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Boundary regularity for solutions of the equation of prescribed Gauss curvature

by

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ABSTRACT. — We study the boundary regularity of convex solutions of the equation of prescribed Gauss curvature in a domain $\Omega \subset \mathbb{R}^n$ in the case that the gradient of the solution is infinite on some relatively open, uniformly convex portion Γ of $\partial\Omega$. Under suitable conditions on the data we show that near $\Gamma \times \mathbb{R}$ the graph of u is a smooth hypersurface (as a submanifold of $\mathbb{R}^{n+1}$) and that $u|_\Gamma$ is smooth. In particular, u is Hölder continuous with exponent 1/2 near Γ.

Key words : Gauss curvature, boundary regularity, free boundary, Legendre transform.

1. INTRODUCTION

In this paper we study the boundary regularity of convex solutions of the equation of prescribed Gauss curvature

$$\det D^2 u = K(x)(1 + |Du|^2)^{(n+2)/2} \tag{1.1}$$

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on a domain $\Omega\subset\mathbb{R}^n$ in the case that

$$(1.2)\qquad |Du|=\infty \quad \text{on } \Gamma$$

for some relatively open, uniformly convex portion Γ of $\partial\Omega$. The boundary condition (1.2) is to be interpreted as meaning

$$(1.3)\qquad \lim_{\substack{x\to y\\ x\in\Omega}} |Du(x)|=\infty$$

for all $y\in\Gamma$. This situation arises in a number of problems connected with equation (1.1), and in fact, proving that (1.2) holds on a suitable portion Γ of $\partial\Omega$ is a key step in the proof of the interior regularity of solutions of (1.1) in the papers [17], [18], [19].

Before stating our results, we recall the problems in which the above situation arises. Let us suppose that Ω is a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$ and $K\in C^{1,1}(\Omega)$ is a positive function. It is easily shown that the condition

$$(1.4)\qquad \int_\Omega K\leqq\omega_n,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$, is necessary for the existence of a convex solution of (1.1). In order to solve the Dirichlet problem for (1.1) with arbitrary boundary data φ of class $C^{1,1}$, it is known from the papers [3], [15], [17], [19] that two conditions are necessary. Namely, we require

$$(1.5)\qquad \int_\Omega K<\omega_n$$

to ensure the validity of an *a priori* bound for the solution u, and in addition

$$(1.6)\qquad K(x)\leqq C\operatorname{dist}(x, \partial\Omega)$$

in order to obtain a boundary gradient estimate. As shown in [8], this condition is not necessary if instead we impose some restrictions on $\|\varphi\|_{C^{1,1}(\partial\Omega)}$. If (1.5) holds, but not (1.6), then the Dirichlet problem may not be solvable in the classical sense (*see* [15], [19]). However, it is shown in [18] that there is a unique convex solution of (1.1) which solves the Dirichlet problem in a certain optimal sense: the solution u is the infimum of all convex supersolutions of (1.1) lying above φ on $\partial\Omega$, or alternatively, u is the supremum of all convex subsolutions of (1.1) lying below φ on $\partial\Omega$. The Dirichlet boundary condition is generally satisfied in the classical sense on some parts of $\partial\Omega$ and not at other parts. At points at which it is not satisfied we have $|Du|=\infty$, provided K satisfies a mild growth restriction near $\partial\Omega$, for example $K\in L^p(\Omega)$ for some $p>n$. In fact, $K\in L^{(n+1)/2}(\Omega)$

is sufficient, by an examination of the proof of Lemma 3.6 of [18] (*see* also [21], Lemma 2.1) and Remark (v) following the proof of Lemma 2.2.

In the extremal case

$$\int_{\Omega} K = \omega_n \tag{1.7}$$

it is shown in [17], [19] that there is a convex solution of (1.1) which is unique up to additive constants, and furthermore, if $K \in L^p(\Omega)$ for some $p > n$ (as above $K \in L^{(n+1)/2}(\Omega)$ suffices), then $|Du| = \infty$ on $\partial\Omega$.

At present the only theorems which yield any information about the regularity of u near Γ are the Hölder estimates of [19]. These are valid much more generally however, and their proof makes no use of the boundary condition (1.2). It is reasonable therefore to expect that better boundary regularity can be obtained by using (1.2).

Our results are in fact valid for more general Gauss curvature equations of the form

$$\det D^2 u = K(x, u, \nu)(1 + |Du|^2)^{(n+2)/2} \tag{1.8}$$

where ν denotes the downward unit normal to the graph of u given by

$$\nu = \frac{(Du, -1)}{(1 + |Du|^2)^{1/2}} \tag{1.9}$$

and $K \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \bar{S}^n_-)$ is a positive function. Here S^n_- denotes the lower hemisphere of the n-dimensional unit sphere $S^n \subset \mathbb{R}^{n+1}$. The situations described above for equation (1.1) also arise for equation (1.8) if we impose a number of hypotheses on K (*see* [18], Theorem 1.1, and [19], Theorem 4.10).

Our main result is the following. As usual, B_R denotes the open ball in $\mathbb{R}^n$ with radius R and centre 0.

THEOREM 1. — *Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that for some $x_0 \in \partial\Omega$, which we may take to be the origin, $\Gamma = \partial\Omega \cap B_{R_0}$ is a connected, $C^{2,1}$, uniformly convex portion of $\partial\Omega$. Suppose that $K \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \bar{S}^n_-)$ is a positive function and $u \in C^2(\Omega)$ is a convex solution of (1.8) satisfying (1.2) and such that*

$$\mu_0 \leqq K(x, u, \nu) \leqq \mu_1 \tag{1.10}$$

and

$$|DK(x, u, \nu)| + |D^2 K(x, u, \nu)| \leqq \mu_2 \tag{1.11}$$

where μ_0, μ_1, μ_2 are positive constants. Then there is a number $\rho \in (0, R_0)$, depending only on n, R_0, Γ, μ_0, μ_1 and μ_2, such that the following hold:

(i) $u \in C^{0,1/2}(\overline{\Omega \cap B_\rho})$, and for any $x, y \in \overline{\Omega \cap B_\rho}$ we have

$$|u(x) - u(y)| \leqq C_1 |x - y|^{1/2} \tag{1.12}$$

where C_1 *depends only on* n, R_0, Γ, μ_0, μ_1 *and* μ_2.

(ii) graph $[u|_{\overline{\Omega \cap B_\rho}})$ is a $C^{2,\alpha}$ *hypersurface for any* $\alpha<1$, *and we have*

$$\|\nu\|_{C^{1,\alpha}(((\overline{\Omega \cap B_\rho}) \times \mathbb{R}) \cap \operatorname{graph} u)} \leq C_2 \tag{1.13}$$

where C_2 *depends only on* n, R_0, Γ, μ_0, μ_1, μ_2 *and* α.

(iii) $u|_{\overline{\Gamma \cap B_\rho}}$ is of class $C^{1,\alpha}$ *for any* $\alpha<1$, *and we have*

$$\|u\|_{C^{1,\alpha}(\overline{\Gamma \cap B_\rho})} \leq C_3 \tag{1.14}$$

where C_3 *depends only on* n, R_0, Γ, μ_0, μ_1, μ_2, α *and* $\inf_\Omega u$.

(iv) *If in addition to the above hypotheses we have*

$$K \in C^{k-1,\alpha}(\bar{\Omega} \times \mathbb{R} \times \bar{S}^n_-)$$

and $\Gamma \in C^{k+1,\alpha}$ *for some integer* $k \geq 2$ *and some* $\alpha \in (0,1)$, *then* graph $(u|_{\overline{\Omega \cap B_\rho}})$ *is a* $C^{k+1,\alpha}$ *hypersurface,* $u|_{\overline{\Gamma \cap B_\rho}}$ *is of class* $C^{k,\alpha}$ *and we have*

$$\|\nu\|_{C^{k,\alpha}(((\overline{\Omega \cap B_\rho}) \times \mathbb{R}) \cap \operatorname{graph} u)} \leq C_4 \tag{1.15}$$

and

$$\|u\|_{C^{k,\alpha}(\overline{\Gamma \cap B_\rho})} \leq C_5, \tag{1.16}$$

where C_4 *and* C_5 *depend only on* n, R_0, Γ, μ_0, μ_1, μ_2, K, k, α *and* $\inf_\Omega u$.

If K *and* Γ *are analytic, then* graph $(u|_{\overline{\Omega \cap B_\rho}})$ *and* $u|_{\overline{\Gamma \cap B_\rho}}$ *are analytic.*

If the hypotheses of Theorem 1 are satisfied with $\Gamma = \partial\Omega$, then we obtain the conclusions of Theorem 1 in $\{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) < \rho\}$ for a suitable $\rho > 0$. We may then apply the interior regularity result [20], Theorem 1.1, to extend the estimates to all of $\bar{\Omega}$. We state this special case separately.

THEOREM 2. – *Suppose* Ω *is a* $C^{2,1}$ *uniformly convex domain in* $\mathbb{R}^n$, $K \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \bar{S}^n_-)$ *is a positive function and* u *is a convex solution of* (1.8) *such that* (1.10) *and* (1.11) *hold, and*

$$|Du| = \infty \quad \text{on } \partial\Omega. \tag{1.17}$$

Then the assertions (i) *to* (iv) *of Theorem* 1 *hold with* $\Omega \cap B_\rho$ *replaced by* Ω *and* Γ *by* $\partial\Omega$. *The dependence of the constants* $C_1, \ldots, C_5$ *on* R_0, Γ *should now be replaced by* Ω.

We shall prove Theorem 1 in the remaining sections of the paper. In Section 2 we shall derive some preliminary estimates for graph u near Γ. We also reformulate our problem as a free boundary problem for a function w, obtained by expressing graph u locally as the graph of a function w over a suitable subdomain of the tangent hyperplane to graph u at $(0, u(0))$. In Section 3 we show how the information obtained in Section 2 can be used to deduce appropriate regularity assertions for the Legendre transform of w, from which the assertions of Theorem 1 follow.

As will become clear later, some of the hypotheses of Theorems 1 and 2 may be weakened slightly. First, u may be assumed to be a generalized solution of (1.8) in the sense of Aleksandrov [2] (*see* also [18]) rather than a classical solution. However, under the conditions of Theorem 1, u is a classical solution on $\Omega \cap B_\rho$ for some $\rho \in (0, R_0)$ depending only on n, R_0, Γ, μ_0 and μ_1 [*see* Remark (ii) following the proof of Lemma 2.2]. Second, the boundary condition (1.2) may be weakened to a measure theoretic condition [*see* (2.20)] which arises quite naturally. In fact, it is this condition which is first shown to be satisfied in the papers [17], [18], [19]; the stronger condition (1.2) is then deduced by arguments similar to those used in the proof of Lemma 2.2.

Analogous boundary regularity questions for solutions of the nonparametric least area problem were studied by Simon [14]. His results were improved and also extended to the mean curvature equation in situations corresponding to those mentioned above for the Gauss curvature equation by Lau and Lin [11] and Lin [12], [13]. Our approach is similar to [11], [12], [13] in that we also reduce the problem to a free boundary problem. In our case, however, the free boundary problem is not *a priori* uniformly elliptic, and considerably more preliminary information is required before we can apply standard results.

It will be evident that the techniques of Section 2 are also applicable to certain other Monge-Ampère equations of the form

$$\det D^2 u = f(x, u, Du), \tag{1.18}$$

for which the boundary condition (1.2) arises in a similar fashion as for the Gauss curvature equation, provided we impose suitable conditions on the function f. In particular, sufficiently fast growth of f with respect to Du is necessary. Furthermore, in the case of equation (1.8) we may also allow some unboundedness or decay to zero of K as x approaches Γ. We shall make some remarks about these extensions later. In these situations, however, we obtain a free boundary problem which in general is necessarily nonuniformly elliptic or degenerate, and the techniques of Section 3 do not yield any further regularity. Nevertheless, even in these cases it is perhaps reasonable to expect somewhat better regularity of $u|_\Gamma$ than is provided by the Hölder estimates of [19].

For Monge-Ampère equations of the general form (1.18) the boundary condition (1.2) also arises if f grows fast enough as x approaches $\partial\Omega$, even without imposing sufficiently rapid growth on f with respect to Du. However, we cannot expect regularity assertions for $u|_\Gamma$ such as those of Theorem 1. For as Cheng and Yau [5] have shown, if Ω is a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$ and $f \in C^{1,1}(\Omega)$ satisfies

$$A \operatorname{dist}(x, \partial\Omega)^{-\alpha} \leqq f(x) \leqq B \operatorname{dist}(x, \partial\Omega)^{-\beta} \tag{1.19}$$

for some constants A, $B>0$, $\alpha>1$ and $\beta<n+1$, then the Dirichlet problem

$$(1.20)\qquad \begin{cases} \det D^2 u = f(x) & \text{in } \Omega, \\ u=\varphi & \text{on } \partial\Omega, \end{cases}$$

has a unique convex solution $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ for any $C^{1,1}$ boundary data φ, and we have $|Du|=\infty$ on $\partial\Omega$. Similar assertions are also valid for Monge-Ampère equations of the general from (1.18), where $f\in C^{1,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ satisfies $f_z\geqq 0$ and

$$(1.21)\qquad A\operatorname{dist}(x,\partial\Omega)^{-\alpha}\leqq f(x,z,p)\leqq B\operatorname{dist}(x,\partial\Omega)^{-\beta}(1+|p|)^{\gamma}$$

for all $(x,z,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$, where A, B, α, β, γ are constants such that A, $B>0$, $\alpha>1$, $0\leqq\gamma<n$ and $1<\beta<n+1-\gamma$ (*see* [16], Theorem 2.19).

2. ESTIMATES FOR GRAPH *u*

The main results of this section are a Hölder estimate for the normal vector field ν to graph u and a strict convexity estimate for graph u, both of these being valid at any point $(x_0, u(x_0))$ with $x_0\in\Gamma\cap B_{R_0/2}$. These results hold under weaker regularity hypotheses than stated in Theorem 1; namely, we need only assume Γ to be uniformly convex and of class $C^{1,1}$, and the bound (1.11) is not needed.

We begin with some preliminary estimates for u. From [19], Corollary 3.11, we have the following preliminary regularity result.

LEMMA 2.1. – *There is a number* $\alpha\in(0,1/2]$, *depending only on n, such that* $u\in C^{0,\alpha}(\overline{\Omega\cap B_{R_0/2}})$, *we have*

$$(2.1)\qquad |u(x)-u(y)|\leqq C|x-y|^{\alpha}$$

where C *depends only on n*, R_0, Γ *and* μ_0.

Remarks. – The estimate (2.1) does not depend on the boundary condition (1.2); it uses only the $C^{1,1}$ regularity of Γ and the lower bound of (1.10), and is independent of any boundary condition. The proof of (2.1) given in [19] yields the exponent $\alpha=1/2n$, but this can be increased slightly by an iteration argument described in [19]. The optimal exponent is not known at present, but for our purposes any positive exponent suffices.

From Lemma 2.1 we see that by replacing R_0 by $R_0/2$ we may assume from the start that $u\in C^{0,\alpha}(\overline{\Omega\cap B_{R_0}})$. It is also convenient to replace Ω by $\Omega\cap B_{R_0}$. This involves no loss of generality and has the advantage that Ω is convex, which makes the proof of the following lemma technically simpler. It is a refinement of the argument used in [17], [18], [19] to show that $|Du|=\infty$ on a suitable portion of $\partial\Omega$.

LEMMA 2.2. — *There are numbers* ε_1, $C>0$, *depending only on* n, R_0, Γ *and* μ_1, *such that for each* $x \in B_{R_0/2} \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon_1\}$ *we have*

$$|Du(x)| \geqq C \operatorname{dist}(x, \partial\Omega)^{-1/2}. \tag{2.2}$$

Proof. — Choose coordinates so that the positive x_n axis points in the direction of the inner normal to $\partial\Omega$ at 0. It suffices to prove (2.2) for $x=(0, \varepsilon^2)$ with $\varepsilon \leqq \varepsilon_1 = \varepsilon_1(n, R_0, \Gamma, \mu_1)$, because a similar argument, with R_0 replaced by $R_0/2$ and $\varepsilon_1/2$ if necessary, will yield the estimate at any point of $B_{R_0/2} \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon_1\}$.

Since Γ is uniformly convex, there is an enclosing ball $B_R(z_0)$ at 0 *i.e.* $B_R(z_0) \cap \partial\Omega = \{0\}$ with the radius R estimated from above by a constant depending only on Γ. Let $B_{2R}(z_1)$ be another enclosing ball at 0, and for $\varepsilon > 0$ consider the set

$$U = U_\varepsilon = \Omega \cap \{x : x_n \leqq \varepsilon^2 + \varepsilon |x'|\}$$

where $x' = (x_1, \ldots, x_{n-1})$. It is clear that for $\varepsilon > 0$ sufficiently small, say $\varepsilon \leqq \varepsilon_0 = \varepsilon_0(\Gamma)$, we have $\operatorname{dist}(\xi_0, \partial\Omega) = \varepsilon^2$ where $\xi_0 = (0, \varepsilon^2)$, and furthermore, letting $\Sigma = \partial B_{2R}(z_1 + \xi_0)$,

$$\Sigma \cap B_{2\theta\varepsilon}(\xi_0) \subset \Sigma \cap \Omega \tag{2.3}$$

and

$$U \subset \Omega \cap (B^{n-1}_{\Lambda_1 \varepsilon}(0) \times (0, \Lambda_2 \varepsilon^2)) \subset \Omega \cap B_{R_0/2} \tag{2.4}$$

for some positive constants θ, Λ_1 and Λ_2 depending only on Γ and R_0. Here $B^{n-1}_R(0)$ denotes the open ball in $\mathbb{R}^{n-1}$ with radius R and centre 0.

Let

$$v(x) = u(x) - u(\xi_0) - Du(\xi_0).(x - \xi_0),$$

so that $v(\xi_0) = 0$ and $v \geqq 0$ in Ω. For $t \geqq 0$ let

$$\Sigma_t = \{x \in \Omega : v(x) = t\} \cup \{x \in \partial\Omega : v(x) \leqq t\}.$$

Then Σ_t is a closed, convex, $(n-1)$-dimensional hypersurface in $\bar{\Omega}$ and ξ_0 is enclosed by Σ_t. Let G_t and G be the generalized Gauss maps of Σ_t and Σ respectively. The meaning of G is clear, since Σ is smooth, while for any $\xi \in \Sigma_t$, $G_t(\xi)$ is defined to be the set of outward unit normals of supporting hyperplanes of Σ_t at ξ, and for any set $E \subset \Sigma_t$,

$$G_t(E) = \bigcup_{\xi \in E} G_t(\xi).$$

Let $\mathscr{C} = \{x : x_n = \varepsilon^2 + \varepsilon |x'|\}$ and let $\tilde{G}$ be the Gauss map of $\mathscr{C}$. Since each Σ_t encloses ξ_0, it is evident that if T is any $(n-1)$-dimensional supporting plane of $\mathscr{C}$ at ξ_0, T can be translated in the negative x_n direction to yield a parallel $(n-1)$-dimensional supporting plane T_t of Σ_t at some point of $\Sigma_t \cap \bar{U}$. Thus

$$\begin{aligned} G_t(\Sigma_t \cap \bar{U}) &\supset \tilde{G}(\mathscr{C}) \\ &\supset G(\Sigma \cap B_{\theta\varepsilon}(\xi_0)) \end{aligned} \tag{2.5}$$

by virtue of (2.3) and an elementary geometric argument.

To proceed further we need to define some set functions. Let

$$\text{(2.6)} \qquad M = \text{boundary of} \{ (x, t) \in \Omega \times \mathbb{R} : v(x) < t \}.$$

Since Ω is convex, M is a complete, convex hypersurface in $\mathbb{R}^{n+1}$. As in [18], we define

$$\text{(2.7)} \quad \tilde{\chi}_v(y) = \{ p \in \mathbb{R}^n : \exists \text{ a supporting hyperplane of M at } (y, v(y)) \text{ with slope } p \}.$$

For any set $E \subset \bar{\Omega}$ we define

$$\text{(2.8)} \qquad \begin{cases} \tilde{\chi}_v(E) = \bigcup_{y \in E} \tilde{\chi}_v(y), \\ \chi_v(E) = \tilde{\chi}_v(E \cap \Omega), \\ \chi_v^*(E) = \tilde{\chi}_v(E \cap \partial\Omega). \end{cases}$$

A result of Aleksandrov [1] (*see* also [5]) asserts that if A, B are disjoint subsets of $\bar{\Omega}$, then $\tilde{\chi}_v(A) \cap \tilde{\chi}_v(B)$ has measure zero. Furthermore, if E is a Borel set, then so are $\tilde{\chi}_v(E)$, $\chi_v(E)$ and $\chi_v^*(E)$ (*see* [18]).

Continuing now with the proof of the lemma, we see from (2.5) that

$$\text{(2.9)} \qquad \tilde{\chi}_v(\bar{U}) \supset \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Sigma \cap B_{\theta\varepsilon}(\xi_0)) \right\},$$

because each nonhorizontal n-dimensional supporting plane T of M yields an $(n-1)$-dimensional supporting plane $T_t = T \cap \{ x : x_{n+1} = t \}$ of Σ_t for each $t \geqq 0$. From (2.8) we have

$$\text{(2.10)} \qquad \tilde{\chi}_v(\bar{U}) = \chi_v(\bar{U}) \cup \chi_v^*(\bar{U} \cap \partial\Omega).$$

However, by (2.4) the second set on the right hand side of (2.10) is contained in $\chi_v^*(\Gamma)$ and so is empty, since $|Du| = \infty$ on Γ. Thus from (2.9) we obtain

$$\text{(2.11)} \qquad \chi_v(\bar{U}) \supset \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Sigma \cap B_{\theta\varepsilon}(\xi_0)) \right\},$$

or recalling the definition of v,

$$\text{(2.12)} \qquad \chi_u(\bar{U}) \supset p_0 + \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Sigma \cap B_{\theta\varepsilon}(\xi_0)) \right\},$$

where $p_0 = Du(x_0)$ and χ_u is defined as above with v replaced by u.

Next, using the fact that u is a convex solution of (1.8), we have

$$\text{(2.13)} \qquad \int_U K = \int_{\chi_u(\bar{U})} h(|p|)\, dp$$

where $h(t)=(1+t^2)^{-(n+2)/2}$. Defining

$$\mathscr{C}_\delta=\left\{p\in\mathbb{R}^n:\frac{p}{|p|}\in G(\Sigma\cap B_{\theta\varepsilon}(\xi_0))\right\}$$

for any $\delta>0$, using (2.12), and estimating the left hand side of (2.13) from above, we obtain

$$\begin{aligned}(2.14)\qquad \mu_1\,\omega_{n-1}(\Lambda_1\varepsilon)^{n-1}\Lambda_2\varepsilon^2 &\geqq \int_{p_0+\mathscr{C}_{\theta\varepsilon}} h(|p|)\,dp\\ &\geqq \int_{p_0+\mathscr{C}_{\theta\varepsilon}} h(|p_0|+|p-p_0|)\,dp\\ &= \int_{\mathscr{C}_{\theta\varepsilon}} h(|p_0|+|p|)\,dp\\ &\geqq C_1\,\varepsilon^{n-1}\int_{1+|p_0|}^{\infty} r^{-3}\,dr\\ &= \frac{C_2\,\varepsilon^{n-1}}{(1+|p_0|)^2}\end{aligned}$$

for suitable constants C_1 and C_2. It follows that

$$(2.15)\qquad |p_0|\geqq C_3\,\varepsilon^{-1}-1\geqq\frac{1}{2}C_3\,\varepsilon^{-1}$$

for ε small enough, say $\varepsilon\leqq\varepsilon_1\leqq\varepsilon_0$. Recalling now that $p_0=Du(\xi_0)$ and $\xi_0=(0,\varepsilon^2)$, we see that we have proved

$$(2.16)\qquad |Du(\xi_0)|\geqq C_4\,\mathrm{dist}(\xi_0,\partial\Omega)^{-1/2},$$

since $\varepsilon^2=\mathrm{dist}(\xi_0,\partial\Omega)$. The proof of the lemma is complete.

Remarks. — (i) The proof of Lemma 2.2 does not require u to be a classical solution of (1.8). All we require is that u is a generalized solution in the sense that

$$(2.17)\qquad \int_{\chi_u(E)}(1+|p|^2)^{-(n+2)/2}\,dp=\int_E K(x,u,\nu)$$

for any Borel set $E\subset\Omega$. Furthermore, if u is not differentiable at x, then in the statement of the lemma we need only replace $Du(x)$ by the slope of any supporting hyperplane of graph u at $(x,u(x))$.

(ii) If u is a generalized solution of (1.8) and the hypotheses of Theorem 1 are satisfied, then u is in fact a classical solution on $\Omega\cap B_\rho$ for some positive $\rho>0$. For if $x_0\in\Omega\cap B_\rho$, then for ρ small enough we have

$$(2.18)\qquad |p_0|\geqq C\,\rho^{-1/2}$$

where p_0 is the slope of any supporting hyperplane T of graph u at $(x_0, u(x_0))$. We assert that $T \cap$ graph u cannot contain a line segment with an endpoint on $\partial\Omega \times \mathbb{R}$. Clearly we cannot have such a line segment with an endpoint on $\Gamma \times \mathbb{R}$, by virtue of the boundary condition (1.2), while if there were such a segment L with an endpoint on $(\partial\Omega - \Gamma) \times \mathbb{R}$, then by the uniform convexity of Γ there would be a point $(x_1, u(x_1)) \in L$ with $\operatorname{dist}(x_1, \partial\Omega) \geqq \delta_0$ for some positive constant δ_0 depending only on R_0 and Γ. But then, since T is also a supporting hyperplane of graph u at $(x_1, u(x_1))$, we have

$$|p_0| \leqq \delta_0^{-1} \operatorname{osc}_{\Omega} u \leqq C \delta_0^{-1}, \tag{2.19}$$

since u is convex. This contradicts (2.18) if ρ is small enough, so our assertion is proved. The C^2 regularity of u on $\Omega \cap B_\rho$ now follows in a standard way, *see* for example [5], [20].

(iii) In the proof of Lemma 2.2 the boundary condition (1.2) was used only to assert that the set $\chi_v^*(\bar{U} \cap \partial\Omega)$ in (2.10) is empty. All that is required for the proof is that this set have measure zero. Thus the boundary condition (1.2) could be replaced by the weaker measure theoretic condition

$$|\chi_u^*(\Gamma)| = 0. \tag{2.20}$$

(iv) Refinements of Lemma 2.2 are possible. For example if we have

$$K(x, u, \nu) \leqq \tilde{K}(x) \tag{2.21}$$

where $K \in L^q(\Omega)$ for some $q > (n+1)/2$, then in place of (2.14) we obtain

$$C_1 \| \tilde{K} \|_{L^q(\Omega)} \varepsilon^{(n+1)(1-1/q)} \geqq \frac{C_2 \varepsilon^{n-1}}{(1+|p_0|)^2}, \tag{2.22}$$

which leads to the estimate

$$|Du(\xi_0)| \geqq C \operatorname{dist}(\xi_0, \partial\Omega)^{-(1-((n+1)/2q))/2}. \tag{2.23}$$

Similarly, if

$$K(x, u, \nu) \leqq C \operatorname{dist}(x, \partial\Omega)^{\beta} \tag{2.24}$$

for some constant $\beta > -1$, we obtain

$$|Du(\xi_0)| \geqq C \operatorname{dist}(\xi_0, \partial\Omega)^{-(1+\beta)/2}. \tag{2.25}$$

(v) From the proof of Lemma 2.2 and (iv) it is evident that the interior regularity result of [18], Theorem 1.1, is valid under the slightly weaker growth restriction $\tilde{g} \in L^q(\mathcal{N})$ with $q \geqq (n+1)/2$ rather than $q > n$. One can similarly improve the interior regularity assertions of [19], Theorems 4.8 and 4.10, and [21], Theorems 1 and 2, in the case that $\partial\Omega \in C^{1,1}$. Improvements along these lines are possible even if $\partial\Omega \notin C^{1,1}$, but we shall not pursue these questions here.

(vi) Minor modifications of the proof of Lemma 2.2, specifically in (2.13) and (2.14), lead to similar results for more general Monge-Ampère equations, for example, equations of the form (1.18) where

$$f(x, z, p) \leqq g(x)(1+|p|)^{\alpha} \tag{2.26}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $g \in L^{\infty}(\Omega)$ and $\alpha > n$ is a constant. Arguing as above we obtain

$$|Du(\xi_0)| \geqq C \operatorname{dist}(\xi_0, \partial\Omega)^{1/(n-\alpha)}. \tag{2.27}$$

Further modifications along the lines of (iv) are also possible, but we shall not state these.

Since $\Gamma \in C^{1,1}$, the distance function d defined by

$$d(x) = \operatorname{dist}(x, \partial\Omega)$$

is of class $C^{1,1}$ near $\Gamma \cap B_{R_0/2}$, say for $x \in B_{R_0/2} \cap \{x \in \Omega : d(x) < \sigma_1\}$ for some $\sigma_1 > 0$ depending only on Γ (*see* [6], Lemma 14.16). By replacing ε_1 from Lemma 2.2 by a smaller constant if necessary, we may assume that $\varepsilon_1 \leqq \sigma_1$. We may extend μ, the outer unit normal vectorfield to Γ, to $B_{R_0/2} \cap \{x \in \Omega : d(x) < \sigma_1\}$ by setting

$$\mu(x) = -Dd(x).$$

The tangential gradient $\delta u = (\delta_1 u, \ldots, \delta_n u)$ of u relative to Γ is defined by

$$\delta_i u = (\delta_{ij} - \mu_i \mu_j) D_j u$$

for all points of $B_{R_0/2} \cap \{x \in \Omega : d(x) < \sigma_1\}$.

Using Lemmas 2.1 and 2.2 we now obtain the following.

LEMMA 2.3. — *There is a number* $\beta > 0$, *depending only on n, such that*

$$|\delta u|^{1+\beta} \leqq C |Du| \tag{2.28}$$

on $B_{R_0/2} \cap \{x \in \Omega : d(x) < \varepsilon_1\}$, *where* C *depends only on n*, R_0, Γ, μ_0 *and* μ_1.

Proof. — From [19], Lemma 3.8, and its refinement to the case $u \in C^{0,\alpha}(\bar{\Omega})$ (*see* [19], equation (3.37)), we have

$$|\delta u(x)| \leqq C\, dist(x, \partial\Omega)^{(\alpha-1)/2} \tag{2.29}$$

for all $x \in B_{R_0/2} \cap \{x \in \Omega : d(x) < \varepsilon_1\}$, where $\alpha > 0$ is the exponent given by Lemma 2.1 and C depends only on n, R_0, Γ, μ_0 and μ_1. The estimate (2.28) with $\beta = \alpha/(1-\alpha)$ now follows by combining (2.29) with Lemma 2.2.

Using Lemmas 2.1 to 2.3 we can obtain a Hölder estimate for the normal vectorfield to graph u and a strict convexity estimate for graph u at any point $(x_0, u(x_0))$ with $x_0 \in \Gamma \cap B_{R_0/2}$. First of all, we observe that since $|Du| = \infty$ on Γ and $\Gamma \in C^{1,1}$, the only supporting hyperplane of

$$M = \text{boundary of } \{(x, t) \in \Omega \times \mathbb{R} : u(x) < t\}$$

at a point $(x_0, u(x_0)) \in \Gamma \times \mathbb{R}$ is a vertical hyperplane tangent to $\Gamma \times \mathbb{R}$. Since M is a convex hypersurface, it follows that the unit normal vectorfield to M is continuous at any point $(x_0, u(x_0)) \in M$ with $x_0 \in \Gamma$. Thus the unit normal vectorfield to graph u given by (1.9) has a continuous extension (which we also denote by ν) to $\overline{\text{graph}\, u} \cap (\Gamma \times \mathbb{R})$, and

$$\nu(x) \equiv \nu(x, u(x)) = \mu(x) \tag{2.30}$$

for $x \in \Gamma$.

LEMMA 2.4. — *There is a number $\gamma > 0$, depending only on n, such that*

$$|\nu(x) - \nu(x_0)| \leqq C|x - x_0|^\gamma \tag{2.31}$$

for all $x_0 \in \Gamma \cap B_{R_0/2}$ *and* $x \in \overline{\Omega \cap B_{R_0/2}}$, *where* C *depends only on* n, R_0, Γ, μ_0 *and* μ_1.

Proof. — It clearly suffices to prove (2.31) for $x_0 = 0$ and $x \in B_{R_0/2} \cap \{x \in \Omega : d(x) < \varepsilon_1\}$; for other x_0 it can be obtained by a similar argument, while the case $d(x) \geqq \varepsilon_1$ is trivial.

Choose coordinates so that the positive x_n axis points in the direction of the inner normal to Γ at 0. Then $\nu = \mu$ on Γ, and in particular, $\nu(0) = -e_n$ where $e_1, \ldots, e_{n+1}$ denote the unit coordinate vectors in $\mathbb{R}^{n+1}$. If $x = (0, x_n) \in B_{R_0/2} \cap \{x \in \Omega : d(x) < \varepsilon_1\}$, then at x we have

$$\begin{aligned} |\nu(x) - \nu(0)| &= \left| (1 + |Du|^2)^{-1/2} \left(\sum_{k=1}^{n} D_k u e_k - e_{n+1} \right) + e_n \right| \\ &= \frac{1 + |\delta u|}{(1 + |Du|^2)^{1/2}} + 1 - \frac{D_\mu u}{(1 + |Du|^2)^{1/2}} \\ &\leqq \frac{1 + |\delta u|}{(1 + |Du|^2)^{1/2}} + \frac{1 + |\delta u|^2}{1 + |Du|^2} \\ &\leqq C \operatorname{dist}(x, \partial\Omega)^\gamma \\ &= C|x|^\gamma \end{aligned}$$

for some $\gamma = \gamma(n) > 0$, by virtue of Lemmas 2.2 and 2.3. In fact, we may take $\gamma = \beta/2(1 + \beta)$, where β is the constant of Lemma 2.3.

Next, if x is any point of $B_{R_0/2} \cap \{x \in \Omega : d(x) < \varepsilon_1\}$, the same argument as above shows that

$$|\nu(x) - \nu(\hat{x})| \leqq C|x - \hat{x}|^\gamma$$

where $\hat{x}$ is the point of Γ nearest to x. It follows that

$$\begin{aligned} |\nu(x) - \nu(0)| &\leqq |\nu(x) - \nu(\hat{x})| + |\nu(\hat{x}) - \nu(0)| \\ &\leqq C_1|x - \hat{x}|^\gamma + C_2|\hat{x}| \\ &\leqq C_3|x|^\gamma \end{aligned}$$

since $\nu=\mu$ on Γ, Γ is of class $C^{1,1}$, and

$$|\hat{x}|\leqq|x-\hat{x}|+|x|\leqq 2|x|.$$

The lemma is proved.

Next we come to the strict convexity estimate.

LEMMA 2.5. – *There are numbers*

$$\delta=\delta(n)\geqq 2 \qquad and \qquad \varepsilon_2=\varepsilon_2(n, R_0, \Gamma, \mu_0, \mu_1)>0$$

such that if T *is the unique tangent hyperplane to* $\overline{\text{graph } u}$ *at* $X_0=(x_0, u(x_0))$ *where* $x_0\in\Gamma\cap B_{R_0/2}$, *then for any point* $X=(x, u(x))\in(\text{graph } u)\cap(B_{\varepsilon_2}(x_0)\times\mathbb{R})$ *we have*

$$\text{(2.32)} \qquad \operatorname{dist}(X, T)\geqq C_0|\Pi(X-X_0)|^{\delta}$$

where Π *is the orthogonal projection onto* T *and* C_0 *depends only on* n, R_0, Γ, μ_0 *and* μ_1.

Proof. – As usual it suffices to prove this for $x_0=0$, the other cases being obtained by a similar argument. We suppose for convenience that $u(0)=0$, and let $\varepsilon_2, C_1, \ldots, C_{10}$ denote various positive constants depending on some or all of the quantities n, R_0, Γ, μ_0 and μ_1.

By Lemma 2.4 there is a number $\varepsilon_2>0$ such that $(\text{graph } u)\cap(B_{\varepsilon_2}(0)\times\mathbb{R})$ can be written as the graph of a function w defined on a suitable subdomain U of T obtained by projecting $(\text{graph } u)\cap(B_{\varepsilon_2}\times\mathbb{R})$ orthogonally onto T. In fact, if we choose new coordinates $y_\alpha=x_\alpha$ for $\alpha<n$, $y_n=-x_{n+1}$ and $y_{n+1}=x_n$, then by Lemma 2.4 we have

$$\text{(2.33)} \qquad U\supset D=D_R=\{y\in\mathbb{R}^n:|y|<R,\ y_n>\omega(y')\}$$

for some $R>0$ depending only on n, R_0, Γ, μ_0 and μ_1, where graph $\omega\subset T$ is the orthogonal projection of $\text{graph}(u|_\Gamma)$ onto T, and $y'=(y_1, \ldots, y_{n-1})$. We may also assume that

$$\text{(2.34)} \qquad \|w\|_{C^1(\bar{D})}\leqq C_1.$$

In addition, by Lemma 2.1 we have

$$\text{(2.35)} \qquad \|\omega\|_{C^{0,\alpha}(\overline{B_R^{n-1}})}\leqq C_2$$

where α is the exponent given in Lemma 2.1 and B_R^{n-1} denotes the open ball in $\mathbb{R}^{n-1}$ with radius R and centre 0.

We also represent $\Gamma\times\mathbb{R}$ near (0, 0) as the graph of a function $\psi:\{y\in\mathbb{R}^n:|y'|<R\}\to\mathbb{R}$. ψ is independent of y_n, and since $\Gamma\in C^{1,1}$ is uniformly convex we have

$$\text{(2.36)} \qquad C_3|y'|^2\leqq\psi(y', y_n)\leqq C_4|y'|^2$$

for all $y \in B_R$. Clearly we also have

$$w(y', y_n) \geqq \psi(y', y_n) = \psi(y', 0) \tag{2.37}$$

for all $(y', y_n) \in D$.

Let $y = (y', y_n) \in D$ and let $\hat{y}$ be the unique point on the lower boundary of D such that $y' = \hat{y}'$; thus

$$\hat{y}_n = \omega(y'), \tag{2.38}$$

$$|\hat{y}_n| \leqq C_2 |y'|^\alpha \tag{2.39}$$

and

$$w(\hat{y}) = \psi(\hat{y}). \tag{2.40}$$

Using Lemma 2.1 we have

$$w(y) \geqq \psi(y', \hat{y}_n) + C_5 |y_n - \hat{y}_n|^\lambda \tag{2.41}$$

where $\lambda = 1/\alpha$ and α is the exponent of Lemma 2.1. If $y_n \geqq 2 C_2 |y'|^\alpha$, then from (2.36), (2.39) and (2.41) we obtain

$$\begin{aligned} w(y) &\geqq C_3 |y'|^2 + C_6 (y_n^\lambda - |\hat{y}_n|^\lambda) \\ &\geqq C_3 |y'|^2 + \frac{1}{2} C_6 y_n^\lambda \\ &\geqq C_7 |y|^\lambda \end{aligned}$$

since $\lambda \geqq 2$, while if $|y_n| \leqq 2 C_2 |y'|^\alpha$ we have

$$\begin{aligned} w(y) &\geqq C_3 |y'|^2 \\ &\geqq \frac{1}{2} C_3 |y'|^2 + C_8 |y_n|^{2\lambda} \\ &\geqq C_9 |y|^{2\lambda}. \end{aligned}$$

Thus in any case we have

$$w(y) \geqq C_{10} |y|^{2\lambda} \tag{2.42}$$

for all $y \in D$. A similar estimate for all $y \in U$ follows by the convexity of w. The estimate (2.32) with $\delta = 2\lambda$ follows immediately from this, so the lemma is proved.

Remark. — Hölder continuity estimates for ν similar to (2.31) can also be obtained if we allow some decay to zero or growth to infinity of K as x approaches Γ. For example, if we have

$$\lambda \operatorname{dist}(x, \partial\Omega)^\theta \leqq K(x, u, \nu) \leqq \tilde{K}(x) \tag{2.43}$$

where $\tilde{K} \in L^q(\Omega)$ for some $q > n$, and $\lambda > 0$ and $\theta \in [0, 1)$ are constants, then by [19], Corollary 3.11, we have

$$|u(x) - u(y)| \leqq C |x - y|^{(1-\theta)/2n} \tag{2.44}$$

for all $x, y \in \Omega \cap B_{R_0/2}$, where C depends only on n, R_0, Γ, λ and θ. This leads to the tangential gradient estimate

$$(2.45)\qquad |\delta u(x)| \leqq C\,\mathrm{dist}(x, \partial\Omega)^{-(1-((1-\theta)/2n)/2}.$$

On the other hand, as we have already observed, the second inequality of (2.43) leads to the estimate (2.23). The combination of (2.23) and (2.45) then leads to a Hölder continuity estimate for ν of the form (2.31) with $\gamma > 0$ and $C > 0$ now depending in addition on θ and q, provided we have

$$(2.46)\qquad \frac{n+1}{q} < \frac{1-\theta}{n}.$$

Strict convexity estimates such as (2.31) may also be proved under these somewhat more general hypotheses. Further refinements of these results are also possible, for solutions of (1.8) as well as for solutions of more general Monge-Ampère equations, but we shall not pursue this here. However, since the exponent α obtained in Lemma 2.1 and its various analogues for other equations is not generally sharp, it is clear that the exponents obtained in Lemmas 2.3, 2.4 and 2.5, and conditions such as (2.46), are not optimal.

To conclude this section we make some further observations about the function w defined in the proof of Lemma 2.5. First of all, from Lemma 2.4 and the fact that $u \in C^2(\Omega)$, we have $w \in C^2(D) \cap C^1(\bar{D})$ and

$$(2.47)\qquad w = \psi, \qquad Dw = D\psi \quad \text{on } \Sigma,$$

where $\Sigma = \{y \in \partial D : y_n = \omega(y')\}$ is the lower boundary of D. In particular, since ψ is independent of y_n, we have

$$(2.48)\qquad D_n w = 0 \quad \text{on } \Sigma.$$

Furthermore, it is clear that w is a convex solution of an equation of the form

$$(2.49)\qquad \det D^2 w = \tilde{K}(y, w, \nu)(1 + |Dw|^2)^{(n+2)/2}$$

where $\tilde{K}$ is obtained from K by the relation

$$(2.50)\qquad \tilde{K}(y_1, \ldots, y_{n+1}, \nu_1, \ldots, \nu_{n+1}) = K(y_1, \ldots, y_{n-1}, y_{n+1}, -y_n, \nu_1, \ldots, \nu_{n-1}, \nu_{n+1}, -\nu_n).$$

Since (2.34) holds, there is no loss of generality in assuming that w in fact satisfies an equation of the form

$$(2.51)\qquad \det D^2 w = f(y, w, Dw) \quad \text{in } D$$

with $f \in C^{1,1}(\bar{D} \times \mathbb{R} \times \mathbb{R}^n)$ satisfying

$$(2.52)\qquad \tilde{\mu}_0 \leqq f(y, z, p) \leqq \tilde{\mu}_1,$$

$$(2.53)\qquad |Df(y, z, p)| + |D^2 f(y, z, p)| \leqq \tilde{\mu}_2$$

for almost all $(y, z, p) \in \bar{D} \times \mathbb{R} \times \mathbb{R}^n$, where $\tilde{\mu}_0$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ depend on μ_0, μ_1 and μ_2 in (1.10), (1.11) and in addition on n and $\sup_D |Dw|$.

We see therefore that w is a solution of a free boundary problem, and the regularity assertions of Theorem 1 are equivalent to suitable regularity assertions for the free boundary Σ and for w near Σ. From Lemmas 2.4 and 2.5 we see that for any $y_0 \in \Sigma$ and any $y \in \bar{D}$ we have the Hölder gradient estimate

$$|Dw(y) - Dw(y_0)| \leqq C |y - y_0|^\gamma \tag{2.54}$$

and the strict convexity estimate

$$w(y) - w(y_0) - Dw(y_0) \cdot (y - y_0) \geqq c_0 |y - y_0|^\delta \tag{2.55}$$

where $\gamma \in (0, 1)$ and $\delta \geqq 2$ depend only on n and C, c_0 depend in addition on R_0, Γ, μ_0 and μ_1. In particular, setting $y_0 = 0$ in (2.55) we obtain

$$w(y) \geqq c_0 |y|^\delta \tag{2.56}$$

for all $y \in \bar{D}$. Using (2.55) and the convexity of w we also obtain

$$|Dw(y) - Dw(y_0)| \geqq c_0 |y - y_0|^{\delta - 1} \tag{2.57}$$

for all $y_0 \in \Sigma$, $y \in \bar{D}$.

The free boundary problem obtained above is similar to those studied by Kinderlehrer and Nirenberg [9], and in fact, their results would imply higher regularity of Σ and w near Σ if we already knew that $\Sigma \in C^1$ and $w \in C^2(D \cup \Sigma)$. Of course, we do not know such regularity at this stage, but nevertheless, the method of [9] can still be used.

3. THE LEGENDRE TRANSFORM

To proceed further in the proof of Theorem 1 we adopt a technique used by Kinderlehrer and Nirenberg [9] and study the free boundary problem for the function w obtained at the end of Section 2 by studying its Legendre transform. In our case however, we may use the full Legendre transform rather than a partial Legendre transform as in [9].

Since $w \in C^2(D) \cap C^1(\bar{D})$ is convex and w solves (2.47), (2.51) with f positive, the mapping $\Psi : y \to Dw(y)$ defines a diffeomorphism of D onto an open subset D* of $\mathbb{R}^n_+ = \{ z \in \mathbb{R}^n : z_n > 0 \}$. In fact, since ψ is of class $C^{2,1}$ and uniformly convex with respect to y', we may assume that $\Psi \in C^1(D) \cap C^0(\bar{D})$ is a homeomorphism of $\bar{D}$ onto $\bar{D}^*$. Using the estimate (2.57) we easily see that for any $y_0 \in \Sigma \cap B_{R/2}$ and any $\varepsilon \in (0, R/4]$ we have

$$\Psi((\bar{D} \cap B_\varepsilon(y_0)) \supset \{ z \in \bar{\mathbb{R}}^n_+ : |z - z_0| \leqq C_1 \varepsilon^{\delta - 1} \} \tag{3.1}$$

and

$$\Psi((\Sigma \cap B_\varepsilon(y_0)) \supset \{ z \in \mathbb{R}^{n-1} : |z - z_0| \leqq C_1 \varepsilon^{\delta-1} \} \tag{3.2}$$

where $z_0 = \Psi(y_0) = Dw(y_0)$, δ is as in (2.57), and C_1 depends only on n, R_0, Γ, μ_0 and μ_1. Using the estimate (2.54) we see that for any $y_0 \in \Sigma \cap B_{R/2}$ and any $\varepsilon \in (0, R/4]$ we have

$$\Psi(\overline{D \cap B_\varepsilon(y_0)}) \subset \{ z \in \bar{\mathbb{R}}^n_+ : |z - z_0| \leqq C_2 \varepsilon^\gamma \} \tag{3.3}$$

and

$$\Psi(\overline{\Sigma \cap B_\varepsilon(y_0)}) \subset \{ z \in \bar{\mathbb{R}}^n_+ : |z - z_0| \leqq C_2 \varepsilon^\gamma \}, \tag{3.4}$$

or alternatively, for all $\sigma > 0$, $z_0 \in \mathbb{R}^{n-1}$ such that σ, $|z_0| \leqq \sigma_0$ we have

$$\Psi^{-1}(\overline{B^+_\sigma(z_0)}) \supset \{ y \in \bar{D} : |y - y_0| \leqq C_3 \sigma^{1/\gamma} \} \tag{3.5}$$

and

$$\Psi^{-1}(\mathbb{R}^{n-1} \cap \overline{B_\sigma(z_0)}) \supset \{ y \in \Sigma : |y - y_0| \leqq C_3 \sigma^{1/\gamma} \}, \tag{3.6}$$

where $y_0 = \Psi^{-1}(z_0)$, γ is as in (2.54), and σ_0, C_2, C_3 are positive constants depending only on n, R_0, Γ, μ_0 and μ_1. In fact, the estimates (3.2), (3.4) and (3.6) can be improved, since $w = \psi$, $Dw = D\psi$ on Σ, and ψ is of class $C^{2,1}$ and uniformly convex with respect to y'.

The Legendre transform of w is the function w^* on $D^* = Dw(D)$ given by

$$w^*(z) = \sum_{k=1}^{n} y_k D_k w(y) - w(y) \tag{3.7}$$

where $z = Dw(y)$. It is easily verified that

$$\frac{\partial w^*}{\partial z_i}(z) = y_i \tag{3.8}$$

and

$$\frac{\partial^2 w^*}{\partial z_i \partial z_j} = w^{ij}(y) \tag{3.9}$$

where $[w^{ij}] = [D^2 w]^{-1}$. Thus $D^2 w^*$ is positive definite in D^*, and we have $w^* \in C^2(D^*) \cap C^{0,1}(\bar{D}^*)$, $w^*(0) = 0$ and $w^* \geqq 0$ in D^* by the convexity of w and the fact that $w(0) = 0$. Since (2.47) holds and $\psi \in C^{2,1}$ is uniformly convex with respect to y', we see that we have

$$\| w^* \|_{C^{2,1}(\Sigma^*)} \leqq C_4 \tag{3.10}$$

and

$$A |z'|^2 \leqq w^*(z', 0) \leqq B |z'|^2 \tag{3.11}$$

for all $(z', 0) \in \Sigma^* = Dw(\Sigma) \subset \mathbb{R}^{n-1}$, where C_4, A, B are controlled positive constants.

Next, using the relations (3.8) and (3.9) it is easily verified that w^* solves the equation

$$\det D^2 w^* = f^*(z, w^*, Dw^*) \quad \text{in } D^* \tag{3.12}$$

where $f^* \in C^{1,1}(\bar{D}^* \times \mathbb{R} \times \mathbb{R}^n)$ is given by

$$f^*(z, t, q) = \frac{1}{f(q, z_k q_k - t, z)} \tag{3.13}$$

for any $(z, t, q) \in \bar{D}^* \times \mathbb{R} \times \mathbb{R}^n$. Thus by (2.52) and (2.53) we have

$$\bar{\mu}_0 \leqq f^*(z, t, q) \leqq \bar{\mu}_1 \tag{3.14}$$

and

$$|Df^*(z, t, q)| + |D^2 f^*(z, t, q)| \leqq \bar{\mu}_2 \tag{3.15}$$

for almost all $(z, t, q) \in \bar{D}^* \times \mathbb{R} \times \mathbb{R}^n$, for some controlled positive constants $\bar{\mu}_0$, $\bar{\mu}_1$ and $\bar{\mu}_2$.

It can be shown that the estimates (2.54) and (2.55) for w imply analogous estimates for w^* at each point of Σ^*. More precisely, from (2.55) we see that $w^* \in C^1(D^* \cup \Sigma^*)$, and, using (2.54) and (2.55), for any sufficiently small $z_0 \in \Sigma^*$ and any $z \in D^*$ we have

$$w^*(z) - w^*(z_0) - Dw^*(z_0).(z - z_0) \geqq C_5 |z - z_0|^{1 + 1/\gamma} \tag{3.16}$$

and

$$|Dw^*(z) - Dw^*(z_0)| \leqq C_6 |z - z_0|^{1/(\delta - 1)}, \tag{3.17}$$

where γ, δ are as in (2.54), (2.55) and C_5, C_6 are controlled positive constants. We shall not prove these estimates because they are not needed to complete the proof.

The following lemma now essentially completes the proof of Theorem 1.

LEMMA 3.1. – *Let $u \in C^2(B_R^+) \cap C^{0,1}(\bar{B}_R^+)$ be a convex solution of the equation*

$$\log \det D^2 u = g(x, u, Du) \quad \textit{in } B_R^+ \tag{3.18}$$

where

$$B_R^+ = \{ x \in \mathbb{R}^n : |x| < R, x_n > 0 \},$$

and suppose that $g \in C^{1,1}(\bar{B}_R^+ \times \mathbb{R} \times \mathbb{R}^n)$ satisfies

$$|g(x, z, p)| + |Dg(x, z, p)| + |D^2 g(x, z, p)| \leqq \beta_0 \tag{3.19}$$

for almost all $(x, z, p) \in B_R^+ \times \mathbb{R} \times \mathbb{R}^n$, where β_0 is a positive constant. Suppose also that

$$\|u\|_{C^{2,1}(T_R)} \leqq \beta_1 \tag{3.20}$$

and

$$D_{\alpha\beta} u(x', 0) \xi_\alpha \xi_\beta \geqq \lambda |\xi|^2 \tag{3.21}$$

for all $(x', 0) \in T_R = \bar{B}_R^+ \cap \mathbb{R}^{n-1}$ *and all* $\xi \in \mathbb{R}^{n-1}$, *where* β_1 *and* λ *are positive constants. Then there is a number* $\rho \in (0, R)$, *depending only on* n, R, β_0, β_1, λ *and* $\|u\|_{C^1(\bar{B}_R^+)}$, *such that for any* $\alpha \in (0, 1)$ *we have*

(3.22) $$\|u\|_{C^{2,\alpha}(\bar{B}_\rho^+)} \leqq C$$

where C *depends on the same quantities as* ρ *and in addition on* $\alpha \in (0, 1)$.

Proof. – If we already knew that $u \in C^{1,1}(\bar{B}_R^+)$, the estimate (3.22), with C depending in addition on $\sup_{B_R^+} |D^2 u|$, would be a consequence of well known global second derivative Hölder estimates for fully nonlinear, uniformly elliptic equations (*see* [6], Theorem 17.26, [10], Theorem 5.5.2). So it suffices to prove

(3.23) $$\sup_{B_\sigma^+} |D^2 u| \leqq C$$

for suitable positive constants $\sigma \in (0, R)$ and C depending only on n, R, β_0, β_1, λ and $\|u\|_{C^1(\bar{B}_R^+)}$.

Let $\tilde{u} = u + A x_n$ where $A > 0$ is fixed so large that $D_n \tilde{u} \geqq 1$ in B_R^+. Choose new coordinates $y_\alpha = x_\alpha$ for $\alpha < n$, $y_n = x_{n+1}$ and $y_{n+1} = -x_n$. Then graph $\tilde{u}$ can be expressed as the graph of a convex function v defined on a subset V of the hyperplane $y_{n+1} = 0$, so we have

$$V \supset U = \{ y \in \mathbb{R}^n : |y| < R, y_n > u(y', 0) \}.$$

Thus the lower boundary $\tilde{T}$ of U, given by $y_n = u(y', 0)$, $|y'| < R$, is uniformly convex and of class $C^{2,1}$, by virtue of (3.20) and (3.21). Furthermore, we clearly have $v|_{\tilde{T}} = 0$ and $v \in C^2(U) \cap C^{0,1}(\bar{U})$ satisfies an equation of the form

(3.24) $$\log \det D^2 v = \tilde{g}(y, v, Dv) \quad \text{in } U$$

where $\tilde{g} \in C^{1,1}(\bar{U} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies

(3.25) $$|\tilde{g}(y, z, p)| + |D\tilde{g}(y, z, p)| + |D^2 \tilde{g}(y, z, p)| \leqq \tilde{\beta}_0$$

for almost all $(y, z, p) \in U \times \mathbb{R} \times \mathbb{R}^n$, where $\tilde{\beta}_0$ depends only on n, β_0 and $\sup_{B_R^+} |Du|$.

Our aim now is to prove that

(3.26) $$\sup_{U \cap B_\rho} |D^2 v| \leqq C$$

for suitable positive constants ρ and C. Once we have this, (3.23) follows immediately and the lemma will be proved.

A bound for $D^2 v$ on $\partial U \cap B_{R/2}$ would follow from the results proved in [4], [6], [7], [15] if we knew that v were of class C^2 on $\bar{U}$. We do not know this yet, so we need to make our estimates on a suitable approximating sequence of solutions. First we note that by [20], Lemma 2.2, there is a constant $\rho_0 \leqq R/2$, depending only on n and $\tilde{\beta}_0$, such that if v_1, v_2 are two convex solutions of (3.24) in any domain $U' \subset U$ with diam $U' \leqq \rho_0$, then

$$(3.27) \qquad \sup_{U'} |v_1 - v_2| \leqq 2 \sup_{\partial U'} |v_1 - v_2|.$$

Let $\{\Omega_k\}$ be an increasing sequence of C^4 uniformly convex subdomains of $U \cap B_{\rho_0}$ with $\Omega_k \subset\subset U \cap B_{\rho_0}$, $\bigcup_{k=1}^{\infty} \Omega_k = U \cap B_{\rho_0}$, such that the principal curvatures of $\partial\Omega_k$ are bounded from below by some positive constant depending only on ρ_0 and U, but not on k, and such that the boundary portions $\partial\Omega_k \cap B_{\rho_1}$ are uniformly bounded in the $C^{2,1}$ sense. (Here for notational covenience we let $\rho_1 = \rho_0/2$, and in general $\rho_l = 2^{-l}\rho_0$.) It is clear that such a sequence of subdomains can be constructed by appropriately smoothing $U \cap B_{\rho_0}$. Next, let $\{\varphi_k\} \subset C^0(\overline{U \cap B_{\rho_0}})$ be a sequence of functions with $\varphi_k \in C^4(\bar{\Omega}_k)$ for each k, such that $\sup_{\Omega_k} |v - \varphi_k| \to 0$ and $\varphi_k = 0$ on $\partial\Omega_k \cap B_{\rho_1}$. It is clear that such a sequence can be constructed. For example, if v is extended to be zero outside $\bar{U}$ and v_h denotes a suitable regularization of v (*see* [6], Section 7.2), then we may take $\varphi_k = v_{h_k} \psi_k$ for a suitable sequence $\{h_k\}$ decreasing to zero, where ψ_k is a C^4 function on $\bar{\Omega}_k$ such that $0 \leqq \psi_k \leqq 1$, $\psi_k = 1$ on $\bar{\Omega}_k - \{x : \text{dist}(x, \partial\Omega_k \cap B_{\rho_1}) < 1/k\}$ and $\psi_k = 0$ on $\partial\Omega_k \cap B_{\rho_1}$.

Let $v_k \in C^2(\bar{\Omega}_k)$ be the unique convex solution of the Dirichlet problem

$$(3.28) \quad \log \det D^2 v_k = \tilde{g}(y, v_k, Dv_k) \quad \text{in } \Omega_k, \qquad v_k = \psi_k \quad \text{on } \partial\Omega_k.$$

Such a function v_k exists by [4], Theorem 7.1 (*see* also [8]), since a convex subsolution of (3.28) can be constructed by adding to φ_k a sufficiently large multiple of a uniformly convex defining function for Ω_k. The uniqueness of v_k is a consequence of (3.27). From (3.27) we also see that v_k converges uniformly to v on any compact subset of $U \cap B_{\rho_0}$. Furthermore, by [6], Theorem 17.4, we have

$$(3.29) \qquad \sup_{\Omega_k} |v_k| \leqq C_1$$

where C_1 depends only on n, $\tilde{\beta}_0$, ρ_0 and $\sup_{U \cap B_{\rho_0}} |v|$, but not on k.

Next, a bound

$$(3.30) \qquad \sup_{\partial\Omega_k \cap B_{\rho_2}} |Dv_k| \leqq C_2$$

with C_2 independent of k, follows easily, since if ξ is any point of $\partial\Omega_k \cap B_{\rho_2}$, there is an enclosing ball, say $B_R(z)$, for Ω_k at ξ with R depending only on ∂U. The function

$$w = d^2 - B\, d,$$

where $d(x)=\operatorname{dist}(x, \partial B_R(z))$ is then a local lower barrier for v_k at ξ, for a suitable positive constant B, depending only on known quantities but not on k. Thus the outer normal derivative of v_k, $D_\mu v_k$, satisfies a bound

$$D_\mu v_k(\xi) \leqq C_3.$$

A lower bound follows easily using the convexity of v_k, so (3.30) is proved. The convexity of v_k then also implies

$$\sup_{\Omega_k \cap B_{\rho_3}} |Dv_k| \leqq C_4. \tag{3.31}$$

We now proceed to show that for some sufficiently small but controlled $\tau \in (0, \rho_3)$ we have

$$\sup_{\Omega_k \cap B_\tau} |D^2 v_k| \leqq C_5 \tag{3.32}$$

for all sufficiently large k, where C_5 is independent of k. First we prove a boundary estimate

$$\sup_{\partial\Omega_k \cap B_{\rho_4}} |D^2 v_k| \leqq C_6. \tag{3.33}$$

The proof of this is similar to the proofs of the boundary second derivative estimates given in [4], [6], [7], [15] so we shall only outline the main points. First, using (3.31) and the fact that the principal curvatures of $\partial\Omega_k \cap B_{\rho_1}$ are bounded above and below by positive constants independent of k, we obtain, since $v_k=0$ on $\partial\Omega_k \cap B_{\rho_1}$,

$$C_7 \leqq D_{\tau\tau} v_k(y) \leqq C_8 \tag{3.34}$$

for any $y \in \partial\Omega_k \cap B_{\rho_4}$ and any direction τ tangential to $\partial\Omega_k$ at y, where C_7 and C_8 are positive constants independent of k. Next, using a suitable barrier argument, together with the $C^{2,1}$ regularity of $\partial\Omega_k \cap B_{\rho_1}$, we obtain

$$|D_{\tau\mu} v_k(y)| \leqq C_9 \tag{3.35}$$

for any $y \in \partial\Omega_k \cap B_{\rho_4}$ and any direction τ as above; here μ denotes the outer unit normal vectorfield to $\partial\Omega_k$. Finally, the estimate

$$D_{\mu\mu} v_k(y) \leqq C_{10} \tag{3.36}$$

for any $y \in \partial\Omega_k \cap B_{\rho_4}$ follows by solving (3.28) for $D_{\mu\mu} v_k$ and using (3.34), (3.35) and the fact that $\tilde{g}$ is bounded.

Next we prove (3.32). We first choose a constant M so large that

$$\eta(y) = 1 - M y_n < 0 \quad \text{in } \Omega_k - B_{\rho_4}.$$

This can be done with M independent of k, since we have a uniform positive lower bound for the principal curvatures of $\partial\Omega_k$. Now for $\beta > 0$ we consider the function

$$W(y, \xi) = \eta\, e^{\beta |Dv_k|^2} D_{\xi\xi} v_k$$

for all $y \in \mathcal{N}_k = \{ y \in \bar{\Omega}_k : \eta(y) \geqq 0 \} \subset \bar{\Omega}_k \cap B_{\rho 4}$ and all directions ξ in $\mathbb{R}^n$. Clearly $\mathcal{N}_k \neq \varnothing$ for k sufficiently large. If W attains its maximum at a point $y_0 \in \partial\Omega_k \cap B_{\rho_4}$ and a direction ξ_0 in $\mathbb{R}^n$, we have an upper bound for W by virtue of (3.33). Otherwise W attains its maximum at an interior point of $\mathcal{N}_k$, and by the argument of [6], Theorem 17.19, we again conclude a bound for W, provided β is fixed sufficiently large. Thus for $\tau > 0$ so small that $\Omega_k \cap B_{2\tau} \subset \mathcal{N}_k$, (3.32) holds for all sufficiently large k.

The second derivative Hölder estimates proved in [6], Theorem 17.26, [10], Theorem 5.5.2, and standard linear theory [6], Lemma 17.16, now imply

$$(3.37) \qquad \| v_k \|_{C^{2,\alpha}(\overline{\Omega_k \cap B_{\tau/2}})} \leqq C$$

for any $\alpha < 1$ and all sufficiently large k, where C depends on α in addition to other known quantities, but not on k. Since v_k converges uniformly on compact subsets of $U \cap B_{\rho_1}$ to v, we conclude that (3.26) holds. This completes the proof of the lemma.

To complete the proof of Theorem 1 we now apply Lemma 3.1 to obtain

$$(3.38) \qquad \| w^* \|_{C^{2,\alpha}(\bar{B}_\rho^+)} \leqq C$$

for any $\alpha < 1$, where ρ and C are controlled positive constants with C depending on α. Thus the mappings $\Psi = Dw$ and $\Psi^{-1} = (Dw)^{-1} = Dw^*$ are of class $C^{1,\alpha}$ near 0 for any $\alpha < 1$, and for suitable $\sigma > 0$ we have the estimates

$$(3.39) \qquad \| \Psi \|_{C^{1,\alpha}(\overline{D \cap B_\sigma})} \leqq C_1$$

and

$$(3.40) \qquad \| \Psi^{-1} \|_{C^{1,\alpha}(\overline{D^* \cap B_\rho})} \leqq C_2.$$

Here and below $C_1, \ldots, C_9$ are positive constants depending only on known quantities and in addition on $\alpha < 1$. Since $\Sigma = \Psi^{-1}(\Sigma^*)$ and Σ is the orthogonal projection of a suitable portion of graph$(u|_{\bar{\Gamma}})$ onto the tangent hyperplane to $\Gamma \times \mathbb{R}$ at $(0, u(0))$, where u is our original solution of (1.8), assertion (iii) of Theorem 1 follows.

Next, since w^* solves (3.12) and (3.14) holds, we see that (3.38) implies

$$(3.41) \qquad C_3 |\xi|^2 \leqq D_{ij} w^*(z) \xi_i \xi_j \leqq C_4 |\xi|^2$$

for all $z \in \bar{B}_\rho^+$ and all $\xi \in \mathbb{R}^n$. Thus using (3.9) we obtain

$$(3.42) \qquad C_4^{-1} |\xi|^2 \leqq D_{ij} w(y) \xi_i \xi_j \leqq C_3^{-1} |\xi|^2$$

for all $y \in \overline{D \cap B_\sigma}$ and all $\xi \in \mathbb{R}^n$. Using (3.8), (3.9), (3.38), (3.41) and (3.42) we then find that

$$(3.43) \qquad \| w \|_{C^{2,\alpha}(\overline{D \cap B_\sigma})} \leqq C_5$$

for any $\alpha < 1$, and

(3.44) $$C_6 |y|^2 \leqq w(y) \leqq C_7 |y|^2$$

for any $y \in \overline{D \cap B_\sigma}$. Assertions (i) and (ii) of Theorem 1 then follow.

Finally we observe that under the additional hypotheses of part (iv) of Theorem 1 we have

(3.45) $$\| w^* \|_{C^{k+1,\alpha}(\bar{B}^+_{\rho/2})} \leqq C_8$$

by virtue of standard elliptic regularity theory [6], Lemma 17.16. Using (3.41) and (3.42) we then obtain

(3.46) $$\| w \|_{C^{k+1,\alpha}(\overline{D \cap B_\tau})} \leqq C_9$$

for some controlled $\tau \in (0, \sigma)$. Assertion (iv) of Theorem 1 follows from (3.45) and (3.46). The case of analytic data follows similarly, so the proof of Theorem 1 is complete.

Remarks. − (i) The proof of Lemma 3.1 remains valid if we assume $u \in C^{0,1}(\bar{B}^+_R)$ is a generalized solution of (3.18).

(ii) Lemma 3.1 can be generalized to the case where B^+_R is replaced by a domain of the form $B_R \cap \{x : x_n > \omega(x')\}$ for certain nonconvex $\omega \in C^{3,1}(\bar{B}^{n-1}_R)$. In this case we generally need to assume that $u|_{B_R \cap \{x : x_n = \omega(x')\}}$ is of class $C^{3,1}$. Coupled with suitable barrier arguments, this leads to existence theorems for globally smooth solutions of the Dirichlet problem for Monge-Ampère equations on nonconvex domains. We intend to treat this more fully in a future paper.

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