ANNALES DE L'I. H. P., SECTION C

K. WYSOCKI

Multiple critical points for variational problems on partially ordered Hilbert spaces

Annales de l'I. H. P., section C, tome 7, nº 4 (1990), p. 287-304
<http://www.numdam.org/item?id=AIHPC_1990__7_4_287_0>

Ann. Inst. Henri Poincaré,

Vol. 7, n° 4, 1990, p. 287-304. *Analyse non linéaire*

Multiple critical points for variational problems on partially ordered Hilbert spaces

by

K. WYSOCKI

Department of Mathematics,
Rutgers University, U. S. A.

ABSTRACT. — We are studying the existence of multiple critical points for functionals whose potential operators preserve an order structure. By using Morse type arguments we prove that the existence of local minima of a functional Φ which are ordered in a special way « forces » Φ to have many additional critical points. We also show how these abstract results apply to a concrete situation.

Key-words: Critical points, order structures, stable transition layers.

RÉSUMÉ. — Nous étudions l'existence de points critiques multiples pour les problèmes variationnels dont les opérateurs potentiels préservent une structure d'ordre en utilisant des arguments du type Morse. Nous démontrons que, pour une fonctionnelle Φ, l'existence de minima locaux qui soient ordonnés d'une manière particulière « force » Φ à avoir beaucoup d'autres points critiques. Nous montrons comment ces résultats abstraits s'appliquent à une situation concrète.

INTRODUCTION

The aim of this paper is to prove existence of multiple critical points of functionals Φ which are defined on ordered Hilbert spaces. More precisely we study $\Phi \in C^2(H, R)$ whose gradient admits the decomposition

Classification A. M. S. : 34B, 58E.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 7/90/04/287/18/\$ 3.80/© Gauthier-Villars

Identity-K with K being compact and increasing with respect to the order structure on H. The first results concerning functionals of this type were obtained by Hofer in [6]. Here we extend some of his results. Our Theorem 2.4 says that an order interval C which contains 2^n local minima of Φ ordered in a special way must contain at least 3^n critical points. Another result, Theorem 2.5, has following interpretation.

Let Σ be the set of all local minima of Φ and draw an edge between $u, v \in \Sigma$ iff u and v are order related and there is no $w \in \Sigma$ such that $u \ll w \ll v$ or $v \ll w \ll u$. In this way we define an abstract graph. Then Theorem 2.5 simply states that subgraphs of a certain type correspond to critical points of Φ. The proofs of these results are based on Morse type arguments adopted to our setting. In section 1, we list some preliminary results; in section 2, we prove our main theorems; in section 3 we apply our results to the problem: $\epsilon^2 u''(t) + u(1 - u)(u - a(t)) = 0$, $u'(0) = u'(1) = 0$. We prove that if $a(t) - 1/2$ has $k-1$ zeros than the least number of solutions of this equation is equal to $\dfrac{2^{k+2} + (-1)^{k-1}}{3}$.

Lastly, we mention that in a forthcomming paper using some ideas of [4] we extend our results to cover problems without variational structure and applicable to PDE's.

1. NOTATION AND PRELIMINARIES

In this section we state some basic tools and results in critical point and Morse theory.

Let H be a Hilbert space with scalar product $(,)$ and norm $\| \ \|$. If $\Phi \in C^1(U, R)$ for some open subset U of H, $\emptyset \neq C \subseteq U$, and $a \in R$, $S \subseteq R$ we set

$$\mathrm{Cr}(\Phi, C, a) := \{ u \in C ; \Phi(u) = a, \Phi'(u) = 0 \}$$

$$\mathrm{Cr}(\Phi, C, S) := \bigcup_{e \in S} \mathrm{Cr}(\Phi, C, e)$$

$$\Phi^a := \Phi^{-1}((-\infty, a]), \qquad \Phi_a := \Phi^{-1}([a, \infty))$$

We say that Φ satisfies $(PS)_C$ (Palais-Smale condition on C), if for every sequence $\{ u_n \} \subseteq C$ such that $\{ \Phi(u_n) \}$ is bounded and $\Phi'(u_n) \to 0$ there is a convergent subsequence $u_{n_k} \to u \in C$.

With $H^*(X, Y)$ we indicate the singular cohomology groups with R-coefficients of the pair of topological spaces (X, Y), If (X, Y) is a pair of topological spaces we write $(X, Y) \in \mathrm{Top}^2$ if $\dim H^i(X, Y) < \infty$ for all $i \in N \cup \{ 0 \}$ and define

$$P(X, Y)(t) = \sum_{i=0}^{\infty} \dim H^i(X, Y) t^i$$

The pair $(X, Y) \in \mathrm{Top}^2$ has a finite cohomology if $P(X, Y) \subseteq Z^+[t]$. It is known that if $(X_0, \ldots, X_n)$ is a $(n+1)$-tuple of topological spaces such that $(X_i, X_{i+1}) \in \mathrm{Top}^2$, $i=0, \ldots, n$, then there is $Q(t) \in Z^+[[t]]$ such that $\sum_{i=0}^{n-1} P(X_i, X_{i+1})(t) = P(X_0, X_n)(t) + (1+t)Q(t)$ and if (X_i, X_{i+1}) has finite cohomology then $Q(t) \in Z^+[t]$.

We need the following definition.

DEFINITION 1.1. — Let u be an isolated critical point of $\Phi \in C^1(U, R)$. The Poincare series of u is defined by

$$P_{\Phi, u}(t) = \sum_{i=0}^{\infty} \dim H^i(\Phi^d \cap C \cap W_u, (\Phi^d \setminus \{u\}) \cap W_u)$$

where $d = \Phi(u)$ and W_u is an open neighborhood of u such that $\mathrm{Cr}(\Phi, U) \cap W_u = \{u\}$.

If C is a closed subset of U then the Poincare series of u relative to C is defined by

$$P_{\Phi, C, u}(t) = \sum_{i=0}^{\infty} \dim H^i(\Phi^d \cap C \cap W_u, (\Phi^d \cap C \setminus \{u\}) \cap W_u)$$

where $d = \Phi(u)$ and W_u is an open neighborhood of u such that $\mathrm{Cr}(\Phi, U) \cap W_u = \{u\}$.

If C is a closed subset of U then the Poincare series of u relative to C is defined·by

$$P_{\Phi, C, u}(t) = \sum_{i=0}^{\infty} \dim H^i(\Phi^d \cap C \cap W_u, (\Phi^d \cap C \setminus \{u\}) \cap W_u)$$

where $d = \Phi(u)$ and W_u is an open neighborhood of u such that $\mathrm{Cr}(\Phi, C) \cap W_u = \{u\}$.

Note that if C has nonempty interior and $u \in \mathrm{Cr}(\Phi, C) \cap \mathrm{int}\, C$ then $P_{\Phi, C, u} = P_{\Phi, u}$. Also if $u \in C$ is an isolated local minimum then $P_{\Phi, C, u}(t) = P_{\Phi, u}(t) = 1$.

The following lemma in the case $U = C = H$ is well-known.

LEMMA 1.2. — Let H be a Hilbert space, U open and $C \subseteq U$ be closed and convex. Suppose that $\Phi \in C^2(U, R)$ satisfies $(PS)_C$ and its gradient Φ' has the decomposition $I - K$ with $K(C) \subseteq C$

a) If $\mathrm{Cr}(\Phi, C, [a, \infty)) = \emptyset$ then $\Phi^a \cap C$ is a deformation retract of C,

b) If $\mathrm{Cr}(\Phi, C, [a, b]) = \emptyset$ then $\Phi^a \cap C$ is a deformation retract of $\Phi^b \cap C$,

c) If $\mathrm{Cr}(\Phi, C, [a, b)) = \emptyset$ then $\Phi^a \cap C$ is a deformation retract of $\Phi^b \cap C \setminus \mathrm{Cr}(\Phi, C, b)$.

Proof. — The proof is the usual one (see [8], Lemma 3.3). To define deformation retractions one can use the positive semiflow η associated to the differential equation $\dot{u} = -\beta(\| \Phi'(u) \|)\Phi'(u)$ where $\beta(t)=1$, $t \leq 1$ and $\beta(t)=1/t$, $t \geq 1$. However, in our setting we must make sure that $\eta(t, \xi) \in C$ for $t \geq 0$, $\xi \in C$. But this follows from the assumptions that C is closed convex, $K(C) \subseteq C$ and the sub-tangential criterion (see [5], Theorem 3.2). $\square$

Using Lemma 1.1 we can proof the following proposition.

PROPOSITION 1.3. — Assume that $\Phi \in C^2(U, R)$, U is open and $C \subseteq U$ is closed and convex. Assume that Φ satisfies $(PS)_C$ and that $\Phi' = I - K$ with $K(C) \subseteq C$. Let $a < b$ be regular values of Φ on C. Suppose that the set Cr $(\Phi, C, (a, b))$ is finite and if $u \in \mathrm{Cr}(\Phi, C, (a, b))$ then $(\Phi^d \cap C, \Phi^d \cap C \setminus \{u\}) \in \mathrm{Top}^2$, $(d = \Phi(u))$. Then

$$\sum_{u \in \mathrm{Cr}(\Phi, C, (a,b))} P_{\Phi,C,u}(t) = P(\Phi^b \cap C, \Phi^a \cap C) + (1 + t)Q(t) \tag{1}$$

where $Q(t) \in Z^+[[t]]$. In particular, if Φ is bounded on C and Cr (Φ, C) is finite then

$$\sum_{u \in \mathrm{Cr}(\Phi, C)} P_{\Phi,C,u}(t) = 1\,(1 + t)Q(t) \tag{2}$$

Proof. — See [3]. We only point out that the arguments of [3] can be carried out in our setting because of Lemma 1.3. To prove (2) we can take $a < \inf_C \Phi$ and b so that Cr $(\Phi, C, [b, \infty)) = \emptyset$. Then using Lemma 1.1 *a*), $P(\Phi^b \cap C, \Phi^a \cap C)(t) = P(\Phi^b \cap C, \emptyset)(t) = P(C, \emptyset)(t) = 1$, because C is convex. $\square$

Before stating the next result we recall the Morse Lemma from [7]. Assume that H is a Hilbert space, $\Phi \in C^2(U, R)$ and $\Phi' = I - K$ where K is compact. Suppose that u_0 is an isolated critical point of Φ and $H = H^- \oplus H^0 \oplus H^+$ is the canonical decomposition of H associated to $\Phi''(u_0)$ via the spectral resolution. Then there is a homeomorphism D defined in a neighborhood of u_0 in H such that $D(u_0) = u_0$ and there is $\Psi \in C^2(H^0, R)$ such that

$$\Phi(Du) = \Phi(u_0) - 1/2 \| u^- - u_0^- \|^2 + 1/2 \| u^+ - u_0^+ \|^2 + \Psi(u^0 - u_0^0) \tag{3}$$

for all $u = u^- + u^0 + u^+ \in H^- \oplus H^0 \oplus H^+$ and $\| u_0 - u \|$ small.

If Φ and u_0 are as above we denote by $m^-(u_0)$ the negative and by $m^0(u_0)$ the zero Morse index i. e. $m^-(u^0) = \dim H^-$, $m^0(u_0) = \dim H^0 = \dim \ker (\Phi''(u_0))$.

In the next proposition we compute the Poincare series of critical points with certain Morse indices.

PROPOSITION 1.4. — Assume $\Phi \in C^2(U, R)$, $\Phi' = I - K$ and K is compact. Assume that $u_0 \in U$ is an isolated critical point of Φ such that $(m^-(u_0), m^0(u_0)) \in (N \cup \{0\}) \times \{0, 1\}$.

a) If $m^0(u_0) = 0$ then $P_{\Phi,u_0}(t) = t^{m^-(u_0)}$.

b) Assume $m^0(u_0) = 1$. Then we have three cases :

i) If u_0^0 is a maximum of Ψ then $P_{\Phi,u_0}(t) = t^{m^-(u_0)+1}$.

ii) If u_0^0 is a minimum of Ψ then $P_{\Phi,u_0}(t) = t^{m^-(u_0)}$.

iii) If u_0^0 is a neither maximum nor minimum of Ψ then $P_{\Phi,u_0}(t) = 0$.

Proof. — We prove *b*), since the part *a*) is a classical result. For simplicity we set $u_0 = 0$ and $\Phi(u_0) = 0$. Also we write $m^- = m^-(0)$ and $m^0 = m^0(0)$. By (3) we can write

$$\Phi(u) = -1/2\,\|x\|^2 + 1/2\,\|z\|^2 + \Psi(y) \tag{4}$$

for $u=x+y+z \in H^- \oplus H^0 \oplus H^+$, $\|u\|$ small. Let $W = W_- \oplus W_0 \oplus W_+$ be such that cl (W) $\subseteq$ U and W_-, W_0, W_+ small balls around 0 in H^-, H^0, H^+. Since $\dim H^0 = 1$ we identify W_0 with $(-\delta, \delta) \subseteq R$. Define G : $[0, 1] \times \Phi^0 \cap W \to \Phi^0 \cap W$

$$(x + y + z) \mapsto x + y + (1 - t)z.$$

Using (4) it is easy to see that G and $G_{|(\Phi^0\setminus\{0\})\cap W}$ defines deformation retractions of $\Phi^0 \cap W$ onto $\Phi^0 \cap (W_- \oplus W_0)$ and $(\Phi^0\setminus\{0\}) \cap W$ onto $(\Phi^0\setminus\{0\}) \cap W_- \oplus W_0$. Thus

$$H^*(\Phi \cap W, \Phi^0\setminus\{0\} \cap W) = H^*(\Phi^0 \cap (W_- \oplus W_0),\ (\Phi^0\setminus\{0\}) \cap W_- \oplus W_0)$$

In the case when 0 is a local minimum of Ψ then $\Psi(y) < 0$, $y \neq 0$, $y \in (-\delta, \delta)$ and using (4)

$$\Phi^0 \cap (W_- \oplus W_0) = W_- \oplus W_0$$

This implies that

$$H^*(\Phi^0 \cap W,\ (\Phi^0\setminus\{0\}) \cap W) = H^*(W_- \oplus W_+, W_- \oplus W_0\setminus\{0\}) \simeq$$

$$\simeq H^*(R^{m^-+1}, R^{m^-+1}\setminus\{0\}) = \begin{cases} R; & \text{if} k = m^- + 1 \\ 0\,; & \text{otherwise} \end{cases}$$

and this proves *i*). Assume now that 0 is a local minimum of Ψ. Then $\Psi(sy) \leq \Psi(y)$ for any $y \in (-\delta, \delta)$ and $s \in [0, 1]$ and if

$$G_1(t, x + y) = x + (1 - t)y$$

for $t \in [0, 1]$, $x + y \in \Phi^0 \cap W_- \oplus W_0$ then $G_1([0, 1] \times \Phi^0 \cap W_- \oplus W_0) \subseteq \Phi^0 \cap (W_- \oplus W_0)$. Again using (4), one can show that G_1 and $G_{1|\Phi^0\setminus\{0\}\cap W_-\oplus W_0}$ define deformation retractions of $\Phi^0 \cap W_- \oplus W_0$ and $\Phi^0\setminus\{0\} \cap W_- \oplus W_0$ onto W_- and $W_-\setminus\{0\}$ respectively. Thus

$$H^*(\Phi^0 \cap W, \Phi^0\setminus\{u\} \cap W) = H^*(W_-, W_-\setminus\{0\}) \simeq$$

$$\simeq H^*(R^{m^-}, R^{m^-}\setminus\{0\}) = \begin{cases} R; & k = m^- \\ 0\,; & \text{otherwise} \end{cases}$$

and *ii*) is obvious. Finally assume that 0 is neither a maximum or a minimum of Ψ. We can assume that Ψ is increasing on $(-\delta, \delta)$. Define

$$G_2 : [0, 1] \times \Phi^0 \cap (W_- \oplus W_0) \mapsto \Phi^0 \cap (W_- \oplus W_0)$$

by

$$G_2 : (t, x + y) = \begin{cases} x + y & ; \quad y \in (-\delta, 0] \\ x + (1 - t)y ; & y \in [0, \delta) \end{cases}$$

The maps G_2 and $G_{2|\Phi^0 \setminus \{0\} \cap W_- \oplus W_0}$ provides deformation retractions of $\Phi^0 \cap (W_- \oplus W_0)$ and $\Phi^0 \setminus \{0\} \cap (W_- \oplus W_0)$ onto $W_- \oplus (-\delta, 0]$ and $W_- \oplus (-\delta, 0] \setminus \{0\}$. Thus

$$H^*(\Phi^0 \cap W, \Phi^0 \setminus \{0\} \cap W) = H^*(W_- \oplus (-\delta, 0], \ W_- \oplus (-\delta, 0] \setminus \{0\}) = 0$$

and *iii*) is immediate. $\square$

The above proposition has interesting consequences. Let C be a closed convex subset of U, int C $\neq \emptyset$ and $u \in \mathrm{Cr}(\Phi, C) \cap \mathrm{int}\, C$. If $m^0(u) = 1$ then u contributes to (2) of Proposition 1.3 as nondegenerate critical point or does not contribute at all.

The following concept will be useful later.

By an ordered Banach space we mean a pair (F, P) where F is a Banach space and P is a closed convex subset of F such that $(-P) \cap P = \{0\}$ and $R^+ \times P \subseteq P$. The set P is called a cone.

Then we can define an ordering on F by

$$x \leq y \Leftrightarrow y - x \in P.$$

Also we write

$$x < y \Leftrightarrow x \leq y \quad \text{and} \quad x \neq y$$
$$x \ll y \Leftrightarrow y - x \in \mathrm{int}\, P$$

If $p, q \in F$ and $p \leq q$ then the set $[p, q] = \{x \in F;\ p \leq x \leq q\}$ is an order interval. We say that p, q are comparable if $p - q \in P \cup -P$; otherwise they are noncomparable. An operator $T : F \to F$ is order preserving if $x \leq y$ implies $Tx \leq Ty$ and strongly order preserving if $x < y$ implies $Tx \ll Ty$.

2. EXISTENCE OF MULTIPLE CRITICAL POINTS

In this section we prove our main results concerning existence of multiple critical points of Φ. For the following we impose the condition.

(Φ) (H, P) is an ordered Hilbert space the cone P has a nonempty interior, $U \subseteq H$ is order- convex i. e. if $u, v \in U$ and $u \leq v$ then $[u, v] \subseteq U$, $\Phi \in C^2(U, R)$ with a gradient Φ' of the form $I - K$ where K is compact and strongly order preserving.

If $u \in \mathrm{Cr}(\Phi, \mathrm{U})$ then $\mathrm{K}'(u)$ is strongly order preserving and that for any $u, v \in \mathrm{U}$, Φ satisfies $(\mathrm{PS})_{[u,v]}$.

We point out that if Φ satisfies (Φ) and $u, v \in \mathrm{Cr}(\Phi, \mathrm{U})$ such that $u < v$ then necessarilly $u \ll v$. Moreover, if $u \in \mathrm{Cr}(\Phi, \mathrm{U})$ then since $\mathrm{K}'(u)$ is self-adjoint we have $r(\mathrm{K}'(u)) = \| \mathrm{K}'(u) \| > 0$ and since $\mathrm{K}'(u)$ is strongly order preserving then by the Krein-Rutman result (see [*1*]) $\| \mathrm{K}'(u) \|$ is an eigenvalue of $\mathrm{K}'(u)$, the corresponding eigenspace is one dimensional and spanned by some $w \gg 0$.

Using that fact we derive:

LEMMA 2.1. — Assume that Φ satisfies (Φ) and that $u \in \mathrm{Cr}(\Phi, \mathrm{U})$ with $m_-(u) > 1$. Then there is $v \gg u$ such that $\| u - v \|$ is small and $\Phi(u) > \Phi(v)$.

Proof. — Since Φ satisfies (Φ) and $m_-(u) \geq 1$ the above remarks imply that the smallest eigenvalue λ of $\Phi''(u)$ is negative and the corresponding eigenvector $w \gg 0$. Then $\Phi'(u + tw) = \lambda tw + o(t)$ and

$$\Phi(u + sw) - \Phi(u) = \int_0^1 (\Phi'(u + \tau w), w) d\tau = (\lambda/2) s^2 \| w \|^2 + o(s^2).$$

Since $\lambda > 0$ we conclude that $\Phi(v) < \Phi(u)$ and $v \ll u$ with $v = u + sw$ for small $s > 0$. $\square$

Let $\mathrm{I}^n = [0, 1]^n$ be the standard n-cube in R^n. By V we denote the set of its vertices. A $(n - 1)$ dimensional face of I^n is the set $\mathrm{I} \times \ldots \times \mathrm{I} \times \{a_k\} \times \mathrm{I} \times \mathrm{I} \ldots \times \mathrm{I}$ for some $k \in \{1, \ldots, n\}$ and a_k equal to 0 or 1. A $(n - 2)$ dimensional face of I^n is a $(n - 2)$ dimensional face of some $(n - 1)$ dimensional face of I^n, and so on. We make the simple observation that if $\mathrm{I}_1, \mathrm{I}_2$ are k dimensional faces of I^n then either $\mathrm{I}_1 = \mathrm{I}_2$ or $\mathrm{I}_1 \cap \mathrm{I}_2 = \emptyset$ or $\mathrm{I}_1 \cap \mathrm{I}_2 = \mathrm{I}_3$ where I_3 is a $(k - 1)$ dimensional face. We also remark that each k dimensional face is completely determined by its vertices and that the number of k dimensional faces of I^n is given by $\binom{n}{k} 2^{n-k}$.

We introduce an ordering on V by

$$(\alpha_1, \ldots, \alpha_n) \leq (\beta_1, \ldots, \beta_n) \Leftrightarrow \alpha_i \leq \beta_i, \quad i = 1, \ldots, n.$$

Now let $\Sigma := \{ v \in \mathrm{U}; v \text{ is a local minimum of } \Phi \}$. With Σ we associate an abstract graph (Σ, Γ) where $\Gamma \subseteq \Sigma \times \Sigma$ is defined by

$$\Gamma := \{ (u, v) \in \Sigma \times \Sigma;\ u \ll v \text{ and there is no } w \in \Sigma : u \ll w \ll v \}.$$

The elements of Σ are vertices and a pair $(u, v) \in \Gamma$ is the edge of (Σ, Γ) with end points u and v. If $\Sigma_0 \subseteq \Sigma$ then $\Gamma_{|\Sigma_0}$ means $\{(u,v) \in \Gamma;\ u, v \in \Sigma_0\}$.

DEFINITION 2.2. — Let $\Sigma^n \subseteq \Sigma$ such that if $u,\ v \in \Sigma^n,\ w \in \Sigma$ and $v \ll w \ll u$ or $u \ll w \ll v$ then $w \in \Sigma^n$.

We call $(\Sigma^n, \Gamma_{\Sigma^n})$ a n-cube if there is an isomorphism $f : \Sigma^n \to \mathrm{V}$ which satisfies

$$v \ll u \Leftrightarrow f(v) < f(u).$$

A pair $(\Sigma^k, \Gamma_{\Sigma^k})$, $\Sigma^k \subseteq \Sigma^n$, is a k-subcube of $(\Sigma^n, \Gamma_{\Sigma^n})$ if $\Sigma^k = f^{-1}(\mathrm{V}^k)$ where V^k is a set of vertices of some k dimensional face of I^n.

From now we will identify the graphs (Σ, Γ), $(\Sigma^n, \Gamma_{\Sigma^n})$ with their sets of vertices Σ, Σ^n with the understanding that edges are defined as above. Note that if Σ^n is a n-cube then Σ^n contains the smallest and the greatest element p and q, namely $p = f^{-1}(0,\ldots,0)$ and $q = f^{-1}(1,\ldots,1)$. With the n-cube Σ^n we associate an order interval $\mathrm{C} = [p, q]$ where p, q are as above. Similarly if Σ^k is a k-subcube of Σ^n we can associate with Σ^k an order interval $\mathrm{C}_1 = [p_1, q_1]$ where p_1 and q_1 are the smallest and the greatest elements of Σ^k. Note that $\mathrm{C}_1 \subseteq \mathrm{C}$.

Moreover.

LEMMA 2.3. — Assume that Σ^n is a n-cube and Σ_1^{n-1}, Σ_2^{n-1} are different $(n-1)$-subcubes of Σ^n and $\mathrm{C} = [p, q]$, $\mathrm{C}_i = [p_i, q_i]$, $(i = 1, 2)$, corresponding order-intervals via f. Then either $\mathrm{C}_1 \cap \mathrm{C}_2 = \emptyset$ or there is exactly one $(n-2)$-subcube $\Sigma^{n-2} = \Sigma_1^{n-1} \cap \Sigma_2^{n-1}$ and if $\mathrm{C}_3 = [p_3, q_3]$ is the corresponding order interval then $\mathrm{C}_3 \subseteq \mathrm{C}_1 \cap \mathrm{C}_2$.

More precisely there are four possibilities:

i) $p = p_1 = p_2 = p_3$, q_1 and q_2 are noncomparable and $q_3 \in \operatorname{int}(\mathrm{C}_1 \cap \mathrm{C}_2)$.
ii) $q = q_1 = q_2 = q_3$, p_1 and p_2 are noncomparable and $p_3 \in \operatorname{int}(\mathrm{C}_1 \cap \mathrm{C}_2)$.
iii) $p = p_1 \ll p_2 = p_3 \ll q_1 = q_3 \ll q_2 = q$.
iv) $p = p_1 \ll p_1 = p_3 \ll q_2 = q_3 \ll q_1 = q$.

Proof. — Let V_i, $i = 1, 2$ be the corresponding via f to Σ_i^{n-1} set of vertices of a $(n-1)$-dimensional face I_i *of* I^n. Since $\Sigma_1^{n-1} \neq \Sigma_2^{n-1}$ then $\mathrm{V}_1 \neq \mathrm{V}_2$. If $\mathrm{V}_1 \cap \mathrm{V}_2 = \emptyset$ then obviously $\mathrm{C}_1 \cap \mathrm{C}_2 = \emptyset$. If $\mathrm{V}_1 \cap \mathrm{V}_2 \neq \emptyset$ then $\mathrm{I}_1 \cap \mathrm{I}_2 = \mathrm{I}_3$. Denote by V_3 the set of vertices of I_3. Then $\mathrm{V}_3 = \mathrm{V}_1 \cap \mathrm{V}_2$ and if C_3 is the corresponding order interval to ν_3 then $\mathrm{C}_3 \subseteq \mathrm{C}_1 \cap \mathrm{C}_2$. The rest of the lemma follows easily by inspecting vertices of I_1, I_2, I_3. □

REMARK. — The part *iii*) and *iv*) simply says that $\mathrm{C}_1 \cap \mathrm{C}_2 \backslash \mathrm{C}_3 \neq \emptyset$, but elements of Σ^n which are in $\mathrm{C}_1 \cap \mathrm{C}_2$ are contained in C_3.

For the rest of the paper we call an isolated critical point of Φ trivial if its Poincare polynomial is equal to 0; otherwise a critical point is non-trivial. Now we can state our result.

THEOREM 2.4. — Suppose that Φ satisfies (Φ) and that Σ^n is a n-cube. Suppose that $\mathrm{C} = [p, q]$ is an order interval corresponding to Σ^n and that Φ is bounded from below on C. Assume that $\mathrm{Cr}(\Phi, \mathrm{C})$ is finite and if $u \in \mathrm{Cr}(\Phi, \mathrm{C})$ then $m^0(u) \leq 1$. Then there exists an odd number of nontrivial critical points in C and that number is at least equal to 3^n. Let $\mathrm{C}^0 = \mathrm{C} \setminus \bigcup_{i=1}^{2n} \mathrm{C}_i$, where the C_i are order intervals corresponding to all $(n-1)$-subcubes of Σ^n. Then the set C^0 contains an odd number of nontrivial critical points which are not minima of Φ.

Proof. — In order to prove that we will apply Proposition 1.3 to order intervals $[r, s]$, $r, s \in \Sigma$. Then $\mathrm{P}_{\Phi,[r,s],r}(t) = \mathrm{P}_{\Phi,[r,s],s}(t) = 1$ and if $u \in \mathrm{Cr}(\Phi, [r, s]) \cap \operatorname{int}[r, s]$ then $\mathrm{P}_{\Phi,[r,s],u} = \mathrm{P}_{\Phi,u}$. Note also that since $m^0(u) \leq 1$ then $\mathrm{P}_{\Phi,[r,s],u}$ is a monomial and the coefficient in front of t^i of the left side of (1) or (2) of Proposition 1.3 is the number of critical points in $[r, s]$ with Poincare polynomials t^i. We prove the theorem by induction.

Let Σ^1 be 1-cube and $\mathrm{C} = [p, q]$ the corresponding order interval. Then p, q are the only local minima in C. Let a_i, $i \in \mathrm{N} \cup \{0\}$ be the number of critical points in C whose Poincare polynomials are t^i. Then $a_0 = 2$ and by (2) of Proposition 1.3

$$2 + \sum_{i=1}^{\infty} a_i t^i = 1 + (1+t)\mathrm{Q}(t)$$

for some $\mathrm{Q}(t) = \sum_{i=0}^{\infty} b_i t^i \in \mathrm{Z}^+[t]$.

After substituting $t = 1$ we get

$$2 + \sum_{i=1}^{\infty} a_i = 1 + 2\sum_{i=0}^{\infty} b_i$$

which implies that the total number $\sum_{i=1}^{\infty} a_i$ of nontrivial critical points contained in $\mathrm{C}^0 = \mathrm{C} \setminus \{p, q\}$ is odd, and at least equal to 1. Hence $\mathrm{C} = [p, q]$ contains at least 3 nontrivial critical points and the number of nontrivial critical points in C is odd.

Assume that the result holds for any k-cube with $k \leq n-1$. Let Σ^n be a n-cube, $\mathrm{C} = [p, q]$ the corresponding order interval and let $\mathrm{C}_1, \ldots, \mathrm{C}_{2n}$ be order intervals corresponding to $(n-1)$-subcubes of Σ^n. By the induction assumption each of C_i contains an odd number of nontrivial critical points in C_i^0 and none of them is a local minimum of Φ. Denote the sets of nontrivial critical points in C_i^0 by $\mathcal{S}_i$, $i = 1, \ldots, 2n$. First we claim that $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $i \neq j$. If $\mathrm{C}_i \cap \mathrm{C}_j = \emptyset$ then this is obvious. Hence assume that $\mathrm{C}_i \cap \mathrm{C}_j \neq \emptyset$. For simplicity we take $i=1$, $j=2$ and we write $\mathrm{C}_1 = [p_1, q_1]$, $\mathrm{C}_2 = [p_2, q_2]$. By the Lemma 2.3 there is an order interval $\mathrm{C}_{12} = [p_3, q_3]$ such that $\mathrm{C}_{12} \subseteq \mathrm{C}_1 \cap \mathrm{C}_2$. By the same lemma there are essentially two cases either $p_1 = p_2 = p_3$, $q_3 \ll q_1$, q_2 and q_1, q_2 are noncomparable or $p_1 \ll p_3 = p_2 \ll q_3 = q_1 \ll q_2$. In the later case $\mathrm{C}_1 \cap \mathrm{C}_2 = \mathrm{C}_{12}$ and then $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ since $\mathcal{S}_1 \cap \mathrm{C}_{12} = \mathcal{S}_2 \cap \mathrm{C}_{12} = \emptyset$. In the contrary case assume that there is $u \in \mathcal{S}_1 \cap \mathcal{S}_2$. Consider $\mathrm{D} := [u, q_1] \cap [u, q_2]$. Then D is closed convex, D has nonempty interior and $\mathrm{K}(\mathrm{D}) \subseteq \mathrm{D}$. Then by Proposition 1 in [6] there is $u_1 \in \mathrm{D}$ such that $\Phi'(u_1) = 0$ and $\inf_{\mathrm{D}} \Phi = \Phi(u_1)$. Since K is strongly order preserving $u_1 \ll q_1\, q_2$. If $m^-(u) \geq 1$ then by Lemma 2.1 we can find $v \gg u$, $v \in \mathrm{D}$ such that $\Phi(u) > \Phi(v)$. Thus $u_1 \neq u$ and $u_1 \in \operatorname{int} \mathrm{D}$ and u_1 is a local minimum of Φ. But then $u_1 \in \mathrm{C}_{12}$ which is impossible since u is not in C_{12}.

If $m^-(u) = 0$ (hence $m^0(u) = 1$, because otherwise u is a local minimum) then since u is a nontrivial critical point we have two cases, either u is a local minimum of Ψ or u is a local maximum of Ψ (see the Morse Lemma (3) for definition of Φ). In the first case u is again a local minimum of Φ. In the second since ker $\Phi''(u)$ is spanned by a positive eigenvector (by the Krein-Rutman result) $\Phi(u) > \Phi(v)$ for some $v \in \mathrm{D}$. As before u_1 is a local minimum of Φ and this is impossible. Thus $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

With a_i, $i \in \mathrm{N} \cup \{ 0 \}$ denoting the number of nontrivial critical points in C whose Poincare polynomial is t^i. By (2) of Proposition 1.3 we have $a_0 = 2^n$ and

$$2^n + \sum_{i=1}^{\infty} a_i = 1 + (1 + t)\mathrm{Q}(t), \ \mathrm{Q}(t) = \sum_{i=1}^{\infty} b_i t^i \in \mathrm{Z}^+[t] .$$

Taking $t = 1$ we get that

$$2^n + \sum_{i=1}^{\infty} a_i = 1 + 2 \sum_{i=0}^{\infty} b_i .$$

As a consequence C contains an odd number of nontrivial critical points.

Furthermore, if L_1, L_2 are different order interval in C corresponding to two k-subcubes of Σ^n, $k \leq n - 1$, then the sets $\mathcal{S}_1$, $\mathcal{S}_2$ of nontrivial critical points of Φ contained in L_1^0, L_2^0, respectively, satisfy $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and then the total number of nontrivial critical points in $\bigcup_{i=1}^{2n} \mathrm{C}_i$ is equal to $b := \Sigma_{k=1}^{n-1} \Sigma_{\Sigma_0} b_{\Sigma_0}$, where $\Sigma_0 =$ k-subcube of Σ^n and b_{Σ_0} is the number of nontrivial critical points in L^0, L is an order interval associated with Σ_0. But by the induction assumption b_{Σ_0} is odd for any subcube $\Sigma_0 \subseteq \Sigma^n$ and since for a given $k \in \{ 0, \ldots, n - 1 \}$ the number of all k-subcubes of Σ^n is $2^{n-k}\binom{n}{2}$ we get that b is even. Since b is even and the number of all nontrivial critical points in C is odd, C^0 must contain an odd number of nontrivial critical points.

Finally, since each order interval corresponding to k-subcube of Σ^n (for $k = 0, \ldots, n$) contains at least one nontrivial critical point in L^0 there are at least $\Sigma_{k=0}^n \binom{n}{k} 2^{n-k} = 3^n$ critical points in C. $\square$

For the next theorem let $\Sigma = \{ u \in \mathrm{U};\ u$ is local minimum of $\Phi \}$ and let $\Gamma \subseteq \Sigma \times \Sigma$ be the set of edges defined as before.

By η_k we denote the cardinality of the set

$$\{(\Sigma_0, \Gamma_{|\Sigma_0}); \ \Sigma_0 \text{ is a } k\text{-cube in } \Sigma\}$$

THEOREM 2.5. — Assume that Φ satisfies (Φ), Cr (Φ, U) is finite and Φ is bounded below on any $[u, v]$, $u, v \in \Sigma$. Suppose that if $u \in$ Cr (Φ, U) then $m^0(u) \leq 1$. Let the cardinality $|\Sigma|$ of Σ be equal to l and let k be the largest integer so that $2^k \leq l$. Then the number of nontrivial critical points of Φ is at least $\Sigma_{i=0}^k \eta_i$.

Proof. — By the previous theorem we know that if Σ^n is a n-cube, $n \geq 1$ then there is a nontrivial critical point which is not a local minimum contained in C^0, where $C = [p, q]$ is the corresponding order interval. Hence it is enough to show that if Σ^n, Σ^k are different n- and k-cubes and C_0, C_1 (with $n \geq k$) are order intervals associated to Σ^n, Σ^k then the above solutions u_0, u_1 are different.

Assume that $u_0 = u_1$ and let $C_i = [p_i, q_i]$, $i = 0, 1$. Then $u = u_0 = u_1 \in C_1 \cap C_2$. If q_0 and q_1 are noncomparable then by considering $D = [u, q_0] \cap [u, q_1]$ we can find $v_1 \in D$ so that $\inf_D \Phi = \Phi(v_1)$ and $\Phi'(v_1) = 0$. Since u is a nontrivial critical point and is not a local minimum then v_1 is a local minimum of Φ and $u \ll v_1 \ll q_0$. But then $v_1 \in \Sigma^n$ and since $u \in C_1^0$, u is noncomparable to any element of $\Sigma \setminus \{p_0, q_0\}$. This implies a contradiction.

Similarly, p_0, p_1 cannot be comparable. Hence we can assume that $q_1 \ll q_0$. Then $p_0 \ll u \ll q_1 \ll q_0$. But then again by the definition of Σ^n, $q_1 \in \Sigma^n$. This implies $u \notin C_1^0$ which again is a contradiction. Thus $u_0 \neq u_1$ and the proof is completed. $\square$

3. APPLICATION

In this section we illustrate the previous result. We apply Proposition 2.5 to the problem recently studied by Angenent, Mallet-Paret and Peletier in [2]. They consider an equation

$$\epsilon^2 u''(t) + f(t, u(t)) = 0 \tag{5}$$

$$u'(0) = u'(1) = 0 \tag{6}$$

in which $f(t, u) = u(1-u)(u - a(t))$. Here a is a C^1-function $[0, 1] \to (0, 1)$ satisfying

i) $a(t) \neq 1/2$, if $t = 0, 1$

ii) $a'(t) \neq 0$ whenever $a(t) = 1/2$.

For definitness we assume $a(0) > 1/2$. We briefly describe their results referring the reader to the original paper for interesting details.

The main result of [2] states that if $Z = \{ t; a(t) = 1/2 \}$ (by *ii*) Z is finite) and $Z_0 \in Z$ is the sequence $0 < t_1 < \ldots < t_k < 1$ then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ there is a stable solution of (5-6) and $a'(t_i)u'(t_i) < 0$ for each i, and u is monotone in a small neighborhood of each t_i and away from t_i either $u(t)$ or $1 - u(t)$ is small. Furthermore all stable solutions are obtained in this way.

Here the stability of the solution u means that the principal eigenvalue μ_ϵ of the linearized problem

$$\epsilon^2 v''(t) + f_\xi(t, u(t))v(t) = \lambda v(t) \tag{7}$$

is nonpositive.

The proof of the existence of stable solutions is based on the method of super- and subsolutions. For a given $Z_0 \subseteq Z$ they construct a subsolution $\underline{u}$ and a supersolution $\bar{u}$ for the problem (5-6) so that $\underline{u}(t) < \bar{u}(t)$, $t \in [0, 1]$. Then there is a stable solution $u \in [\underline{u}, \bar{u}]$ (see [1]).

Moreover, for small $\epsilon > 0$ an order interval $[\underline{u}, \bar{u}]$ contains exactly one stable solution u and the principal eigenvalue μ_ϵ of (7) at u is strictly less than zero. Also note that the problem (5-6) has two obvious solutions $u \equiv 0$ and $u \equiv 1$ and all solutions have their values in [0, 1].

We will be interested in the following question; assume that a is as above and $| Z | = k$. Let ϵ be sufficiently small so that their theorem holds. We ask what is the least number of solutions of (5-6). For the sake of simplicity we assume that $\epsilon = 1$. Let

$$H = H^1 = \{ u; u \text{ is absolutely continues and } u' \in L^2[0, 1] \}.$$

We equip H with the inner product

$$(u, v)_\lambda := \int_0^1 u'v' + \lambda \int_0^1 uv, \lambda > 0.$$

Let $P = \{ u \in H; u(t) \geq 0, t \in [0, 1]) \}$.

It is easy to show that (H, P) is an ordered Hilbert space and that P has nonempty interior.

We define

$$f_\lambda(t, \xi) = f(t, \xi) + \lambda\xi$$

and

$$F_\lambda(t, \xi) = \int_0^\xi f_\lambda(t, s)ds$$

The number λ is chosen so that $f_\lambda(t, \blacksquare)$ is increasing and $f_{\lambda,\xi}(t, \xi) > 0$ for $t \in [0, 1]$, $\xi \in [0, 1]$. Let

$$\Phi(u) := 1/2 \| u \|_\lambda^2 - \int_0^1 F_\lambda(t, u(t))dt \quad \text{for} \quad u \in H.$$

Then

$$\Phi'(u)h = (u, h)_\lambda - \int_0^1 f_\lambda(t, u(t))dt = (u - \mathrm{K}(u), h)_\lambda ,$$

$$(\Phi''(u)h, v)_\lambda = (h, v)_\lambda - \int_0^1 f_{\lambda,\xi}(t, u(t))v(t)h(t)dt = \\ = (h - \mathrm{K}'(u)h, v)_\lambda \quad \text{for} \quad u, v, h \in \mathrm{H}.$$

The critical points of Φ are the C^2-solutions of (1). It is an easy exercise to show that K, $\mathrm{K}'(u)$ are compact, that dim ker $\mathrm{K}'(u) \leq 1$ and that Φ is bounded below on $[u_0, u_1]$ and satisfies $(\mathrm{PS})_{[u_0, u_1]}$, where $u_0 \equiv 0$, $u_1 \equiv 1$.

The maximum principle implies that K and $\mathrm{K}'(u)$ are strongly order preserving and if $\underline{u} < \bar{u}$ are supersolution and subsolution of (5-6) then $\mathrm{K}([\underline{u}, \bar{u}]) \subseteq [\underline{u}, \bar{u}]$. We remark that the stability of a solution u of (5-6) means that $(\Phi''(u)h, h)_\lambda \geq 0$ for any $h \in \mathrm{H}$. Furthermore, if u is a stable solution of (5-6) then the negativity of the principal eigenvalue μ_1 (see (7)) implies

$$\inf_{h \in \mathrm{H}\setminus\{0\}} \frac{(\Phi''(u), h)_\lambda}{\| h \|_\lambda^2} = \rho > 0 .$$

Hence, if u is a stable solution of (5-6) then $(\Phi''(u)h, h)_\lambda \geq \rho \| h \|_\lambda^2$, $h \in \mathrm{H}$ and that means u is a strict local minimum of Φ. On the other hand, if u is a local minimum of Φ then $(\Phi''(u)h, h)_\lambda \geq 0$ and u must be a stable solution. Thus all stable solutions found in [2] are all local minima of Φ.

We denote the set of all local minima of Φ by Σ. Let $t_1 < \ldots < t_{k-1}$ be the zeros of $a(t) - 1/2$ and $t_0 = 0$, $t_k = 1$. With each point $u \in \Sigma$ we associate a k-tuplet of numbers $\alpha(u) = (\alpha_1, \ldots, \alpha_k)$ in the following way:

$$\alpha_i = 0 \ \text{ if } \ u\left(\frac{t_{i-1} + t_i}{2}\right) < 1/2 \qquad \text{and} \qquad \alpha_i = 1 \ \text{ if } \ u\left(\frac{t_{i-1} + t_i}{2}\right) > 1/2$$

Moreover from the fact that $a'(t_i)u'(t_i) < 0$ for $u \in \Sigma$, $i = 1, \ldots, k$ it follows that $\alpha(u) = (\alpha_1, \ldots, \alpha_k)$ satisfies: $\alpha_i = 1$ for i odd implies $\alpha_{i+1} = 1$ and $\alpha_i = 0$ for i even implies $\alpha_{i+1} = 0$.

A sequence of k numbers $(\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \{0, 1\}$ with the above property will be called admissible of the length k.

Let $\mathfrak{A} = \mathfrak{A}(k)$ be the set of all admissible sequences of the length k. We introduce an ordering on the set $\mathfrak{A}$ by

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i, \qquad i = 1, \ldots, k.$$

The examination of the construction in [2] of subsolution $\underline{u}$ and supersolution $\bar{u}$ shows that if $\alpha(u) \leq \alpha(v)$ then $\underline{u} \leq \underline{v}$ and $\bar{u} \leq \bar{v}$. We also have that $\alpha: \Sigma \to \mathfrak{A}$ is order preserving.

LEMMA 3.1. — If $u, v \in \Sigma$ then $u \ll v \Leftrightarrow \alpha(u) < \alpha(v)$ and if there is $i \in \{1, \ldots, k\}$ such that

$$\alpha(u)_i = 0, \qquad \alpha(v)_i = 1 \qquad \text{and} \qquad \alpha(u)_j = \alpha(v)_j, \qquad j \neq i$$

then the order interval $[u, v]$ does not contain any element of Σ different than u or$v$.

Proof. — It is obvious that $u \ll v$ implies $\alpha(u) \leq \alpha(v)$. The construction of subsolution and supersolution $\underline{u}, \bar{u}$ shows that $\alpha(u) = \alpha(v)$ gives $u = v$ (by the uniqueness proved in [2]). If $u < v$ then we must have $u \ll v$ and $\alpha(u) < \alpha(v)$ because otherwise $v \leq u$. Assume that $\alpha(u) < \alpha(v)$ but u and v are noncomparable. Since $\alpha(u) \leq \alpha(v)$ implies $\underline{u} \leq \underline{v}$ we have $\underline{u} \ll u$ and $\underline{u} \ll v$. Consider $D := [\underline{u}, u] \cap [\underline{u}, v]$. We have $K(D) \subseteq D$ and Φ is bounded below on D. Thus there is a critical point w of Φ such that $\inf_D \Phi = \Phi(w)$. Hence w is a classical solution of (5-6) and $\underline{u} \ll w \ll \bar{u}$. But $[\underline{u}, \bar{u}]$ contains exactly one minimum of Φ, namely u and this gives a contradiction. Thus $\alpha(u) < \alpha(v)$ implies $u \ll v$. The last part is the simple consequence of the previous. □

As in the section 2 we associate with $\mathfrak{A}$ an abstract graph $(\mathfrak{A}, \Gamma')$ where the set of edges Γ' is defined as

$$\Gamma' = \left\{ (\alpha, \beta);\ \alpha, \beta \in \mathfrak{A},\ \alpha \leq \beta \quad \text{and} \quad \sum_{i=1}^{\infty} |\beta_i - \alpha_i| = 1 \right\}.$$

Lemma 3.1 says that the graphs $(\mathfrak{A}, \Gamma')$ and (Σ, Γ), where Γ is defined as in section 2, are isomorphic.

According to Theorem 2.5 the total number of nontrivial critical points of Φ is at least equal

$$\sum_n |\{ (\mathfrak{A}^n, \Gamma_{|\mathfrak{A}^n});\ (\mathfrak{A}^n, \Gamma_{|\mathfrak{A}^n}) \text{ is a } n\text{-cube in } \mathfrak{A} \}|.$$

Hence we have to find that number.

THEOREM 3.2. — Let $t_1 < \ldots < t_{k-1}$ be the set of all solution of $a(t) = 1/2$. Then the least number of solutions of (5-6) is

$$\frac{2^{k+2} + (-1)^{k-1}}{3}.$$

In order to prove that result let $N_l(k)$ be the number of all l-cubes in the graph $(\mathfrak{A}, \Gamma')$ where $\mathfrak{A} = \mathfrak{A}(k)$ is the set of all admissible sequences of length k. We set $N_l(k) = 0$ if $k \leq 0$ and $N_l(k) = 0$ if $k \geq l > 1$ (because $|\mathfrak{A}(k)| < 2^k$ and the points of $\mathfrak{A}(k)$ cannot form l-cubes if $k \geq l > 1$).

First we prove:

LEMMA 3.3. — The numbers $N_l(k)$ satisfy the following recurrent formulas:

$$N_0(1) = 2, \qquad N_0(2) = 3, \qquad N_1(1) = 1, \qquad N_1(2) = 2$$
$$N_0(k) = N_0(k-1) + N_0(k-2)$$
$$N_l(k) = N_l(k-1) + N_l(k-2) + N_{l-1}(k-2) \quad \text{for} \quad l \in \mathrm{N}, \quad k > 2$$

Proof. — For $l = 0$ and $k = 1$ there are two admissible sequences namely (0) and (1). If $k = 2$ the only admissible sequences are (0, 0), (0, 1) and (1, 1). Thus $N_0(1) = 2$ and $N_0(2) = 3$. Taking $l = 1$ and $k = 1$ we see that there is exactly one 1-cube, namely an edge ((0), (1)), and if $l = 1$ and $k = 2$ there are two edges ((0, 0), (0, 1)) and ((0, 1), (1, 1)). Hence $N_1(1) = 1$ and $N_1(2) = 2$.

Now let $l \geq 0$, $k \geq 2$ and let $\mathcal{A}_l(k)$ be the set of all l-cubes in $\mathcal{A}(k)$. We can write

$$\mathcal{A}_l(k) = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$$

where $\mathcal{B}_0$, $\mathcal{B}_1$ is the set of all l-cubes in $\mathcal{A}(k)$ whose sets of vertices have the last entry equal to 0 and 1, respectively, and a l-cube W is an element of $\mathcal{B}_2$ if it contains vertices with last entry equal to 0 and also vertices with last entry equal to 1.

The sets $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{B}_2$ are disjoint and thus

$$N_l(k) = |\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2|.$$

Let $l = 0$. Then $\mathcal{B}_2 = \emptyset$ and $\mathcal{B}_i$ are the sets of admissible sequences with last entry equal to i, $i = 0, 1$. If k is odd and $(\alpha_1, \ldots, \alpha_k) \in \mathcal{B}_0$ then $\alpha_k = 0$, α_{k-1} can be equal to 0 or 1, and $(\alpha_1, \ldots, \alpha_{k-1}) \in \mathcal{A}(k-1)$. Then $|\mathcal{B}_0| = N_0(k-1)$. If $(\alpha_1, \ldots, \alpha_k) \in \mathcal{B}_1$ then $\alpha_k = 1$, $\alpha_{k-1} = 1$, and $(\alpha_1, \ldots, \alpha_{k-2}) \in \mathcal{A}(k-2)$. Hence $|\mathcal{B}_1| = N_0(k-2)$. This shows that $N_0(k) = N_0(k-1) + N_0(k-2)$ if k is odd. The case k is even similar.

Now let $l \geq 1$ and k is odd. The other case is the same. Assume that $W \in \mathcal{B}_0$ is a l-cube in $\mathcal{A}(k)$. Then if $\alpha = (\alpha_1, \ldots, \alpha_k) \in W$, $\alpha_k = 0$, $(\alpha_1, \ldots, \alpha_{k-1}) \in \mathcal{A}(k-1)$ and if $W' = \{(\alpha_1, \ldots, \alpha_{k-1}); (\alpha_1, \ldots, \alpha_k) \in W\}$ then W' is a l-cube in $\mathcal{A}(k-1)$. Thus $|\mathcal{B}_0| = N_l(k-1)$.

Similarly if $W \in \mathcal{B}_1$ then any $(\alpha_1, \ldots, \alpha_k) \in W$ must satisfy $\alpha_{k-1} = \alpha_k = 1$ and $(\alpha_1, \ldots, \alpha_{k-2}) \in \mathcal{A}(k-2)$, and $W' = \{(\alpha_1, \ldots, \alpha_{k-2}); (\alpha_1, \ldots, \alpha_k) \in W\}$ is a l-cube in $\mathcal{A}(k-2)$. Thus $|\mathcal{B}_1| = N_l(k-2)$. Hence we have to prove that $|\mathcal{B}_2| = N_{l-1}(k-2)$. Note that it is enough to show that if $W \in \mathcal{B}_2$ then $W = \{(\alpha, 1, 1), (\alpha, 1, 0); \alpha \in W'\}$ where W' is the set of vertices of some $(l-1)$-cube in $\mathcal{A}(k-2)$. We prove the above claim by induction.

Let $l = 1$ and $W \in \mathcal{B}_2$ be a 1-cube. Then obviously $W = \{(\alpha, 1, 1), (\alpha, 1, 0)\}$, for some $\alpha \in \mathcal{A}(k-2)$.

Assume that our claim holds for any r-cube $\in \mathcal{B}_2$ for $r \leq l-1$. Let $W \in \mathcal{B}_2$ be a l-cube in $\mathcal{A}(k)$. First we show that W does not contain vertices of the form $(\alpha', 0, 0)$ with $\alpha' \in \mathcal{A}(k-2)$. Assume it does. Fix $\alpha = (\alpha', 0, 0) \in W$ and let W_1 be a $(l-1)$-subcube of W which contains $\alpha = (\alpha', 0, 0)$. Then W_1 does not contain a vertex of the form $(\beta, 1, 1)$, because if $(\beta, 1, 1) \in W_1$ then also $(\beta, 1, 0) \in W_1$ and by the induction assumption $W_1 = \{(\gamma, 1, 1), (\gamma, 1, 0), \gamma \in W_1'\}$ where W_1' is a $(l-2)$-cube in $\mathcal{A}(k-2)$. Hence all vertices of W_1 have last entry 0. But $\alpha = (\alpha', 0, 0)$ belongs to l, $(l-1)$-subcubes of W, and consequently all of these $(l-1)$-subcubes have all vertices with last entry 0. The number of vertices of $(l-1)$-subcubes of W which contain $\alpha = (\alpha', 0, 0)$ is equal to $2^l - 1$. On the other hand, since $W \in \mathcal{B}_2$ there is a vertex in W whose last entry is 1 and again applying our induction assumption we can find another vertex in W with last entry 1. But then the number of elements of W is greater than $|W| = 2^l$ which gives a contradiction. Hence the vertices of W have the form $(\alpha, 1, 0)$, $(\beta, 1, 1)$. Therefore the l-cube W necessarily contains a $(l-1)$-subcube W_1 such that

$$W_1 = \{(\alpha, 1, 1), (\alpha, 1, 0);\ \alpha \in W_1'\}$$

where W_1' is a $(l-2)$-cube of $\mathcal{A}(k-2)$.

Let $Y = \{(\alpha, 1, 1); (\alpha, 1, 1) \in W\}$. We will show that Y is a $(l-1)$-cube in $\mathcal{A}(k)$. Fix $(\alpha, 1, 1) \in W_1$. Then $(\alpha, 1, 1)$ can be connected by an edge to l vertices of W; to $(\alpha, 1, 0)$, to $(\beta, 1, 1)$ with $\beta \notin W_1'$ and to $(l-2)$ vertices which are of the form $(\gamma, 1, 1)$, $\gamma \in W_1'$. Let W_2 be a $(l-1)$-cube which contains vertices $(\gamma, 1, 1)$, $\gamma \in W_1'$ and $(\beta, 1, 1)$. Note that by the induction assumption W_2 does not contain any vertex of the form $(\sigma, 1, 0)$.

Then $W_2 \subseteq Y$ and $2^{l-1} = |W_2| \leq |Y|$. In a similar way, if $Y_1 = \{(\alpha, 1, 0); (\alpha, 1, 0) \in W\}$ we can find a $(l-1)$-cube W_3 with vertices of the form $(\gamma, 1, 0)$, $\gamma \in \mathcal{A}(k-2)$ such that $W_3 \subseteq Y_1$ and $2^{l-1} = |W_3| \leq |Y_1|$. But then since $|W| = 2^l$ and $Y \cap Y_1 = \emptyset$ we must have $Y = W_2$ and $Y_1 = W_3$. This implies that Y and Y_1 are $(l-1)$-cubes in $\mathcal{A}(k)$ and that

$$W = \{(\alpha, 1, 1);\ \alpha \in Y'\} \cup \{(\alpha, 1, 0);\ \alpha \in Y_1'\}$$

where Y' and Y_1' are $(l-1)$-cubes in $\mathcal{A}(k-2)$. Now we have to show that $Y' = Y_1'$. Since W is a l-cube there are $l2^{l-1}$ edges in W; $(l-1)2^{l-2}$ of them between elements $(\alpha, 1, 1)$, $(\beta, 1, 1)$, $\alpha, \beta \in Y'$, $(l-1)2^{l-2}$ between $(\alpha, 1, 0)$, $(\beta, 1, 0)$, $\alpha, \beta \in Y_1'$, and the rest 2^{l-1} between $(\alpha, 1, 1)$, $(\beta, 1, 0)$, $\alpha \in Y'$, $\beta \in Y_1'$. But the last is possible if $\alpha = \beta$. Hence $Y' = Y_1'$ and our claim is proved. □

Now we can prove Theorem 2.5.

Proof. — Using the same notation as in the lemma we see that the last number of solutions of (5-6) is $N(k) = \Sigma_{l \geq 0} N_l(k)$. First we claim that

$N(k) = 2N(k-2) + N(k-1)$, $k \geq 3$. Since $N_l(k) = N_l(k-1) + N_l(k-2) + N_{l-1}(k-2)$ for $l \geq 1$, $k > 2$ we have

$$\sum_{l \geq 1} N_l(k) = \sum_{l \geq 1} N_l(k-1) + \sum_{l \geq 1} N_l(k-2) + \sum_{l \geq 1} N_{l-1}(k-2) =$$
$$= \sum_{l \geq 1} N_l(k-1) + \sum_{l \geq 1} N_l(k-2) + \sum_{l \geq 0} N_l(k-2).$$

But $N_0(k) = N_0(k-1) + N_0(k-2)$ and hence

$$N(k) = \sum_{l \geq 0} N_l(k) = 2\sum_{l \geq 0} N_l(k-2) + \sum_{l \geq 0} N_l(k-1) = 2N(k-2) + N(k-1).$$

Next we claim that

$$N(k) = \frac{2^{k-1} + 2(-1)^{k-1}}{3} N(1) + \frac{2^{k-1} + (-1)^k}{3} N(2) \quad \text{for} \quad k \geq 3.$$

This equality follows easily by induction, where we use the fact $N(k) = 2N(k-2) + N(k-1)$. Now since $N(1) = N_0(1) + N_1(1) = 3$ and $N(2) = N_0(2) + N_1(2) + N_2(2) = 5$ we get

$$N(k) = \frac{2^{k-1} + 2(-1)^{k-1}}{3} 3 + \frac{2^{k-1} + (-1)^k}{3} 5 = \frac{2^{k+2} + (-1)^{k-1}}{3} \quad \text{for } k \geq 3.$$

Hence the proof is completed. □

The following table shows comparison between the number of stable solutions $N_0(k)$ and the least number, $N(k)$, of all solutions of (5-6) for some values of k.

k	$N_0(k)$	$N(k)$
1	2	3
2	3	5
5	13	43
10	144	1 365
20	17 711	1 398 101
25	196 418	44 739 243

ACKNOWLEDGEMENTS

I wish to thank Prof. H. Hofer for useful discussions and suggestions.

REFERENCES

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *Siam Rev.*, t. **18**, 1976, p. 620-709.

[2] S. B. Angenent, J. Mallet-Paret, L. A. Peletier, Stable transition layers in simi-linear boundary value problem. *J. Diff. Eq.*, t. **62**, 1986, p. 427-442.

[3] R. Bott, Lectures on Morse theory, old and new. *Bull. Am. Math. Soc.*, t. **7**, 1982, p. 331-358.

[4] E. N. Dancer, Multiple fixed points of positive mappings. *J. Reine aug. Math.*, t. **371**, 1986, p. 46-66.

[5] K. Deimling, Ordinary Differential Equations in Banach Spaces. *Lect. Notes Math.*, vol. 596, Springer 1977.

[6] H. Hofer, Variational and topological methods in partially ordered Hilbert spaces. *Math. Ann.*, t. **261**, 1982, p. 493-514.

[7] H. Hofer, The topological degree at a critical point of moutain-pass type. *Proceedings of Symposia in Pure Mathematics,* vol. 45, 1986, p. 501-509.

[8] A. Marino, G. Prodi, Metodi perturbativi nella teoria di Morse. *Boll. Un. Math. Ital.*, Suppl., 3, 1975, p. 1-32.

(Manuscrit received December 4, 1988)