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Homogenization of renormalized solutions of elliptic equations

by

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ABSTRACT. – We study in this paper the stability of the renormalized nonlinear elliptic equation

$$\begin{aligned} & u \in H_0^1(\Omega) \\ & -\operatorname{div}(h(u)\,A \operatorname{grad} u) + h'(u)\,A \operatorname{grad} u \operatorname{grad} u \\ & -\operatorname{div}(h(u)\,\Phi(u)) + h'(u)\,\Phi(u) \operatorname{grad} u = fh(u) \quad \text{in } \mathscr{D}'(\Omega) \\ & \forall h \in C_c^1(\mathbb{R}) \end{aligned}$$

where $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function which is not assumed to satisfy any growth condition. The above renormalized formulation differs from the usual weak one by the fact that the test functions $h(u)\varphi$ (which depend on the solution itself) are used in place of the usual test functions $\varphi \in \mathscr{D}(\Omega)$.

Consider a sequence of renormalized solutions u^ε relative to a fixed right-hand side f, to a fixed function Φ and to a sequence of matrices A^ε which converges in the homogenization's sense to A^0. We prove that a subsequence of u^ε weakly converges in $H_0^1(\Omega)$ to a renormalized solution of the equation relative to f, Φ and A^0.

We also consider a sequence of renormalized solutions u^ε relative to a fixed matrix A, to a fixed function Φ and to a sequence of right-hand sides f^ε which weakly converges to f^0 in $H^{-1}(\Omega)$. Under a special equi-integrability assumption on f^ε we prove that a subsequence of u^ε weakly

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converges in $H_0^1(\Omega)$ to a renormalized solution of the equation relative to A, Φ and f^0.

Key words : Homogenization, Renormalized solutions, Nonlinear elliptic equations.

RÉSUMÉ. — Nous étudions dans cet article la stabilité de l'équation elliptique non linéaire renormalisée

$$\begin{aligned}
&u \in H_0^1(\Omega)\\
&-\operatorname{div}(h(u)\,A \operatorname{grad} u) + h'(u)\,A \operatorname{grad} u \operatorname{grad} u\\
&-\operatorname{div}(h(u)\,\Phi(u)) + h'(u)\,\Phi(u) \operatorname{grad} u = fh(u) \quad \text{dans } \mathscr{D}'(\Omega)\\
&\forall\, h \in C_c^1(\mathbb{R})
\end{aligned}$$

où $\Phi : \mathbb{R} \to \mathbb{R}^N$ est une fonction continue à laquelle aucune hypothèse de croissance n'est imposée. La formulation renormalisée ci-dessus diffère de la formulation faible habituelle par le fait qu'on y utilise des fonctions test $h(u)\varphi$ (qui dépendent de la solution elle-même) et non seulement les fonctions test habituelles $\varphi \in \mathscr{D}(\Omega)$.

Considérons une suite de solutions renormalisées u^ε relatives à un second membre fixé f, à une fonction fixée Φ et à une suite de matrices A^ε qui converge au sens de l'homogénéisation vers A^0. Nous démontrons qu'une sous-suite de u^ε converge faiblement dans $H_0^1(\Omega)$ vers une solution renormalisée de l'équation relative à f, Φ et A^0.

Nous considérons également une suite de solutions renormalisées u^ε relatives à une matrice fixée A, à une fonction fixée Φ et à une suite de second membres f^ε qui converge faiblement dans $H^{-1}(\Omega)$ vers f^0. Sous une hypothèse d'équi-intégrabilité spéciale sur les f^ε nous montrons qu'une sous-suite de u^ε converge faiblement dans $H_0^1(\Omega)$ vers une solution renormalisée de l'équation relative à A, Φ et f^0.

To the memory of Ron DiPerna

1. INTRODUCTION

In a joint paper [BDGM1] with Lucio Boccardo, Ildefonso Diaz and Daniela Giachetti, we have considered the nonlinear elliptic problem

$$-\operatorname{div}(A \operatorname{grad} u) - \operatorname{div}(\Phi(u)) = f \quad \text{in } \Omega \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ (no smoothness is assumed on $\partial\Omega$), A is a coercive matrix with L^∞ coefficients, *i. e.* satisfies

$$A \in (L^\infty(\Omega))^{N \times N} \tag{1.3}$$
$$A(x)\xi\xi \geqq \alpha |\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N \tag{1.4}$$

for some $\alpha > 0$; the right-hand side f is assumed to satisfy

$$f \in H^{-1}(\Omega), \quad i.e.\ f = -\operatorname{div} g \text{ for some } g \in (L^2(\Omega))^N \tag{1.5}$$

and the nonlinearity $-\operatorname{div}(\Phi(u))$ is defined from a function Φ which only satisfies

$$\Phi \in (C^0(\mathbb{R}))^N. \tag{1.6}$$

The main feature of this problem is the fact that *no growth condition is assumed on* Φ. In view of (1.5) it is very natural to look for a solution u of (1.1), (1.2) which only belongs to $H_0^1(\Omega)$ and not to a space of more regular functions. Therefore there is no ground for the measurable function $\Phi(u)$ to belong to $(L^1(\Omega))^N$, and for $-\operatorname{div}(\Phi(u))$ to be defined in distributional sense.

This is the reason why we considered in [BDGM1] a weaker formulation of (1.1), (1.2):

$$u \in H_0^1(\Omega) \tag{1.7}$$
$$\begin{gathered} -\operatorname{div}(h(u) A \operatorname{grad} u) + h'(u) A \operatorname{grad} u \operatorname{grad} u \\ -\operatorname{div}(h(u)\Phi(u)) + h'(u)\Phi(u) \operatorname{grad} u = f h(u) \quad \text{in } \mathcal{D}'(\Omega) \\ \forall h \in C_c^1(\mathbb{R}) \end{gathered} \tag{1.8}$$

for which we were able to prove the existence of at least one solution. We then extended in [BDGM2] this existence result to the case where $-\operatorname{div}(A \operatorname{grad} u)$ is replaced by a Leray-Lions operator

$$-\operatorname{div}(a(x, u, \operatorname{grad} u))$$

defined from $W_0^{1,p}(\Omega)$ into its dual.

Note that each term of (1.8) makes sense as a distribution whenever u belongs to $H_0^1(\Omega)$, since $h\Phi$ and $h\Phi'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$ for any h in $C_c^1(\mathbb{R})$, while the right hand-side of (1.8) read as

$$f h(u) = -\operatorname{div}(h(u) g) + h'(u) g \operatorname{grad} u. \tag{1.9}$$

Equation (1.8) is obtained formally by a pointwise multiplication of (1.1) by $h(u)$ and a convenient rewriting of the various terms.

Equation (1.8) is actually a generalization of the usual weak formulation of equation (1.1), since two different test functions have now to be used: the usual test function in $\mathcal{D}(\Omega)$ and a new one, $h(u)$ with $h \in C_c^1(\mathbb{R})$, which depends on the solution u itself. This is exactly the concept of *renormalized solution* introduced by Ronald DiPerna and Pierre-Louis Lions in [DL1], [DL2] to study the Boltzmann equation.

The present paper consists of two variations on the theme introduced above: Section 2 deals with the homogenization of the renormalized equations (1.7), (1.8), *i.e.* with the case where a sequence of matrices A^ε is considered which is bounded in $(L^\infty(\Omega))^{N\times N}$ and satisfies (1.4) for some fixed $\alpha>0$. We prove that if A^ε H-converges to A^0 (*see* the definition of this convergence in Appendix A below), a subsequence of the sequence of the solutions of the renormalized equations (1.8) relative to A^ε weakly converges in $H_0^1(\Omega)$ to a solution of the renormalized equation relative to A^0. The notion of renormalized equation is thus robust in the sense that it is stable under the H-convergence of the matrices, which is the "weakest possible" convergence for the corresponding operators. It is also worth noticing that the proof that we will present in Section 3 to prove this homogenization result of renormalized solutions is closed to the proof we used in [BDGM1], [BDGM2] to obtain the existence of renormalized solutions. This illustrates the robustness of the method.

The robustness of both the notion of renormalized solution and of the method of proof is emphasized by the stability result of renormalized solutions with respect to variations of the right-hand side which is given in Section 4: consider a sequence of renormalized solutions u^ε of (1.7), (1.8) relative to the same matrix A and to a sequence of right-hand sides f^ε which converges weakly to f^0 in $H^{-1}(\Omega)$. Under a special assumption of equi-integrability on f^ε a subsequence of the sequence u^ε is proved to converge weakly in $H_0^1(\Omega)$ to a solution of the renormalized equation relative to the right hand-side f^0.

2. STABILITY OF THE RENORMALIZED SOLUTIONS WITH RESPECT TO HOMOGENIZATION

Consider a given bounded open subset Ω of $\mathbb{R}^N$, a fixed right-hand side f and a fixed nonlinearity Φ such that

$$f \in H^{-1}(\Omega) \tag{2.1}$$

$$\Phi \in (C^0(\mathbb{R}))^N. \tag{2.2}$$

Consider also a sequence of matrices $A^\varepsilon \in (L^\infty(\Omega))^{N\times N}$ which satisfy for some $\alpha>0$ and $\beta>0$

$$A^\varepsilon(x)\xi\xi \geqq \alpha|\xi|^2 \quad \text{a.e. } x\in\Omega, \quad \forall \xi\in\mathbb{R}^N \tag{2.3}$$

$$(A^\varepsilon)^{-1}(x)\xi\xi \geqq \beta^{-1}|\xi|^2 \quad \text{a.e. } x\in\Omega, \quad \forall \xi\in\mathbb{R}^N \tag{2.4}$$

as well as

$$A^\varepsilon \xrightarrow{H} A^0; \tag{2.5}$$

for the definition of the H-convergence (2.5), see if necessary Definition A.1 in Appendix A below. Note that the matrix A^0 also belongs to $(L^\infty(\Omega))^{N\times N}$ and satisfies (2.3) and (2.4). Note also that matrices A^ε satisfying (2.4) are bounded in $(L^\infty(\Omega))^{N\times N}$ since the choice $\xi = A^\varepsilon(x)\lambda$ in (2.4) and Cauchy Schwartz' inequality lead to

$$|A^\varepsilon(x)\lambda| \leqq \beta|\lambda| \quad \text{a.e. } x\in\Omega, \quad \forall\lambda\in\mathbb{R}^N. \tag{2.6}$$

In view of the existence result of [BDGM1], Theorem 1.1, there exists at least (*) one solution u^ε of the renormalized equation relative to A^ε, f and φ, *i.e.* some u^ε which satisfies

$$u^\varepsilon \in H_0^1(\Omega) \tag{2.7}$$

$$\begin{aligned}&-\operatorname{div}(h(u^\varepsilon)A^\varepsilon \operatorname{grad} u^\varepsilon) + h'(u^\varepsilon)A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon\\ &-\operatorname{div}(h(u^\varepsilon)\Phi(u^\varepsilon)) + h'(u^\varepsilon)\Phi(u^\varepsilon)\operatorname{grad} u^\varepsilon = fh(u^\varepsilon) \quad \text{in } \mathscr{D}'(\Omega)\\ &\forall h\in C_c^1(\mathbb{R}).\end{aligned} \tag{2.8}$$

Moreover Theorem 1.3 of [BDGM1] asserts that

$$\int_\Omega A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon\, dx = \langle f, u^\varepsilon\rangle. \tag{2.9}$$

This equality can be obtained *formally* by taking $h=1$ in (2.8), then multiplying by u^ε, integrating par parts, and using Stokes' Theorem which formally implies that

$$\int_\Omega \Phi(u^\varepsilon)\operatorname{grad} u^\varepsilon\, dx = 0;$$

note that although this heuristic computation is not licit, in particular since 1 does not belong to $C_c^1(\mathbb{R})$ and since there is no ground for $\Phi(u^\varepsilon)\operatorname{grad} u^\varepsilon$ to belong to $L^1(\Omega)$. Nevertheless the result (2.9) can be proved rigourously by using some convenient approximation of $h=1$, *see* [BDGM1].

From (2.9) and the uniform coerciveness assumption (2.3) we infer

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leqq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{2.10}$$

THEOREM 2.1. – *Assume that* (2.1)-(2.5) *hold true and extract* (*) *a subsequence* ε' *such that* [see (2.10) *and use Rellich's compactness Theorem*]

$$u^{\varepsilon'} \rightharpoonup u \textit{ weakly in } H_0^1(\Omega) \textit{ and a.e. } x\in\Omega. \tag{2.11}$$

(*) *See* the *Note added in proof.*

The weak limit u is a solution of the renormalized equation relative to A^0, f *and* φ, *i. e. u satisfies*

$$u \in H_0^1(\Omega) \tag{2.12}$$

$$\begin{aligned}&-\operatorname{div}(h(u)\,A^0 \operatorname{grad} u) + h'(u)\,A^0 \operatorname{grad} u \operatorname{grad} u\\ &-\operatorname{div}(h(u)\,\Phi(u)) + h'(u)\,\Phi(u) \operatorname{grad} u = fh(u) \quad \textit{in } \mathscr{D}'(\Omega)\\ &\forall h \in C_c^1(\mathbb{R}). \quad \square\end{aligned} \tag{2.13}$$

Note that the extraction of a subsequence in (2.11) is due to the fact that we do not know if the solution of the renormalized equation (2.12) (2.13) is unique or not (*).

In order to prove Theorem 2.1 let us introduce the solution z^ε of the linear problem

$$\left.\begin{aligned}-\operatorname{div}(A^\varepsilon \operatorname{grad} z^\varepsilon) = -\operatorname{div}(A^0 \operatorname{grad} u) \quad \text{in } \mathscr{D}'(\Omega)\\ z^\varepsilon \in H_0^1(\Omega)\end{aligned}\right. \tag{2.14}$$

where u is defined by (2.11). Note that in view of hypotheses (2.3), (2.4), (2.5) on A^ε and of the Definition A.1 of H-convergence we have

$$\left.\begin{aligned}&\| z^\varepsilon \|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \| A^0 \operatorname{grad} u \|_{(L^2(\Omega))^N}\\ &z^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),\\ &A^\varepsilon \operatorname{grad} z^\varepsilon \rightharpoonup A^0 \operatorname{grad} u \quad \text{weakly in } (L^2(\Omega))^N.\end{aligned}\right. \tag{2.15}$$

The main step in the proof of Theorem 2.1 is the following

THEOREM 2.2. – *Define*

$$r^\varepsilon = u^\varepsilon - z^\varepsilon. \tag{2.16}$$

One has the strong convergence

$$r^{\varepsilon'} \to 0 \ \textit{strongly in } H^1_{loc}(\Omega) \tag{2.17}$$

for the subsequence ε' extracted (*) *in* (2.11). *The convergence* (2.17) *takes place in* $H_0^1(\Omega)$ (*and not only locally in* Ω) *when* $\partial\Omega$ *is sufficiently smooth.* $\square$

3. PROOF OF THE HOMOGENIZATION RESULT (THEOREMS 2.2 AND 2.1)

The proof follows along the lines of [BDGM1], [BDGM2], and is in some sense very closed to the proofs used there to obtain the existence of a renormalized solution of (1.7), (1.8). On the other hand, the idea which

(*) *See* the *Note added in proof.*

consists in introducing the solution z^ε equation (2.14) and in proving the strong convergence (2.17) has already been used in [B], [BeBM2].

Since Theorem 2.1 appears as a simple consequence of Theorem 2.2, we begin with the proof of the latest, which uses non linear (with respect to u^ε) test functions closely related to those used in [BeBM1] for the study of another elliptic equation.

Proof of Theorem 2.2

For any positive real number k, denote by $\mathrm{T}_k : \mathbb{R} \to \mathbb{R}$ the "truncation at the height k" defined by

$$\mathrm{T}_k(s) = \begin{cases} s & \text{if } \quad |s| \leqq k \\ k\, s/|s| & \text{if } \quad |s| \geqq k, \end{cases} \tag{3.1}$$

and denote by S_k the "surplus" of this truncation, defined by

$$\mathrm{S}_k(s) = s - \mathrm{T}_k(s), \qquad \forall s \in \mathbb{R}.$$

First step. – Since S_k is a Lipschitz-continuous, piecewise $\mathrm{C}^1(\mathbb{R})$ function such that $\mathrm{S}_k(0)=0$, Theorem 3.1 of [BDGM1] applies to $\mathrm{S}_k(u^\varepsilon)$ and (2.7), (2.8), and yields

$$\int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} \mathrm{S}_k(u^\varepsilon)\, dx = \langle f, \mathrm{S}_k(u^\varepsilon) \rangle; \tag{3.2}$$

this equality can be obtained formally by an heuristic computation similar to that described above to obtain (2.9).

Defining E_k^ε as the set

$$\mathrm{E}_k^\varepsilon = \{ x \in \Omega;\ |u^\varepsilon(x)| \geqq k \} \tag{3.3}$$

we deduce from (3.2) and from the uniform coerciveness (2.3) of A^ε that

$$\alpha \int_{\mathrm{E}_k^\varepsilon} |\operatorname{grad} u^\varepsilon|^2\, dx \leqq \langle f, u^\varepsilon - \mathrm{T}_k(u^\varepsilon) \rangle. \tag{3.4}$$

Extracting (*) a subsequence ε' such that (2.11) holds true, we thus have for any fixed $k>0$

$$\limsup_{\varepsilon' \to 0} \alpha \int_{\mathrm{E}_k^{\varepsilon'}} |\operatorname{grad} u^{\varepsilon'}|^2\, dx \leqq \langle f, u - \mathrm{T}_k(u) \rangle, \tag{3.5}$$

(*) *See* the *Note added in proof.*

and since $u-\mathrm{T}_k(u)$ tends to 0 in $\mathrm{H}_0^1(\Omega)$ when k tends to infinity, we have proved that

$$\limsup_{\varepsilon'\to 0} \alpha \int_{\mathrm{E}_k^{\varepsilon'}} |\operatorname{grad} u^{\varepsilon'}|^2\, dx \leqq \omega_k \tag{3.6}$$

where $\omega_k \to 0$ if $k \to \infty$.

Second Step. — Consider in (2.8) the test function

$$w^\varepsilon = \mathrm{T}_i(u^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \in \mathrm{H}_0^1(\Omega) \tag{3.7}$$

where z^ε is defined by (2.14) and the truncations T_i and T_j are defined by (3.1). Since $|u^\varepsilon(x)| \geqq i+j$ implies $|u^\varepsilon(x) - \mathrm{T}_j(z^\varepsilon(x))| \geqq i$ we have

$$\operatorname{grad} w^\varepsilon = 0 \quad \text{a. e. on } \{x \in \Omega;\ |u^\varepsilon(x)| \geqq i+j\}.$$

An alternative to Theorem 1.3 of [BDGM1] (*see* Theorem 4 of [BDGM2] or Theorem B.1 in Appendix B below) then asserts that

$$\int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} w^\varepsilon\, dx + \int_\Omega \Phi_{i+j}(u^\varepsilon) \operatorname{grad} w^\varepsilon\, dx = \langle f, w^\varepsilon \rangle \tag{3.8}$$

where Φ_{i+j} is the $(\mathrm{C}^0(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R}))^\mathrm{N}$ function defined by

$$\Phi_{i+j}(s) = \Phi(\mathrm{T}_{i+j}(s)), \qquad \forall s \in \mathbb{R}.$$

Define $\mathrm{F}_{ij}^\varepsilon$ as the set

$$\mathrm{F}_{ij}^\varepsilon = \{x \in \Omega;\ |u^\varepsilon(x) - \mathrm{T}_j(z^\varepsilon(x))| \leqq i\}. \tag{3.9}$$

Substracting to both hand sides of (3.8) the quantity

$$\int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} \mathrm{T}_j(z^\varepsilon) \operatorname{grad} w^\varepsilon\, dx$$
$$= \int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} z^\varepsilon \operatorname{grad} w^\varepsilon\, dx - \int_\Omega \mathrm{A}^\varepsilon \operatorname{grad}(z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \operatorname{grad} w^\varepsilon\, dx,$$

and noticing that in view of (2.14)

$$\int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} z^\varepsilon \operatorname{grad} w^\varepsilon\, dx = \int_\Omega \mathrm{A}^0 \operatorname{grad} u \operatorname{grad} w^\varepsilon\, dx,$$

we obtain using the uniform coerciveness (2.3) of A^ε:

$$\begin{aligned} &\alpha \int_{\mathrm{F}_{ij}^\varepsilon} |\operatorname{grad}(u^\varepsilon - \mathrm{T}_j(z^\varepsilon))|^2\, dx + \int_\Omega \Phi_{i+j}(u^\varepsilon) \operatorname{grad} w^\varepsilon\, dx \\ &\leqq \langle f, w^\varepsilon \rangle - \int_\Omega \mathrm{A}^0 \operatorname{grad} u \operatorname{grad} w^\varepsilon\, dx \\ &+ \int_\Omega \mathrm{A}^\varepsilon \operatorname{grad}(z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \operatorname{grad} w^\varepsilon\, dx. \end{aligned} \tag{3.10}$$

Since $w^{\varepsilon'}$ tends weakly to $T_i(u-T_j(u))$ in $H_0^1(\Omega)$ [*see* (2.11), (2.15)] and since $\Phi_{i+j}(u^{\varepsilon'})$ tends strongly to $\Phi_{i+j}(u)$ in $[L^2(\Omega)^N]$ by Lebesgue's dominated convergence Theorem, we have

$$\int_\Omega \Phi_{i+j}(u^{\varepsilon'})\operatorname{grad} w^{\varepsilon'}\,dx \to \int_\Omega \Phi_{i+j}(u)\operatorname{grad} T_i(u-T_j(u))\,dx = 0; \tag{3.11}$$

in order to obtain the latest equality we used the identity $\int_\Omega \theta(u)\operatorname{grad} u\,dx = 0$ which results from Stokes' Theorem for any piecewise-continuous, bounded function $\theta:\mathbb{R}\to\mathbb{R}^N$ and any u in $H_0^1(\Omega)$ (*see* Lemma 2.1 of [BDGM1] if necessary).

Consider now the right-hand side of (3.10). It is straightforward to pass to the limit in the first two terms and to obtain

$$\begin{aligned}&\langle f, w^{\varepsilon'}\rangle - \int_\Omega A^0 \operatorname{grad} u \operatorname{grad} w^{\varepsilon'}\,dx\\ &\to \langle f, T_i(u-T_j(u))\rangle - \int_\Omega A^0 \operatorname{grad} u \operatorname{grad} T_i(u-T_j(u))\,dx.\end{aligned} \tag{3.12}$$

For the last term we use the bound (2.6) and the coerciveness (2.3) of A^ε to obtain the estimate

$$\begin{aligned}&\left|\int_\Omega A^\varepsilon \operatorname{grad}(z^\varepsilon - T_j(z^\varepsilon))\operatorname{grad} w^\varepsilon\,dx\right|\\ &\leqq \beta\,\|\operatorname{grad}(z^\varepsilon - T_j(z^\varepsilon))\|_{(L^2(\Omega))^N}\,\|\operatorname{grad} w^\varepsilon\|_{(L^2(\Omega))^N}\\ &\leqq \beta\frac{1}{\sqrt{\alpha}}\left(\int_\Omega A^\varepsilon \operatorname{grad}(z^\varepsilon - T_j(z^\varepsilon))\operatorname{grad}(z^\varepsilon - T_j(z^\varepsilon))\,dx\right)^{1/2}\|\operatorname{grad} w^\varepsilon\|_{(L^2(\Omega))^N}\end{aligned} \tag{3.13}$$

Since the truncations decrease the $H_0^1(\Omega)$ norm we have in view of (3.7) and (2.10)

$$\begin{aligned}\|\operatorname{grad} w^\varepsilon\|_{(L^2(\Omega))^N} &\leqq \|\operatorname{grad} u^\varepsilon\|_{(L^2(\Omega))^N} + \|\operatorname{grad} z^\varepsilon\|_{(L^2(\Omega))^N}\\ &\leqq \frac{1}{\alpha}(\|f\|_{H^{-1}(\Omega))} + \|A^0 \operatorname{grad} u\|_{(L^2(\Omega))^N}).\end{aligned} \tag{3.14}$$

On the other hand, using first the property that $T_j'(s)=1$ when $|s|<j$ and 0 elsewhere, then using $z^\varepsilon - Tj(z^\varepsilon)$ as test function in (2.14) and passing

to the limit in ε imply that

$$\begin{aligned}
&\int_\Omega \mathrm{A}^\varepsilon \operatorname{grad}(z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \operatorname{grad}((z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \, dx \\
&= \int_\Omega \mathrm{A}^\varepsilon \operatorname{grad} z^\varepsilon \operatorname{grad}(z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \, dx \\
&= \int_\Omega \mathrm{A}^0 \operatorname{grad} u \operatorname{grad}(z^\varepsilon - \mathrm{T}_j(z^\varepsilon)) \, dx \\
&\to \int_\Omega A^0 \operatorname{grad} u \operatorname{grad}(u - \mathrm{T}_j(u)) \, dx.
\end{aligned} \tag{3.15}$$

Combining (3.10)-(3.15) we have proved that for any fixed i and j:

$$\begin{aligned}
&\limsup_{\varepsilon' \to 0} \alpha \int_{\mathrm{F}_{ij}^{\varepsilon'}} |\operatorname{grad}(u^{\varepsilon'} - \mathrm{T}_j(z^{\varepsilon'}))|^2 \, dx \\
&\leqq \langle f, \mathrm{T}_i(u - \mathrm{T}_j(u)) \rangle - \int_\Omega \mathrm{A}^0 \operatorname{grad} u \operatorname{grad} \mathrm{T}_i(u - \mathrm{T}_j(u)) \, dx \\
&+ \frac{\beta}{\sqrt{\alpha}} \left(\int_\Omega \mathrm{A}^0 \operatorname{grad} u \operatorname{grad}(u - \mathrm{T}_j(u)) \, dx \right)^{1/2} \\
&\qquad \frac{1}{\alpha} (\|f\|_{\mathrm{H}^{-1}(\Omega)} + \|\mathrm{A}^0 \operatorname{grad} u\|_{(\mathrm{L}^2(\Omega))^\mathrm{N}}).
\end{aligned} \tag{3.16}$$

Since the truncation T_i decreases the $\mathrm{H}_0^1(\Omega)$ norm and since $u - \mathrm{T}_j(u)$ tends to 0 in $\mathrm{H}_0^1(\Omega)$ when j tends to infinity, this implies that for any fixed i and j

$$\limsup_{\varepsilon' \to 0} \int_{\mathrm{F}_{ij}^{\varepsilon'}} |\operatorname{grad}(u^{\varepsilon'} - \mathrm{T}_j(z^{\varepsilon'}))|^2 \, dx \leqq \omega_j \quad \text{where } \omega_j \to 0 \text{ if } j \to \infty. \tag{3.17}$$

Third step. — Consider a fixed measurable set K with $\bar{\mathrm{K}} \subset \Omega$. [When $\partial\Omega$ is sufficiently smooth, the choice $\mathrm{K} = \Omega$ becomes licit and this will prove the strong convergence of $r^{\varepsilon'}$ in $\mathrm{H}_0^1(\Omega)$.] From now on we will assume that

$$i > j.$$

Since $|u^\varepsilon(x) - \mathrm{T}_j(z^\varepsilon(x))| > i$ implies

$$|u^\varepsilon(x)| \geqq |u^\varepsilon(x) - \mathrm{T}_j(z^\varepsilon(x))| - |\mathrm{T}_j(z^\varepsilon(x))| \geqq i - j,$$

one has [*see* the definitions (3.9) and (3.3) of $\mathrm{F}_{ij}^\varepsilon$ and E_k^ε]

$$\Omega \setminus \mathrm{F}_{ij}^\varepsilon \subset \mathrm{E}_{i-j}^\varepsilon. \tag{3.18}$$

Thus

$$\begin{aligned}
&\int_{K} |\operatorname{grad}(u^\varepsilon - z^\varepsilon)|^2\,dx \\
&\leqq \int_{K\cap F^\varepsilon_{ij}} |\operatorname{grad}(u^\varepsilon - z^\varepsilon)|^2\,dx + \int_{K\cap E^\varepsilon_{i-j}} |\operatorname{grad}(u^\varepsilon - z^\varepsilon)|^2\,dx \\
&\leqq 2\int_{F^\varepsilon_{ij}} |\operatorname{grad}(u^\varepsilon - T_j(z^\varepsilon))|^2\,dx + 2\int_{\Omega} |\operatorname{grad}(T_j(z^\varepsilon) - z^\varepsilon)|^2\,dx \\
&\qquad + 2\int_{E^\varepsilon_{i-j}} |\operatorname{grad} u^\varepsilon|^2\,dx + 2\int_{K\cap E^\varepsilon_{i-j}} |\operatorname{grad} z^\varepsilon|^2\,dx.
\end{aligned} \tag{3.19}$$

In view of the results (3.17) and (3.6) of two first steps of the present proof, the lim sup in ε' of the first and third terms of the right-hand side of (3.19) are small whenever j and $i-j$ are large. Similarly (3.15) implies that the lim sup in ε of the second term of the right-hand side of (3.19) is small when j is large. The proof of Theorem 2.2 will thus be complete if we prove that for any fixed measurable set K with $\bar{K} \subset \Omega$ (the choice $K=\Omega$ being licit if $\partial\Omega$ is sufficiently smooth) we have

$$\int_{K\cap E^\varepsilon_k} |\operatorname{grad} z^\varepsilon|^2\,dx \leqq \omega_k(K) \quad \text{independently of } \varepsilon \tag{3.20}$$

where $\omega_k(K) \to 0$ if $k \to \infty$.

Since u^ε is bounded in $L^2(\Omega)$ [*see* (2.10)] one has in view of the definition (3.3) of E^ε_k

$$k^2 |E^\varepsilon_k| \leqq \int_{\Omega} |u^\varepsilon_k|^2\,dx \leqq C, \tag{3.21}$$

which implies that the measure $|E^\varepsilon_k|$ of E^ε_k is small independently of ε when k is large. Proposition A.4 (which is a consequence of Meyers' regularity theorem) then implies (3.20). Theorem 2.1 is proved. □

Proof of Theorem 2.1

Since $u^{\varepsilon'}$ tends to u weakly in $H^1_0(\Omega)$ and almost everywhere in Ω since $h\Phi$ and $h'\Phi$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$ we have

$$\begin{aligned}
&h(u^{\varepsilon'})\Phi(u^{\varepsilon'}) \rightharpoonup h(u)\Phi(u) \text{ weakly} * \text{in } (L^\infty(\Omega))^N \text{ and a.e. in } \Omega \\
&h'(u^{\varepsilon'})\Phi(u^{\varepsilon'}) \rightharpoonup h'(u)\Phi(u) \text{ weakly} * \text{in } (L^\infty(\Omega))^N \text{ and a.e. in } \Omega.
\end{aligned} \tag{3.22}$$

This allows one to pass to the limit in ε' in distributional sense in the two last terms of the left-hand side of (2.8) for any fixed h in $C^1_c(\mathbb{R})$.

For what concerns the right-hand side we use the formulation (1.9), *i.e.*

$$fh(u^{\varepsilon'}) = -\operatorname{div}(h(u^{\varepsilon'})g) + h'(u^{\varepsilon'})g \operatorname{grad} u^{\varepsilon'}, \tag{3.23}$$

in which it is easy to pass to the limit in ε' in distributional sense since $h(u^{\varepsilon'})$ and $h'(u^{\varepsilon'})g$ respectively tend to $h(u)$ and $h'(u)g$ strongly in $L^2(\Omega)$ and in $(L^2(\Omega))^N$.

Passing to the limit in ε' for any fixed h in $C_c^1(\mathbb{R})$ in the first two terms of (2.8) needs to use Theorem 2.2. For what concerns the first term of (2.8) we have

$$\begin{aligned} h(u^{\varepsilon'})\,A^{\varepsilon'}\operatorname{grad} u^{\varepsilon'} &= h(u^{\varepsilon'})\,A^{\varepsilon'}\operatorname{grad} z^{\varepsilon'} + h(u^{\varepsilon'})\,A^{\varepsilon'}\operatorname{grad} r^{\varepsilon'} \\ &\rightharpoonup h(u)\,A^0\operatorname{grad} u \ \text{ in } \mathscr{D}'(\Omega); \end{aligned} \tag{3.24}$$

indeed $h(u^{\varepsilon'})$ converges strongly to $h(u)$ in $L^2(\Omega)$ while $A^\varepsilon \operatorname{grad} z^\varepsilon$ tends weakly to $A^0 \operatorname{grad} u$ in $(L^2(\Omega))^N$ by the definition of H-convergence (see (2.15)); on the other hand $h(u^{\varepsilon'})$ and A^ε are respectively bounded in $L^\infty(\Omega)$ and in $(L^\infty(\Omega))^{N\times N}$ [*see* (2.6)] while $\operatorname{grad} r^{\varepsilon'}$ strongly converges to 0 in $(L^2_{\mathrm{loc}}(\Omega))^N$ by Theorem 2.2.

For what concerns the second term of (2.8) we have

$$\begin{aligned} & h'(u^{\varepsilon'})\,A^{\varepsilon'}\operatorname{grad} u^{\varepsilon'}\operatorname{grad} u^{\varepsilon'} \\ & = A^{\varepsilon'}\operatorname{grad} z^{\varepsilon'}\operatorname{grad} h(u^{\varepsilon'}) + A^{\varepsilon'}\operatorname{grad} r^{\varepsilon'}\operatorname{grad} h(u^{\varepsilon'}) \\ & \rightharpoonup A^0\operatorname{grad} u\operatorname{grad} h(u) \quad \text{in } \mathscr{D}'(\Omega); \end{aligned} \tag{3.25}$$

indeed $A^{\varepsilon'}\operatorname{grad} r^{\varepsilon'}\operatorname{grad} h(u^{\varepsilon'})$ tends to 0 in distributional sense since $A^{\varepsilon'}$ and $\operatorname{grad} h(u^{\varepsilon'})$ are respectively bounded in $(L^\infty(\Omega))^{N\times N}$ and in $(L^2(\Omega))^N$ while $\operatorname{grad} r^{\varepsilon'}$ strongly tends to 0 in $(L^2_{\mathrm{loc}}(\Omega))^N$ by Theorem 2.2; the convergence in distributional sense of $A^{\varepsilon'}\operatorname{grad} z^{\varepsilon'}\operatorname{grad} h(u^{\varepsilon'})$ to $A^0\operatorname{grad} u\operatorname{grad} h(u)$ is easily obtained by using $\varphi\, h(u^{\varepsilon'})$ [with φ in $\mathscr{D}(\Omega)$] as test function in (2.14) and then passing to the limit in ε' with the help of (2.15) [from another standpoint this is just an application of the theory of compensated compactness (*see* [T2], [M2]) since the divergence of $A^{\varepsilon'}\operatorname{grad} z^{\varepsilon'}$ is fixed while the curl of $\operatorname{grad} h(u^{\varepsilon'})$ is identically zero].

In conclusion, we passed to the limit (in distributional sense) in ε' for any fixed $h \in C_c^1(\mathbb{R})$ in each term of (2.8), obtaining the corresponding terms of (2.13). The proof of Theorem 2.1 is thus complete. □

4. STABILITY OF THE RENORMALIZED SOLUTIONS WITH RESPECT TO THE VARIATION OF THE RIGHT-HAND SIDE

In this Section we fix a matrix A satisfying (1.3) and (1.4) and we consider a sequence of right-hand sides f^ε satisfying

$$f^\varepsilon \rightharpoonup f^0 \quad \text{weakly in } H^{-1}(\Omega). \tag{4.1}$$

Let u^ε be a solution of the renormalized equation associated to A and f^ε, *i.e.*

$$u^\varepsilon \in H_0^1(\Omega) \tag{4.2}$$

$$\begin{gathered}-\operatorname{div}(h(u^\varepsilon)\,\mathrm{A}\operatorname{grad}u^\varepsilon)+h'(u^\varepsilon)\,\mathrm{A}\operatorname{grad}u^\varepsilon\operatorname{grad}u^\varepsilon\\ -\operatorname{div}(h(u^\varepsilon)\,\Phi(u^\varepsilon))+h'(u^\varepsilon)\,\Phi(u^\varepsilon)\operatorname{grad}u^\varepsilon=f^\varepsilon h(u^\varepsilon)\quad\text{in }\mathscr{D}'(\Omega)\\ \forall h\in C_c^1(\mathbb{R}).\end{gathered}\tag{4.3}$$

Theorem 1.3 of [BDGM 1] asserts that

$$\int_\Omega \mathrm{A}\operatorname{grad}u^\varepsilon\operatorname{grad}u^\varepsilon\,dx=\langle f^\varepsilon, u^\varepsilon\rangle \tag{4.4}$$

[*see* the comment after (2.9) above for a formal proof of this equality]. In view of (4.1), (4.4) implies that u^ε is bounded in $\mathrm{H}_0^1(\Omega)$ and extracting a subsequence (*) we have

$$u^{\varepsilon'}\rightharpoonup u\quad\text{weakly in }\mathrm{H}_0^1(\Omega)\text{ and a.e. }x\in\Omega. \tag{4.5}$$

Denote by y^ε the solution of

$$\left.\begin{aligned}-\operatorname{div}(\mathrm{A}\operatorname{grad}y^\varepsilon)=f^\varepsilon-f^0\quad\text{in }\mathscr{D}'(\Omega)\\ y^\varepsilon\in\mathrm{H}_0^1(\Omega)\end{aligned}\right. \tag{4.6}$$

and by Y_k the number

$$\mathrm{Y}_k=\limsup_{\varepsilon'\to 0}\int_{|y^{\varepsilon'}|\geq k}|\operatorname{grad}y^{\varepsilon'}|^2\,dx. \tag{4.7}$$

THEOREM 4.1. – *Assume that*

$$\mathrm{Y}_k\to 0\quad as\,k\to\infty. \tag{4.8}$$

Then u is a solution of the renormalized equation relative to f^0, i.e. u *satisfies*

$$u\in\mathrm{H}_0^1(\Omega) \tag{4.9}$$

$$\begin{gathered}-\operatorname{div}(h(u)\,\mathrm{A}\operatorname{grad}u)+h'(u)\,\mathrm{A}\operatorname{grad}u\operatorname{grad}u\\ -\operatorname{div}(h(u)\,\Phi(u))+h'(u)\,\Phi(u)\operatorname{grad}u=f^0 h(u)\quad\text{in }\mathscr{D}'(\Omega)\\ \forall h\in C_c^1(\mathbb{R}).\quad\square\end{gathered}\tag{4.10}$$

Hypothesis (4.8) is nothing but to assume a special equi-integrability property in $(\mathrm{L}^2(\Omega))^\mathrm{N}$ on $\operatorname{grad}y^\varepsilon$ [actually the hypothesis that $\operatorname{grad}y^\varepsilon$ is equi-integrable in $(\mathrm{L}^2(\Omega))^\mathrm{N}$ implies (4.8)], *i.e.* a special property on the right-hand sides f^ε. This hypothesis is in particular satisfied when the right-hand sides are "equi-integrable in $\mathrm{H}^{-1}(\Omega)$" and $\partial\Omega$ is smooth, see Remark 4.3 below. We do not know if the result of Theorem 4.1 still holds true without assuming (4.8).

The proof of Theorem 4.1 is simple when f^ε strongly converges to f^0 in $\mathrm{H}^{-1}(\Omega)$: in such case it is sufficient to follow along the lines of the

(*) *See* the *Note added in proof.*

proof of the existence result of [BDGM 1] (proofs of Theorems 1.1 and 2.1) to obtain the result; this proof is nothing but the proof of Theorems 2.1, 2.2 above, when A^ε coincides with A and z^ε with u.

When only the weak convergence of f^ε and (4.8) are is assumed, the main step in the proof of Theorem 4.1 is the following

THEOREM 4.2. – *Assume that* (4.1)-(4.8) *hold true. Then one has the following strong convergence*

$$\int_{|u^{\varepsilon'}|\leqq k} |\operatorname{grad}(u^{\varepsilon'}-u-y^{\varepsilon'})|^2\,dx \to 0 \quad \text{as } \varepsilon' \to 0 \tag{4.11}$$

for any fixed k. □

Proof of Theorem 4.1

Theorem 4.1 is easily deduced of Theorem 4.2. Indeed in view of the definition (4.6) of y^ε one has

$$\begin{aligned} f^\varepsilon h(u^\varepsilon) &= (f^0 - \operatorname{div}(A \operatorname{grad} y^\varepsilon))\,h(u^\varepsilon) \\ &= f^0 h(u^\varepsilon) - \operatorname{div}(h(u^\varepsilon) A \operatorname{grad} y^\varepsilon) + h'(u_\varepsilon) A \operatorname{grad} y^\varepsilon \operatorname{grad} u^\varepsilon, \end{aligned} \tag{4.12}$$

and (4.3) can thus be rewritten as

$$\begin{aligned} &\operatorname{div}(h(u^\varepsilon)\,\varphi(u^\varepsilon)) - h'(u^\varepsilon)\,\varphi(u^\varepsilon) \operatorname{grad} u^\varepsilon \\ &= -\operatorname{div}(h(u^\varepsilon) A \operatorname{grad} u^\varepsilon) + h'(u^\varepsilon) A \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon - f^\varepsilon h(u^\varepsilon) \\ &= -\operatorname{div}(h(u^\varepsilon) A \operatorname{grad} u) + h'(u^\varepsilon) A \operatorname{grad} u \operatorname{grad} u^\varepsilon - f^0 h(u^\varepsilon) \\ &- \operatorname{div}(h(u^\varepsilon) A \operatorname{grad}(u^\varepsilon - u - y^\varepsilon)) + h'(u^\varepsilon) A \operatorname{grad}(u^\varepsilon - u - y^\varepsilon) \operatorname{grad} u^\varepsilon. \end{aligned} \tag{4.13}$$

It is then easy to pass to the limit in ε' in distributional sense in each term of (4.13), using (4.11) for the two last terms and (4.5) for the other ones. Note that we take here advantage of the presence of the "cut-off functions" $h(u^\varepsilon)$ and $h'(u^\varepsilon)$ in the delicate last two terms of (4.13), using only the strong convergence of $u^{\varepsilon'}-u-y^{\varepsilon'}$ to 0 "in the area where $|u^{\varepsilon'}|$ is bounded by k". □

REMARK 4.3. – If hypothesis (4.8) is replaced by the stronger hypothesis

$$\operatorname{grad} y^{\varepsilon'} \quad \text{is equi-integrable in } (L^2(\Omega))^N \tag{4.14}$$

one can actually prove that

$$u^{\varepsilon'} - u - y^{\varepsilon'} \to 0 \quad \text{strongly in } H_0^1(\Omega). \tag{4.15}$$

Convergence (4.15), which improves (4.11), is easily proved by combining (4.11), (4.14) and the estimate

$$\limsup_{\varepsilon' \to 0} \int_{|u^{\varepsilon'}|\geqq k} |\operatorname{grad} u^{\varepsilon'}|^2\,dx \leqq \omega_k \tag{4.16}$$

where $\omega_k \to 0$ if $k \to \infty$.

This latest estimate is obtained by multiplying (4.3) by $u^\varepsilon - T_k(u^\varepsilon)$ as in the first step of the proof of Theorem 2.2 above: one obtains

$$\int_\Omega A \operatorname{grad} u^\varepsilon \operatorname{grad}(u^\varepsilon - T_k(u^\varepsilon))\,dx = \langle f^\varepsilon, u^\varepsilon - T_k(u^\varepsilon)\rangle$$
$$= \langle f_0, u^\varepsilon - T_k(u^\varepsilon)\rangle + \int A \operatorname{grad} y^\varepsilon \operatorname{grad}(u^\varepsilon - T_k(u^\varepsilon))\,dx$$

and one uses the equi-integrability assumption (4.14) since the last term is controlled by some constant multiplied by $\left(\int_{|u^\varepsilon|\geqq k} |\operatorname{grad} y^\varepsilon|^2\,dx\right)^{1/2}$.

Note also that hypothesis (4.14) [and thus hypothesis (4.8)] is satisfied when $\partial\Omega$ is sufficiently smooth in order for Meyers' Theorem to hold true and when f^ε is "equi-integrable in $H^{-1}(\Omega)$", *i.e.* satisfies

$$f^\varepsilon - f^0 = -\operatorname{div} g^\varepsilon \quad \text{with } g^\varepsilon \text{ equi-integrable in } (L^2(\Omega))^N. \tag{4.17}$$

Indeed denoting by $T_k(g^\varepsilon)$ the $(L^2(\Omega))^N$ function obtained by applying the (scalar) truncation T_k to each component of g^ε, hypothesis (4.17) is equivalent to

$$\left.\begin{aligned} &\|g^\varepsilon - T_k(g^\varepsilon)\|_{(L^2(\Omega))^N} \leqq \varpi_k \quad \text{uniformly in } \varepsilon \\ &\text{where } \varpi_k \to 0 \text{ if } k \to \infty. \end{aligned}\right. \tag{4.18}$$

Decompose g^ε and y^ε as follows

$$g^\varepsilon = T_k(g^\varepsilon) + g^\varepsilon - T_k(g^\varepsilon)$$
$$y^\varepsilon = y_k^\varepsilon + \bar{y}_k^\varepsilon$$

where y_k^ε and $\bar{y}_k^\varepsilon$ are defined by

$$\left.\begin{aligned} -\operatorname{div}(A \operatorname{grad} y_k^\varepsilon) &= -\operatorname{div}(T_k(g^\varepsilon)) \quad \text{in } \mathscr{D}'(\Omega) \\ y_k^\varepsilon &\in H_0^1(\Omega) \end{aligned}\right. \tag{4.19}$$

$$\left.\begin{aligned} -\operatorname{div}(A \operatorname{grad} \bar{y}_k^\varepsilon) &= -\operatorname{div}(g^\varepsilon - T_k(g^\varepsilon)) \quad \text{in } \mathscr{D}'(\Omega) \\ \bar{y}_k^\varepsilon &\in H_0^1(\Omega). \end{aligned}\right. \tag{4.20}$$

In view of (4.18), (4.20) we have

$$\alpha \|\operatorname{grad} \bar{y}_k^\varepsilon\|_{(L^2(\Omega))^N} \leqq \varpi_k \quad \text{uniformly in } \varepsilon.$$

On the other hand, Meyers' regularity result (*see* e.g. Theorem A.5 in Appendix A below) ensures that for k fixed $\operatorname{grad} y_k^\varepsilon$ is bounded in $(L^p(\Omega))^N$ for some $p>2$ and is thus equi-integrable in $(L^2(\Omega))^N$. Combining these two results proves that (4.17) implies (4.14) when $\partial\Omega$ is sufficiently smooth. □

Proof of Theorem 4.2

Consider the test function

$$w^\varepsilon = \mathrm{T}_i(u^\varepsilon - \mathrm{T}_j(u) - \mathrm{T}_m(y^\varepsilon)) \tag{4.21}$$

where T_k is the truncation at the height k defined by (3.1). The test function w^ε belongs to $\mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and satisfies

$$\operatorname{grad} w^\varepsilon = 0 \quad \text{a.e. on } \{x \in \Omega;\ |u^\varepsilon(x)| \geqq i+j+m\}.$$

An alternative to Theorem 1.3 of [BDGM 1] (*see* Theorem 4 of [BDGM 2] or Theorem B.1 in Appendix B below) asserts that:

$$\int_\Omega \mathrm{A} \operatorname{grad} u^\varepsilon \operatorname{grad} w^\varepsilon \, dx + \int_\Omega \Phi_{i+j+m}(u^\varepsilon) \operatorname{grad} w^\varepsilon \, dx = \langle f^\varepsilon, w^\varepsilon \rangle \tag{4.22}$$

where Φ_{i+j+m} is the $(\mathrm{C}^0(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R}))^\mathrm{N}$ function defined by

$$\Phi_{i+j+m}(s) = \Phi(\mathrm{T}_{i+j+m}(s)), \qquad \forall s \in \mathbb{R}.$$

From (4.22) we deduce that

$$\begin{aligned} &\int_\Omega \mathrm{A} \operatorname{grad}(u^\varepsilon - \mathrm{T}_j(u) - \mathrm{T}_m(y^\varepsilon)) \operatorname{grad} w^\varepsilon \, dx \\ &= \langle f^\varepsilon - f^0, w^\varepsilon \rangle + \langle f^0, w^\varepsilon \rangle - \int_\Omega \Phi_{i+j+m}(u^\varepsilon) \operatorname{grad} w^\varepsilon \, dx \\ &- \int_\Omega \mathrm{A} \operatorname{grad} \mathrm{T}_j(u) \operatorname{grad} w^\varepsilon \, dx \\ &+ \int_\Omega \mathrm{A} \operatorname{grad}(y^\varepsilon - \mathrm{T}_m(y_\varepsilon)) \operatorname{grad} w^\varepsilon \, dx - \int_\Omega \mathrm{A} \operatorname{grad} y^\varepsilon \operatorname{grad} w^\varepsilon \, dx. \end{aligned} \tag{4.23}$$

Note that the first and the last terms of the right-hand side of (4.23) cancel in view of the definition (4.6) of y^ε. On the other hand since $w^{\varepsilon'}$ weakly tends to $\mathrm{T}_i(u - \mathrm{T}_j(u))$ in $\mathrm{H}_0^1(\Omega)$ it is easy to pass to the limit in the second, third and fourth terms of the right-hand side of (4.23), obtaining

$$\begin{aligned} \langle f^0, \mathrm{T}_i(u - \mathrm{T}_j(u)) \rangle - \int_\Omega \Phi_{i+j+m}(u) \operatorname{grad} \mathrm{T}_i(u - \mathrm{T}_j(u)) \, dx \\ - \int_\Omega \mathrm{A} \operatorname{grad} \mathrm{T}_j(u) \operatorname{grad} \mathrm{T}_i(u - \mathrm{T}_j(u)) \, dx = 0. \end{aligned} \tag{4.24}$$

In (4.24) the third term is zero since $\operatorname{grad} \mathrm{T}_j(u) \operatorname{grad} \mathrm{T}_i(u - \mathrm{T}_j(u)) = 0$ a.e. $x \in \Omega$. The second term in zero too, since defining ψ by

$$\psi(s) = \int_0^s \Phi_{i+j+m}(r) \mathrm{T}_i'(r - \mathrm{T}_j(r))(1 - \mathrm{T}_j'(r)) \, dr$$

[note that ψ is a Lipschitz-continuous, piecewise $(C^1(\mathbb{R}))^N$ function] we have using Stokes' Theorem:

$$\int_\Omega \Phi_{i+j+m}(u)\,\mathrm{grad}\,T_i(u-T_j(u))\,dx=\int_\Omega \mathrm{div}(\psi(u))\,dx=\int_{\partial\Omega}\psi(u)\,n\,ds=0.$$

Finally the fifth term of the right hand-side of (4.23) is estimated by

$$\begin{aligned}&\left|\int_\Omega A\,\mathrm{grad}(y^\varepsilon-T_m(y_\varepsilon))\,\mathrm{grad}\,w^\varepsilon\,dx\right|\\&\leqq \|A\|_{(L^\infty(\Omega))^{N\times N}}\left(\int_{|y^\varepsilon|\geqq m}|\mathrm{grad}\,y^\varepsilon|^2\,dx\right)^{1/2}\|\mathrm{grad}\,w^\varepsilon\|_{(L^2(\Omega))^N}\\&\leqq \|A\|_{(L^\infty(\Omega))^{N\times N}}C_0(Y_m)^{1/2}\end{aligned}\tag{4.25}$$

where Y_m is the number defined by (4.7) and where C_0 is a constant such that (recall that the truncation reduces the L^2 norm)

$$\begin{aligned}&\|\mathrm{grad}\,w^\varepsilon\|_{(L^2(\Omega))^N}\\&\leqq\|\mathrm{grad}\,u^\varepsilon\|_{(L^2(\Omega))^N}+\|\mathrm{grad}\,u\|_{(L^2(\Omega))^N}+\|\mathrm{grad}\,y^\varepsilon\|_{(L^2(\Omega))^N}\leqq C_0.\end{aligned}\tag{4.26}$$

Using the coerciveness of the matrix A in the right-hand side of (4.23) we have thus proved that

$$\begin{aligned}&\limsup_{\varepsilon'\to 0}\alpha\int_{|u^{\varepsilon'}-T_j(u)-T_m(y^{\varepsilon'})|\leqq i}|\mathrm{grad}(u^{\varepsilon'}-T_j(u)-T_m(y^{\varepsilon'}))|^2\,dx\\&\qquad\leqq\langle f^0, T_i(u-T_j(u))\rangle+\|A\|_{(L^\infty(\Omega))^{N\times N}}C_0(Y_m)^{1/2}\end{aligned}\tag{4.27}$$

Fix now k and define for j and m large

$$i=k+j+m;$$

the following inclusion holds true:

$$\{x\in\Omega;\ |u^\varepsilon(x)|\leqq k\}\subset\{x\in\Omega;\ |u^\varepsilon(x)-T_j(u(x))-T_m(y^\varepsilon(x))|\leqq i\}.\tag{4.28}$$

Since

$$\begin{aligned}\mathrm{grad}(u^{\varepsilon'}-u-y^{\varepsilon'})&=\mathrm{grad}(u^{\varepsilon'}-T_j(u)-T_m(y^{\varepsilon'}))\\&\quad+\mathrm{grad}(T_j(u)-u)+\mathrm{grad}(T_m(y^{\varepsilon'})-y^{\varepsilon'}),\end{aligned}$$

we have in view of (4.28),

$$\begin{aligned}&\int_{|u^{\varepsilon'}|\leqq k}|\mathrm{grad}\,u^{\varepsilon'}-u-y^{\varepsilon'})|^2\,dx\\&\leqq 3\int_{|u^{\varepsilon'}-T_j(u)-T_m(y^{\varepsilon})|\leqq i}|\mathrm{grad}(u^{\varepsilon'}-T_j(u)-T_m(y^{\varepsilon'})|^2\,dx\\&+3\int_\Omega|\mathrm{grad}(T_j(u)-u)|^2\,dx+3\int_\Omega|\mathrm{grad}(T_m(y^{\varepsilon'})-y^{\varepsilon'})|^2\,dx.\end{aligned}\tag{4.29}$$

Since

$$\omega_j=\|u-T_j(u)\|_{H_0^1(\Omega)}\to 0\quad\text{when } j\to\infty\tag{4.30}$$

we deduce from (4.29), (4.27) and (4.30) that

$$\begin{aligned}&\limsup_{\varepsilon'\to 0}\int_{|u^{\varepsilon'}|\leqq k}|\operatorname{grad}(u^{\varepsilon'}-u-y^{\varepsilon'})|^2\,dx\\&\leqq\frac{3}{\alpha}\{\|f^0\|_{H^{-1}(\Omega)}\,\omega_j+\|A\|_{(L^\infty(\Omega))^{N\times N}}C_0(Y_m)^{1/2}\}+3\,\omega_j^2+3\,Y_m.\end{aligned}\tag{4.31}$$

Since the right hand-side of (4.31) tends to zero when k is fixed while j and m tend to infinity, Theorem 4.2 is proved. □

APPENDIX A
WHAT DO YOU NEED TO KNOW ABOUT H-CONVERGENCE IN ORDER TO READ SECTIONS 2 AND 3

Recall that in the present paper ε denotes a sequence of strictly positive real numbers which converges to zero.

Define for a given bounded open subset Ω of $\mathbb{R}^N$ (no smoothness is assumed on $\partial\Omega$) and two real numbers α and β satisfying $0<\alpha<\beta$ the set of matrices:

$$\begin{aligned}\mathscr{M}(\alpha,\beta;\Omega)=\{&A\in(L^\infty(\Omega))^{N\times N};\\&A(x)\xi\xi\geqq\alpha|\xi|^2,\ A^{-1}(x)\xi\xi\geqq\beta^{-1}|\xi|^2,\\&\text{a. e. } x\in\Omega,\ \forall\xi\in\mathbb{R}^N\}.\end{aligned}\tag{A.1}$$

This set is bounded in $(L^\infty(\Omega))^{N\times N}$ since any element of $\mathscr{M}(\alpha,\beta;\Omega)$ satisfies [*see* (2.6)]

$$\|A\|_{(L^\infty(\Omega))^{N\times N}}\leqq\beta.\tag{A.2}$$

DEFINITION A.1. – *A sequence* A^ε *of* $\mathscr{M}(\alpha,\beta;\Omega)$ *is said to* H-*converge to a matrix* $A^0\in\mathscr{M}(\alpha,\beta;\Omega)$, *and this convergence is denoted by*

$$A^\varepsilon\overset{H}{\rightharpoonup}A^0$$

if and only if for any g *in* $H^{-1}(\Omega)$ *the sequence of the solutions* v^ε *of*

$$\begin{aligned}&-\operatorname{div}(A^\varepsilon\operatorname{grad}v^\varepsilon)=g\quad\textit{in }\mathscr{D}'(\Omega)\\&v^\varepsilon\in H_0^1(\Omega)\end{aligned}\tag{A.3}$$

satisfies

$$\begin{aligned}&v^\varepsilon\rightharpoonup v\quad\textit{weakly in } H_0^1(\Omega)\\&A^\varepsilon\operatorname{grad}v^\varepsilon\rightharpoonup A^0\operatorname{grad}v\quad\textit{weakly in }(L^2(\Omega))^N\end{aligned}\tag{A.4}$$

where v^0 *is the solution of*

$$\begin{aligned}&-\operatorname{div}(A^0\operatorname{grad}v)=g\quad\textit{in }\mathscr{D}'(\Omega)\\&v\in H_0^1(\Omega).\quad\square\end{aligned}\tag{A.5}$$

The above definition was introduced by S. Spagnolo [S] (under the name of G-convergence) in the case of symmetric matrices, and by L. Tartar [T 1] and F. Murat [M 1] in the non symmetric case. An extensive litterature on the topic in now available: *see* e. g. the books of A. Bensous-
san,
J.-L. Lions and G. Papanicolaou [BeLiP] and of E. Sanchez-Palencia [Sa] [which deal with the important case of periodic coefficients, *i. e.* the case where $A^\varepsilon(x)=A(x/\varepsilon)$ for some periodic matrix A defined on $\mathbb{R}^N$], as well as the survey paper of V. V. Zhikov, S. M. Kozlov, O. A. Oleinik and K. T. Ngoan [ZKON]; for a summary of the basic results of H-convergence, and specially for the corrector results (a topic which will not be discussed here) one can consult Section 2 of S. Brahim-Otsmane, G. A. Francfort and F. Murat [BrFM].

The above definition A. 1 of H-convergence is motivated by the following compactness results, due to S. Spagnolo [S] and L. Tartar [T 1] (*see* also [M 1]):

THEOREM A. 2. — *Any sequence of* $\mathcal{M}(\alpha, \beta; \Omega)$ *has a subsequence which* H-*converges to an element of* $\mathcal{M}(\alpha, \beta; \Omega)$. □

We emphasize in this Appendix the property of "convergence of the energy".

THEOREM A. 3. — *Consider a fixed right-hand side* $g\in H^{-1}(\Omega)$ *and a sequence* A^ε *of* $\mathcal{M}(\alpha, \beta; \Omega)$ *which* H-*convergence to a matrix* $A^0\in\mathcal{M}(\alpha, \beta; \Omega)$. *Defining respectively* v^ε *and* v *to be the solution of* (A. 3) *and* (A. 5) *we have*

$$A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon \rightharpoonup A^0 \operatorname{grad} v \operatorname{grad} v \quad \textit{weakly in } L^1_{\mathrm{loc}}(\Omega). \qquad (A.6)$$

This convergence takes place in $L^1(\Omega)$ (*and not only locally in* Ω) *when* $\partial\Omega$ *is sufficiently smooth.* □

The fact that $A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon$ converges in distributional sense to $A^0 \operatorname{grad} v \operatorname{grad} v$ easily results from (A. 4) and from integrations by parts in (A. 3) and (A. 5) after multiplication by φv^ε and φv, with φ in $\mathcal{D}(\Omega)$ [from another standpoint, this convergence in distributional sense is the simplest application of the theory of compensated compactness (*see* [T 2], [M 2]) since the divergence of $A^\varepsilon \operatorname{grad} v^\varepsilon$ is fixed while the curl of $\operatorname{grad} v^\varepsilon$ is identically zero]. The fact that the convergence (A. 6) takes place in the weak topology of $L^1_{\mathrm{loc}}(\Omega)$ (and not only in distributional sense) is actually an immediate consequence of the

PROPOSITION A. 4. — *Let* h *be fixed in* $H^{-1}(\Omega)$ *and let* w *be defined by*

$$\left.\begin{aligned} -\operatorname{div}(A \operatorname{grad} w) &= h \quad \textit{in } \mathcal{D}'(\Omega), \\ w &\in H^1_0(\Omega). \end{aligned}\right\} \qquad (A.7)$$

When A *varies in* $\mathcal{M}(\alpha, \beta; \Omega)$ (*with* $0<\alpha<\beta$ *fixed*) *the family of functions* $\operatorname{grad} w$ *is uniformly equi-integrable in* $(L^2_{loc}(\Omega))^N$, *i.e. for any fixed set* K *with* $\bar{K}\subset\Omega$, $\int_{K\cap E} |\operatorname{grad} w|^2\,dx$ *is small independently of* A *in* $\mathcal{M}(\alpha, \beta; \Omega)$ *and of the measurable set* E *whenever the measure* $|E|$ *is small. The choice* $K=\Omega$ *becomes licit when* $\partial\Omega$ *is sufficiently smooth: in such case* $\operatorname{grad} w$ *is uniformly equi-integrable in* $(L^2(\Omega))^N$. □

Proposition A.4 results from the easy estimate $\left[\right.$*see* (A.8) below for the definition of $\bar{w}$; here r is defined by $\frac{2}{p}+\frac{1}{r}=1\left.\right]$

$$\int_{K\cap E} |\operatorname{grad} w|^2\,dx \leqq 2\int_\Omega |\operatorname{grad}(w-\bar{w})|^2\,dx + 2\int_{K\cap E} |\operatorname{grad}\bar{w}|^2\,dx$$
$$\leqq \frac{2}{\alpha^2}\,\|h-\bar{h}\|^2_{H^{-1}(\Omega)} + 2\,\|\operatorname{grad}\bar{w}\|^2_{L^p(K)}\,|E|^{1/r}$$

and from an important regularity result due to N. G. Meyers [Me] (*see* also [BeLiP], Chapter 1, Section 25); a local version of this result can be stated as follows.

THEOREM A.5 (*N. G. Meyers*). — *For any measurable set* K *with* $\bar{K}\subset\Omega$ *there exists some number* $p>2$ *and some constant* C (*which only depend on* Ω, K, α *and* β) *such that for any* $\bar{h}$ *in* $W^{-1,p}(\Omega)$ *and any* A *in* $\mathcal{M}(\alpha, \beta; \Omega)$ *the solution* $\bar{w}$ *of*

$$\begin{cases} -\operatorname{div}(A\operatorname{grad}\bar{w})=\bar{h} \quad \text{in } \mathcal{D}'(\Omega) \\ \bar{w}\in H^1_0(\Omega) \end{cases} \tag{A.8}$$

belongs to $W^{1,p}(K)$ *and satisfies*

$$\|\bar{w}\|_{W^{1,p}(K)} \leqq C\,\|\bar{h}\|_{W^{-1,p}(\Omega)}. \tag{A.9}$$

The choice $K=\Omega$ *becomes licit when* $\partial\Omega$ *is sufficiently smooth.* □

APPENDIX B
USING SPECIAL TEST FUNCTIONS IN RENORMALIZED EQUATIONS

The following Theorem is in some sense an alternative to Theorem 1.3 of [BDGM 1] (*see* also Theorem 4 of [BDGM 2]).

THEOREM B.1. — *Assume that* (1.3)-(1.6) *hold true. Consider a solution* u *of the renormalized equation* (1.7), (1.8) *and a function* w *which satisfies*

$$\begin{array}{l} w \in H_0^1(\Omega) \\ \operatorname{grad} w = 0 \quad \text{a.e. on } \{x \in \Omega;\ |u(x)| \geq k\} \\ \text{for some positive number } k. \end{array} \tag{B.1}$$

The following equality then holds:

$$\int_\Omega A \operatorname{grad} u \operatorname{grad} w\, dx + \int_\Omega \Phi(T_k(u)) \operatorname{grad} w\, dx = \langle f, w \rangle \tag{B.2}$$

where T_k *is the "truncation at the height* k*" defined by* (3.1). □

Note that $\Phi \circ T_k$ belongs to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$; thus $\Phi(T_k(u))$ is an element of $(L^\infty(\Omega))^N$ and each term of (B.2) makes sense.

Proof of Theorem B.1

This proof is a variation of the proof of Theorem 1.3 of [BDGM 1].

First step. — Equation (1.8) is understood in distributional sense. Consider however a test function w which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Using a sequence of functions of $\mathscr{D}(\Omega)$ which converges to w in $H_0^1(\Omega)$ and remains bounded in $L^\infty(\Omega)$, it can be easily proved that if u is a solution of the renormalized equation (1.7), (1.8), then

$$\begin{aligned} &\int_\Omega h(u) A \operatorname{grad} u \operatorname{grad} w\, dx + \int_\Omega w h'(u) A \operatorname{grad} u \operatorname{grad} u\, dx \\ &+ \int_\Omega h(u) \Phi(u) \operatorname{grad} w\, dx + \int_\Omega w h'(u) \Phi(u) \operatorname{grad} u\, dx \\ &= \langle f, h(u) w \rangle = \int_\Omega h(u) g \operatorname{grad} w\, dx + \int_\Omega w h'(u) g \operatorname{grad} u\, dx \end{aligned} \tag{B.3}$$

for any w in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and any h in $C_c^1(\mathbb{R})$.

Note that each term makes sense in (B.3).

Second step. — Assume first that further to hypothesis (B.1), w also belongs to $L^\infty(\Omega)$.

Consider a function H in $C_c^1(\mathbb{R})$ such that

$$\begin{gathered} H(s) = 1 \quad \text{if } |s| \leq 1, \qquad H(s) = 0 \quad \text{if } |s| \geq 2 \\ 0 \leq H(s) \leq 1, \quad |H'(s)| \leq 2, \qquad \forall s \in \mathbb{R} \end{gathered}$$

and define $h_n(s)$ by

$$h_n(s) = \begin{cases} H(s+n) & \text{if } s \leq -n \\ 1 & \text{if } |s| \leq n \\ H(s-n) & \text{if } s \geq n \end{cases} \tag{B.4}$$

Since w belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$ and h_n to $C_c^1(\mathbb{R})$ we have in view of (B.3)

$$\begin{aligned}&\int_\Omega h_n(u)\,\mathrm{A}\,\mathrm{grad}\,u\,\mathrm{grad}\,w\,dx+\int_\Omega wh_n'(u)\,\mathrm{A}\,\mathrm{grad}\,u\,\mathrm{grad}\,w\,dx\\&+\int_\Omega h_n(u)\,\Phi(u)\,\mathrm{grad}\,w\,dx+\int_\Omega wh_n'(u)\,\Phi(u)\,\mathrm{grad}\,u\,dx\\&=\langle f,\,h(u)\,w\rangle=\int_\Omega h_n(u)\,g\,\mathrm{grad}\,w\,dx+\int_\Omega wh_n'(u)\,g\,\mathrm{grad}\,u\,dx.\end{aligned}\tag{B.5}$$

It is easy to pass to the limit in each term of (B.5) when n tends to infinity, mostly using Legesgue's dominated convergence Theorem: in the third term $\Phi(u)$ can be replaced by $\Phi(\mathrm{T}_k(u))$ [which belongs to $(L^\infty(\Omega))^N$] in view of hypothesis (B.1); the fourth term is zero whenever $n\geqq k$, since defining

$$\psi_n(s)=\int_0^s h_n'(r)\,\Phi(r)\,dr$$

which belongs to $(C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R}))^N$, we have using Stokes' Theorem and then hypothesis (B.1) combined with the fact that $\psi_n(s)=0$ when $|s|\leqq k\leqq n$:

$$\int_\Omega wh_n'(u)\,\Phi(u)\,\mathrm{grad}\,u\,dx=\int_\Omega w\,\mathrm{grad}\,\psi_n(u)\,dx=-\int_\Omega \psi_n(u)\,\mathrm{grad}\,w\,dx=0.$$

This proves that

$$\left.\begin{aligned}&\int_\Omega \mathrm{A}\,\mathrm{grad}\,u\,\mathrm{grad}\,w\,dx+\int_\Omega \Phi(\mathrm{T}_k(u))\,\mathrm{grad}\,w\,dx=\langle f,\,w\rangle\\&\text{for any } w \text{ in } L^\infty(\Omega) \text{ satisfying (B.1).}\end{aligned}\right.\tag{B.6}$$

Third step. — Consider now the general case where w satisfying (B.1) is not assumed to belong to $L^\infty(\Omega)$. The function $w_m=\mathrm{T}_m(w)$ obtained by the truncation of w to the height m is an admissible test function in (B.6). It is now easy to pass to the limit with m tending to infinity, recovering (B.2). □

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In a forthcoming joint paper with Pierre-Louis Lions we will prove that the renormalized solution of (1.7), (1.8) is unique if φ is assumed to be locally Lipschitz-continuous and if a zero order term $+\lambda uh(u)$ with $\lambda>0$ is added to the left-hand side of (1.8). In such a setting the extraction of a sub-sequence ε' in the statements of Theorems 2.1 and 2.2 [respectively in (4.5) and in the statement of Theorem 4.2] becomes unnecessary, since this uniqueness result of the renormalized solution of (2.7), (2.8) [respectively of (4.9), (4.10)] implies the convergences for the whole sequence ε.

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