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Maximum principles and a priori estimates for a class of problems from nonlinear elasticity

by

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ABSTRACT. — We consider smooth solutions, $\mathcal{U}$, to the nonlinear elliptic system associated with a two dimensional elastic material which has energy functional

$$\mathcal{W}(\mathcal{U})=\int_{\Omega}\left(\frac{|D\mathcal{U}|^2}{2}+H(\det D\mathcal{U})\right)dX.$$

The function $H(d)$ is nonnegative, convex and unbounded in a neighborhood of zero. Two maximum principles are proved for $D\mathcal{U}$ and we show that if $\Omega' \subset\subset \Omega$ then $\|D\mathcal{U}\|_{C^\alpha(\Omega')}$ and $\|D\mathcal{U}^{-1}\|_{L^\infty(\Omega')}$ are bounded *a priori* in terms of $\|D\mathcal{U}\|_{L^p(\Omega)}$ and $\mathcal{W}(\mathcal{U})$ for some $p=p(H)$.

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RÉSUMÉ. — On considère une solution régulière $\mathscr{U}$ du système elliptique non linéaire associé à la fonctionnelle d'énergie

$$\mathscr{W}(\mathscr{U}) = \int_\Omega \left(\frac{|D\mathscr{U}|^2}{2} + H(\det D\mathscr{U}) \right) dX$$

en dimension 2, la fonction H étant positive, convexe, et $H(t) \to +\infty$ quand $t \to 0_+$. On démontre deux principes du maximum et une estimation de $D\mathscr{U}$ à l'intérieur de Ω.

1. INTRODUCTION

In this paper we derive several *a priori* estimates for classical solutions of certain problems in two-dimensional compressible nonlinear elasticity.

We consider a two-dimensional elastic body occupying a reference configuration Ω in $\mathbb{R}^2$ where Ω is an open bounded set with smooth boundary. We define a *smooth deformation* of the body as a diffeomorphism,

$$\mathscr{U}(x, y) = (u(x, y), v(x, y)) \qquad \text{for} \quad (x, y) \in \bar{\Omega}$$

which satisfies $\mathscr{U} \in C^3(\Omega; \mathbb{R}^2) \cap C^1(\bar{\Omega}, \mathbb{R}^2)$ with $\det D\mathscr{U} = u_x v_y - u_y v_x > 0$ in $\bar{\Omega}$. We assume that the body is composed of a compressible neo-Hookean material. Its mechanical properties are then described by a stored energy function of the form

$$\sigma(F) = \frac{|F|^2}{2} + H(\det F) \tag{1.1}$$

for $F \in M_+^{2\times 2} \equiv \{ 2\times 2 \text{ real matrices with } \det F > 0 \}$. The function H is assumed to satisfy the following hypotheses:

1. $H \in C^3((0, \infty))$
2. $H \geqq 0$ and $H'' > 0$
3. For some positive constants s, c_1, c_2, and d_0,
$$c_1 t^{-s-k} \leqq (-1)^k \frac{d^k}{dt^k} H(t) \leqq c_2 t^{-s-k} \qquad \text{for} \quad 0 < t < d_0 \tag{1.2}$$
and $k = 0, 1, 2, 3$.
4. For some constant τ and positive constants c_3, c_4, and d_1,
$$c_3 t^\tau \leqq \frac{d^2}{dt^2} H(t) \leqq c_4 t^\tau \qquad \text{for} \quad t \geqq d_1.$$

Assumption 3 implies that $H(t)$ is proportional to t^{-s} as $t \to 0^+$. Thus σ satisfies the growth condition

$$(1.3) \qquad \sigma(F) \to \infty \quad \text{as } \det F \to 0^+.$$

This condition expresses the notion that it takes an infinite amount of energy to extend or compress a finite volume of material into zero volume. (*See* [1] for a detailed discussion of this condition.)

Under a smooth deformation, a point X in Ω is displaced to a point $\mathcal{U}(X)$. The total stored energy (neglecting body forces) is given by

$$\mathcal{W}(\mathcal{U}) = \int_\Omega \sigma(D\mathcal{U})\, dX$$

where $D\mathcal{U}$ denotes the gradient of $\mathcal{U}$. By our assumptions on $\mathcal{U}$, it follows that $\mathcal{U} + \varepsilon\Phi$ is also a smooth deformation for any $\Phi \in C_0^\infty(\Omega; \mathbb{R}^2)$ and ε sufficiently small. Thus one can compute the first variation of $\mathcal{W}$ at $\mathcal{U}$:

$$\frac{d}{d\varepsilon}\mathcal{W}(\mathcal{U} + \varepsilon\Phi)\Big|_{\varepsilon=0} = \int_\Omega \frac{\partial\sigma}{\partial F_{ij}}(D\mathcal{U})\, \Phi^i_{x_j}\, dX.$$

A *classical equilibrium solution* is defined to be a smooth deformation $\mathcal{U}$ whose first variation is zero. This gives the Euler-Lagrange equations

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{\partial\sigma}{\partial F_{ij}}(D\mathcal{U}(X))\right) = 0 \qquad \text{for} \quad i = 1, 2.$$

For σ as in (1.1) we obtain the system

$$(1.4) \qquad \left\{ \begin{array}{l} \Delta u + v_y . H'(d)_x - v_x . H'(d)_y = 0 \\ \Delta v - u_y . H'(d)_x + u_x . H'(d)_y = 0 \end{array} \right\} \qquad \text{for} \quad (x, y) \in \Omega$$

where $d \equiv \det D\mathcal{U} = u_x v_y - u_y v_x$. This system is elliptic since the strict Legendre-Hadamard condition

$$(1.5) \qquad \sum_{i, j, k, l=1}^{2} \frac{\partial^2 \sigma}{\partial F_{ij}\partial F_{kl}}(F)\, \lambda_i \lambda_k \pi_j \pi_l \geqq |\lambda|^2 |\pi|^2$$

holds for all λ, $\pi \in \mathbb{R}^2$ and all $F \in M_+^{2\times 2}$. The ellipticity is not uniform, however, since $D^2\sigma$ becomes singular at the boundary of $M_+^{2\times 2}$, *i. e.*,

$$\sup_{|\lambda|, |\pi|=1} \left(\sum_{i, j, k, l=1}^{2} \frac{\partial^2 \sigma(F)}{\partial F_{ij}\partial F_{kl}} \lambda_i \lambda_k \pi_j \pi_l \right) \geqq 1 + \frac{|F|^2}{2} . H''(\det F)$$

and the corresponding infimum is equal to 1 for all F in $M_+^{2\times 2}$.

Significant progress has been made in finding deformations that solve elliptic boundary value problems for stored energy functions γ whose structure is compatible with compressible nonlinear elasticity theory. Here γ has two important properties: first, it has the singular behavior described in (1.3) and second, it is frame-indifferent; that is, a rotation following a deformation leaves γ unchanged, so that

$$\gamma(\mathrm{QF})=\gamma(\mathrm{F}) \quad \text{for all } \mathrm{Q}\in \mathrm{SO}(2). \tag{1.6}$$

A class of functions which permits these properties is that of polyconvex functions defined by Ball [1]. Ball and Murat have shown (*see* [1] and [3]) that if γ is polyconvex and satisfies certain growth conditions, there exists a minimizer of $\int_\Omega \gamma(\mathrm{D}\mathscr{U})\,d\mathrm{X}$ among all functions $\mathscr{U}$ in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^2)$ satisfying $\det \mathrm{D}\mathscr{U}>0$ almost everywhere and taking on prescribed boundary values. (*See* Giaquinta, Modica and Souček [7] for an alternative approach to such problems.) The function σ of (1.1) is polyconvex and satisfies (1.6). Moreover, Ball and Murat's existence theorems apply to $\mathscr{W}(\mathscr{U})=\int_\Omega \sigma(\mathrm{D}\mathscr{U})\,d\mathrm{X}$. However, there are no regularity results to show that the minimizer lies in a smoother class of functions or that it is in fact a weak solution to the Euler-Lagrange equations.

The regularity theory for elliptic variational problems in two space dimensions is developed mainly in the case where γ is defined and finite-valued at all $\mathrm{F}\in\mathrm{M}^{2\times 2}=\{2\times 2 \text{ real matrices}\}$ and γ is convex. For instance if γ is C^2, $\mathrm{D}^2\gamma$ is uniformly positive definite and $|\mathrm{D}^2\gamma|$ is bounded, then it is known that any minimizer has Hölder continuous first derivatives in Ω. (*See* [6].) From this point, linear elliptic theory implies that $\mathscr{U}$ is as smooth as γ allows; e. g., if γ is $\mathrm{C}^{k,\alpha}$ then the minimizer is $\mathrm{C}^{k,\alpha}$ for $k\geq 2$ and $0<\alpha<1$. For the same problem in n space dimensions where $n\geq 3$ there are partial regularity results, *i. e.*, any minimizer is smooth on an open subset Ω_0 of Ω with $\mathscr{H}^p(\Omega\setminus\Omega_0)=0$ for some $p<n-2$. The condition that γ be convex is too restrictive for elasticity since it is not compatible with the principle of frame indifference (1.6). In fact, for σ as in (1.1), $\mathrm{D}^2\sigma$ is not positive definite on $\mathrm{M}_+^{2\times 2}$.

In recent work of Evans [4] (*see* also Evans and Gariepy [5]), the convexity of γ is replaced by a weaker condition related to polyconvexity and a partial regularity result is obtained. However, it is required that γ be continuous and finite-valued on $\mathrm{M}^{2\times 2}$ which rules out the singular behavior of σ in (1.3).

In all the works cited above where γ is locally bounded the main idea is to estimate the gradient of the solution, namely $\mathrm{D}\mathscr{U}$. On the other hand for solutions related to σ as in (1.1) one must simultaneously estimate $\mathrm{D}\mathscr{U}$ and $(\mathrm{D}\mathscr{U})^{-1}$. Our investigation shows that the special structure of σ allows one to deduce such bounds. As a result, we are able to get *a priori* estimates on classical equilibrium solutions of (1.4). In particular, we show that if $\Omega' \subset\subset \Omega$ then

$$\left\| \frac{1}{\det D\mathcal{U}} \right\|_{L^\infty(\Omega')} \leqq c \qquad \text{and} \qquad \| D\mathcal{U} \|_{C^\alpha(\Omega')} \leqq c$$

where c and α depend only on Ω, Ω', H, $\mathcal{W}(\mathcal{U})$, and $\| D\mathcal{U} \|_{L^p(\Omega)}$ for some $p=p(\mathrm{H})$ with $2<p<\infty$. (*See* Theorems 4.2 and 5.2) Moreover, we show that the functions, $d \equiv \det D\mathcal{U}$ and $z \equiv \frac{|D\mathcal{U}|^2}{2} + f(\det D\mathcal{U})$, where $f(d) = d\mathrm{H}'(d) - \mathrm{H}(d)$, are super and sub-solutions, respectively, for certain elliptic equations. As a result, they satisfy classical maximum principles in Ω.

It is our hope that the estimates presented here will help to produce a regularity theory for minimizers of $\mathcal{W}(\mathcal{U})$ or aid in establishing the existence of classical equilibrium solutions by a different approach.

Our paper is organized as follows. Assume $\mathcal{U}$ is a classical equilibrium solution in Ω and let $d=\det D\mathcal{U}$. In Section 2 we show that higher integrability of d^{-s} can be obtained from higher integrability of $|D\mathcal{U}|^2$. Recall that $\mathrm{H}(d) \sim d^{-s}$ for d near zero and hence

$$\| d^{-s} \|_{L^1(\Omega)} \leqq c.(1+\mathcal{W}(\mathcal{U})).$$

For any p with $1<p<\infty$ and $\Omega' \subset\subset \Omega$, we prove that

$$(1.7) \quad \| d^{-s} \|_{L^p(\Omega')} \leqq c_1(\mathrm{H}, \Omega').(1+\| f(d) \|_{L^p(\Omega')}) \\ \leqq c_2(\mathrm{H}, p, \Omega, \Omega').(1+\mathcal{W}(\mathcal{U}) + \| |D\mathcal{U}|^2 \|_{L^p(\Omega)}).$$

(*See* Corollary 2.3.)

In section 3 we prove maximum principles which give global bounds on $|D\mathcal{U}|$ and $|(D\mathcal{U})^{-1}|$. Let $\nu_1(\mathrm{X})$ and $\nu_2(\mathrm{X})$ be the singular values of $D\mathcal{U}$ at X, *i. e.*, the eigenvalues of $[D\mathcal{U}(D\mathcal{U})^{\mathrm{T}}]^{1/2}$ with $0<\nu_1 \leqq \nu_2 < \infty$. We have

$$(1.8) \quad |D\mathcal{U}| = (\nu_1^2+\nu_2^2)^{1/2} \qquad \text{and} \qquad |(D\mathcal{U})^{-1}| = \left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} \right)^{1/2}.$$

Thus it suffices to bound $\left(\frac{1}{\nu_1(\mathrm{X})} + \nu_2(\mathrm{X}) \right)$. We show that the functions $d(\mathrm{X})$ and $z(\mathrm{X}) \equiv \frac{|D\mathcal{U}|^2}{2} + f(d)$, where $f(d) = d\mathrm{H}'(d) - \mathrm{H}(d)$, satisfy super and subelliptic inequalities, respectively. (*See* Theorems 3.2 and 3.6.) Thus

$$\inf_{\Omega} d \geqq \inf_{\partial\Omega} d$$

and

$$\sup_{\Omega} z \leqq \sup_{\partial\Omega} z.$$

From our hypotheses on H it follows that

$$\sup_{\Omega}\left(\frac{1}{\nu_1}+\nu_2\right)\leqq c\,(\inf_{\partial\Omega} d,\ \sup_{\partial\Omega} z).$$

In section 4 we prove interior estimates. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and let $p=\max\left\{16, 12+\frac{16}{s}\right\}$. Then

$$\sup_{\Omega'}\frac{1}{d}\leqq c\,(\Omega', \Omega'', \|f(d)\|_{L^{p/2}(\Omega'')}, \|D\mathscr{U}\|_{L^p(\Omega'')})$$

and

$$\sup_{\Omega'} z\leqq c\,(\Omega', \Omega'', \|f(d)\|_{L^{p/2}(\Omega'')}, \|D\mathscr{U}\|_{L^p(\Omega'')}).$$

(*See* Theorems 4.2 and 4.4.) It follows from (1.7) that

$$\sup_{\Omega'}\left(\frac{1}{\nu_1}+\nu_2\right)\leqq c\,(\Omega, \Omega', \Omega'', H, \mathscr{W}(\mathscr{U}), \|D\mathscr{U}\|_{L^p(\Omega)}). \tag{1.9}$$

(*See* Theorem 4.5.)

Finally in section 5 we prove a Hölder estimate of $D\mathscr{U}$ depending only on $\beta_1=\inf_{\Omega}\nu_1$ and $\beta_2=\sup_{\Omega}\nu_2$. Since the system (1.4) is elliptic this is the estimate needed to get higher order ($C^{k,\alpha}$) interior estimates. We prove the following Caccioppoli inequality on $D\mathscr{U}$:

There exists a constant $c_1(\beta_1, \beta_2)$ so that for any ball $B_{2r}(X_0)\subset\Omega$ we have

$$\int_{B_r(X_0)}|D^2\mathscr{U}|^2\,dX\leqq\frac{c_1}{r^2}\int_{B_{2r}(X_0)}|D\mathscr{U}-\overline{D\mathscr{U}}_{x_0, 2r}|^2\,dX$$

where $\overline{D\mathscr{U}}_{X_0, 2r}\equiv\fint_{B_{2r}(X_0)} D\mathscr{U}\,dX$. (*See* Lemma 5.1.) It then follows from (1.9) and elliptic theory that

$$\|D\mathscr{U}\|_{C^\alpha(\Omega')}\leqq c_2$$

for some $\alpha>0$ where α and c_2 depend only on Ω, Ω', H, $\mathscr{W}(\mathscr{U})$, and $\|D\mathscr{U}\|_{L^p(\Omega)}$. (*See* Theorem 5.2.) From this and elliptic theory we obtain

$$\|\mathscr{U}\|_{C^{k,\beta}(\Omega')}\leqq c_3$$

for any $k\geqq 2$ and $0<\beta<1$ such that $H\in C^{k,\beta}_{\mathrm{loc}}(\mathbb{R}^+)$ where c_3 depends only on k, β, Ω, Ω', H, $\mathscr{W}(\mathscr{U})$, $\|\mathscr{U}\|_{L^2(\Omega)}$ and $\|D\mathscr{U}\|_{L^p(\Omega)}$. (*See* Theorem 5.4.)

2. L^p-ESTIMATES OF $\frac{1}{d}$

In this section we show that if $\mathcal{U}$ is a classical equilibrium solution, the L^p_{loc} norm of $d^{-s} \equiv \det D\mathcal{U})^{-s}$ is bounded *a priori* by the energy $\mathcal{W}(\mathcal{U})$ and the L^p-norm of $|D\mathcal{U}|^2$. (*See* Corollary 2.3.) To prove this we use the following system of partial differential equations which is equivalent to (1.4):

$$(2.1) \qquad \begin{cases} u_x \Delta u + v_x \Delta v + d.H'(d)_x = 0, \\ u_y \Delta u + v_y \Delta v + d.H'(d)_y = 0. \end{cases}$$

The first equation above is the sum of the equations obtained by multiplying $(1.4)_1$ by u_x and $(1.4)_2$ by v_x. The second equation is the sum of the equations obtained by multiplying $(1.4)_1$ by u_y and $(1.4)_2$ by u_y.

Define $f(d) = d.H'(d) - H(d)$ and note that $f'(d) = d.H''(d)$. Let $z = z(X) \equiv \frac{1}{2}|D\mathcal{U}|^2 + f(d)$. From (2.1) we deduce:

LEMMA 2.1. – *Assume $\mathcal{U}$ is a classical equilibrium solution. Then* $\Delta z = 2(u_{xy}^2 - u_{xx}.u_{yy}) + 2(v_{xy}^2 - v_{xx}.v_{yy})$.

Proof. – Differentiating $(2.1)_1$ with respect to x, $(2.1)_2$ with respect to y, and adding, we obtain

$$(2.2) \quad \begin{aligned} 0 = \{ (u_x \Delta u)_x + (v_x \Delta v)_x + (u_y \Delta u)_y + (v_y \Delta v)_y \} \\ + [d.H'(d)_x]_x + [d.H'(d)_y]_y = (u_x \Delta u_x + v_x \Delta v_x + u_y \Delta u_y + v_y \Delta v_y) \\ + (u_{xx} \Delta u + v_{xx} \Delta v + u_{yy} \Delta u + v_{yy} \Delta v) + \Delta f(d). \end{aligned}$$

Now

$$\begin{aligned} \Delta z \equiv \frac{1}{2} \Delta (u_x^2 + v_x^2 + u_y^2 + v_y^2 + \Delta f(d) \\ = (u_x \Delta u_x + v_x \Delta v_x + u_y \Delta u_y + v_y \Delta v_y) \\ + (|\nabla u_x|^2 + |\nabla v_x|^2 + |\nabla u_y|^2 + |\nabla v_y|^2) + \Delta f(d). \end{aligned}$$

Combining this with (2.2) we have

$$\begin{aligned} \Delta z = -(u_{xx} \Delta u + v_{xx} \Delta v + u_{yy} \Delta u + v_{yy} \Delta v) \\ + (|\nabla u_x|^2 + |\nabla v_x|^2 + |\nabla u_y|^2 + |\nabla v_y|^2) \\ = 2(u_{xy}^2 - u_{xx}.u_{yy}) + 2(v_{xy}^2 - v_{xx}.v_{yy}). \quad \square \end{aligned}$$

We are now in a position to prove:

THEOREM 2.2. – *Assume $\mathcal{U}$ is a classical equilibrium solution and* $B_{3r} \equiv B_{3r}(X_0) \subset \Omega$. *For any p in* $(1, \infty)$,

$$\begin{aligned} \| f(d) \|_{L^p(B_r)} \leqq c_1 . (\| 1 + |D\mathcal{U}|^2 + H(d) \|_{L^1(B_{3r})} + \| |D\mathcal{U}|^2 \|_{L^p(B_{3r})}) \\ \leqq c_2 . (1 + \mathcal{W}(\mathcal{U}) + \| |D\mathcal{U}|^2 \|_{L^p(B_{3r})}) \end{aligned}$$

where c_1 and c_2 are constants depending only on r, p, and H.

Proof. — By direct calculation,

$$2(u_{xy}^2 - u_{xx}.u_{yy}) + 2(v_{xy}^2 - v_{xx}.v_{yy}) = (u_x^2 + v_x^2)_{yy} - 2(u_x u_y + v_x v_y)_{xy} + (u_y^2 + v_y^2)_{xx}$$

for any C^3-functions, u and v. Combining this with Lemma 2.1, we obtain:

$$\Delta f(d) = \sum_{i,j=1}^{2} (e_{ij})_{x_i x_i} \tag{2.3}$$

where $X=(x, y)=(x_1, x_2)$ and $|e_{ij}| \leqq |D\mathcal{U}|^2$ for $1 \leqq i, j \leqq 2$.

Now choose $\eta \in C_c^\infty(B_{3r})$ with $0 \leqq \eta \leqq 1$, $\eta = 1$ on B_{2r}, $|\nabla \eta| \leqq \frac{c}{r}$, and $|D^2 \eta| \leqq \frac{c}{r^2}$ where c is independent of r. Let $g \in C_c^\infty(B_r)$ and set

$$w(X) = \frac{1}{2\pi} \cdot \int_{B_r} \log|X-Z| \, . g(Z) \, dZ.$$

Then

$$\begin{aligned}
\int_{B_r} f(d).g\,dX &= \int_{\mathbb{R}^2} f(d).\eta.\Delta w\,dX \\
&= -\int_{B_{3r}-B_{2r}} f(d).w.\Delta\eta\,dX - 2\int_{B_{3r}-B_{2r}} f(d).\langle \nabla w, \nabla \eta \rangle\,dX \\
&\quad + \int_{B_{3r}} \left(\sum_{i,j=1}^{2} e_{ij}.(\eta\, w)_{x_i x_j} \right) dX \\
&= \int_{B_{3r}-B_{2r}} \Bigg\{ -f(d).w.\Delta\eta - 2f(d).\langle \nabla w, \nabla \eta \rangle \\
&\quad + \sum_{i,j=1}^{2} e_{ij}.(w.\eta_{x_i x_j} + w_{x_i}\eta_{x_j} + w_{x_j}\eta_{x_i}) \Bigg\} dX \\
&\quad + \int_{B_{3r}} \eta . \sum_{i,j=1}^{2} e_{ij}.w_{x_i x_j}\,dX \equiv \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{2.4}$$

By Hölder's inequality and the definition of w,

$$|w| + |\nabla w| \leqq c_1(r, p).\|g\|_{L^q(B_r)} \quad \text{in } B_{3r} - B_{2r}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence

$$|\mathrm{I}| \leqq c_2(r,p).\left\{ \|f(d)\|_{L^1(B_{3r})} + \sum_{i,j=1}^{2} \|e_{ij}\|_{L^1(B_{3r})} \right\} . \|g\|_{L^q(B_r)}.$$

From our hypotheses on H [*See* (1.2)], we have

$$\begin{aligned} |f(d)| &\leqq |\mathrm{H}(d)| + |d.\mathrm{H}'(d)| \\ &\leqq \mathrm{M}_1 + \mathrm{M}_2 . d + \mathrm{M}_3 . |\mathrm{H}(d)| \end{aligned}$$

for $d>0$. Since $d \leqq |\mathrm{D}\mathscr{U}|^2$ and $|e_{ij}| \leqq |\mathrm{D}\mathscr{U}|^2$, it follows that

$$(2.5) \qquad |\mathrm{I}| \leqq c_3 . \| 1 + |\mathrm{D}\mathscr{U}|^2 + \mathrm{H}(d) \|_{\mathrm{L}^1(\mathrm{B}_{3r})} . \| g \|_{\mathrm{L}^q(\mathrm{B}_r)}$$

where c_3 depends on r, p, and H.

To estimate II, we note that by the Calderon-Zygmund inequality,

$$\begin{aligned} \| \mathrm{D}^2 w \|_{\mathrm{L}^q(\mathbb{R}^2)} &\leqq c_4 . \| g \|_{\mathrm{L}^q(\mathbb{R}^2)} \\ &= c_4 . \| g \|_{\mathrm{L}^q(\mathrm{B}_r)} \end{aligned}$$

and hence

$$(2.6) \qquad \begin{aligned} |\mathrm{II}| &\leqq \| \eta \|_{\mathrm{L}^\infty(\mathrm{B}_{3r})} . \sum_{i,j=1}^{2} \| e_{ij} \|_{\mathrm{L}^p(\mathrm{B}_{3r})} . \| \mathrm{D}^2 w \|_{\mathrm{L}^q(\mathbb{R}^2)} \\ &\leqq c_5 . \| |\mathrm{D}\mathscr{U}|^2 \|_{\mathrm{L}^p(\mathrm{B}_{3r})} . \| g \|_{\mathrm{L}^q(\mathrm{B}_r)} \end{aligned}$$

where c_4 and c_5 depend only on p.

By (2.4), (2.5), and (2.6), we have

$$\begin{aligned} \left| \int_{\mathrm{B}_r} f(d) . g \, d\mathrm{X} \right| &\leqq c_1(r, p, \mathrm{H}) \\ &\times \{ \| 1 + |\mathrm{D}\mathscr{U}|^2 + \mathrm{H}(d) \|_{\mathrm{L}^1(\mathrm{B}_{3r})} + \| |\mathrm{D}\mathscr{U}|^2 \|_{\mathrm{L}^p(\mathrm{B}_{3r})} \} . \| g \|_{\mathrm{L}^q(\mathrm{B}_r)} \\ &\leqq c_2(r, p, \mathrm{H}) . (1 + \mathscr{W}(\mathscr{U}) + \| |\mathrm{D}\mathscr{U}|^2 \|_{\mathrm{L}^p(\mathrm{B}_{3r})}) . \| g \|_{\mathrm{L}^q(\mathrm{B}_r)} \end{aligned}$$

and the theorem follows. □

Our hypothesis (1.2) implies that

$$(2.7) \qquad d^{-s} \leqq c |f(d)| \qquad \text{for} \quad 0 < d \leqq d_0.$$

Indeed on this interval $\mathrm{H}'(d) < 0$ so by $(1.2)_3$

$$d^{-s} \leqq c \mathrm{H}(d) \leqq c |d\mathrm{H}'(d) - \mathrm{H}(d)| = c |f(d)|.$$

From this and Theorem 2.2 we have:

COROLLARY 2.3. — *Assume $\mathscr{U}$ is a classical equilibrium solution and $\Omega' \subset\subset \Omega$. If $1 < p < \infty$, then*

$$\begin{aligned} \| d^{-s} \|_{\mathrm{L}^p(\Omega')} &\leqq c_1 . (1 + \| f(d) \|_{\mathrm{L}^p(\Omega')}) \\ &\leqq c_2 . (1 + \mathscr{W}(\mathscr{U}) + \| |\mathrm{D}\mathscr{U}|^2 \|_{\mathrm{L}^p(\Omega)}) \end{aligned}$$

where c_1 and c_2 depend on Ω, Ω', H, *and p.*

Our proof of Theorem 2.2 used duality and hence it requires only that $\mathscr{U}$ satisfy (2.3) in the sense of distributions. As a result, Theorem 2.2 can be extended to a weaker class of equilibrium solutions. We conclude this section by defining the notion of weak equilibrium solutions (due to Ball) and showing that Theorem 2.2 holds for such solutions.

Our definition is based on the following two results:

THEOREM 2.4 (See *Ball and Murat* [3].) — *Let*

$$\mathcal{A}=\{\mathcal{U}\in W^{1,2}(\Omega;\mathbb{R}^2):\det D\mathcal{U}>0 \textit{ a. e. in } \Omega \textit{ and } \mathcal{W}(\mathcal{U})<+\infty\}.$$

Suppose $\mathcal{U}_0\in\mathcal{A}$ and set

$$\mathcal{A}(\mathcal{U}_0)=\{\mathcal{U}\in\mathcal{A}:\mathcal{U}-\mathcal{U}_0\in W_0^{1,2}(\Omega;\mathbb{R}^2)\}.$$

Then $\mathcal{W}(\mathcal{U})$ *attains its minimum in* $\mathcal{A}(\mathcal{U}_0)$.

The minimizer of $\mathcal{W}$ in $\mathcal{A}(U_0)$ satisfies a system of partial differential equations which reduces to (1.4) for sufficiently smooth solutions. This follows from:

THEOREM 2.5 (*Ball*). — *Assume* $\mathcal{U}=(u^1, u^2)\in\mathcal{A}$. *Then for each* Φ *in* $C_0^\infty(\Omega;\mathbb{R}^2)$ *there is an* $\varepsilon_0>0$ *such that* $\mathcal{U}_\varepsilon(x;\Phi)\equiv\mathcal{U}(X+\varepsilon\Phi(X))\in\mathcal{A}$ *for* $|\varepsilon|\leqq\varepsilon_0$, $\left.\frac{d}{d\varepsilon}\mathcal{W}(\mathcal{U}_\varepsilon)\right|_{\varepsilon=0}$ exists and

$$\left.\frac{d}{d\varepsilon}\mathcal{W}(\mathcal{U}_\varepsilon)\right|_{\varepsilon=0}=\int_\Omega\left(-\sigma.\delta_k^j+u_{x_k}^i.\frac{\partial\sigma}{\partial u_{x_j}^i}\right).\Phi_{x_j}^k\,dX$$

where $\sigma\equiv\sigma(D\mathcal{U})=\dfrac{|D\mathcal{U}|^2}{2}+H(\det D\mathcal{U})$ *and* $X=(x_1, x_2)$. *In particular if* $\mathcal{U}$ *minimizes* $\mathcal{W}$ *in* $\mathcal{A}(\mathcal{U}_0)$, *then*

$$\left(-\sigma.\delta_k^j+u_{x_k}^i.\frac{\partial\sigma}{\partial u_{x_j}^i}\right)_{x_j}=0 \quad \textit{in } \Omega$$

in the sense of distributions for $k=1,2$.

The proof of this result is described briefly in [2]. We prove it an detail in Appendix A, and we also show that the above system of partial differential equations simplifies to:

$$(2.8)\qquad \begin{aligned} f(d)_x&=\frac{1}{2}(u_y^2+v_y^2-u_x^2-v_x^2)_x-(u_x u_y+v_x v_y)_y,\\ f(d)_y&=\frac{1}{2}(u_x^2+v_x^2-u_y^2-v_y^2)_y-(u_x u_y+v_x v_y)_x \end{aligned}$$

in Ω in the sense of distributions where $\mathcal{U}=(u^1, u^2)\equiv(u, v)$ and $X=(x_1, x_2)\equiv(x, y)$.

Based on these results, we make the following definition.

DEFINITION 2.6. — Suppose $\mathcal{U}\in\mathcal{A}$. Then $\mathcal{U}$ is said to be a *weak equilibrium solution* in Ω if $0=\left.\frac{d}{d\varepsilon}\mathcal{W}(\mathcal{U}_\varepsilon(X;\Phi))\right|_{\varepsilon=0}$ for all Φ in $C_0^\infty(\Omega;\mathbb{R}^2)$, or equivalently, if (2.8) holds in the sense of distributions.

By Theorems 2.4 and 2.5, weak equilibrium solutions in $\mathcal{A}(\mathcal{U}_0)$ always exist. Differentiating the first equation in (2.8) with respect to x, the second with respect to y, and adding, we get (2.3) in $\mathcal{D}'(\Omega)$. From the proofs of Theorem 2.2 and Corollary 2.3 we conclude:

THEOREM 2.7. – *Let* $\mathcal{U}$ *be a weak equilibrium solution in* Ω *with* $|D\mathcal{U}|^2 \in L^p_{loc}(\Omega)$ *for some* $p \in (1, \infty)$. *Then* $d^{-s} \in L^p_{loc}(\Omega)$ *and if* $\Omega' \subset\subset \Omega'' \subset \Omega$,

$$\begin{aligned}\|d^{-s}\|_{L^p(\Omega')} &\leqq c_1 \cdot (1 + \|f(d)\|_{L^p(\Omega')}) \\ &\leqq c_2 \cdot (1 + \mathcal{W}(\mathcal{U}) + \||D\mathcal{U}|^2\|_{L^p(\Omega'')})\end{aligned}$$

where c_1 *and* c_2 *depend on* Ω', Ω'', H *and* p.

3. MAXIMUM PRINCIPLES FOR z AND d

In this section we prove via maximum principles that if $\mathcal{U}$ is a classical equilibrium solution, the functions $\frac{1}{d} \equiv \frac{1}{\det D\mathcal{U}}$ and $z \equiv \frac{|D\mathcal{U}|^2}{2} + f(d)$ attain their maxima on the boundary of Ω. As a result, we obtain global bounds on $|D\mathcal{U}|$ and $|D\mathcal{U}^{-1}|$ in terms of their boundary values in Ω.

Our proof is based on showing that z and d are sub and super solutions, respectively, for certain elliptic equations. We use the fact that $\sigma(F)$ is invariant under rotations in the reference and current configurations.

Assume that P and Q are in SO(2), *i. e.* P and Q are real, orthogonal matrices with $\det P = \det Q = 1$. Assume $V \in C^2(B_r(X_0); \mathbb{R}^2)$ with $\det DV > 0$. Define $\tilde{V}(X')$ on $B_r \equiv B_r(X_0)$ by

$$\begin{aligned}\tilde{V}(X') &= P . V(X_0 + Q(X' - X_0)) \\ &= P . V(X)\end{aligned}$$

where $X = X_0 + Q(X' - X_0)$. It follows easily that

$$\begin{aligned}&|D_{X'} \tilde{V}(X')| = |D_X V(X)|, \\ &\det D_{X'} \tilde{V}(X') = \det D_X V(X), \\ &\text{and } |D^2_{X'} \tilde{V}(X')| = |D^2_X V(X)|.\end{aligned} \tag{3.1}$$

Hence

$$\int_{B_r} \sigma(D_{X'} \tilde{V}(X'))\, dX' = \int_{B_r} \sigma(D_X V(X))\, dX.$$

From this we obtain:

PROPOSITION 3.1. – *Assume* $\mathcal{U}$ *is a classical equilibrium solution in* $B_r \equiv B_r(X_0) \subset \Omega$. *Then* $\tilde{\mathcal{U}}(X') \equiv (\tilde{u}(x', y'), \tilde{v}(x', y'))$ *satisfies equations* (1.4) *in the variables* $X' = (x', y')$ *in* B_r.

Proof. — From our calculations on σ, we have:

$$0=\frac{d}{d\varepsilon}\int_{B_r}\sigma(D_X(\mathscr{U}+\varepsilon\Phi))\,dX\Big|_{\varepsilon=0}$$

for all Φ in $C_0^1(B_r;\,\mathbb{R}^2)$ and hence

$$0=\frac{d}{d\varepsilon}\int_{B_r}\sigma(D_{X'}(\tilde{\mathscr{U}}+\varepsilon\tilde{\Phi}))\,dX'\Big|_{\varepsilon=0}. \quad \square$$

For $\mathscr{U}$ as above and $B_r=B_r(X_0)\subset\Omega$, we may choose P and Q so that $D\tilde{\mathscr{U}}(X_0)$ is diagonal. Indeed, since $\det D\mathscr{U}(X_0)>0$, it has a polar decomposition: $D\mathscr{U}(X_0)=CR$, where C is symmetric and positive definite and $R\in SO(2)$. Hence $D\mathscr{U}(X_0)=P^T\Lambda PR$ where $P\in SO(2)$ and Λ is diagonal and positive definite. Setting $Q=R^TP^T$ we have $D_{X'}\tilde{\mathscr{U}}(X_0)=\Lambda$ where $X'=X_0+Q^T(X-X_0)$ in B_r. We use this to prove:

THEOREM 3.2. — *Let $\mathscr{U}$ be a classical equilibrium solution and set*

$$z(X)=z(X;\,\mathscr{U})=\frac{1}{2}|D\mathscr{U}(X)|^2+f(\det D\mathscr{U}(X)).$$

Then $\Delta z\geqq -H''(d).|\nabla z|^2$ in Ω.

Proof. — Fix $B_r\equiv B_r(X_0)\subset\Omega$. By (3.1),

$$z(X)=\frac{1}{2}|D\tilde{\mathscr{U}}(X')|^2+f(\det D\tilde{\mathscr{U}}(X'))=z(X';\,\tilde{\mathscr{U}})\equiv\varphi(X')\quad\text{in } B_r.$$

Since the Laplacian is invariant under translations and rotations, $\Delta_X z(X_0)=\Delta_{X'}\varphi(X_0)$. From this, (3.1), and Proposition 3.1 we may assume without loss of generality that

$$D\mathscr{U}(X_0)=\begin{bmatrix}u_x(X_0) & 0\\ 0 & v_y(X_0)\end{bmatrix}$$

with $u_x(X_0)>0$ and $v_y(X_0)>0$.

Consider the partial differential equation for z found in Lemma 2.1, namely

$$\Delta z=2(v_{xy}^2-u_{xx}.u_{yy})+2(u_{xy}^2-v_{xx}.v_{yy}).$$

At X_0:

$$(3.2)\quad \begin{aligned} v_{xy}^2-u_{xx}.u_{yy}&=v_{xy}^2+u_{xx}(u_{xx}+H'(d)_x.v_y)\\ &=v_{xy}^2+u_{xx}^2+H''(d).(v_y^2.u_{xx}^2+v_y.u_x.u_{xx}.v_{xy})\end{aligned}$$

where we used (1.4) for the first equality and

$$d_x(X_0)=[u_{xx}.v_y+u_x.v_{xy}](X_0)$$

for the second. Now

$$z_x=u_x.u_{xx}+v_y.v_{xy}+u_x.v_yH''(d).(v_y.u_{xx}+u_xv_{xy})\quad\text{at } X_0.$$

Solving for v_{xy} in this equation we obtain

$$(3.3)\qquad v_{xy}=\frac{z_x-u_x\,.\,u_{xx}(1+v_y^2\,.\,\mathrm{H}''(d))}{v_y(1+u_x^2\,.\,\mathrm{H}''(d))}\quad \text{at } \mathrm{X}_0.$$

By (3.2) and (3.3), we have

$$\begin{aligned}
v_{xy}^2-u_{xx}\,.\,u_{yy}&\\
&=v_{xy}^2+u_{xx}^2\,.\,(1+v_y^2\,.\,\mathrm{H}''(d))+\mathrm{H}''(d)\,.\,v_y\,.\,u_x\,.\,u_{xx}\,.\,v_{xy}\\
&=v_{xy}^2+u_{xx}^2\,.\,(1+v_y^2\,.\,\mathrm{H}''(d))\\
&\quad+v_y\,.\,u_x\,.\,\mathrm{H}''(d)\,.\,u_{xx}\,.\,\frac{[z_x-u_x\,.\,u_{xx}\,.\,(1+v_y^2\,.\,\mathrm{H}''(d))]}{v_y(1+u_x^2\,.\,\mathrm{H}''(d))}\\
&=v_{xy}^2+(1+v_y^2\,.\,\mathrm{H}''(d))\,.\,u_{xx}^2\,.\left[1-\frac{u_x^2\,.\,\mathrm{H}''(d)}{1+u_x^2\,.\,\mathrm{H}''(d)}\right]\\
&\quad+\frac{u_x\,.\,\mathrm{H}''(d)}{(1+u_x^2\,.\,\mathrm{H}''(d))}\,.\,z_x\,.\,u_{xx}\\
&=v_{xy}^2+\frac{[(1+v_y^2\,.\,\mathrm{H}''(d))\,.\,u_{xx}^2+u_x\,.\,\mathrm{H}''(d)\,.\,u_{xx}\,.\,z_x]}{(1+u_x^2\,.\,\mathrm{H}''(d))}\quad \text{at } \mathrm{X}_0.
\end{aligned}$$

Hence

$$\begin{aligned}
(3.4)\quad v_{xy}^2-u_{xx}\,.\,u_{yy}&\geqq v_{xy}^2+\frac{(u_{xx}^2+u_x\,.\,\mathrm{H}''(d)\,.\,u_{xx}\,.\,z_x)}{(1+u_x^2\,.\,\mathrm{H}''(d))}\\
&=v_{xy}^2+\frac{[(u_{xx}+(u_x\,.\,\mathrm{H}''(d)/2)\,.\,z_x)^2-(z_x^2\,.\,\mathrm{H}''(d)/4)\,.\,u_x^2\,.\,\mathrm{H}''(d)]}{(1+u_x^2\,.\,\mathrm{H}''(d))}\\
&\geqq-\frac{z_x^2\,.\,\mathrm{H}''(d)}{4}\quad \text{at } \mathrm{X}_0.
\end{aligned}$$

A similar argument gives

$$(3.5)\qquad \begin{aligned}
u_{xy}^2-v_{xx}\,.\,v_{yy}&\geqq u_{xy}^2+\frac{(v_{yy}^2+v_y\,.\,\mathrm{H}''(d)\,.\,v_{yy}\,.\,z_y)}{(1+v_y^2\,.\,\mathrm{H}''(d))}\\
&\geqq-\frac{z_y^2\,.\,\mathrm{H}''(d)}{4}\quad \text{at } \mathrm{X}_0.
\end{aligned}$$

Thus $\Delta z\geqq-\mathrm{H}''(d)\,.\,|\nabla z|^2$. □

By definition of classical equilibrium solutions, we have $f(d)$, $\mathrm{H}''(d)$, $|\nabla f(d)|$, and $|\nabla z|$ locally bounded in Ω. From this and Theorem 3.2 we obtain:

THEOREM 3.3. — *Assume* $\mathcal{U}$ *is a classical equilibrium solution. Then* $z(\mathrm{X})$ *satisfies the strong maximum principle*, i. e.

$$z(\mathrm{X})\leqq\sup_{\partial\Omega} z$$

for each X *in* Ω *with equality holding if and only if* $z\equiv$ *constant. Moreover if* $z\equiv$ *constant then* $\mathcal{U}$ *is affine*, i. e. $\mathcal{U}(\mathrm{X})=\mathrm{AX}+\mathbf{b}$.

Proof. — The strong maximum principle follows from the fact that z is a subsolution for the elliptic equation, $\Delta z + \mathrm{H}''(d).|\nabla z|^2 = 0$. To prove the second assertion, it suffices to show that if $z \equiv$ constant then $\mathrm{D}^2 \mathcal{U} = 0$ in Ω. Fix X_0 in Ω. From (3.1) it follows without loss of generality that we may assume $\mathrm{D}\mathcal{U}(\mathrm{X}_0)$ is diagonal. If $z \equiv$ constant, then $z_x \equiv z_y \equiv 0$. By (3.4) and (3.5), we have

$$0 = \frac{1}{2}.\Delta z \geqq v_{xy}^2 + \frac{u_{xx}^2}{(1 + u_x^2.\mathrm{H}''(d))} + u_{xy}^2 + \frac{v_{yy}^2}{(1 + v_y^2.\mathrm{H}''(d))} \quad \text{at } \mathrm{X}_0.$$

Hence $v_{xy} = u_{xx} = u_{xy} = v_{yy} = 0$ at X_0. It follows that

$$d_x(\mathrm{X}_0) = [u_{xx} v_y + u_x v_{yx}](\mathrm{X}_0) = 0$$

and

$$d_y(\mathrm{X}_0) = [u_{xy} v_y + u_x v_{yy}](\mathrm{X}_0) = 0.$$

From this and (1.4) we have $\mathrm{D}^2 u(\mathrm{X}_0) = \mathrm{D}^2 v(\mathrm{X}_0) = 0$. $\square$

We now proceed to obtain an elliptic equation for which $d(\mathrm{X}) = \det \mathrm{D}\mathcal{U}(\mathrm{X})$ is a supersolution. Let F_1 and F_2 be linear operators defined by

$$\mathrm{F}_1(w) = (v_y w)_x - (v_x w)_y,$$
$$\mathrm{F}_2(w) = (u_x w)_y - (u_y w)_x,$$

where $\mathcal{U} = (u, v)$ is a given chassical equilibrium solution. Applying F_1 to $(1.4)_1$, F_2 to $(1.4)_2$, and adding we get:

$$\begin{aligned} &v_y \Delta u_x + u_x \Delta v_y - v_x \Delta u_y - u_y \Delta v_x \\ &\quad + [(v_y^2 + u_y^2).\mathrm{H}'(d)_x]_x - [(u_x u_y + v_x v_y).\mathrm{H}'(d)_x]_y \\ &\qquad - [(u_x u_y + v_x v_y).\mathrm{H}'(d)_y]_x + [(v_x^2 + u_x^2).\mathrm{H}'(d)_y]_y = 0. \end{aligned}$$

Adding $2.\{\langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle\}$ to both sides and using the identity,

$$\Delta d = v_y.\Delta u_x + u_x.\Delta v_y - v_x.\Delta u_y - u_y.\Delta v_x + 2.\{\langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle\},$$

we obtain

$$\begin{aligned} (3.6)\quad \mathrm{L}_1(d) \equiv \Delta d &+ [(u_y^2 + v_y^2).\mathrm{H}''(d).d_x]_x \\ &- [(u_x u_y + v_x v_y).\mathrm{H}''(d).d_y]_x - [(u_x u_y + v_x v_y).\mathrm{H}''(d).d_x]_y \\ &+ [(u_x^2 + v_x^2).\mathrm{H}''(d).d_y]_y = 2.\{\langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle\}. \end{aligned}$$

Note that $\mathrm{L}_1(d)$ is an elliptic operator. In fact, we have:

PROPOSITION 3.4. — *Assume $\mathcal{U}$ is a classical equilibrium solution. Define $[a_{ij}]$ by $\mathrm{L}_1(d) \equiv \sum_{i,j=1}^{2} (a_{ij}(\mathrm{X}).d_{x_i})_{x_j}$ where $\mathrm{X} = (x, y) \equiv (x_1, x_2)$. Let $\lambda_1 \leqq \lambda_2$ be the eigenvalues of $[a_{ij}]$. Then $\lambda_1 = 1 + \mathrm{v}_1^2.\ \mathrm{H}''(d)$ and $\lambda_2 = 1 + \mathrm{v}_2^2.\ \mathrm{H}''(d)$*

where $0<\nu_1 \leqq \nu_2$ *are the singular values of* $D\mathcal{U}$. *In particular,*

$$(3.7) \qquad [1+\nu_1^2 . H''(d)] . |\xi|^2 \leqq \sum_{i,\, j=1}^{2} a_{ij} . \xi_i \xi_j \leqq [1+\nu_2^2 . H''(d)] . |\xi|^2$$

for all $\xi \in \mathbb{R}^2$ *and* L_1 *is an elliptic operator.*

Proof. – By direct calculation,

$$tr\,[a_{ij}] = 2 + (u_y^2 + v_y^2 + u_x^2 + v_x^2) . H''(d) = 2 + |D\mathcal{U}|^2 . H''(d)$$

and

$$\det [a_{ij}] = 1 + (u_x^2 + v_x^2 + u_y^2 + v_y^2) . H''(d) + [(u_y^2 + v_y^2)(u_x^2 + v_x^2) - (u_x u_y + v_x v_y)^2] . H''(d)^2 = 1 + |D\mathcal{U}|^2 . H''(d) + d^2 . H''(d)^2.$$

Since a_{ij} is a symmetric 2×2 matrix, its eigenvalues are uniquely determined by these quantities. Hence

$$\lambda_1 = 1 + \nu_1^2 . H''(d) \qquad \text{and} \qquad \lambda_2 = 1 + \nu_2^2 . H''(d). \quad \square$$

Our maximum principle for d follows from an equation derived from (3.6). We shall need:

LEMMA 3.5. – *Assume* P *and* Q *are in* $SO(2)$ *and* $V \in C^2(B_r(X_0))$. *Define* $\tilde{V}$ *on* $B_r(X_0)$ *by* $\tilde{V}(X') = P . V(X)$ *where* $X = X_0 + Q(X' - X_0)$. *Let* $V(X) = (u(x, y), v(x, y))$ *and* $\tilde{V}(X') = (\tilde{u}(x', y'), \tilde{v}(x', y'))$. *Then*

$$\langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle = \langle \nabla \tilde{u}_{x'}, \nabla \tilde{v}_{y'} \rangle - \langle \nabla \tilde{u}_{y'}, \nabla \tilde{v}_{x'} \rangle.$$

Proof. – Since $DV = P^T . D\tilde{V} . Q^T$, we have

$$\frac{\partial v^i}{\partial x_j} = \sum_{k,\, l=1}^{2} p_{ki} . \frac{\partial \tilde{v}^k}{\partial x'_l} . q_{jl}$$

and

$$\nabla v^i_{x_j} = \sum_{k,\, l=1}^{2} p_{ki} . \nabla \tilde{v}^k_{x'_l} . Q^T . q_{jl},$$

where

$$(v^1, v^2) = (u, v), \qquad (x_1, x_2) = (x, y), \qquad (\tilde{v}^1, \tilde{v}^2) = (\tilde{u}, \tilde{v}),$$

and $(x'_1, x'_2) = (x', y')$. Hence

$$\begin{bmatrix} \nabla u_x & \nabla u_y \\ \nabla v_x & \nabla v_y \end{bmatrix} = P^T . \begin{bmatrix} \nabla \tilde{u}_{x'} . Q^T & \nabla \tilde{u}_{y'} . Q^T \\ \nabla \tilde{v}_{x'} . Q^T & \nabla \tilde{v}_{y'} . Q^T \end{bmatrix} . Q^T$$

where the matrices in brackets are 2×2 "block matrices" whose entries are in $\mathbb{R}^2$, and the multiplication on the right is defined as in matrix multiplication of three 2×2 matrices. Let the product of two vectors in $\mathbb{R}^2$ be their inner product. Then

$$\begin{aligned}\langle \nabla u_x, \nabla v_y\rangle - \langle \nabla u_y, \nabla v_x\rangle &= \det\begin{bmatrix}\nabla u_x & \nabla u_y\\ \nabla v_x & \nabla v_y\end{bmatrix}\\ &= \det\left\{\mathrm{P}^T.\begin{bmatrix}\nabla \tilde{u}_{x'}.\mathrm{Q}^T & \nabla \tilde{u}_{y'}.\mathrm{Q}^T\\ \nabla \tilde{v}_{x'}.\mathrm{Q}^T & \nabla \tilde{v}_{y'}.\mathrm{Q}^T\end{bmatrix}.\mathrm{Q}^T\right\}\\ &= \det\begin{bmatrix}\nabla \tilde{u}_{x'}.\mathrm{Q}^T & \nabla \tilde{u}_{y'}.\mathrm{Q}^T\\ \nabla \tilde{v}_{x'}.\mathrm{Q}^T & \nabla \tilde{v}_{y'}.\mathrm{Q}^T\end{bmatrix}\\ &= \langle \nabla \tilde{u}_{x'}, \nabla \tilde{v}_{y'}\rangle - \langle \nabla \tilde{u}_{y'}, \nabla \tilde{v}_{x'}\rangle. \quad \square\end{aligned}$$

We can now prove:

THEOREM 3.6. – *Assume $\mathcal{U}$ is a classical equilibrium solution and define $[a_{ij}]$ as in Proposition 3.4. Then*

$$(3.8)\quad \mathrm{L}_1(d) \equiv \sum_{i,\, j=1}^{2} (a_{ij} d_{x_i})_{x_j} \leqq -\frac{1}{4}.\frac{\nu_1}{\nu_2}.|\mathrm{D}^2\mathcal{U}|^2 + \frac{1}{d}.[2+\mathrm{M}_1.|\mathrm{D}\mathcal{U}|^4.\mathrm{H}''(d)^2].|\nabla d|^2$$

and

$$(3.9)\quad \mathrm{L}_2(d) \equiv \sum_{i,\, j=1}^{2} a_{ij}.d_{x_i x_j} \leqq \frac{1}{d}.[2+\mathrm{M}_2.|\mathrm{D}\mathcal{U}|^4.\mathrm{H}''(d)^2 + \mathrm{M}_3.d.|\mathrm{H}'''(d)|.|\mathrm{D}\mathcal{U}|^2].|\nabla d|^2$$

in Ω where M_1, M_2, and M_3 are universal constants (independent of H, $\mathcal{U}$, *and* Ω).

Proof. – First we prove (3.8). By (3.6) above,

$$\mathrm{L}_1(d) = 2.\{\langle \nabla u_x, \nabla v_y\rangle - \langle \nabla u_y, \nabla v_x\rangle\} \equiv \mathrm{I}.$$

We wish to estimate I from above and we do this in terms of $|\nabla d|^2$. Fix an arbitrary X_0 in Ω and choose $r>0$ so that $\mathrm{B}_r(\mathrm{X}_0)\subset\Omega$. Recall that there exists P and Q in SO (2) (depending on X_0) so that if $\tilde{\mathcal{U}}(\mathrm{X}') = \mathrm{P}.\mathcal{U}(\mathrm{X})$ with $\mathrm{X} = \mathrm{X}_0 + \mathrm{Q}(\mathrm{X}'-\mathrm{X}_0)$ then $\mathrm{D}\tilde{\mathcal{U}}(\mathrm{X}_0)$ is diagonal and positive define. Note that $\tilde{d}(\mathrm{X}') \equiv \det(\mathrm{D}\tilde{\mathcal{U}}(\mathrm{X}')) = \det(\mathrm{D}\mathcal{U}(\mathrm{X})) \equiv d(\mathrm{X})$ and $|\nabla\tilde{d}(\mathrm{X}')| = |\nabla d(\mathrm{X})|$. From this, Proposition 3.1, and Lemma 3.5, we may assume without loss of generality that $u_y = v_x = 0$ and $\{u_x, v_y\} = \{\nu_1, \nu_2\}$ at X_0. By direct calculation,

$$\begin{aligned}\mathrm{I} &= 4\langle \nabla u_x, \nabla v_y\rangle - 2\{\langle \nabla u_x, \nabla v_y\rangle + \langle \nabla u_y, \nabla v_x\rangle\}\\ &= 4\langle \nabla u_x, \nabla v_y\rangle - 2.(v_{xy}.\Delta u + u_{xy}.\Delta v).\end{aligned}$$

Combining with (1.4) we obtain

$$\mathrm{I}=4\langle\nabla u_x, \nabla v_y\rangle+2\, v_{xy}.v_y.\mathrm{H}'(d)_x+2\, u_{xy}.u_x.\mathrm{H}'(d)_y \quad \text{at } \mathrm{X}_0.$$

Now

$$\nabla d=v_y.\nabla u_x+u_x.\nabla v_y-u_y.\nabla v_x-v_x.\nabla u_y.$$

Evaluating at X_0, we have:

$$\nabla d=v_y.\nabla u_x+u_x.\nabla v_y,$$
$$|\nabla d|^2=v_y^2.|\nabla u_x|^2+2\, d.\langle\nabla u_x, \nabla v_y\rangle+u_x^2.|\nabla v_y|^2,$$

and hence

$$(3.10)\quad \mathrm{I}=\frac{2}{d}.\{|\nabla d|^2-v_y^2.|\nabla u_x|^2-u_x^2.|\nabla v_y|^2\}$$
$$+2\, v_y.v_{xy}.\mathrm{H}''(d).d_x+2\, u_x.u_{xy}.\mathrm{H}''(d).d_y \quad \text{at } \mathrm{X}_0.$$

From $(1.4)_1$ we have $u_{yy}=-u_{xx}-v_y.\mathrm{H}''(d).d_x$ at X_0. Thus $u_{yy}^2\leqq 2.u_{xx}^2+2.v_y^2.\mathrm{H}''(d)^2.d_x^2$ and we get

$$|\nabla u_x|^2\geqq\frac{|\nabla u_x|^2}{2}+\frac{u_{xx}^2}{2}$$
$$\geqq\frac{|\nabla u_x|^2}{2}+\frac{u_{yy}^2}{4}-\frac{v_y^2.\mathrm{H}''(d)^2.d_x^2}{2}$$
$$\geqq\frac{|\mathrm{D}^2 u|^2}{4}-\frac{v_y^2.\mathrm{H}''(d)^2.|\nabla d|^2}{2} \quad \text{at } \mathrm{X}_0.$$

In the same manner we find

$$|\nabla v_y|^2\geqq\frac{|\mathrm{D}^2 v|^2}{4}-\frac{u_x^2.\mathrm{H}''(d)^2.|\nabla d|^2}{2} \quad \text{at } \mathrm{X}_0.$$

By (3.10) we obtain

$$\mathrm{I}\leqq\frac{2}{d}\left\{\left[1+\frac{(v_y^4+u_x^4)}{2}.\mathrm{H}''(d)^2\right].|\nabla d|^2-\frac{v_y^2}{4}.|\mathrm{D}^2 u|^2-\frac{u_x^2}{4}.|\mathrm{D}^2 v|^2\right\}$$
$$+2\,(v_y+u_x).|\mathrm{D}^2\mathscr{U}|.\mathrm{H}''(d).|\nabla d| \quad \text{at } \mathrm{X}_0.$$

Since $\{u_x, v_y\}=\{\nu_1, \nu_2\}$ at X_0 we have $\nu_1=\min\{u_x, v_y\}$ and $\nu_2=\max\{u_x, v_y\}$. Thus

$$\mathrm{I}\leqq\frac{2}{d}\left\{(1+\nu_2^4.\mathrm{H}''(d)^2)|\nabla d|^2-\frac{\nu_1^2}{4}.|\mathrm{D}^2\mathscr{U}|^2\right\}$$
$$+4.\nu_2.|\mathrm{D}^2\mathscr{U}|.\mathrm{H}''(d).|\nabla d|$$
$$=-\frac{1}{2}.\frac{\nu_1}{\nu_2}|\mathrm{D}^2\mathscr{U}|^2+\frac{2}{d}.(1+\nu_2^4.\mathrm{H}''(d)^2)|\nabla d|^2$$
$$+4.\nu_2.|\mathrm{D}^2\mathscr{U}|.\mathrm{H}''(d).|\nabla d| \quad \text{at } \mathrm{X}_0.$$

The last term on the right is dominated by

$$\frac{1}{4}\frac{\nu_1}{\nu_2}|D^2\mathcal{U}|^2+4.\frac{\nu_2}{\nu_1}.4\,\nu_2^2.H''(d)^2.|\nabla d|^2.$$

Hence

$$I\leqq-\frac{1}{4}.\frac{\nu_1}{\nu_2}.|D^2\mathcal{U}|^2+\frac{1}{d}.\{2+18.\nu_2^4.H''(d)^2\}.|\nabla d|^2 \quad \text{at } X_0.$$

This proves (3.8).

To prove (3.9) we note that

$$\begin{aligned}|L_1(d)-L_2(d)|&=\left|\sum_{i,\,j=1}^{2}(a_{ij})_{x_j}.d_{x_i}\right|\\ &\leqq c_1.|D^2\mathcal{U}|.|D\mathcal{U}|.H''(d).|\nabla d|+c_2.|D\mathcal{U}|^2.|H'''(d)|.|\nabla d|^2\\ &\leqq\frac{c_1}{8}.\frac{\nu_1}{c_1\nu_2}.|D^2\mathcal{U}|^2+8.\frac{c_1\nu_2}{\nu_1}.c_1.|D\mathcal{U}|^2.H''(d)^2.|\nabla d|^2\\ &\qquad+c_2.|D\mathcal{U}|^2.|H'''(d)|.|\nabla d|^2\\ &\leqq\frac{1}{8}.\frac{\nu_1}{\nu_2}.|D^2\mathcal{U}|^2+8.c_1^2.\frac{1}{d}.|D\mathcal{U}|^4.H''(d)^2.|\nabla d|^2\\ &\qquad+c_2.|D\mathcal{U}|^2.|H'''(d)|.|\nabla d|^2.\end{aligned}$$

Combining this with (3.8) we obtain (3.9). □

The above theorem implies that d is a supersolution for an elliptic equation in Ω. As a consequence we have:

THEOREM 3.7. – *If $\mathcal{U}$ is a classical equilibrium solution, then*

$$d(X)\geqq\inf_{\partial\Omega} d$$

for each X in Ω with equality holding if and only if d is constant. Moreover, in the latter case $\mathcal{U}$ is affine.

Proof. – Since d satisfies (3.8), the first assertion follows from the stong maximum principle. If d is constant, it follows from (3.8) that $D^2\mathcal{U}\equiv 0$ in Ω and hence $\mathcal{U}$ is affine. □

We note that by (1.8), upper bounds on $|D\mathcal{U}|$ and $|D\mathcal{U}^{-1}|$ exist if and only if $\left(\frac{1}{\nu_1}+\nu_2\right)$ is bounded from above. The latter bound follows from Theorems 3.3 and 3.7. More precisely, we have:

THEOREM 3.8. – *If $\mathcal{U}$ is a classical equilibrium solution, then*

$$\sup_{\Omega}\left(\frac{1}{\nu_1}+\nu_2\right)\leqq\theta$$

where θ *is a positive number depending only on* f, $\inf_{\partial\Omega} \nu_1$, *and* $\sup_{\partial\Omega} \nu_2$.

Proof. – Assume $d \geqq \underline{d} > 0$ in Ω and $z \leqq \bar{z} < \infty$ in Ω, where $d = \det(\mathrm{D}\mathscr{U})$ and $z = \dfrac{|\mathrm{D}\mathscr{U}|^2}{2} + f(d)$. Let $\underline{\nu} = \inf_{\partial\Omega} \nu 1$ and $\bar{\nu} = \sup_{\partial\Omega} \nu_2$. Since

$$f'(d) = d.\mathrm{H}''(d) > 0, f(d) \geqq f(\underline{d}) \quad \text{in } \Omega.$$

Hence

$$\frac{\nu_2^2}{2} \leqq z - f(d) \leqq \bar{z} - f(\underline{d}) \quad \text{in } \Omega.$$

It follows that

$$\frac{1}{\nu_1^2} = \frac{\nu_2^2}{d^2} \leqq \frac{2\,[\bar{z} - f(\underline{d})}{\underline{d}^2} \quad \text{in } \Omega.$$

By Theorems 3.3 and 3.7 we may assume without loss of generality that $\underline{d} \geqq \nu^2$ and $\bar{z} \leqq \bar{\nu}^2 + f(\bar{\nu}^2)$. Thus

$$\left(\frac{1}{\nu_1} + \nu_2\right) \leqq \theta\,(\underline{\nu}, \bar{\nu}) \quad \text{in } \Omega. \quad \square$$

4. INTERIOR ESTIMATES OF z AND $\frac{1}{d}$

In this section we prove L^∞ estimates of z and $\frac{1}{d}$ $\left(\text{and hence of } \frac{1}{\nu_1} \text{ and } \nu_2\right)$ in subdomains $\Omega' \subset\subset \Omega$ in terms of L^p estimates of $|\mathrm{D}\mathscr{U}|$ in Ω. Our approach is based on an application of the Aleksandrov maximum principle to local estimates for nonliear elliptic equations due to Trudinger. (*See* [9].)

We being by recalling the estimate of Aleksandrov in the two-dimensional case. Let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^2$. Let $[b_{ij}(\mathrm{X})]$ be a symmetric, positive definite, 2×2 matrix defined for X in $\mathscr{D}$. Set $\mathscr{B}(\mathrm{X}) = \det[b_{ij}(\mathrm{X})]$. If $\sigma \in \mathrm{C}(\bar{\mathscr{D}})$ define $\Gamma(\varphi)$ to be the upper contact set of $\varphi(\mathrm{X})$:

$$\Gamma(\varphi) = \{\mathrm{Y} \in \mathscr{D} : \varphi(\mathrm{X}) \leqq \varphi(\mathrm{Y}) + \langle \mathrm{P}, \mathrm{X} - \mathrm{Y} \rangle \text{ for all X in } \mathscr{D} \text{ and some } \mathrm{P} = \mathrm{P}(\mathrm{Y}) \in \mathbb{R}^2\}.$$

The *Aleksandrov Maximum Principle* asserts the following: Let $\varphi \in C^2(\mathscr{D}) \cap C_0(\bar{\mathscr{D}})$ and assume that $\sum_{i,j=1}^{2} b_{ij}.\varphi_{x_i x_j} \geqq \psi$ in $\mathscr{D}$. Then

$$\sup_{\mathscr{D}} \varphi \leqq c_0.(\operatorname{diam} \mathscr{D}).\left(\int_{\Gamma(\varphi)} \frac{\psi^2}{\mathscr{B}}\, dX\right)^{1/2}$$

where c_0 is a universal constant. (*See* Section 9.1 in [8].)

We are interested in interior estimates of C^2 solutions satisfying an inequality of the form

$$(4.1) \qquad \sum_{i,j=1}^{2} b_{ij}.w_{x_i x_j} \geqq g(X).|\nabla w|^2$$

in Ω. To this end, let $\mathscr{D} = B_{2r} \equiv B_{2r}(X_0) \subset \Omega$ and define $\eta(X) = \left(4 - \left|\frac{X - X_0}{r}\right|^2\right)^2$ for X in B_{2r}. If $w \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\varphi = \eta w$, then $\varphi \in C^2(B_{2r}) \cap C_0(\overline{B_{2r}})$ and

$$(4.2) \quad \sum_{i,j=1}^{2} b_{ij}.\varphi_{x_i x_j} \geqq \eta.g.|\nabla w|^2 + w.\sum_{i,j=1}^{2} b_{ij}.\eta_{x_i x_j} + 2.\sum_{i,j=1}^{2} b_{ij}.\eta_{x_i}.w_{x_j}$$

in B_{2r}. Now

$$|\nabla \eta| \leqq \frac{c_0}{r}.\eta^{1/2} \qquad \text{and} \qquad |D^2 \eta| \leqq \frac{c_0}{r^2}.$$

Moreover, Trudinger observed that

$$|\nabla w| \leqq \frac{c_0}{r}.\eta^{-1/2}.w \quad \text{on } \Gamma(\varphi).$$

[*See* inequality (10) of [9].] Now let $b(X)$ be the largest eigenvalue of $[b_{ij}(X)]$. By (4.1) and the above inequalities, we have

$$\sum_{i,j=1}^{2} b_{ij}.\varphi_{x_i x_j} \geqq -\frac{c_1}{r^2}.(|g|.w^2 + b.w) \quad \text{on } \Gamma(\varphi)$$

where c_1 is a universal constant. Since the Aleksandrov maximum principle requires such an estimate only on $\Gamma(\varphi)$, we obtain:

$$\sup_{B_{2r}} \varphi \leqq \frac{c_2}{r}.\left(\int_{\Gamma(\varphi)} \frac{(|g|.w^2 + b.w)^2}{\mathscr{B}}\, dx\right)^{1/2}.$$

Thus

$$(4.3)\quad \sup_{B_r} w \leqq \frac{c_3}{r} \cdot \left(\int_{\Gamma(\varphi)} \frac{(|g| \cdot w^2 + b \cdot w)^2}{\mathscr{B}} dx \right)^{1/2} \\ \leqq \frac{c_4}{r} \cdot \left(\int_{\Gamma(\varphi)} \frac{(g^2 \cdot w^4 + b^2 \cdot w^2)}{\mathscr{B}} dx \right)^{1/2}.$$

where c_4 is a universal constant.

We use this inequality in two instances: first with $w = \frac{1}{d} - \frac{1}{d_0}$ where d_0 is the constant defined in (1.2) and second with $w = z = \frac{|D\mathscr{U}|^2}{2} + f(d)$. As we have seen in Section 3, supremum bounds on these two functions provide such bounds on $\frac{1}{\nu_1}$ and ν_2. In both applications we show that the integrand in (4.3) can be estimated in terms of $\frac{1}{d}$ and $|D\mathscr{U}|$.

LEMMA 4.1. – *Let $\mathscr{U}$ be a classical equilibrium solution in Ω and assume that $B_{2r} \equiv B_{2r}(X_0) \subset \Omega$. Then*

$$\sup_{B_r} \frac{1}{d} \leqq \frac{1}{d_0} + \frac{c}{r} \cdot \left(\int_{B_{2r}} |D\mathscr{U}|^6 \cdot d^{-3s-8} \, dx \right)^{1/2}$$

where c is determined by the constants in (1.2).

Proof. – We use c for all constants determined by (1.2). Observe that

$$L_2\left(\frac{1}{d}\right) \equiv \sum_{i,j=1}^{2} a_{ij} \cdot \left(\frac{1}{d}\right)_{x_i x_j} \\ = -\frac{1}{d^2} \cdot \sum_{i,j=1}^{2} a_{ij} \cdot d_{x_i x_j} + \frac{2}{d^3} \cdot \sum_{i,j=1}^{2} a_{ij} \cdot d_{x_i} \cdot d_{x_j}.$$

By (3.7) and (3.9), we have

$$L_2\left(\frac{1}{d}\right) \geqq -\frac{|\nabla d|^2}{d^3} \\ \times [2 + M_2 \cdot |D\mathscr{U}|^4 \cdot H''(d)^2 + M_3 \cdot d \cdot |H'''(d)| \cdot |D\mathscr{U}|^2] \\ + 2 \cdot \frac{|\nabla d|^2}{d^3} \cdot [1 + \nu_1^2 \cdot H''(d)] \\ \geqq -\left|\nabla\left(\frac{1}{d}\right)\right|^2 \cdot d \cdot [M_2 \cdot |D\mathscr{U}|^4 \cdot H''(d)^2 + M_3 \cdot d \cdot |H'''(d)| \cdot |D\mathscr{U}|^2].$$

From (1.2) we obtain

$$L_2\left(\frac{1}{d}\right) \geqq -c.\left|\nabla\left(\frac{1}{d}\right)\right|^2.|D\mathcal{U}|^2.d^{-s-1}[d^{-s-2}.|D\mathcal{U}|^2+1] \quad \text{on } \{d \leqq d_0\}.$$

Now $d \leqq |D\mathcal{U}|^2$ and so

$$(4.4) \qquad |D\mathcal{U}|^2.d^{-s-2} \geqq d^{-s-1} \geqq c \quad \text{on } \{d \leqq d_0\}.$$

Thus if $w = \frac{1}{d} - \frac{1}{d_0}$, we have

$$(4.5) \qquad L_2(w) - c.|D\mathcal{U}|^4.d^{-2s-3}.|\nabla w|^2 \quad \text{on } \{w \geqq 0\}.$$

Let $\mathcal{D} = B_{2r}$, $\varphi = \eta w$, and $[b_{ij}] = [a_{ij}]$. By definition of $\Gamma(\varphi)$ and since $\varphi = 0$ on $\partial\mathcal{D} = \partial B_{2r}$, it follows that $\Gamma(\varphi) \subset \{w \geqq 0\}$. Hence $\Gamma(\varphi) \subset \{d \leqq d_0\}$ and

$$(4.6) \qquad w^2 \leqq d^{-2} \quad \text{on } \Gamma(\varphi).$$

By Proposition 3.4,

$$\begin{aligned} \mathcal{B}(X) &= [1 + v_1^2.H''(d)].[1 + v_2^2.H''(d)] \\ &= 1 + |D\mathcal{U}|^2.H''(d) + d^2.H''(d)^2. \end{aligned}$$

Hence on $\Gamma(\varphi)$,

$$\begin{aligned} \mathcal{B}(X) &\geqq c.(|D\mathcal{U}|^2.d^{-s-2} + d^{-2s-2}) \\ &= c.d^{-s-2}.(|D\mathcal{U}|^2 + d^{-s}). \end{aligned}$$

Also by Proposition 3.4,

$$\begin{aligned} b(X) &= 1 + v_2^2.H''(d) \\ &\leqq 1 + c.|D\mathcal{U}|^2.d^{-s-2} \\ &\leqq c.|D\mathcal{U}|^2.d^{-s-2} \quad \text{on } \Gamma(\varphi). \end{aligned}$$

From this (4.3), (4.5), and (4.6) we have

$$\begin{aligned} \sup_{B_r} w &\leqq \frac{c}{r}\left[\int_{\Gamma(\varphi)} \frac{(|D\mathcal{U}|^8.d^{-4s-6}.w^4 + |D\mathcal{U}|^4.d^{-2s-4}.w^2)}{d^{-s-2}.(|D\mathcal{U}|^2 + d^{-s})}\, dX\right]^{1/2} \\ &\leqq \frac{c}{r}.\left[\int_{\Gamma(\varphi)} \frac{(|D\mathcal{U}|^8.d^{-3s-8} + |D\mathcal{U}|^4.d^{-s-4}.w^2)}{(|D\mathcal{U}|^2 + d^{-s})}\, dX\right]^{1/2} \\ &\leqq \frac{c}{r}\left[\int_{\Gamma(\varphi)} (|D\mathcal{U}|^6.d^{-3s-8} + |D\mathcal{U}|^2.d^{-s-4})\, dx\right]^{1/2}. \end{aligned}$$

By (4.4),

$$|D\mathcal{U}|^2.d^{-s-4} \leqq c.|D\mathcal{U}|^6.d^{-3s-8} \quad \text{on } \Gamma(\varphi).$$

Thus

$$\sup_{B_r} \frac{1}{d} \leqq \frac{1}{d_0} + \frac{c}{r}.\left[\int_{\Gamma(\varphi)} |D\mathcal{U}|^6.d^{-3s-8}\, dX\right]^{1/2}. \quad \square$$

The above lemma and Theorem 2.2 can be combined to prove:

THEOREM 4.2. – *Assume $\mathcal{U}$ is a classical equilibrium solution in Ω and $\Omega' \subset\subset \Omega$. Then*

$$\sup_{\Omega'} \frac{1}{d} \leqq c.\left[1+\mathcal{W}(\mathcal{U})^{6+(8/s)}+\int_{\Omega} |D\mathcal{U}|^{12+(16/s)}\, dX\right]^{1/2}$$

where c is a constant depending on Ω', Ω, and the constants in (1.2).

Proof. – Suppose $B_{2r} = B_{2r}(X_0) \subset \Omega$. By Lemma 4.1 it follows that

$$\left(\sup_{B_r} \frac{1}{d}\right)^2 \leqq \frac{c}{r^2}.\left[\int_{B_{2r}} (1+|D\mathcal{U}|^6.d^{-3s-8})\, dX\right]$$

where c depends on the constants in (1.2). Applying Young's inequality, namely $|ab| \leqq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ with $p=\frac{6s+8}{3s}$ and $q=\frac{6s+8}{3s+8}$, we obtain

$$\left(\sup_{B_r} \frac{1}{d}\right)^2 \leqq \frac{c}{r^2}.\left[\int_{B_{2r}} \left\{1+\frac{1}{p}.|D\mathcal{U}|^{12+(16/s)}+\frac{1}{q}.(d^{-s})^{6+(8/s)}\right\} dX\right].$$

Combining this with Corollary 2.3 gives the desired conclusion. □

We do a similar analysis to obtain local supremum estimates of z (and hence of $|D\mathcal{U}|$) in terms of L^p_{loc} estimates of $|D\mathcal{U}|$:

LEMMA 4.3. – *Suppose $\mathcal{U}$ is a classical equilibrium solution. If $B_{2r} \equiv B_{2r}(X_0) \subset \Omega$, the function $z(X)=\frac{1}{2}|D\mathcal{U}|^2+f(d)$ satisfies*

$$\sup_{B_r} z \leqq \frac{c}{r}.\left[\int_{B_{2r}} (z^4.H''(d)^2+z^2)\, dX\right]^{1/2}$$

where c is a universal constant.

Proof. – By Theorem 3.2, z satisfies

$$\Delta z \geqq -H''(d).|\nabla z|^2 \quad \text{in } \Omega.$$

Applying (4.3) with $[b_{ij}]=[\delta_{ij}]$ gives the desired inequality. □

THEOREM 4.4. – *Assume $\mathcal{U}$ is a classical equilibrium solution and $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Then*

$$\sup_{\Omega'} z \leqq c_1.\left[1+\int_{\Omega''} |D\mathcal{U}|^{16}+f(d)^{6+(8/s)})\, dX\right]^{1/2}$$

$$\leqq c_2.\left[1+\mathcal{W}(\mathcal{U})^{6+(8/s)}+\int_{\Omega} |D\mathcal{U}|^p\, dX\right]^{1/2}$$

where $p=\max\left(16,\ 12+\dfrac{16}{s}\right)$ *and the constants* c_1 *and* c_2 *depend on* Ω, Ω', Ω'', *and* H.

Proof. – Since $f'(d)=d.\mathrm{H}''(d)$ we have

$$f(d)-f(a)=\int_a^d \zeta.\mathrm{H}''(\zeta)\,d\zeta.$$

Now $\mathrm{H}''(d)\sim d^{-s-2}$ for $d\leqq d_0$ and $\mathrm{H}''(d)\sim d^{\tau}$ for $d\geqq d_1$. It follows that $|f(d)|\geqq c_1.d^2.\mathrm{H}''(d)-c_2$ for $d\leqq d_0$ or $d\geqq d_1$, where c_1 and c_2 are positive constants depending on H. Since $\mathrm{H}''>0$ and $f'>0$, we obtain:

$$d^2\,\mathrm{H}''(d)\leqq c\,(|f(d)|+1)$$

for all $d>0$.

Now assume that $\mathrm{B}_{6r}\equiv\mathrm{B}_{6r}(\mathrm{X}_0)\subset\Omega$. By Lemma 4.3 and the above estimate on $\mathrm{H}''\,d)$,

$$(4.7)\quad (\sup_{B_r} z^+)^2\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}(z^4.\mathrm{H}''(d)^2+z^2)\,d\mathrm{X}$$
$$\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}\{[|\mathrm{D}\mathscr{U}|^8+f(d)^4].[1+f(d)^2].d^{-4}+|\mathrm{D}\mathscr{U}|^4+f(d)^2\}\,d\mathrm{X}$$

where $z^+=\max\{z,0\}$. In $\mathrm{B}_{2r}\cap\{d\geqq d_0\}$, d^{-4} is bounded above by a constant. Hence

$$\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}\cap\{d\geqq d_0\}}([|\mathrm{D}\mathscr{U}|^8+f(d)^4]$$
$$\times[1+f(d)^2].d^{-4}+|\mathrm{D}\mathscr{U}|^4+f(d)^2)\,d\mathrm{X}$$
$$\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}(1+|\mathrm{D}\mathscr{U}|^{16}+f(d)^6)\,d\mathrm{X}$$
$$\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}(1+|\mathrm{D}\mathscr{U}|^{16}+|f(d)|^{6+(8/s)}\,d\mathrm{X}.$$

In $\mathrm{B}_{2r}\cap\{d\leqq d_0\}$, we have $c_3\leqq d^{-s}\leqq c_4.|f(d)|$. [*See* (2.7)]. Hence $d^{-8}\leqq c.|f(d)|^{8/s}$ and

$$\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}\cap\{d\leqq d_0\}}([|\mathrm{D}\mathscr{U}|^8+f(d)^4]$$
$$\times[1+f(d)^2].d^{-4}+|\mathrm{D}\mathscr{U}|^4+f(d)^2)\,d\mathrm{X}$$
$$\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}(|\mathrm{D}\mathscr{U}|^{16}+[1+f\,d)^6].d^{-8})\,d\mathrm{X}$$
$$\leqq\frac{c}{r^2}\cdot\int_{\mathrm{B}_{2r}}(|\mathrm{D}\mathscr{U}|^{16}+1+|f(d)|^{6+(8/s)})\,d\mathrm{X}$$

where c depends on H. By (4.7) and the above estimates,

$$(4.8)\qquad (\sup_{B_r} z^+)^2 \leqq \frac{c}{r^2}\cdot\int_{B_{2r}} (1+|D\mathscr{U}|^{16}+|f(d)|^{6+(8/s)})\,dX.$$

Now by Theorem 2.2, the term on the right is bounded by

$$\frac{c}{r^2}\cdot\left[1+\mathscr{W}(\mathscr{U})^{6+(8/s)}+\int_{B_{6r}} (1+|D\mathscr{U}|^{16}+|D\mathscr{U}|^{12+(16/s)})\,dX\right].$$

The conclusion of the theorem follows from this and (4.8). □

Recall that $z=\frac{1}{2}(\nu_1^2+\nu_2^2)+f(\nu_1\nu_2)$ and $\frac{1}{d}=\frac{1}{\nu_1\nu_2}$. It follows from Theorems 4.2 and 4.4 that $\frac{1}{\nu_1}+\nu_2$ (and hence $|D\mathscr{U}|$ and $|D\mathscr{U}^{-1}|$) are bounded on compact subdomains of Ω by constants depending on H, $\mathscr{W}(\mathscr{U})$, and $\|D\mathscr{U}\|_{L^p(\Omega)}$ where $p=\max\left\{16,\ 12+\frac{16}{s}\right\}$. More precisely, we have:

THEOREM 4.5. – *If $\mathscr{U}$ is a classical equilibrium solution and $\Omega' \subset\subset \Omega$, then*

$$\sup_{\Omega'}\left(\frac{1}{\nu_1}+\nu_2\right)\leqq\theta$$

where θ is a constant depending only on $\mathscr{W}(\mathscr{U})$, $|D\mathscr{U}|_{L^p(\Omega)}$, H, Ω', and Ω.

Proof. – Let

$$c_1=\left[1+\mathscr{W}(\mathscr{U})^{6+(8/s)}+\int_\Omega |D\mathscr{U}|^p\,dX\right]^{1/2}$$

where

$$p=\max\left\{16,\ 12+\frac{16}{s}\right\}.$$

By Theorems 4.2 and 4.4 there exists a constant $c_2>0$ depending only on Ω, Ω', and H such that

$$\sup_{\Omega'} z\leqq c_1 . c_2 \qquad \text{and} \qquad \sup_{\Omega'}\frac{1}{d}\leqq c_1 . c_2.$$

Let $c=2\,c_1\,c_2$ and $\tilde{c}=\frac{1}{c_1\,c_2}$. Then

$$\nu_1^2+\nu_2^2+2.f(\nu_1\nu_2)\equiv 2\,z\leqq c \qquad \text{and} \qquad \nu_1\nu_2\equiv d\geqq\tilde{c}\quad \text{in } \Omega'.$$

Since f is increasing it follows that

$$\nu_2^2 \leqq c - 2 . f(\nu_1 \nu_2) \leqq c - 2 . f(\tilde{c}) \qquad \text{and} \qquad \frac{1}{\nu_1} = \frac{\nu_2}{d} \leqq c . \sqrt{c - 2 f(\tilde{c})}.$$

The theorem follows if we set $\theta = (1 + c) . \sqrt{c - 2 f(\tilde{c})}$. □

5. $C^{k,\alpha}$ ESTIMATES OF CLASSICAL EQUILIBRIUM SOLUTIONS

In this section we prove a Caccioppoli inequality on $D\mathcal{U}$ (Lemma 5.1). As a consequence, we obtain *a priori* Hölder estimates of $D\mathcal{U}$ on sub-domains $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ in terms of $\frac{1}{\beta_1} \equiv \sup_{\Omega''} \frac{1}{\nu_1}$ and $\beta_2 \equiv \sup_{\Omega''} \nu_2$ [and hence in terms of $\mathcal{W}(\mathcal{U})$ and $\| D\mathcal{U} \|_{L^p(\Omega)}$]. Classical elliptic theory then provides estimates of $\| \mathcal{U} \|_{C^{k,\alpha}(\Omega')}$ for $k \geqq 2$ in terms of β_1, β_2, $\| H \|_{C^{k,\alpha}_{loc}(\mathbb{R}^+)}$, $\| \mathcal{U} \|_{L^2(\Omega)}$, and $\| D\mathcal{U} \|_{C^\alpha(\Omega'')}$.

We denote by $\bar{F}_{X_0, r}$ the average of F over B_r, namely $⨍_{B_r(X_0)} F\, dX$.

LEMMA 5.1 (CACCIOPPOLI INEQUALITY ON $D\mathcal{U}$). – *Let $\mathcal{U}$ be a classical equilibrium solution and assume that $B_{2r} \equiv B_{2r}(X_0) \subset \Omega''$. Then*

$$\int_{B_r} | D^2 \mathcal{U} |^2 dX \leqq \frac{c}{r^2} . \int_{B_{2r}} | D\mathcal{U} - \overline{D\mathcal{U}}_{X_0, 2r} |^2 dX$$

where $c = c(\beta_1, \beta_2)$.

Proof. – We use M for universal constants and c_i for constants depending only on β_1 and β_2. Let $\tilde{d} = \det \overline{D\mathcal{U}}_{X_0, 2r}$ and set

$$\varphi = \eta^2 . (e^{-kd} - e^{-k\tilde{d}}),$$

where k is a positive constant to be determined and $\eta \in C_0^1(B_{2r})$ with $\eta = 1$ on B_r and $| \nabla \eta | \leqq \frac{M}{r}$. By (3.6), $L_1(d) = 2 . \{ \langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle \}$ in B_{2r}. Multiplying by φ and integrating by parts, we obtain

$$\int_{B_{2r}} L_1(d) . \varphi\, dX = k . \int_{B_{2r}} (a_{ij} d_{xi} d_{xj}) . e^{-kd} . \eta^2 dX - \int_{B_{2r}} 2 a_{ij} . d_{xi} . \eta_{xj} . \eta . (e^{-kd} - e^{-k\tilde{d}})\, dX.$$

The second term on the right is bounded in absolute value by

$$\int_{B_{2r}} | \nabla d |^2 . \eta^2 . e^{-kd} dX + \frac{c_1}{r^2} . \int_{B_{2r}} | e^{-kd} - e^{-k\tilde{d}} |^2 . e^{kd} dX.$$

Hence

$$\int_{B_{2r}} L_1(d).\varphi\, dX$$

$$\geqq k.\int_{B_{2r}} \{|\nabla d|^2 + H''(d).[(u_y^2+v_y^2).d_x^2 - 2.(u_x u_y + v_x v_y) \times d_x d_y + (v_x^2+u_x^2).d_y^2]\}.e^{-kd}.\eta^2\, dX - \int_{B_{2r}} |\nabla d|^2 \eta^2 e^{-kd}\, dX - \frac{c_1}{r^2}.\int_{B_{2r}} |e^{-kd} - e^{-k\tilde{d}}|^2.e^{kd}\, dX.$$

Setting

$$E = [(u_y^2+v_y^2).d_x^2 - 2(u_x u_y + v_x v_y).d_x d_y + (u_x^2+v_x^2).d_y^2],$$

we obtain

$$(5.1)\quad k.\int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX + k.\int_{B_{2r}} H''(d).E.e^{-kd}.\eta^2\, dX \leqq \int_{B_{2r}} L_1(d).\varphi\, dX + \int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX + \frac{c_1}{r^2}\int_{B_{2r}} |e^{-kd} - e^{-k\tilde{d}}|^2.e^{kd}\, dX.$$

Now

$$\int_{B_{2r}} L_1(d).\varphi\, dX = \int_{B_{2r}} L_1(d).e^{-kd}.\eta^2\, dX - \int_{B_{2r}} L_1(d).e^{-k\tilde{d}}.\eta^2\, dX \equiv I + II.$$

By Theorem 3.6,

$$(5.2)\qquad I \leqq c_2.\int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX.$$

To estimate II, we use the fact that

$$L_1(d) = 2.\{\langle \nabla u_x, \nabla v_y\rangle - \langle \nabla u_y, \nabla v_x\rangle\} = 2\langle (\nabla u - \overline{\nabla u}_{X_0,2r}), \nabla v_y\rangle_x - 2\langle (\nabla u - \overline{\nabla u}_{X_0,2r}), \nabla v_x\rangle_y.$$

Hence

$$|II| = 4e^{-k\tilde{d}}.\left|\int_{B_{2r}} \{\langle (\nabla u - \overline{\nabla u}_{X_0,2r}), \nabla v_y\rangle.\eta_x.\eta - \langle (\nabla u - \overline{\nabla u}_{X_0,2r}), \nabla v_x\rangle.\eta_y.\eta\}\, dX\right|$$

$$\leqq \varepsilon.\int_{B_{2r}} |D^2\mathscr{U}|^2.\eta^2\, dX + \frac{M}{\varepsilon r^2}.e^{-2k\tilde{d}}.\int_{B_{2r}} |D\mathscr{U} - \overline{D\mathscr{U}}_{X_0,2r}|^2\, dX$$

for any $\varepsilon > 0$.

From this estimate and (5.2) we get

$$\int_{B_{2r}} L_1(d).\varphi\, dX$$
$$\leqq c_2.\int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX+\varepsilon.\int_{B_{2r}} |D^2\mathscr{U}|^2.\eta^2\, dX$$
$$+\frac{M}{\varepsilon r^2}.e^{-2k\tilde{d}}.\int_{B_{2r}} |D\mathscr{U}-\overline{D\mathscr{U}}_{X_0,2r}|^2\, dX.$$

Combining this inequality and (5.1) we have

$$(5.3)\quad k.\int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX$$
$$+k.\int_{B_{2r}} H''(d).E.e^{-kd}.\eta^2\, dX$$
$$\leqq(1+c_2).\int_{B_{2r}} |\nabla d|^2.e^{-kd}.\eta^2\, dX+\frac{c_1}{r^2}$$
$$\times\int_{B_{2r}} e^{kd}.|e^{-kd}-e^{-k\tilde{d}}|^2\, dX+\varepsilon.\int_{B_{2r}} |D^2\mathscr{U}|^2.\eta^2\, dX$$
$$+\frac{M}{\varepsilon r^2}.e^{-2k\tilde{d}}.\int_{B_{2r}} |D\mathscr{U}-\overline{D\mathscr{U}}_{X_0,2r}|^2 dX.$$

Now set $k=1+c_2$. Then $k=k(\beta_1,\beta_2)$ which implies that

$$e^{kd}+e^{-2k\tilde{d}}\leqq c_3 \quad\text{and}\quad |e^{-kd}-e^{-k\tilde{d}}|\leqq c_4.|D\mathscr{U}-\overline{D\mathscr{U}}_{X_0,2r}| \quad\text{in } B_{2r}.$$

Our estimate (5.3) simplifies to

$$(5.4)\quad k.\int_{B_{2r}} H''(d).E.e^{-kd}.\eta^2\, dX$$
$$\leqq\left(c_5+\frac{c_6}{\varepsilon}\right).\frac{1}{r^2}.\int_{B_{2r}} |D\mathscr{U}-\overline{D\mathscr{U}}_{X_0,2r}|^2\, dX$$
$$+\varepsilon.\int_{B_{2r}} |D^2\mathscr{U}|^2.\eta^2\, dX.$$

Observe that if we solve $(1.4)_1$ for Δu and $(1.4)_2$ for Δv, square each equation, and add the results, we obtain

$$H''(d)^2.E=(\Delta u)^2+(\Delta v)^2.$$

Choose $\theta \equiv \theta(\beta_1, \beta_2) = \inf_{\beta_2^2 \geqq d \geqq \beta_1^2} \frac{ke^{-kd}}{H''(d)} > 0$. By (5.4) we have

$$\theta . \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] . \eta^2 \, dX$$
$$\leqq \left(c_5 + \frac{c_6}{\varepsilon} \right) . \frac{1}{r^2} . \int_{B_{2r}} | D\mathscr{U} - \overline{D\mathscr{U}}_{X_0, 2r} |^2 \, dX$$
$$+ \varepsilon . \int_{B_{2r}} | D^2 \mathscr{U} |^2 . \eta^2 \, dX.$$

It is shown in Appendix B that

$$\int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] . \eta^2 \, dX$$
$$\geqq \frac{1}{2} \int_{B_{2r}} | D^2 \mathscr{U} |^2 . \eta^2 \, dX - \frac{M}{r^2} . \int_{B_{2r}} | D\mathscr{U} - \overline{D\mathscr{U}}_{X_0, 2r} |^2 \, dX.$$

Hence

$$\frac{\theta}{2} . \int_{B_{2r}} | D^2 \mathscr{U} |^2 . \eta^2 \, dX$$
$$\leqq \left(c_7 + \frac{c_6}{\varepsilon} \right) . \frac{1}{r^2} . \int_{B_{2r}} | D\mathscr{U} - \overline{D\mathscr{U}}_{X_0, 2r} |^2 \, dX$$
$$+ \varepsilon . \int_{B_{2r}} | D^2 \mathscr{U} |^2 . \eta^2 \, dX.$$

Setting $\varepsilon = \frac{\theta}{4}$ we have our assertion. □

The above lemma and the Sobolev-Poincaré inequality imply the following:

THEOREM 5.2. — *Assume $\mathscr{U}$ is a classical equilibrium solution and $\Omega' \subset\subset \Omega$. Then*

$$\| D\mathscr{U} \|_{C^\alpha(\Omega')} \leqq c$$

where α and c are positive constants depending only on Ω, Ω', H, $\mathscr{W}(\mathscr{U})$, and $\| D\mathscr{U} \|_{L^p(\Omega)}$ with $p = \max\left(16, 12 + \frac{16}{s}\right)$.

Proof. — Without loss of generality assume that Ω' is a Lipschitz domain. (If not, we can replace Ω' with a Lipschitz domain, Ω_0, satisfying $\Omega' \subset \Omega_0 \subset\subset \Omega$.) Choose Ω'' and Ω''' so that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and assume

$B_{4r} \equiv B_{4r}(X_0) \subset \Omega''$. By the Sobolev-Poincaré inequality,

$$\int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}_{X_0,\,2r}|^2\, dX \leqq c_0 \cdot \left(\int_{B_{2r}} |D^2 \mathcal{U}|\, dX \right)^2$$

where c_0 is independent of r. Combining this with Lemma 5.1, we obtain the following "reverse-Hölder" type of inequality:

$$\left(\int_{B_r} |D^2 \mathcal{U}|^2\, dX \right)^{1/2} \leqq c_1 \cdot \int_{B_{2r}} |D^2 \mathcal{U}|\, dX$$

where $c_1 = c_1(\beta_1, \beta_2)$, $\beta_1 = \inf_{\Omega'''} \nu_1$ and $\beta_2 = \sup_{\Omega'''} \nu_2$. It follows (by Proposition 1.1 of Chapter V in [6] and Lemma 5.1) that there exist constants $q > 2$ and c_2 and $c_3 > 0$ depending only on Ω', Ω'', Ω''', β_1 and β_2 such that

$$\| D^2 \mathcal{U} \|_{L^q(\Omega')} \leqq c_2 \cdot \| D^2 \mathcal{U} \|_{L^2(\Omega'')} \leqq c_3.$$

By the Sobolev imbedding theorem and the above inequality,

$$\| D\mathcal{U} \|_{C^\alpha(\Omega')} \leqq c . \| D\mathcal{U} \|_{W^{1,q}(\Omega')} \leqq c_4 \tag{5.5}$$

where $\alpha = 1 - \dfrac{2}{q}$ and c_4 depends only on Ω', Ω'', Ω''', β_1 and β_2. By Theorem 4.5,

$$c_4 \leqq c_5 \tag{5.6}$$

where c_5 depends only on Ω, Ω', Ω'', Ω''', H, $\mathcal{W}(\mathcal{U})$ and $\| D\mathcal{U} \|_{L^p(\Omega)}$. The theorem now follows from (5.5) and (5.6). □

We conclude this section with the following results concerning higher order *a priori* estimates.

LEMMA 5.3. – *Assume $\mathcal{U}$ is a classical equilibrium solution, $\Omega' \subset\subset \Omega$, and α is the exponent defined in Theorem* 5.2. *Then*

$$\| \mathcal{U} \|_{C^{2,\alpha}(\Omega')} \leqq c$$

where c depends on Ω, Ω', H, $\mathcal{W}(\mathcal{U})$, $\| \mathcal{U} \|_{L^2(\Omega)}$, and $\| D\mathcal{U} \|_{L^p(\Omega)}$ with $p = \max\left\{ 16, 12 + \dfrac{16}{s} \right\}$.

Proof. – We observe that $\mathcal{U}$ satisfies (1.4) which can be written as

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\frac{\partial \sigma}{\partial F_{ij}} (D\mathcal{U}) \right) = 0 \qquad \text{for} \quad i = 1, 2$$

where $(x_1, x_2)=(x, y)$, $(u_1, u_2)=(u, v)=\mathcal{U}$ and $F_{ij}=\dfrac{\partial u_i}{\partial x_j}$. Differentiating this system with respect to x_q, we have

$$(5.7)\qquad \sum_{j,m,n=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \sigma}{\partial F_{mn}\,\partial F_{ij}}(D\mathcal{U})\frac{\partial^2 u_m}{\partial x_q\,\partial x_n}\right)=0 \qquad \text{for } i=1,2.$$

If we set $A_{im}^{jn}(X)=\dfrac{\partial^2 \sigma}{\partial F_{mn}\,\partial F_{ij}}(D\mathcal{U}(X))$ and $v_m=\dfrac{\partial u_m}{\partial x_q}$, equation (5.7) becomes

$$(5.8)\qquad \sum_{j,m,n=1}^{2} \frac{\partial}{\partial x_j}\left(A_{im}^{jn}(X)\frac{\partial v_m}{\partial x_n}\right)=0 \qquad \text{for } i=1,2.$$

Choose Ω'' so that $\Omega'\subset\subset\Omega''\subset\subset\Omega$. Then

$$\| A_{im}^{jn} \|_{C^\alpha(\Omega'')} \leqq c_1$$

where c_1 depends on Ω'', Ω, H, $\mathcal{W}(\mathcal{U})$, and $D\mathcal{U}\|_{L^p(\Omega)}$. Indeed we have $\sigma(F)=\dfrac{|F|^2}{2}+H(\det F)$. Now $H\in C^3(\mathbb{R}^+)$, $\mathcal{U}\in C^{1,\alpha}(\Omega'')$, and Theorems 4.2 and 5.2 imply that

$$0<\underline{d}\leqq \det D\mathcal{U}(X)\leqq \bar{d}<\infty \quad \text{on } \Omega''$$

where $\underline{d}$, $\bar{d}$, and $\| D\mathcal{U}\|_{C^\alpha(\Omega'')}$ depend on Ω, Ω', H, $\mathcal{W}(\mathcal{U})$, and $\| D\mathcal{U}\|_{L^p(\Omega)}$. Hence

$$\| A_{im}^{jn} \|_{C^\alpha(\Omega'')} = \left\| \frac{\partial^2 \sigma}{\partial F_{mn}\,\partial F_{ij}}(D\mathcal{U})\right\|_{C^\alpha(\Omega'')} \leqq c_1.$$

Moroeover the strict-Legendre Hadamard condition holds:

$$\sum_{i,j,m,n=1}^{2} A_{im}^{jn}(X)\,\lambda_i\lambda_m\pi_j\pi_n \geqq |\lambda|^2\,|\pi|^2$$

for all λ, $\pi\in\mathbb{R}^2$. [*See* (1.5)] Thus we can apply the regularity theory for linear elliptic systems with Hölder continuous coefficients (*see* Proposition 2.1 and Theorem 3.2 of Chapter III in [6]) to conclude that $\| v_m \|_{C^{1,\alpha}(\Omega')} \leqq c_2 \,.\, \| v \|_{L^2(\Omega'')}$ and hence

$$\| \mathcal{U} \|_{C^{2,\alpha}(\Omega')} \leqq c_3 \,.\, \| \mathcal{U} \|_{W^{1,2}(\Omega'')}.$$

Since $\| D\mathcal{U} \|^2_{L^2(\Omega'')} \leqq 2\,.\,\mathcal{W}(\mathcal{U})$, the theorem follows. □

THEOREM 5.4. — *Let $\mathcal{U}$ be a classical equilibrium solution and assume* [*in addition to* (1.2)] *that* $H\in C^{k,\beta}_{\mathrm{loc}}(\mathbb{R}^+)$ *where* $k\geqq 2$ *and* $0<\beta<1$. *Then* $\mathcal{U}\in C^{k,\beta}_{\mathrm{loc}}(\Omega)$ *and for any* $\Omega'\subset\subset\Omega$

$$\| \mathcal{U} \|_{C^{k,\beta}(\Omega')} \leqq c$$

where c depends on k, β, Ω, Ω', H, $\mathscr{W}(\mathscr{U})$, $\|\mathscr{U}\|_{L^2(\Omega)}$, and $\|D\mathscr{U}\|_{L^p(\Omega)}$ for $p=\max\left\{16, 12+\frac{16}{s}\right\}$.

Proof. – Let l be such that $1\leqq l\leqq k-1$ with Ω'' as in the previous lemma. Note that if $\mathscr{U}\in C^{l,\beta}(\Omega'', \mathbb{R}^2)$ then $A_{im}^{jn}\in C^{l-1,\beta}(\Omega'')$.

By (5.8) and Theorem 3.3 of Chapter III in [6] it follows that $v_m\equiv\frac{\partial u_m}{\partial x_q}\in C^{l,\beta}(\Omega')$ for $q=1$, 2 and

$$\|\mathscr{U}\|_{C^{l+1,\beta}(\Omega')}\leqq c_l \tag{5.9}$$

where c_l depends on Ω', Ω'', β, H, l and $\|\mathscr{U}\|_{C^{l,\beta}(\Omega'')}$. From Lemma 5.3 (for $l=0$) we have

$$\|\mathscr{U}\|_{C^{1,\beta}(\Omega')}\leqq c_0$$

where c_0 depends on Ω, Ω', H, $\mathscr{W}(\mathscr{U})$, $\|\mathscr{U}\|_{L^2(\Omega)}$, and $\|D\mathscr{U}\|_{L^p(\Omega)}$. The assertion follows by iterating (5.9) on nested subdomains. □

APPENDIX

A. Let $\mathscr{A}=\{\mathscr{U}\in W^{1,2}(\Omega; \mathbb{R}^2): \det D\mathscr{U}>0$ a.e. in Ω and $\mathscr{W}(\mathscr{U})<\infty\}$ where

$$\mathscr{W}(\mathscr{U})=\int_\Omega \gamma(D\mathscr{U})\,dX$$

and Ω is a bounded domain in $\mathbb{R}^2$. Assume $\mathscr{U}\in\mathscr{A}$ and $\gamma(F)$ is a differentiable function defined on $M_+^{2\times 2}$. In general it is impossible to take a first variation of $\mathscr{W}$ at $\mathscr{U}$ with respect to a linear perturbation, that is, to obtain $\frac{d}{d\varepsilon}\mathscr{W}(\mathscr{U}+\varepsilon\Phi)|_{\varepsilon=0}$ where $\Phi\in C_0^1(\Omega; \mathbb{R}^2)$. In fact it may happen that for each $\varepsilon\neq 0$ the set $\{X\in\Omega: D(\mathscr{U}+\varepsilon\Phi)(X)\notin M_+^{2\times 2}\}$ has positive measure. Hence $\mathscr{W}(\mathscr{U}+\varepsilon\Phi)$ is undefined for $\varepsilon\neq 0$. On the other hand, Ball observed that if one considers nonlinear perturbations which amount to deformations of the interior of Ω, then under certain conditions on γ the first variation is well defined. (*See* [2].)

We prove Ball's result in Theorem A.1 below. In Lemma A.2 we check that the hypotheses of Ball's theorem hold for

$$\gamma(F)=\sigma(F)\equiv\frac{1}{2}|F|^2+H(\det F).$$

Finally in Theorem A.3, we show that equations (2.8) hold for weak equilibrium solutions when $\gamma=\sigma$.

Let γ satisfy the following hypotheses:

$$(A.1)\quad \begin{array}{ll} (1) & \gamma \geqq 0 \quad \text{on } M_+^{2\times 2}. \\ (2) & \gamma \in C^1(M_+^{2\times 2}). \\ (3) & \text{For some } \theta>0 \text{ there exists a constant N such that} \\ & \quad |F^T . D\gamma(FC)| \leqq N.[1+\gamma(F)] \\ & \text{for all } F, C \in M_+^{2\times 2} \text{ with} \\ & \quad |C-I| \leqq \theta. \end{array}$$

THEOREM A.1. – *Assume* $\mathscr{U} \in \mathscr{A}$ *and* γ *satisfies* (A.1). *For each* Φ *in* $C_0^1(\Omega; \mathbb{R}^2)$ *there exists* $\varepsilon_0>0$ *so that* $\mathscr{U}_\varepsilon(X) \equiv \mathscr{U}(X+\varepsilon.\Phi(X)) \in \mathscr{A}$ *for* $|\varepsilon| \leqq \varepsilon_0$, $\frac{d}{d\varepsilon}\mathscr{W}(\mathscr{U}_\varepsilon)|_{\varepsilon=0}$ *exists, and*

$$(A.2)\qquad \frac{d}{d\varepsilon}\mathscr{W}(\mathscr{U}_\varepsilon)|_{\varepsilon=0} = \int_\Omega \left(-\gamma.\delta_k^j + u_{x_k}^i.\frac{\partial\gamma}{\partial u_{x_j}^i}\right).\Phi_{x_j}^k\, dX$$

where $\gamma \equiv \gamma(D\mathscr{U})$, $(u^1, u^2) = \mathscr{U}$ *and* $(x_1, x_2) = X$. *Moreover for each* j *and* k *with* $1 \leqq j, k \leqq 2$ *the term in parentheses is summable.*

Proof. – Let $Z_\varepsilon = Z_\varepsilon(X) = X + \varepsilon.\Phi(X)$. For ε sufficiently small $Z_\varepsilon(.)$ is a C^1 diffeomorphism from $\bar{\Omega}$ onto itself. We begin by showing that $\mathscr{U}_\varepsilon = \mathscr{U}(Z_\varepsilon) \in \mathscr{A}$ for ε small. Now $\mathscr{U}_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ and

$$\det D\mathscr{U}_\varepsilon(X) = \det[D\mathscr{U}(Z_\varepsilon(X))].\det DZ_\varepsilon(X).$$

Since $\mathscr{U} \in \mathscr{A}$ and $\det DZ_\varepsilon > 0$ in Ω for ε sufficiently small we conclude that $\det D\mathscr{U}_\varepsilon > 0$ a.e. in Ω. Thus $\mathscr{U}_\varepsilon \in \mathscr{A}$ if $\mathscr{W}(\mathscr{U}_\varepsilon) < \infty$ and ε is sufficiently small, say $|\varepsilon| \leqq \varepsilon_1$.

Now

$$\begin{aligned} \mathscr{W}(\mathscr{U}_\varepsilon) &= \int_\Omega \gamma(D\mathscr{U}_\varepsilon)\, dX \\ &= \int_\Omega \gamma(D\mathscr{U}(Z_\varepsilon).DZ_\varepsilon(X))\, dX \\ &= \int_\Omega \gamma(D\mathscr{U}(Z).DZ_\varepsilon(X_\varepsilon)).\det DZ_\varepsilon^{-1}\, dZ \end{aligned}$$

for $|\varepsilon| \leqq \varepsilon_1$ where $X_\varepsilon = X_\varepsilon(Z) \equiv Z_\varepsilon^{-1}(Z)$ since $Z = Z_\varepsilon(X)$ is invertible and hence $X = Z_\varepsilon^{-1}(Z)$. Now chose $\varepsilon_0 \leqq \varepsilon_1$ so that $|DZ_\varepsilon(X) - I| \leqq \theta$ for all $|\varepsilon| \leqq \varepsilon_0$ and all X in Ω, where θ is the constant defined in (A.1). Without loss of generality assume θ is so small that $|C| + |C^{-1}| \leqq 4$ whenever $|C-I| \leqq \theta$. Thus $\frac{1}{16} \leqq \det DZ_\varepsilon^{-1} \leqq 16$ for $|\varepsilon| \leqq \varepsilon_0$ and

$$(A.3)\qquad \mathscr{W}(\mathscr{U}_\varepsilon) \leqq 16.\int_\Omega \gamma(D\mathscr{U}(Z).DZ_\varepsilon(X_\varepsilon))\, dZ.$$

We bound this by estimating $\gamma(\mathrm{FC})$ assuming $|\mathrm{C}-\mathrm{I}|\leqq\theta$. Note that

$$\gamma(\mathrm{FC})-\gamma(\mathrm{F})=\int_0^1 \frac{d}{dt}[\gamma(\mathrm{F}.\mathrm{C}(t))]\,dt$$

where $\mathrm{C}(t)=(1-t)\mathrm{I}+t\mathrm{C}$ and

$$\begin{aligned}\left|\frac{d}{dt}[\gamma(\mathrm{F}.\mathrm{C}(t))]\right|&=\left|\sum_{i,j,k=1}^{2}\frac{\partial\gamma}{\partial \mathrm{F}_{ij}}(\mathrm{F}.\mathrm{C}(t)).\mathrm{F}_{ik}.[\mathrm{C}-\mathrm{I}]_{kj}\right|\\&=\left|\sum_{k,j=1}^{2}[\mathrm{F}^{\mathrm{T}}.\mathrm{D}\gamma(\mathrm{F}.\mathrm{C}(t))]_{kj}.[\mathrm{C}-\mathrm{I}]_{kj}\right|\\&\leqq|\mathrm{F}^{\mathrm{T}}.\mathrm{D}\gamma(\mathrm{F}.\mathrm{C}(t))|.|\mathrm{C}-\mathrm{I}|.\end{aligned}$$

Since $|\mathrm{C}(t)-\mathrm{I}|\leqq\theta$ for $0\leqq t\leqq 1$ we have from $(\mathrm{A}.1)_3$ that

$$|\mathrm{F}^{\mathrm{T}}.\mathrm{D}\gamma(\mathrm{F}.\mathrm{C}(t))|\leqq\mathrm{N}.[1+\gamma(\mathrm{F})].$$

Hence

$$\begin{aligned}|\gamma(\mathrm{FC})-\gamma(\mathrm{F})|&\leqq\mathrm{N}.[1+\gamma(\mathrm{F})].|\mathrm{C}-\mathrm{I}|\\&\leqq\mathrm{N}\,\theta.[1+\gamma(\mathrm{F})].\end{aligned}\tag{A.4}$$

From this and (A.3) we have

$$\mathscr{W}(\mathscr{U}_\varepsilon)\leqq\mathrm{M}_1[\mathscr{W}(\mathscr{U})+|\Omega|]<\infty$$

for $|\varepsilon|\leqq\varepsilon_0$ where M_1 is a fixed constant. Thus $\mathscr{U}_\varepsilon\in\mathscr{A}$ when $|\varepsilon|\leqq\varepsilon_0$.

Next consider

$$\begin{aligned}&\frac{1}{\varepsilon}[\mathscr{W}(\mathscr{U}_\varepsilon)-\mathscr{W}(\mathscr{U})]\\&\quad=\frac{1}{\varepsilon}\int_\Omega[\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}).\mathrm{DZ}_\varepsilon(\mathrm{X}_\varepsilon)).\det\mathrm{DZ}_\varepsilon^{-1}-\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}))]\,d\mathrm{Z}\\&\quad=\int_\Omega\frac{1}{\varepsilon}.[\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}).\mathrm{DZ}_\varepsilon(\mathrm{X}_\varepsilon))-\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}))].\det\mathrm{DZ}_\varepsilon^{-1}\,d\mathrm{Z}\\&\qquad+\int_\Omega\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z})).\frac{1}{\varepsilon}.[\det\mathrm{DZ}_\varepsilon^{-1}-1]\,d\mathrm{Z}.\end{aligned}$$

We apply the dominated convergence theorem to let $\varepsilon\to 0$ in each of the above integrals. For the first integral recall that $0<\det\mathrm{DZ}_\varepsilon^{-1}\leqq 16$ and $|\mathrm{DZ}_\varepsilon-\mathrm{I}|\leqq\theta$ when $|\varepsilon|\leqq\varepsilon_0$. From (A.4) we have

$$\begin{aligned}&\left|\frac{1}{\varepsilon}[\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}).\mathrm{DZ}_\varepsilon(\mathrm{X}_\varepsilon))-\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}))]\right|.\det\mathrm{DZ}_\varepsilon^{-1}(\mathrm{Z})\\&\qquad\leqq 16\,\mathrm{N}.[1+\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}))].\frac{1}{\varepsilon}|\mathrm{DZ}_\varepsilon(\mathrm{X}_\varepsilon)-\mathrm{I}|\\&\qquad\qquad\leqq 16\,\mathrm{N}.[1+\gamma(\mathrm{D}\mathscr{U}(\mathrm{Z}))].\|\mathrm{D}\Phi\|_{\mathrm{L}^\infty(\Omega)}\end{aligned}$$

where the rightland side is integrable by hypothesis. Thus we can pass to the limit under the integral. To evaluate the limit we use $(A.1)_2$ and the fact that $DZ_\varepsilon = I + \varepsilon . D\Phi$. For any F in $M_+^{2\times 2}$ and X in Ω, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\gamma(F.DZ_\varepsilon(X)) - \gamma(F)] = \frac{\partial \gamma}{\partial F_{ij}}(F).F_{ik}.\Phi^k_{x_j}(X)$$

where the convergence is uniform for all X in Ω. Since $X_\varepsilon \equiv Z_\varepsilon^{-1}(Z) \to Z$ as $\varepsilon \to 0$ for each Z in Ω, we conclude that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\gamma(F.DZ_\varepsilon(X_\varepsilon)) - \gamma(F)] = \frac{\partial \gamma}{\partial F_{ij}}(F).F_{ik}.\Phi^k_{x_j}(Z)$$

for all Z in Ω. Hence

$$\text{(A.5)} \quad \lim_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon}[\gamma(D\mathscr{U}(Z).DZ_\varepsilon(X_\varepsilon)) - \gamma(D\mathscr{U}(Z))].\det DZ_\varepsilon^{-1}\, dZ = \int_\Omega \frac{\partial \gamma}{\partial u^i_{x_j}}(D\mathscr{U}(Z)).u^i_{x_k}(Z).\Phi^k_{x_j}(Z)\, dZ.$$

For the second integral we note that

$$\frac{1}{\varepsilon}[\det DZ_\varepsilon(X) - 1] \to \Phi^1_{x_1}(X) + \Phi^2_{x_2}(X) \quad \text{as } \varepsilon \to 0$$

where the convergence is uniform for all X in Ω. Thus

$$\frac{1}{\varepsilon}[\det DZ_\varepsilon^{-1}(Z) - 1] \to -[\Phi^1_{x_1}(Z) + \Phi^2_{x_2}(Z)] \quad \text{as } \varepsilon \to 0$$

for all Z in Ω and

$$\lim_{\varepsilon \to 0} \int_\Omega \gamma(D\mathscr{U}).\frac{1}{\varepsilon}.[\det DZ_\varepsilon^{-1} - 1]\, dZ = -\int_\Omega \gamma(D\mathscr{U}).[\Phi^1_{x_1} + \Phi^2_{x_2}]\, dZ.$$

From this and (A.5) we conclude that the first variation $\frac{d}{d\varepsilon}\mathscr{W}(\mathscr{U}_\varepsilon)|_{\varepsilon=0}$ exists and it is given by (A.2).

Finally we point out that

$$\left(-\gamma.\delta^j_k + u^i_{x_k}.\frac{\partial \gamma}{\partial u^i_{x_j}}\right) \in L^1(\Omega)$$

for each j and k. Indeed, this is just $[-\gamma.I + D\mathscr{U}^T.D\gamma(D\mathscr{U})]_{kj}$; from $(A.1)_3$ with C=I it follows that this is integrable if $\mathscr{W}(\mathscr{U}) < \infty$. □

Next we consider

$$\sigma(F) = \frac{|F|^2}{2} + H(\det F)$$

with H as described in (1.2).

LEMMA A.2. – *Let* $\gamma(F)=\sigma(F)$ *for all* F *in* $M_+^{2\times 2}$. *Then* γ *satisfies* (A.1).

Proof. – By (1.2) properties $(A.1)_1$ and $(A.1)_2$ hold. Thus we need only to establish $(A.1)_3$.

Choose $\theta>0$ so that $|C|+|C^{-1}|\leqq 4$ and $\frac{1}{4}\leqq \det C\leqq 4$ whenever $C\in M_+^{2\times 2}$ and $|C-I|\leqq\theta$. We have

$$D\sigma(F)=F+H'(\det F).\begin{bmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{bmatrix}$$

for all F in $M_+^{2\times 2}$. Hence

$$F^T.D\sigma(F)=F^T F+(\det F).H'(\det F).I$$

and

$$\begin{aligned} F^T.D\sigma(FC)&=(C^T)^{-1}.(FC)^T.D\sigma(FC)\\ &=(C^T)^{-1}.[(FC)^T.FC+(\det FC).H'(\det FC).I]\\ &=F^T FC+\det C.\det F.H'(\det C.\det F).(C^{-1})^T \end{aligned}$$

for all C and F in $M_+^{2\times 2}$ with $|C-I|\leqq\theta$. It follows that

$$F^T.D\sigma(FC)|\leqq M_1.(|F|^2+\det C.\det F.|H'(\det C.\det F)|)$$

where M_1 is a fixed constant. Since $\det F\leqq|F|^2$, $(A.1)_3$ follows if we prove: There exists $M_2>0$ depending on H so that

$$rd.|H'(rd)|\leqq M_2.[1+d+H(d)] \tag{A.6}$$

whenever $\frac{1}{4}\leqq r\leqq 4$ and $d>0$.

We prove this inequality in two cases, $\tau\neq -1$ and $\tau=-1$ where τ is defined in (1.2). Fix r and d as above and assume $\tau\neq -1$. By (1.2) and elementary calculus it follows that

$$rd.|H'(rd)|\leqq c_1.(d^{-s}+d^{\tau+2}+d) \tag{A.7}$$

where c_1 depends on H. For d sufficiently small or d sufficiently large,

$$d^{-s}+d^{\tau+2}\leqq c_2.[1+d+H(d)].$$

Hence

$$d^{-s}+d^{\tau+2}\leqq c_3.[1+d+H(d)] \tag{A.8}$$

for all $d>0$ where c_3 depends on H. By (A.7) and (A.8) we have (A.6) in the case $\tau\neq -1$.

If $\tau=-1$ we note that by (1.2) and elementary calculus,

$$rd.|H'(rd)|\leqq c_4.[d^{-s}+d.(\ln d)^+ +d]$$

and

$$d^{-s}+d.(\ln d)^{+} \leqq c_5.[1+d+\mathrm{H}(d)]$$

for all $d>0$ and $\frac{1}{4} \leqq r \leqq 4$ where c_5 depends on H. Hence (A.6) holds in the case $\tau=-1$. $\square$

Finally we point out the specific form of our equations when $\gamma(\mathrm{F})=\sigma(\mathrm{F})=\frac{|\mathrm{F}|^2}{2}+\mathrm{H}(\det \mathrm{F})$.

THEOREM A.3. – *Assume $\mathscr{U} \in \mathscr{A}$, $\gamma=\sigma$, and $\frac{d}{d\varepsilon} \mathscr{W}(\mathscr{U}_\varepsilon)|_{\varepsilon=0}=0$. Then $\mathscr{U}$ satisfies* (2.8).

Proof. – By (A.2) of Theorem A.1 we have

$$\left[-\sigma(\mathrm{D}\mathscr{U}).\delta_k^j+u_{x_k}^i.\frac{\partial \sigma}{\partial u_{x_j}^i}(\mathrm{D}\mathscr{U})\right]_{x_j}=0$$

for $k=1, 2$. This can be expressed as

$$\operatorname{div}[-\sigma(\mathrm{D}\mathscr{U}).\mathrm{I}+\mathrm{D}\mathscr{U}^{\mathrm{T}}.\mathrm{D}\sigma(\mathrm{D}\mathscr{U})]=0.$$

Since

$$\sigma(\mathrm{F})=\frac{|\mathrm{F}|^2}{2}+\mathrm{H}(\det \mathrm{F})$$

and

$$\mathrm{F}^{\mathrm{T}}.\mathrm{D}\sigma(\mathrm{F})=\mathrm{F}^{\mathrm{T}}\mathrm{F}+\det \mathrm{F}.\mathrm{H}'(\det \mathrm{F}).\mathrm{I},$$

we have

$$\operatorname{div}\left[\left(-\frac{|\mathrm{D}\mathscr{U}|^2}{2}.\mathrm{I}+\mathrm{D}\mathscr{U}^{\mathrm{T}}.\mathrm{D}\mathscr{U}\right)+(-\mathrm{H}(d)+d.\mathrm{H}'(d)).\mathrm{I}\right]=0$$

where $d=\det \mathrm{D}\mathscr{U}$. Setting $f(d)=-\mathrm{H}(d)+d.\mathrm{H}'(d)$ and $(u, v)=\mathscr{U}$ we get

$$\left[\frac{u_x^2+v_x^2-u_y^2-v_y^2}{2}+f(d)\right]_x+[u_x u_y+v_x v_y]_y=0$$

and

$$[u_x u_y+v_x v_y]_x+\left[\frac{u_y^2+v_y^2-u_x^2-v_x^2}{2}+f(d)\right]_y=0. \quad \square$$

B. In this section we prove the following result which was used in the proof of Lemma 5.1.

THEOREM B.1. — *Assume* $u \in W^{2,2}(\Omega)$, $B_{2r}X_0) \equiv B_{2r} \subset \Omega$, *and* $\eta \in C_0^1(B_{2r})$ *with* $|\nabla \eta| \leqq \frac{c_0}{r}$ *on* B_r. *Then*

$$\int_{B_{2r}} (\Delta u)^2 . \eta^2 \, dX \geqq \frac{1}{2} . \int_{B_{2r}} |D^2 u|^2 . \eta^2 \, dX - \frac{c_1}{r^2} . \int_{B_{2r}} |\nabla u - \overline{\nabla u}_{X_0, 2r}|^2 \, dX$$

where c_1 *depends only on* c_0.

Proof. — By approximation we may assume that $u \in C^3(B_{2r})$. Consider $(\Delta u)^2 = \sum_{i,j} u_{x_i x_i} u_{x_j x_j}$. Integrating by parts twice, we have

$$\begin{aligned}\int_{B_{2r}} u_{x_i x_i} . u_{x_j x_j} . \eta^2 \, dX &= \int_{B_{2r}} - u_{x_i} (u_{x_i x_j x_j} . \eta^2 + u_{x_j x_j} . 2\eta . \eta_{x_i}) \, dX \\ &= \int_{B_{2r}} [u_{x_i x_j} . (u_{x_i x_j} . \eta^2 + u_{x_i} . 2\eta . \eta_{x_j}) - u_{x_i} . u_{x_j x_j} . 2\eta . \eta_{x_i}] \, dX \\ &\geqq \int_{B_{2r}} u_{x_i x_j}^2 . \eta^2 \, dX - 4 . \int_{B_{2r}} |D^2 u| . \eta . |\nabla u| . \frac{c_0}{r} \, dX\end{aligned}$$

for any i and j. Hence for any $\varepsilon > 0$.

$$\begin{aligned}\int_{B_{2r}} u_{x_i x_i} . u_{x_j x_j} . \eta^2 \, dX \geqq & \int_{B_{2r}} u_{x_i x_j}^2 . \eta^2 \, dX \\ & - 2\varepsilon . \int_{B_{2r}} |D^2 u|^2 . \eta^2 \, dX - \frac{2}{\varepsilon} . \left(\frac{c_0}{r}\right)^2 . \int_{B_{2r}} |\nabla u|^2 \, dX.\end{aligned}$$

Summing on i and j we have

$$\int_{B_{2r}} (\Delta u)^2 . \eta^2 \, dX \geqq (1 - 8\varepsilon) . \int_{B_{2r}} |D^2 u|^2 . \eta^2 \, dX - \frac{8}{\varepsilon} . \left(\frac{c_0}{r}\right)^2 . \int_{B_{2r}} |\nabla u|^2 \, dX.$$

Setting $\varepsilon = \frac{1}{16}$ we obtain

$$(B.1) \quad \int_{B_{2r}} (\Delta u)^2 . \eta^2 \, dX \geqq \frac{1}{2} . \int_{B_{2r}} |D^2 u|^2 . \eta^2 \, dX - \frac{128 . c_0^2}{r^2} . \int_{B_{2r}} |\nabla u|^2 \, dX.$$

Now let $w(X)=u(X)-\langle X, \overline{\nabla u}_{X_0, 2r}\rangle$. Then $D^2 w=D^2 u$ and $\nabla w=\nabla u-\overline{\nabla u}_{X_0, 2r}$. Applying (B.1) to w we get

$$\int_{B_{2r}} (\Delta u)^2 . \eta^2 \, dX \geqq \frac{1}{2} . \int_{B_{2r}} |D^2 u|^2 . \eta^2 \, dX - \frac{128 . c_0^2}{r^2} . \int_{B_{2r}} |\nabla u - \overline{\nabla u}_{X_0, 2r}|^2 \, dX$$

which proves our assertion. □

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