \end{aligned}$$

can be extended in a unique way to an isometry between \mathscr{W} and a closed subspace of $(L^p(T))^n$, which makes \mathscr{W} a reflexive Banach space.

Now, let us fix the frequencies λ , with $\lambda_r^1, \dots, \lambda_r^{m_r}$ linearly independent on \mathbf{Z} for every $r = 1, \dots, n$, and let us fix $a \in M_{\mathbf{R}^n}$ such that

$$a(\cdot, \xi) \in \text{QP}(\lambda) \quad \text{for every } \xi \in \mathbf{R}^n. \tag{3.3}$$

By definition it follows that there exists a unique function $\tilde{a}(\cdot, \xi): \mathbf{R}^N \rightarrow \mathbf{R}^n$ such that $a(x, \xi) = \tilde{a}(j(x), \xi)$ for every $x \in \mathbf{R}^N$ and for every $\xi \in \mathbf{R}^n$, $\tilde{a}(\cdot, \xi)$ is T -periodic and continuous for every $\xi \in \mathbf{R}^n$. It follows that $\tilde{a}(y, \cdot)$ satisfies (1.1) and (1.2) for every $y \in \mathbf{R}^N$. This is obvious for $y = j(x)$, $x \in \mathbf{R}^N$, whereas the conclusion for a general $y \in \mathbf{R}^N$ comes from Kronecker's lemma (see Appendix A) and the continuity of $\tilde{a}(\cdot, \xi)$.

The next proposition is an extension to the case $p \neq 2$ of Lemma 1 in [36].

PROPOSITION 3.3. — *Let $\xi \in \mathbf{R}^n$ be fixed, let $a \in M_{\mathbf{R}^n}$ satisfying (3.3) and let \tilde{a} be the corresponding T -periodic function. Then, there exists a unique*

solution $w^\xi \in \mathcal{W}$ to the problem

$$\left. \begin{aligned} \frac{1}{|\mathbf{T}|} \int_{\mathbf{T}} (\tilde{a}(y, \partial w^\xi + \xi), \partial h) dy = 0 \quad \text{for every } h \in \mathcal{W}, \\ w^\xi \in \mathcal{W}. \end{aligned} \right\} \quad (3.4)$$

Furthermore, for any $\delta > 0$ there exist $u_\delta^\xi \in \text{Trig}_0(\lambda)$ and a vector function $g_\delta^\xi \in (\text{QP}(\lambda))^n$ such that

$$-\text{div}(a(x, Du_\delta^\xi + \xi)) = -\text{div} g_\delta^\xi \quad \text{in } \mathcal{D}'(\mathbf{R}^n), \quad (3.5)$$

$$\lim_{\delta \rightarrow 0+} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}} = 0, \quad (3.6)$$

$$\lim_{\delta \rightarrow 0+} \int |g_\delta^\xi|^q dx = 0 \quad (3.7)$$

hold.

Proof. - Given $\xi \in \mathbf{R}^n$, let $\tilde{\mathcal{A}}^\xi: \mathcal{W} \rightarrow \mathcal{W}'$ be the operator defined by

$$\langle \tilde{\mathcal{A}}^\xi w, h \rangle = \frac{1}{|\mathbf{T}|} \int_{\mathbf{T}} (\tilde{a}(y, \partial w + \xi), \partial h) dy$$

for every $w, h \in \mathcal{W}$, where $\langle \dots \rangle$ denotes here the duality pairing between \mathcal{W} and its dual space \mathcal{W}' . By the coerciveness and continuity properties of $\tilde{\mathcal{A}}^\xi$ on \mathcal{W} , the theory of monotone operators on reflexive Banach spaces (see, for example [27]) implies immediately that there exists a unique $w^\xi \in \mathcal{W}$ satisfying $\tilde{\mathcal{A}}^\xi w^\xi = 0$, which implies (3.4). By the density of $J(\text{Trig}_0(\lambda))$ in \mathcal{W} there exists $u_\delta^\xi \in \text{Trig}_0(\lambda)$ such that (3.6) is satisfied. By using (3.4) and the equicontinuity assumption on \tilde{a} we get

$$\begin{aligned} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} &= \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi) - \tilde{\mathcal{A}}^\xi w^\xi\|_{\mathcal{W}'} \\ &\leq c(1 + |\xi| + \|J(u_\delta^\xi)\|_{\mathcal{W}} + \|w^\xi\|_{\mathcal{W}})^{p-1-\alpha} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}}^\alpha. \end{aligned}$$

By (3.6) it follows that

$$\lim_{\delta \rightarrow 0+} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} = 0. \quad (3.8)$$

Since $x \mapsto a(x, Du_\delta^\xi + \xi)$ belongs to $(\text{QP}(\lambda))^n$, by Remark 3.2 there exist a function $f_\delta^\xi \in (\text{Trig}(\lambda))^n$ and a quasiperiodic function $h_\delta^\xi \in (\text{QP}(\lambda))^n$ such that

$$a(x, Du_\delta^\xi + \xi) = f_\delta^\xi + h_\delta^\xi, \quad (3.9)$$

and

$$\lim_{\delta \rightarrow 0+} \int |h_\delta^\xi|^q dx = 0. \quad (3.10)$$

Since $a(x, Du_\delta^\xi + \xi) = \tilde{a}(j(x), \partial J(u_\delta^\xi)(j(x)) + \xi)$, by (3.8) and (3.10) we get

$$\lim_{\delta \rightarrow 0+} \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = 0, \quad (3.11)$$

where $(F_\delta^\xi)_r$ is $J((f_\delta^\xi)_r)$, and

$$\left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = \sup_{\|v\|_{\mathcal{W}} \leq 1} \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} (F_\delta^\xi)_r \partial_r v dy.$$

Being $\text{div} f_\delta^\xi \in \text{Trig}_0(\lambda)$, we can write

$$\text{div} f_\delta^\xi(x) = \text{Re} \left(\sum_k c_k \exp \left(i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

where the sum runs over a finite set of non-zero vectors $k \in \mathbb{Z}^N$. It turns out that the function

$$w_\delta^\xi(x) = \text{Re} \left(\sum_k b_k \exp \left(i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

with $b_k = -c_k / (\sum_r (\sum_l k_r^l \lambda_r^l)^2)$, is the unique solution in $\text{Trig}_0(\lambda)$ to

$$\Delta w_\delta^\xi = \text{div} f_\delta^\xi \quad \text{on } \mathbb{R}^n.$$

If $W_\delta^\xi = J(w_\delta^\xi) \in \text{Trig}(\mathbb{T})$, then

$$\sum_{r=1}^n \partial_r^2 W_\delta^\xi = \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \quad \text{in } \mathbb{R}^N$$

holds. By elliptic regularity (see Appendix, Section B)

$$\left(\frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} |\partial_r W_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'},$$

with $c > 0$ independent of ξ and δ . We have then by Birkhoff's theorem

$$\left(\int |\mathbb{D}w_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'}. \tag{3.12}$$

Hence, by setting

$$g_\delta^\xi = \mathbb{D}w_\delta^\xi + h_\delta^\xi$$

we have

$$\text{div} (a(x, \mathbb{D}u_\delta^\xi + \xi)) = \text{div} g_\delta^\xi$$

in the sense of distributions on \mathbb{R}^n , proving (3.5). Furthermore, by (3.10)-(3.12) we obtain (3.7). \square

Let us consider the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\text{div} \left(a \left(\frac{x}{\varepsilon_h}, \mathbb{D}u_h \right) \right) &= f \quad \text{on } \Omega, \\ u_h &\in H_0^1 \cdot p(\Omega), \end{aligned} \right\} \tag{3.13}$$

where $f \in H^{-1,q}(\Omega)$ and (ε_h) is a sequence of positive real numbers converging to 0.

In this section we prove the convergence, as (ε_h) tends to 0^+ , of the solutions u_h of (3.13) to the solution u of the homogenized problem

$$\left. \begin{aligned} -\operatorname{div}(b(Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \quad (3.14)$$

Furthermore, we give an asymptotic formula for the homogenized function b in terms of the solutions v_s^ξ to the following Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(z), \\ v_s^\xi &\in H_0^{1,p}(Q_s(z)). \end{aligned} \right\} \quad (3.15)$$

The convergence result above mentioned will follow from the next two theorems.

THEOREM 3.4. — *Let $a \in M_{\mathbf{R}^n}$ satisfying (3.3) and let \tilde{a} be the corresponding T-periodic function. Let $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the function defined by*

$$b(\xi) = \frac{1}{|T|} \int_T \tilde{a}(y, \partial w^\xi + \xi) dy,$$

where w^ξ is the solution to (3.4). Then, for any family $(z_s)_{s>0}$ in \mathbf{R}^n , we have

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx,$$

where v_s^ξ is the unique solution to (3.15) with $z = z_s$.

THEOREM 3.5. — *Let $a \in M_{\mathbf{R}^n}$ satisfying (3.3). Let (ε_h) be a sequence of positive real numbers converging to 0 and let $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$. Then (a_h) G-converges to b , where b is the function defined in Theorem 3.4.*

The proofs of these theorems are quite technical and are therefore given at the end of the section as a consequence of the next proposition and some remarks stated in the sequel.

For every $s > 0$, and for every family (z_s) in \mathbf{R}^n , let us define

$$g_s(\xi) = \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx, \quad (3.16)$$

where v_s^ξ is the solution to (3.15) on $Q_s(z_s)$. By (iii) and (iv) in Definition 1.1, using Young's and Hölder's inequalities, it turns out that the function v_s^ξ satisfies the following estimates

$$\left. \begin{aligned} \|Dv_s^\xi + \xi\|_{L^p(Q_s(z_s))}^p &\leq c s^n (1 + |\xi|^p), \\ \|a(x, Dv_s^\xi + \xi)\|_{L^q(Q_s(z_s))}^q &\leq c s^n (1 + |\xi|^p) \end{aligned} \right\} \quad (3.17)$$

with c independent of z_s . Here and henceforth, we will denote by c any constant depending at most on $c_1, c_2, \alpha, \beta, n, p$, that can change from line to line. By (3.17), we have

$$|g_s(\xi)| \leq c(1 + |\xi|^{p/q}).$$

PROPOSITION 3.6. — *Let $\xi \in \mathbf{R}^n$ be fixed. Let u_δ^ξ be as in Proposition 3.3 and let $v_s^\xi \in H_0^{1,p}(Q_s(z_s))$ be the unique solution to (3.15) with z replaced by z_s . Then*

$$\lim_{\delta \rightarrow 0+} \limsup_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx = 0 \tag{3.18}$$

for every family $(z_s)_{s>0}$ in \mathbf{R}^n .

Proof. — Given a family $(z_s)_{s>0}$ we consider

$$\begin{aligned} I_{s,\delta} &\equiv \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi) - a(x, Du_\delta^\xi + \xi), Dv_s^\xi - Du_\delta^\xi) dx \\ &= \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Dv_s^\xi) dx \\ &\quad + \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Dv_s^\xi) dx \\ &\equiv I_s^1 + I_{s,\delta}^2 - I_{s,\delta}^3 - I_{s,\delta}^4. \end{aligned}$$

By (3.15), $I_s^1 = 0$. On the other hand,

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \int (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx.$$

By Birkhoff's theorem we get

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \frac{1}{|\Gamma|} \int_\Gamma (\tilde{a}(y, \partial J(u_\delta^\xi) + \xi), \partial J(u_\delta^\xi)) dy = \langle \tilde{\mathcal{A}}^\xi J(u_\delta^\xi), J(u_\delta^\xi) \rangle,$$

where $\tilde{\mathcal{A}}^\xi$ is the operator defined in the proof of Proposition 3.3. Since by (3.6) we have

$$J(u_\delta^\xi) \rightarrow w^\xi \text{ strongly in } \mathcal{W},$$

the continuity of $\tilde{\mathcal{A}}^\xi$ and the equality $\tilde{\mathcal{A}}^\xi w^\xi = 0$ implies that

$$\lim_{\delta \rightarrow 0+} \lim_{s \rightarrow \infty} I_{s,\delta}^2 = 0.$$

Since

$$\frac{1}{|Q_s(z_s)|} \int_{Q_s(z_s)} |Du_s^\xi(x)|^p dx \rightarrow \int |Du_s^\xi(x)|^p dx,$$

and $w_s(x) = \frac{1}{s} u_s^\xi(sx + z_s)$ converges to 0, as $s \rightarrow +\infty$, we obtain that the sequence (w_s) converges weakly to 0 in $H^{1,p}(Q_1(0))$. Applying Lemma 1.7, we obtain that the sequence of functions

$$\zeta_s(x) = (a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi), Du_s^\xi(sx + z_s))$$

converges to 0 in $\mathcal{D}'(Q_1(0))$ as $s \rightarrow +\infty$. On the other hand, by (1.6) and (3.17) we have

$$\|Dv_s^\xi(sx + z_s) + \xi\|_{(L^{p+n}(Q_1(0)))^n} \leq c(1 + |\xi|^p)^{(p+n)/p};$$

by (1.3) it follows that the sequence of functions

$$x \mapsto a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi)$$

is uniformly bounded in some $(L^\tau(Q_1(0)))^n$ for some $\tau > q$. This implies that there exists $\sigma > 1$ such that

$$\|\zeta_s\|_{L^\sigma(Q_1(0))} \leq c,$$

where c is independent of s . Since (ζ_s) converges to 0 in $\mathcal{D}'(Q_1(0))$, the above inequality implies that (ζ_s) converges to 0 weakly in $L^\sigma(Q_1(0))$, as $s \rightarrow +\infty$. This fact gives then

$$\lim_{s \rightarrow \infty} I_{s, \delta}^3 = 0.$$

Let us show that

$$\lim_{\delta \rightarrow 0^+} \limsup_{s \rightarrow \infty} I_{s, \delta}^4 = 0.$$

Let (g_δ^ξ) be as in Proposition 3.3. By applying Hölder's inequality and (3.17) we obtain

$$\begin{aligned} |I_{s, \delta}^4| &\leq c \left(\frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} \left(\frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi|^p dx \right)^{1/p} \\ &\leq c \left(\frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} (1 + |\xi|). \end{aligned}$$

Now, by taking the limit first as s tends to $+\infty$ and then as δ tends to 0^+ , (3.7) implies that

$$\lim_{\delta \rightarrow 0} \limsup_{s \rightarrow \infty} |I_{s, \delta}^4| = 0.$$

Since the equicoerciveness assumption in Definition 1.1 (iii) guarantees

$$|I_{s, \delta}| \geq c_2 \frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^{p-\beta} |Dv_s^\xi - Du_\delta^\xi|^\beta dx,$$

we obtain immediately

$$c |I_{s, \delta}|^{p/\beta} \left(\frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^p dx \right)^{(\beta-p)/\beta} \geq \left(\frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx \right).$$

Hence, by passing to the limit first as s tends to $+\infty$, and then as $\delta \rightarrow 0^+$, we get (3.18) and the proof of Proposition 3.6 is accomplished. \square

Proof of Theorem 3.5. — Let $x_0 \in \Omega$, and let $\xi \in \mathbb{R}^n$. Consider

$$\Psi_h(\xi, Q_\rho(x_0)) = \int_{Q_\rho(x_0)} a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi(x) + \xi\right) dx,$$

where $z_{h,\rho}^\xi$ is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi(x) + \xi\right)\right) &= 0 \quad \text{on } Q_\rho(x_0), \\ z_{h,\rho}^\xi &\in H_0^{1,p}(Q_\rho(x_0)). \end{aligned} \right\}$$

It follows immediately that

$$\Psi_h(\xi, Q_\rho(x_0)) = (\varepsilon_h)^n \int_{Q_{\rho/\varepsilon_h}(x_0/\varepsilon_h)} a(x, Dw_{h,\rho}^\xi + \xi) dx,$$

where $w_{h,\rho}^\xi$ is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(x, Dw_{h,\rho}^\xi(x) + \xi\right)\right) &= 0 \quad \text{on } Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right), \\ w_{h,\rho}^\xi &\in H_0^{1,p}\left(Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right)\right). \end{aligned} \right\}$$

By Theorem 3.4 we conclude that

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, Q_\rho(x_0)) = \rho^n b(\xi)$$

for every $\rho > 0$. Hence, the limit

$$\lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, Q_\rho(x_0))}{\rho^n} = b(\xi)$$

is independent on ρ . This implies by Theorem 2.3 that (a_h) G-converges to b and concludes the proof. \square

4. HOMOGENIZATION OF ALMOST PERIODIC OPERATORS

In this section we prove the homogenization theorem for general almost periodic monotone operators defined by functions of the class $M_{\mathbf{R}^n}$.

DEFINITION 4.1. — A function $f \in L^1_{loc}(\mathbf{R}^n)$ is *almost periodic* (in the sense of Besicovitch [3]) if there exists a sequence of trigonometric polynomials $P_h: \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\lim_{h \rightarrow \infty} \int |P_h(x) - f(x)| dx = 0. \quad (4.1)$$

It is easy to see that for every almost periodic function f and for every $z \in \mathbf{R}^n$ we have

$$\int f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx.$$

The limit is called the *mean value* of f (on \mathbf{R}^n) and is, in general, not uniform with respect to z .

THEOREM 4.2. — Let $a \in M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for all $\xi \in \mathbf{R}^n$ and let (ε_h) be a sequence of positive real numbers converging to 0. Let us define $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every $x \in \Omega$ and $\xi \in \mathbf{R}^n$. Denote by v_s^ξ the solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(0), \\ v_s^\xi &\in H_0^{1,p}(Q_s(0)). \end{aligned} \right\} \quad (4.2)$$

Then, for every $\xi \in \mathbf{R}^n$ there exists the limit

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(0)} a(x, Dv_s^\xi + \xi) dx. \quad (4.3)$$

Moreover, the map b belongs to $M(\alpha, \beta, c_1, c_2)$ and (a_h) G-converges to b .

The proof of this theorem follows from the homogenization Theorem 3.5 for quasiperiodic functions, by means of an approximation result and a closure lemma, which are stated below.

With a slight change of notation we will write

$$b(x, \xi) = G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right)$$

meaning that $b(x, \xi)$ is the G-limit of $a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every sequence (ε_h) which tends to 0^+ .

The following lemma states that homogenization is preserved under passage to the limit in the mean value.

LEMMA 4.3. — *Let (a_h) be a sequence of functions in $M_{\mathbf{R}^n}$, such that for every $h \in \mathbf{N}$ the limit*

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi) \quad (4.4)$$

exists and is independent of x , and let $a \in M_{\mathbf{R}^n}$ such that for every $R > 0$

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \quad (4.5)$$

Then, the limit

$$b(\xi) = \lim_{h \rightarrow \infty} b_h(\xi)$$

exists and

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi).$$

The proof of Theorem 4.2 will be completed by the following approximation result.

LEMMA 4.4. — *Let a be a function of the class $M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for every $\xi \in \mathbf{R}^n$. Then, there exists a sequence (a_h) in $M_{\mathbf{R}^n}$ of quasiperiodic functions satisfying (3.3) (with λ possibly depending on h) such that for every $R \geq 0$ we have*

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \quad (4.6)$$

Throughout this section the letter c will denote a positive constant depending at most on $p, n, c_1, c_2, \alpha, \beta$, and possibly on a fixed vector $\xi \in \mathbf{R}^n$. Its value can vary from line to line.

We begin by proving Lemma 4.4.

Proof of Lemma 4.4.

Step 1 (discretization of the function a on bounded sets). — Let (γ_h) be a sequence of natural numbers, and (μ_h) a sequence of positive real numbers. For every $h \in \mathbf{N}$, let us set

$$I_h = \{z = (z_1, \dots, z_n) \in \mathbf{Z}^n : -h\gamma_h \leq z_j < h\gamma_h \text{ for all } j = 1, \dots, n\}.$$

For every $z \in I_h$, let us define

$$z_z^h = \frac{1}{\gamma_h} z, \quad Q_z^h = z_z^h + \left[0, \frac{1}{\gamma_h}\right]^n, \quad a_z^h(x) = a(x, \xi_{z_z^h}),$$

and let us choose a trigonometric polynomial $P_z^h: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that

$$\int |P_z^h(x) - a_z^h(x)| dx \leq \frac{1}{\mu_h}. \quad (4.7)$$

The choice of the sequences $(\gamma_h) \subset \mathbf{N}$ and $(\mu_h) \subset \mathbf{R}$ will be made in the sequel. Then, taking into account that

$$\bigcup_{z \in I_h} Q_z^h = [-h, h]^n,$$

we define the function $\alpha_h: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ in the following way:

$$\alpha_h(x, \xi) = \begin{cases} P_z^h(x) & \text{if there exists } z \in I_h \text{ such that } \xi \in Q_z^h, \\ 0 & \text{if } \xi \notin [-h, h]^n. \end{cases}$$

Then, for every $\xi \in \mathbf{R}^n$, the function $\alpha_h(\cdot, \xi)$ is quasiperiodic.

Step 2 (projection of the function α_h on $M_{\mathbf{R}^n}$). — For every $h \in \mathbf{N}$, let us define the function $f_h: [0, +\infty[\rightarrow [0, +\infty[$ by

$$f_h(\rho) = \min(1, e^{h!(h-\rho)}).$$

We also define the Hilbert space L_h^2 of measurable functions $u: \mathbf{R}^n \rightarrow \mathbf{R}^n$, provided with the norm

$$\|u\|_h = \left(\int_{\mathbf{R}^n} |u(\xi)|^2 f_h^2(|\xi|) d\xi \right)^{1/2}.$$

The set $M(\alpha, \beta, c_1, c_2)$ is a closed convex subset of L_h^2 . We can define the projection $\pi_h: L_h^2 \rightarrow M(\alpha, \beta, c_1, c_2)$, and for every $x \in \mathbf{R}^n$ the function

$$a_h(x, \cdot) = \pi_h(\alpha_h(x, \cdot)). \quad (4.8)$$

Step 3 (quasiperiodicity of the function a_h). — Let us consider for every $y \in \mathbf{R}^n$, $N = N(h) \in \mathbf{N}$, the function $\tilde{a}_h(y, \cdot): \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by

$$\tilde{a}_h(y, \cdot) = \pi_h(\tilde{\alpha}_h(y, \cdot))$$

where, for every $\xi \in \mathbf{R}^n$, $\tilde{\alpha}_h(\cdot, \xi)$ is the periodic function given by the quasiperiodicity of $\alpha_h(\cdot, \xi)$. It follows clearly that

$$a_h(x, \xi) = \tilde{a}_h(j(x), \xi),$$

where j is the function related to the quasiperiodicity of a_h as in Section 3. Since the function $\tilde{\alpha}_h(\cdot, \xi)$ is a periodic function, such is also $\tilde{a}_h(\cdot, \xi)$. Moreover, being $\tilde{a}_h(y, \cdot)$ uniformly continuous with respect to ξ , it remains only to prove that $\tilde{a}_h(\cdot, \xi)$ is continuous for every $\xi \in \mathbf{R}^n$.

Let $\xi, \eta \in \mathbf{R}^n$; for all $u, v \in M(\alpha, \beta, c_1, c_2)$ we have

$$|u(\xi) - v(\xi)| \leq 2c_1(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha + |u(\eta) - v(\eta)|.$$

Let $0 < \delta \leq 1$; then an integration over $B_\delta(\xi)$ gives

$$\begin{aligned}
 |u(\xi) - v(\xi)| B_\delta(\xi) &\leq 2c_1(1 + 2|\xi| + \delta)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\
 &\quad + \int_{B_\delta(\xi)} |u(\eta) - v(\eta)| d\eta \\
 &\leq c(1 + |\xi|)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\
 &\quad + \left(\int_{B_\delta(\xi)} |u(\eta) - v(\eta)|^2 f_h^2(|\eta|) d\eta \right)^{1/2} \left(\int_{B_\delta(\xi)} (f_h(|\eta|))^{-2} d\eta \right)^{1/2},
 \end{aligned}$$

so that we obtain

$$|u(\xi) - v(\xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right].$$

In particular, since π_h is a Lipschitz function with constant 1 in L_h^2 , for all $u, v \in L_h^2$ and for all $\xi \in \mathbf{R}^n$ we obtain

$$|\pi_h u(\xi) - \pi_h v(\xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right]. \tag{4.9}$$

Let us remark that the function $\tilde{\alpha}_h = \tilde{\alpha}_h(y, \xi)$ is uniformly continuous in y , uniformly with respect to ξ . Fixed $\varepsilon > 0$, let $\rho > 0$ such that if $|\tau| < \rho$, then

$$|\tilde{\alpha}_h(y + \tau, \xi) - \tilde{\alpha}_h(y, \xi)| < \varepsilon$$

for all $y \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$. Then, for such a τ we have

$$\begin{aligned}
 \|\tilde{\alpha}_h(y + \tau, \cdot) - \tilde{\alpha}_h(y, \cdot)\|_h &\leq \left(\int_{|y-\eta| \leq h} |\tilde{\alpha}_h(y + \tau, \eta) - \tilde{\alpha}_h(y, \eta)|^2 d\eta \right)^{1/2} \leq \varepsilon (2h)^{n/2},
 \end{aligned}$$

so that, by (4.9) [with $u(\xi) = \tilde{\alpha}_h(y + \tau, \xi)$ and $v(\xi) = \tilde{\alpha}_h(y, \xi)$]

$$|\tilde{a}_h(y + \tau, \xi) - \tilde{a}_h(y, \xi)| \leq c \left[(1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\varepsilon (2h)^{n/2}}{\delta^{n/2} f_h(|\xi| + 1)} \right] \tag{4.10}$$

for all $\xi \in \mathbf{R}^n$. Since ε and δ can be chosen independently arbitrarily small, (4.10) proves that $\tilde{a}_h(\cdot, \xi)$ is uniformly continuous.

Step 4 [proof of (4.6)]. – Fixed $R \geq 0$, let us consider for every $x \in \mathbf{R}^n$ and $h \in \mathbf{N}$ the function

$$g_h(x, R) = \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)|. \tag{4.11}$$

Since we are interested in the limit as h tends to $+\infty$, we will suppose $h > R + 1$, so that $f_h(|\xi|) = 1$ for $|\xi| \leq R$.

Let us remark that $a(x, \cdot) \in M(\alpha, \beta, c_1, c_2)$ for almost every $x \in \mathbf{R}^n$, since $a \in M_{\mathbf{R}^n}$, so that

$$a(x, \xi) = \pi_h(a(x, \xi))$$

for a. e. $x \in \mathbf{R}^n$, for every $\xi \in \mathbf{R}^n$, and for every $h \in \mathbf{N}$. By (4.9), we have for $|\xi| \leq \mathbf{R}$ and $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| = |\pi_h(\alpha_h(x, \xi)) - \pi_h(a(x, \xi))| \leq c[(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + \delta^{-n/2} \|\alpha_h(x, \cdot) - a(x, \cdot)\|_h]. \quad (4.12)$$

Thus, we have to estimate

$$\|\alpha_h(x, \cdot) - a(x, \cdot)\|_h^2 \leq \int_{[-h, h]^{n_1}} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta + \int_{\{|\eta| > h\}} |\alpha_h(x, \eta) - a(x, \eta)|^2 f_h^2(|\eta|) d\eta \equiv J_h^1 + J_h^2. \quad (4.13)$$

By (1.3) we have

$$J_h^2 \leq c \int_{\{|\eta| > h\}} |\eta|^{2p-2} f_h^2(|\eta|) d\eta \leq c \int_h^{+\infty} \rho^{2p+n-3} f_h^2(\rho) d\rho, \quad (4.14)$$

so that $\lim_{h \rightarrow \infty} J_h^2 = 0$. By the definition of α_h , (1.3), and the Hölder inequality we obtain

$$\begin{aligned} J_h^1 &= \sum_{z \in I_h} \int_{Q_z^h} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta = \sum_{z \in I_h} \int_{Q_z^h} |P_z^h(x) - a(x, \eta)|^2 d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a(x, \xi_z^h)|^2 + |a(x, \xi_z^h) - a(x, \eta)|^2) d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a_z^h(x)|^2 + [c_1(1 + |\xi_z^h| + |\eta|)^{p-1-\alpha} |\xi_z^h - \eta|^\alpha]^2) d\eta \\ &\leq \sum_{z \in I_h} 2 \left(\frac{1}{\gamma_h} \right)^n |P_z^h(x) - a_z^h(x)|^2 + ch^{n+2p-2-2\alpha} \left(\frac{1}{\gamma_h} \right)^{2\alpha}. \end{aligned} \quad (4.15)$$

Taking (4.12)-(4.15) into account, we obtain for $|\xi| \leq \mathbf{R}$ and $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| \leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + c\delta^{-n/2} (h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} |P_z^h(x) - a_z^h(x)| + (J_h^2)^{1/2});$$

an integration gives

$$\begin{aligned} \int g_h(x, \mathbf{R}) dx &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c\delta^{-n/2} \left(h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} \int |P_z^h - a_z^h| dx + (J_h^2)^{1/2} \right) \\ &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c\delta^{-n/2} \left(h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + h^n \frac{(\gamma_h)^{n/2}}{\mu_h} + (J_h^2)^{1/2} \right). \end{aligned} \quad (4.16)$$

Now, we can choose the sequences (γ_h) and (μ_h) in such a way that

$$\lim_{h \rightarrow \infty} h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} = \lim_{h \rightarrow \infty} h^n \frac{(\gamma_h)^{n/2}}{\mu_h} = 0$$

(for example, $\gamma_h = h!$ and $\mu_h = h^{nh}$). Then, (4.16) gives

$$\limsup_{h \rightarrow \infty} \int g_h(x, R) dx \leq c(1+R)^{p-1-\alpha} \delta^\alpha.$$

By the arbitrariness of δ , the proof is completed. \square

Proof of Lemma 4.3. – By the representation Theorem 2.3 it is enough to prove that for every $\xi \in \mathbf{R}^n$ and for every cube Q in \mathbf{R}^n the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \tag{4.17}$$

exists and is independent of Q , where v_ε is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon &\in H_0^{1,p}(Q). \end{aligned} \right\} \tag{4.18}$$

Given a cube Q in \mathbf{R}^n and $\xi \in \mathbf{R}^n$ let us prove that

$$\lim_{h \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx = 0, \tag{4.19}$$

where v_ε^h is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon^h &\in H_0^{1,p}(Q). \end{aligned} \right\} \tag{4.20}$$

For the sake of simplicity we drop in the notation any explicit dependence on ξ of v_ε and v_ε^h throughout this section. We can write

$$\begin{aligned} &\int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \\ &\leq \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \\ &\quad + \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \equiv J_{\varepsilon,h}^1 + J_{\varepsilon,h}^2. \end{aligned}$$

The estimate of $J_{\varepsilon,h}^1$ and $J_{\varepsilon,h}^2$ will be carried out in the following four steps.

Step 1. – Let us fix $\xi \in \mathbf{R}^n$. As in (3.17) one gets

$$\| Dv_\varepsilon^h + \xi \|_{(L^p(Q))^n} \leq c |Q| (1 + |\xi|^p).$$

This estimate will be used frequently in the sequel. By the Meyers estimate (1.6) we have then

$$\|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n} \leq c.$$

Let us fix $R \geq 0$, $h \in \mathbb{N}$ and $\varepsilon > 0$, and let us define the set

$$U_R = \{x \in Q : |Dv_\varepsilon^h(x) + \xi| > R\}.$$

We have then

$$|U_R| R^p \leq \int_{U_R} |Dv_\varepsilon^h + \xi|^p dx,$$

so that, using the Hölder inequality, we obtain the estimate

$$\begin{aligned} \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx &\leq |U_R| + |U_R|^{\eta/(p+\eta)} \|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n}^p \\ &\leq c(R^{-p} + R^{-p\eta/(p+\eta)}). \end{aligned} \quad (4.21)$$

Step 2 (estimate of $J_{\varepsilon, h}^2$). — Given $R > 0$ by (1.3) and (4.12) we have

$$\begin{aligned} (J_{\varepsilon, h}^2)^q &\leq c \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &= c \int_{U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\quad + c \int_{Q \setminus U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c \int_{Q \setminus U_R} g_h\left(\frac{x}{\varepsilon}, R\right)^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c(1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx. \end{aligned}$$

By Step 1 we conclude then

$$J_{\varepsilon, h}^2 \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{1/q}. \quad (4.22)$$

Step 3 (estimate of $\|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}$). — By (4.18) and (4.20) we have

$$\int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx = 0,$$

so that by (iii) and (iv) in Definition 1.1 and the Hölder inequality we have

$$0 = \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx$$

$$\begin{aligned}
& + \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx \\
& \geq c_2 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-\beta} |Dv_\varepsilon - Dv_\varepsilon^h|^\beta dx \\
& - \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| |Dv_\varepsilon - Dv_\varepsilon^h| dx \\
& \geq c (|Q|^{1/p} + \|Dv_\varepsilon + \xi\|_{(L^p(Q))^n} \\
& \quad + \|Dv_\varepsilon^h + \xi\|_{(L^p(Q))^n})^{p-\beta} \|Dv_\varepsilon - Dv_\varepsilon^h\|_{(L^p(Q))^n}^\beta \\
& - \left(\int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \right)^{1/q} \\
& \quad \times \left(\int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{1/p},
\end{aligned}$$

so that by (4.22) we obtain

$$\|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}^\beta \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right). \quad (4.23)$$

Step 4 (estimate of $J_{\varepsilon, h}^1$). — By the equicontinuity condition of a and Hölder's inequality we have

$$\begin{aligned}
J_{\varepsilon, h}^1 & = \left| \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right) dx \right| \\
& \leq c_1 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-1-\alpha} |Dv_\varepsilon - Dv_\varepsilon^h|^\alpha dx \\
& \leq c_1 |Q|^{1/p} \left(\int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^p dx \right)^{(p-1-\alpha)/p} \\
& \quad \times \left(\int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{\alpha/p}.
\end{aligned}$$

Hence, by (4.23) we get

$$J_{\varepsilon, h}^1 \leq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{\alpha/(\beta-1)}. \quad (4.24)$$

Taking into account that there exists $s > 0$ such that $Q \subset Q_s(0)$ and

$$\begin{aligned}
\int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx & \leq \int_{Q_s(0)} g_h\left(\frac{x}{\varepsilon}, R\right) dx \\
& \leq \varepsilon^n \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx \leq \frac{c}{|Q_{s/\varepsilon}(0)|} \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx,
\end{aligned}$$

by passing in (4.22) and (4.24) first to the limit as ϵ tends to 0^+ , then as h tends to $+\infty$, and eventually as R tends to $+\infty$ we obtain (4.19). By (4.4) and Theorem 2.3 this implies that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\epsilon}, Dv_\epsilon + \xi\right) dx \\ = \lim_{h \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a_h\left(\frac{x}{\epsilon}, Dv_\epsilon^h + \xi\right) dx \\ = \lim_{h \rightarrow \infty} b_h(\xi) = b(\xi). \end{aligned} \tag{4.25}$$

Since the first limit does not depend on Q , the proof of Lemma 4.3 can be concluded by applying again the representation Theorem 2.3. \square

Proof of Theorem 4.2. — Let $a \in M_{\mathbf{R}^n}$ such that $a(\cdot, \xi)$ is almost periodic for every $\xi \in \mathbf{R}^n$. By Lemma 4.4 there exists a sequence (a_h) in $M_{\mathbf{R}^n}$ satisfying (3.3) (with λ possibly depending on h) such that condition (4.5) is satisfied. By the homogenization Theorem 3.5 we obtain that for every $h \in \mathbf{N}$ the limit

$$G-\lim_{\epsilon \rightarrow 0^+} a_h\left(\frac{x}{\epsilon}, \xi\right) = b_h(\xi)$$

exists and is independent of x . Hence, by Lemma 4.3 the limit

$$G-\lim_{\epsilon \rightarrow 0^+} a\left(\frac{x}{\epsilon}, \xi\right) = b(\xi)$$

exists. Finally, the representation formula follows from (4.25). \square

APPENDIX

A. Birkhoff's theorem for quasiperiodic functions

Before we give a direct proof of Birkhoff's Ergodic theorem in the quasiperiodic case, we state Kronecker's lemma (see, for instance, [30] 3.1).

KRONECKER'S LEMMA. — *The set of vectors which are equivalent modulo T to vectors of the form $j(x)$, with $x \in \mathbf{R}^n$, is dense in \mathbf{R}^N .*

THEOREM A. — *Let $f: \mathbf{R}^n \rightarrow \mathbf{R}$ be a quasiperiodic function in $QP(\lambda)$ (see Section 3). Then*

$$\int f(x) dx = \frac{1}{|T|} \int_T F(y) dy,$$

where T is defined by (3.2).

Proof. — Let $X = j(\mathbf{R}^n)$ and $Z = X^\perp$. Let us identify $\mathbf{R}^N = X \oplus Z$. Moreover, let us introduce

$$P_s^N(0) = j(Q_s(0)) \times B_s^Z(0),$$

where $B_s^Z(0)$ is the ball in Z of radius s and center 0 , and define the set S by $S = \{z \in B_s^Z(0) : z \text{ is equivalent modulo } T \text{ to vectors of the type } j(\tau) \text{ with } \tau \in \mathbf{R}^n\}$. By Kronecker's lemma the set S is dense in $B_s^Z(0)$. Then, given $z \in S$ there exists $\tau = \tau(z) \in \mathbf{R}^n$ such that

$$F(j(x) + z) = f(x + \tau)$$

for every $x \in \mathbf{R}^n$. Since the limit

$$\lim_{s \rightarrow +\infty} \frac{1}{s^n} \int_{Q_s(\tau)} f(x) dx = \int f(x) dx$$

exists uniformly with respect to τ (see [30] 2. 3), given $\varepsilon > 0$ we have

$$\left| \frac{1}{s^n} \int_{Q_s(0)} f(x + \tau) dx - \int f(x) dx \right| < \varepsilon$$

for every $\tau \in \mathbf{R}^n$, for s sufficiently large, so that for $z \in S$

$$\left| \frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) dx - \int f(x) dx \right| < \varepsilon \tag{A. 1}$$

for s sufficiently large. By the uniform continuity of F and the density of S , the estimate (A. 1) holds for every $z \in B_s^Z(0)$. By Fubini's theorem we have

$$\begin{aligned} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy &= \frac{1}{s^n} \int_{Q_s(0)} \left(\frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} F(j(x) + z) dz \right) dx \\ &= \frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} \left(\frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) dx \right) dz, \end{aligned}$$

which by (A. 1) yields that

$$\left| \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy - \int f(x) dx \right| < \varepsilon.$$

This implies together with the periodicity of F that

$$\frac{1}{|T|} \int_T F(y) dy = \lim_{s \rightarrow +\infty} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) dy = \int f(x) dx$$

and concludes the proof. \square

B. Regularity of the quasiperiodic solutions of $\Delta u = \text{div } f$

Remark B.1. — Let \mathcal{W} be the completion of $\text{Trig}_0(\mathbb{T})$ with respect to the norm $\|\cdot\|_{\mathcal{W}}$ introduced in Section 3. Let us denote by \mathcal{W}' its dual. It turns out that for every $S \in \mathcal{W}'$ there exists $G \in (L^q(\mathbb{T}))^n$ such that

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}. \tag{B.1}$$

In fact, since by the map ∂ the space \mathcal{W} is isometric to a subspace of $(L^p(\mathbb{T}))^n$, by the Hahn-Banach and Riesz theorems there exists a function $G \in (L^q(\mathbb{T}))^n$ such that (B.1) and

$$\|S\|_{\mathcal{W}'} = \left(\frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} |G_r|^q dy \right)^{1/q}$$

hold. We will write $\|S\|_{\mathcal{W}'} = \left\| \sum_{r=1}^n \partial_r G_r \right\|_{\mathcal{W}'}$.

THEOREM B. — *Let $F \in (\text{Trig}(\mathbb{T}))^n$ and let $W \in \text{Trig}_0(\mathbb{T})$ be the unique solution to*

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r F_r \quad \text{in } \mathbf{R}^N. \tag{B.2}$$

Then,

$$\left(\frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} |\partial_r W|^q dy \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r F_r \right\|_{\mathcal{W}'}, \tag{B.3}$$

with $c = c(\mathbb{T}) > 0$ and $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$ as defined in Section 3.

Proof. — Let us define $S \in \mathcal{W}'$ by

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} F_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}.$$

By Remark B.1 there exists $G \in (L^q(\mathbb{T}))^n$ such that

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}$$

and

$$\|G\|_{(L^q(\mathbb{T}))^n} = \|S\|_{\mathcal{W}'}. \tag{B.4}$$

By extending G on \mathbf{R}^N by periodicity we have that

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r G_r \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{B.5}$$

Let us introduce on \mathbf{R}^N a new orthogonal base $(E_l)_{1 \leq l \leq N}$ with

$$\begin{aligned} E_1 &= (\underbrace{1, \dots, 1}_{m_1}, 0, \dots, 0) \\ &\vdots \\ E_n &= (0, \dots, 0, \underbrace{1, \dots, 1}_{m_n}). \end{aligned}$$

We shall denote the new coordinates by

$$(x, z) = (x_1, \dots, x_n, z_1, \dots, z_{N-n}) \in \mathbf{R}^N.$$

It turns out that

$$D_{x_l} u(x, z) = \partial_l u(y) \quad \text{for every } u \in \mathcal{W}, \quad \text{for every } l = 1, \dots, n,$$

where $y = (x, z)$. Hence by (B.5) the function $W \in \mathcal{C}^\infty(\mathbf{R}^n \times \mathbf{R}^{N-n})$ satisfies for almost every $z \in \mathbf{R}^{N-n}$

$$\Delta_x W(\cdot, z) = \text{div}_x G(\cdot, z) \quad \text{in } \mathcal{D}'(\mathbf{R}^n).$$

By elliptic regularity (see, for instance [14]) for every $A' \subset\subset A \subset\subset \mathbf{R}^n$ there exists a constant $c = c(A', A)$, such that

$$\int_{A'} |D_x W(x, z)|^q dx \leq c \int_A |G(x, z)|^q dx.$$

By Fubini's theorem we get

$$\int_B \int_{A'} |D_x W(x, z)|^q dx dz \leq c \int_B \int_A |G(x, z)|^q dx dz$$

for every $B \subset\subset \mathbf{R}^{N-n}$. Choosing A', A and B such that

$$T \subset B \times A' \subset B \times A \subset \prod_{i,r} \left] -\frac{2\pi}{\lambda_r^i}, \frac{4\pi}{\lambda_r^i} \right],$$

and using finally the periodicity of G , we get

$$\int_T |D_x W(x, z)|^q dx dz \leq c \int_T |G(x, z)|^q dx dz.$$

By (B.4) we obtain (B.3), and we can conclude the proof. \square

REFERENCES

- [1] N. S. BAKHVALOV and G. P. PANASENKO, *Averaged Processes in Periodic Media*, Nauka, Moscow, 1984.
- [2] A. BENSOUSSAN, J. L. LIONS and G. C. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.
- [3] A. BESICOVITCH, *Almost Periodic Functions*, Cambridge, 1932.
- [4] L. BOCCARDO and Th. GALLOUET, Homogenization with Jumping Nonlinearities, *Ann. Mat. Pura Appl.*, (4), Vol. **138**, 1984, pp. 211-221.
- [5] L. BOCCARDO and F. MURAT, *Homogénéisation de problèmes quasi linéaires*, Studio di problemi-limite della Analisi Funzionale (Bressanone, 1081), 13-51, Pitagora Editrice, Bologna, 1982.
- [6] L. BOCCARDO and F. MURAT, Remarques sur l'homogénéisation de certains problèmes quasi-linéaires, *Portugal. Math.*, Vol. **41**, 1982, pp. 535-562.
- [7] A. BRAIDES, Omogeneizzazione di integrali non coercivi, *Ricerche Math.*, Vol. **32**, 1983, pp. 347-368.
- [8] A. BRAIDES, Homogenization of Some Almost Periodic Coercive Functionals, *Rend. Accad. Naz. XL*, Vol. **9**, 1985, pp. 313-322.
- [9] A. BRAIDES, A Homogenization Theorem for Weakly Almost Periodic Functionals, *Rend. Accad. Naz. XL*, Vol. **10**, 1986, pp. 261-281.
- [10] A. BRAIDES, Correctors for the Homogenization of Almost Periodic Monotone Operators, *Asympt. Anal.*, Vol. **5**, 1991, pp. 47-74.
- [11] A. BRAIDES, Almost Periodic Methods in the Theory of Homogenization, *Applicable Anal.* (to appear).
- [12] E. CABIB and C. DAVINI, *On the Variational Convergence with a Null Divergence Condition*, Preprint, Università di Udine, 1989.
- [13] E. CABIB, C. DAVINI and C. Q. RU, A Problem in the Optimal Design of Networks Under Transverse Loading, *Quart. Appl. Math.*, Vol. **48**, 1990, pp. 251-263.
- [14] S. CAMPANATO, *Sistemi ellittici in forma divergenza. Regolarità all'interno*. Quaderni Scuola Normale Sup., Pisa, 1980.
- [15] L. CARBONE and C. SBORDONE, Some Properties of Γ -Limits of Integral Functionals, *Ann. Mat. Pura Appl.*, (4), Vol. **122**, 1979, pp. 1-60.
- [16] V. CHIADÒ PIAT, G. DAL MASO and A. DEFRANCESCHI, G-Convergence of Monotone Operators, *Ann. Inst. H. Poincaré. Anal. Non Linéaire*, Vol. **7**, 1990, pp. 123-160.
- [17] V. CHIADÒ PIAT and A. DEFRANCESCHI, Homogenization of Monotone Operators, *Nonlinear Anal.*, Vol. **14**, 1990, pp. 717-732.
- [18] G. DAL MASO and A. DEFRANCESCHI, Correctors for the Homogenization of Monotone Operators, *Diff. Integral Eq.*, Vol. **3**, 1990, pp. 1151-1166.
- [19] G. DAL MASO and L. MODICA, Integral Functionals Determined by their Minima, *Rend. Sem. Mat. Univ. Padova*, Vol. **76**, 1986, pp. 255-267.
- [20] R. DE ARCANGELIS, A General Homogenization Result for Almost Periodic Functionals, *J. Math. Anal. Appl.*, Vol. **156**, (2), 1991, pp. 358-380.
- [21] E. DE GIORGI and S. SPAGNOLO, Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine, *Boll. Un. Mat. Ital.*, (4), Vol. **8**, 1973, pp. 391-411.
- [22] T. DEL VECCHIO, On the Homogenization of a Class of Pseudomonotone Operators in Divergence Form, *Boll. Un. Mat. Ital.*, Vol. **5-B**, 1991, pp. 369-388.
- [23] J. FRANČU, Homogenization and Correctors for Nonlinear Elliptic Equations, *Comment. Math. Univ. Carolin.* (to appear).
- [24] N. FUSCO and G. MOSCARIELLO, L^2 -Lower Semicontinuity of Functionals of Quadratic Type, *Ann. Mat. Pura Appl.*, (4), Vol. **129**, 1981, pp. 306-326.

- [25] N. FUSCO and G. MOSCARIELLO, Further Results on the Homogenization of Quasilinear Operators, *Ricerche Mat.*, Vol. **35**, 1986, pp. 231-246.
- [26] N. FUSCO and G. MOSCARIELLO, On the Homogenization of Quasilinear Divergence Structure Operators, *Ann. Mat. Pura Appl.*, Vol. **146**, 1987, pp. 1-13.
- [27] D. KINDERLEHRER and S. STAMPACCHIA, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.
- [28] S. M. KOZLOV, Averaging Differential Operators with Almost-Periodic Rapidly Oscillating Coefficients, *Math. USSR-Sb.*, Vol. **35**, 1979, pp. 481-498.
- [29] S. M. KOZLOV, Reducibility of Quasiperiodic Differential Operators and Averaging, *Trans. Moscow Math. Soc.*, Vol. **2**, 1984, pp. 101-126.
- [30] B. M. LEVITAN and V. V. ZHIKOV, *Almost Periodic Functions and Differential Equations*, Cambridge Univ. Press, Cambridge, 1982.
- [31] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.
- [32] P. MARCELLINI, Periodic Solutions and Homogenization of Non Linear Variational Problems, *Ann. Mat. Pura Appl.*, (4), Vol. **117**, 1978, pp. 139-152.
- [33] N. G. MEYERS and A. ELCRAT, Of Non-Linear Elliptic Systems and Quasi-Regular Functions, *Duke Math. J.*, Vol. **42**, 1975, pp. 121-136.
- [34] F. MURAT, *H-convergence*, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, 1977.
- [35] F. MURAT, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (4), Vol. **5**, 1978, pp. 489-507.
- [36] O. A. OLEINIK and V. V. ZHIKOV, On the Homogenization of Elliptic Operators with Almost-Periodic Coefficients, *Rend. Sem. Mat. Fis. Milano*, Vol. **52**, 1982, pp. 149-166.
- [37] U. E. RAITUM, On the G-Convergence of Quasilinear Elliptic Operators with Unbounded Coefficients, *Soviet Math. Dokl.*, Vol. **24**, 1981, pp. 472-476.
- [38] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
- [39] E. SANCHEZ-PALENCIA, Non Homogeneous Media and Vibration Theory, *Lect. Notes Phys.*, 127, Springer-Verlag, Berlin, 1980.
- [40] S. SPAGNOLO, Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (3), Vol. **21**, 1967, pp. 657-699.
- [41] S. SPAGNOLO, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (3), Vol. **22**, 1968, pp. 577-597.
- [42] S. SPAGNOLO, Convergence in Energy for Elliptic Operators, *Proc. Third Symp. Numer. Solut. Partial Diff. Equat.*, College Park, 1975, pp. 469-498, Academic Press, New York, 1976.
- [43] L. TARTAR, *Cours Peccot*, Collège de France, Paris, 1977.
- [44] L. TARTAR, Quelques remarques sur l'homogénéisation, *Proc. of the Japan-France seminar 1976 "Functional analysis and numerical analysis"*, pp. 469-482, Japan Society for the Promotion of Science, 1978.
- [45] L. TARTAR, Compensated Compactness and Partial Differential Equations, *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, Vol. **IV**, pp. 136-212, Pitman, London, 1979.
- [46] L. TARTAR, *Homogénéisation d'opérateurs monotones*, Manuscript, 1981.
- [47] V. V. ZHIKOV, S. M. KOZLOV, O. A. OLEINIK and KHA T'EN NGOAN, Averaging and G-Convergence of Differential Operators, *Russian Math. Surveys*, Vol. **34**, 1979, pp. 69-147.

(Manuscript received September 29, 1990;
revised October 15, 1991.)