ANNALES DE L'I. H. P., SECTION C

CHANG KUNG-CHING

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Annales de l'I. H. P., section C, tome 6, nº 5 (1989), p. 363-395

<http://www.numdam.org/item?id=AIHPC_1989__6_5_363_0>

Ann. Inst. Henri Poincaré,
Vol. 6, n° 5, 1989, p. 363-395. *Analyse non linéaire*

Heat flow and boundary value problem for harmonic maps

by

CHANG KUNG-CHING
Dept. of Math. Peking University, PRC,
Courant Institute of Math. Sci. N.Y.U., U.S.A.

ABSTRACT. — We prove an existence result for the heat flow, and apply minimax principles to deduce existence and multiplicity results for harmonic map.

Key words : Heat equation, harmonic maps, minimax principle, Linsternik-Schnirelman theory.

RÉSUMÉ. — On démontre un théorème d'existence pour l'équation de la chaleur, et on déduit des résultats d'existence et de multiplicité pour divers problèmes aux limites associés aux applications harmoniques.

INTRODUCTION

Let (M, g), (N, h) be two compact Riemannian manifolds. For any smooth map $u: M \to N$,

$$e(u)=\frac{1}{2}|\nabla u|^2$$

is called the energy density, in local coordinates, it is expressed as

$$e(u)=\frac{1}{2}g^{ij}(x)\,h_{\alpha\beta}(u)\,u^{\alpha}_{,i}\,u^{\beta}_{,j},$$

where i, j run over $1, 2, \ldots, m=\dim M$, and α, β run over $1, 2, \ldots, n=\dim N$. The energy of u is defined as an integral:

$$E(u)=\int_M e(u)\,dVg$$

where Vg is the volume element over M.

The critical points of the energy are called harmonic maps. $\forall\, \varphi(x)\in T_{u(x)}(N)$, the tangent bundle of N,

$$\langle dE(u), \varphi\rangle = \int_M -\langle \tau_u, \varphi\rangle\, dVg,$$

where τ is the tension field; in local coordinates, it is expressed by

$$\tau_u(x)^{\alpha}=\Delta_M u^{\alpha}(x)+g^{ij}(x)^N\Gamma^{\alpha}_{\beta\gamma}(u)\,u^{\beta}_{,i}\,u^{\gamma}_{,j},$$

where ${}^N\Gamma^{\alpha}_{\beta\gamma}$ denotes the Christoffel symbol of the manifold N, and Δ_M is the Laplacian with respect to the metric g. In the following, we use the shorthand notations

$$\Gamma(u)(\nabla u, \nabla u)=g^{ij}\,{}^N\Gamma^{\alpha}_{\beta\gamma}\,u^{\beta}_{,i}\,u^{\gamma}_{,j},$$

and

$$\underline{\Delta}u=\{\tau_u(x)^{\alpha},\ \alpha=1, \ldots, n\}.$$

Thus, the harmonic maps are solutions of the equation

$$\underline{\Delta}u=0.$$

The evolution equation associated with the harmonic maps is defined as follows: Find $f:[0, \infty)\times M\to N$ satisfying

$$\partial_t f(t, .)=\underline{\Delta}f(t, .).$$

The motivation in studying the evolution equation for harmonic maps is twofold:

(1) The existence of harmonic maps. In some sense, the asymptotic limit of the heat flow converges to a harmonic map, one may prove the existence of the harmonic maps by the associated heat flow. Actually the first existence result in harmonic maps was due to G. Eells and J. H. Sampson [ES1] 1964, in case $\partial M=\emptyset$ and Riem $N\leq 0$ by this method. R. Hamilton [H1] studied the boundary value problems in case $\partial M\neq\emptyset$, under the same curvature condition (1975). J. Jost [J1], Von Wahl [VW1] slightly improved his result replacing the curvature condition by a small range condition.

(2) The multiplicity of harmonic maps. It is well known that the minimax principles as well as the Morse theory provide tools in the study of multiply critical points. These theories are based on a deformation lemma, which provides a deformation from a level set of a given functional to another, if there are no critical points between these two levels. However, in many problems, the deformation is obtained under the Palais-Smale condition. Unfortunately, the energy functional for maps in $W_2^1(M, N)$ does not satisfy the Palais Smale condition. This is the reason that Sacks-Uhlenbeck [SU1] and Uhlenbeck [U1] studied a family of perturbed functionals and established a perturbed Morse theory for harmonic maps.

In this paper we shall use the heat flow as a deformation, and then apply the well known minimax principle and the category theory directly.

The heat flow for harmonic maps was recently studied by M. Struwe [S1], [S2], [S3]. The theorem, which is related to our results, reads as follows:

In case M is a Riemann surface with $\partial M = \varnothing$, for any smooth initial data, there exists a global distribution solution $f(t, x)$ of the evolution equation, which is regular on $[0, \infty) \times M$, with the exception of at most finitely many points $(t_1, x_1), \ldots, (t_l, x_l)$, $0 < t_j \leqq \infty$. And at a singular point $(\bar{t}, \bar{x})$, there is a harmonic map $\bar{f}: S^2 \to N$ such that for suitable $\gamma_m \to 0$, $t_m \to \bar{t}$, $x_m \to \bar{x}$ in local coordinates

$$f(t_m, x_m + \gamma_m x) \to \tilde{f} \quad \text{in } H_{\text{loc}}^{2,2}(\mathbf{R}^2, N)$$

and $\tilde{f}$ has an extension to $\bar{f}$.

Furthermore, $f(t, .) \rightharpoonup f_\infty$, a harmonic map $M \to N$, in the weak $W_2^1(M, N)$ topology, if all $t_j < \infty$, $1 \leqq j \leqq l$, $f(t, .) \to f_\infty$ even strongly in $H^{2,2}(M, N)$.

We study the following initial boundary value problem for the evolution equation.

$$(E) \qquad \begin{aligned} \partial_t f(t, x) &= \underline{\Delta} f(t, x) \quad &&\text{for } (t, x) \in [0, \infty) \times M \\ f(0, x) &= \varphi(x), &&\forall x \in M \\ f|_{[0, \infty) \times \partial M} &= \chi(x), &&\forall (t, x) \in (0, \infty) \times \partial M, \end{aligned}$$

where we assume that M is a Riemann surface with smooth boundary ∂M,

$$\chi \in C^{2+\gamma}(\partial M, N),$$

and

$$\varphi \in C_\chi^{2+\gamma}(\bar{M}, N) = \{ u \in C^{2+\gamma}(\bar{M}, N) \mid u|_{\partial M} = \chi \}$$

for some $\gamma > 0$.

Our main results are as follows:

THEOREM 1.1. — *Let*

$$m = \inf \{ E(u) \mid u \in W_2^1(M, N),\ u|_{\partial M} = \chi \}$$

and

$$b=\inf\{E(v)\mid v:S^2\to N \text{ is minimal and nonconst.}\}$$

Let E *be a homotopy class of maps from* M *to* N *with boundary value* χ. *If* $\varphi\in F$ *satisfies*

$$E(\varphi)<b+m,$$

then

(1) *The global solution* $f\in W_p^{1,2}\cap C_\chi^{1+(\gamma/2),2+\gamma}((0,\infty)\times\bar{M},N)$ *exists*, $\forall p>4/(1-\gamma)$.

(2) *There exists a constant* C_0 *such that*

$$\sup_{[0,\infty)\times M}|\nabla f(t,x)|\leqq C_0.$$

(3) *There exists a harmonic map* $\tilde{u}\in C_\chi^{2+\gamma}(\bar{M},N)$ *and a sequence* $t_j\nearrow+\infty$ *such that*

$$f(t_j,x)\to\tilde{u}(x)\quad\text{in } C^1(\bar{M},N)$$

(4) $\forall T>0$, the map

$$\varphi\mapsto f(t,.)$$

is continuous from $C_\chi^{2+\gamma}(\bar{M},N)$ *to* $W_p^{1,2}\cap C_\chi^{1+(\gamma/2),2+\gamma}((0,T]\times\bar{M},N)$ $p>4/(1-\gamma)$, *and the energy along the flow is nonincreasing*, i. e.

$$E(f(t,.))\searrow \text{as a function of } t$$

Furthermore, if $\pi_2(N)=0$ *is assumed, then we may use*

$$m_E=\inf\{E(u)\mid u\in E\}$$

to replace m in the assumption, i. e. *the same conclusions hold if*

$$E(\varphi)<b+m_E.$$

Remark. — The same discussion applies for the Neumann boundary problem.

As a consequence, we have the following by-products.

COROLLARY 1 (Sacks-Uhlenbeck, Lemaire [SU], [L1]). — *If* $\pi_2(N)=0$, *then for any homotopy class* E *of maps from* M *to* N *with the prescribed boundary value* χ in case $\partial M\neq\varnothing$, *there is a harmonic map.*

Remark. — A "heat-flow proof" for this Corollary in case $\partial M=\varnothing$ was also given by Struwe [S2].

COROLLARY 2 (Breziz-Coron, Jost [BrC1], [J2]). — *Suppose that* N *is a sphere homeomorphic to* S^2, *and that* $\chi\in C^{2+\gamma}(\partial M,N)$ *is nonconstant. Then there exist at least two homotopically different harmonic maps.*

The mountain pass lemma and its high dimensional link analog, as well as the minimax principle applying on homotopy classes or homology classes of the Banach manifold $E\subset C_\chi^{2+\gamma}(\bar{M},N)$ all can be applied to the energy function below the level $m+b$ (or m_E+b) in the case M being a Riemann surface [and $\pi_2(N)=0$ respectively].

The following corollary generalizes slightly a result due to Benci and Coron [BeC1].

COROLLARY 3. — *For any Riemann surface* M *with boundary* ∂M, *if* $\chi \in C^{2+\gamma}(\partial M, S^n)$ *is not a constant map, then there exist at least two harmonic maps from* M *to* S^n, $n \geq 3$, *in any homotopy class* **E**, *with the prescribed boundary value* χ.

The Ljusternik and Schnirelman category theory and the Morse theory for the energy function below the level $m+b$ (or m_E+b) also hold. Namely we have

THEOREM 1.2. — *Suppose that* (M, g) *is a Riemann surface, and that* (N, h) *is a compact Riemannian manifold with* $\pi_2(N)=0$. *Let* **E** *be a homotopy class of mappings from* M *to* N *with boundary value* χ. *Let* $\mathbf{F}_k$ *be a heat flow invariant family of relative closed subsets of* $C_\chi^{2+\gamma}(\bar{M}, N)$ *endowed with the* W_p^2*-topology*, $p>4/(1-\gamma)$, *which possess category* $\geq k$, *and let*

$$c_k = \inf_{A \in \mathbf{F}_k} \sup_{u \in A} E(u), \qquad k=1, 2, \ldots$$

Suppose that

$$c = c_{s+1} = \ldots = c_{s+r} < m_E + b.$$

Then $\operatorname{cat}_{W_p^2}(K_c) \geq r$, *where* K_c *is the set of harmonic maps in* $C_\chi^{2+\gamma}$ *with energy* c.

The paper is organized as follows. Section 2 contains preparational material, the anisotropic Sobolev spaces, Besov spaces and Nikolski spaces and their embedding theorems, some basic relations for harmonic maps are also studied. Section 3 is the local existence of the heat flow. The main theorem is proved in Section 5. A basic estimate, which is an analogy for the perturbed functional studied by Sack-Uhlenbeck is given in section 4. In section 6, a general minimax principle for harmonic maps is discussed. Finally, in section 7, the Ljusternik Schnirelman multiplicity theorem, *i. e.* Theorem 1.2, and the Corollary 3 are proved.

I am grateful to Prof. J. Eells and Prof. M. Struwe for their valuable comments.

II. PRELIMINARIES

The anisotropic function spaces are used in the study of parabolic equations. Here we only write down the definitions and embedding properties of these spaces defined on $\mathbf{R}^{1+m}$, it is easy to transfer all these results to function spaces defined on $[0, T] \times M$.

Let $\vec{\gamma}=(\gamma_0, \gamma_1)$, and let $1 \leqq p \leqq \infty$. In case γ_0, γ_1 are integers, and $p \in (1, \infty)$, $W_p^{\vec{\gamma}}$ is the space of functions f such that the norms

$$\| f \|_{W_p^{\vec{\gamma}}} = \| f \|_{L^p} + \left\| \frac{\partial^{\gamma_0}}{\partial t^{\gamma_0}} f \right\|_{L^p} + \sum_{j=1}^{m} \left\| \frac{\partial^{\gamma_1} f}{\partial x_j^{\gamma_1}} \right\|_{L^p}$$

are finite.

For any positive γ_i, we decompose it into

$$\gamma_i = \bar{\gamma}_i + \alpha_i,$$

where $\bar{\gamma}_i$ is an integer, and $0 < \alpha_i \leqq 1$, $i=0, 1$; $B_p^{\vec{\gamma}}$ is the space of functions f such that the norms

$$\| f \|_{B_p^{\vec{\gamma}}} = \| f \|_{L^p} + \left(\int_0^\infty \tau^{-1-\alpha_0 p} \| f(t, x) - f(t+\tau, x) \|_{L^p}^p \, d\tau \right)^{1/p} + \sum_{j=1}^{m} \left(\int_0^\infty \tau^{-1-\alpha_1 p} \| f(t, x) - f(t, x+\tau e_j) \|_{L^p}^p \, d\tau \right)^{1/p}$$

are finite. In case $\alpha_0 = 1$ (or $\alpha_1 = 1$), the term $f(t, x) - f(t+\tau, x)$ [or $f(t, x) - f(t, x+\tau e_j)$] is replaced by $2 f(t, x) - f(t+\tau, x) - f(t-\tau, x)$ [or $2 f(t, x) - f(t, x+e_j) - f(t, x-e_j)$, where e_j is the j-th unit vector in $\mathbf{R}^n$].

And $H_p^{\vec{\gamma}}$ is the space consisting of functions such that the norms

$$\| f \|_{H_p^{\vec{\gamma}}} = \| f \|_{L_p} + \sup_{\tau \geqq 0} \| f(t, x) - f(t+\tau, x) \|_{L^p} \tau^{-\alpha_0} \times \sum_{j=1}^{m} \sup_{\tau \geqq 0} \| f(t, x) - f(t, x+\tau e_j) \|_{L^p} \tau^{-\alpha_1}$$

are finite. In case $\alpha_i = 1$, we make the same change as above.

As a special case, $p=\infty$, $H_\infty^{\vec{\gamma}}$ is the Hölder class $C^{\vec{\gamma}}$.

These classes of function spaces have the following connections, *cf.* [N1].

(1) For any $\vec{\varepsilon}=(\varepsilon_0, \varepsilon_1)$ with $\varepsilon_0, \varepsilon_1 > 0$,

$$H_p^{\vec{\gamma}+\vec{\varepsilon}} \subsetneq H_p^{\vec{\gamma}}.$$

If $\vec{\gamma}$ has integer components, and $1 < p < \infty$, then

$$H_p^{\vec{\gamma}+\vec{\varepsilon}} \subsetneq W_p^{\vec{\gamma}} \subsetneq H_p^{\vec{\gamma}}, \qquad H_p^{\vec{\gamma}+\vec{\varepsilon}} \subsetneq B_p^{\vec{\gamma}} \subsetneq H_p^{\vec{\gamma}} \quad (1 \leqq p \leqq \infty),$$

and

$$B_p^{\vec{\gamma}} \subsetneq W_p^{\vec{\gamma}}, \qquad 1 < p \leqq 2,$$
$$W_p^{\vec{\gamma}} \subsetneq B_p^{\vec{\gamma}}, \qquad 2 \leqq p < \infty.$$

(2) (embedding theorem) $\forall f \in B_p^{\vec{\gamma}}$, $(H_p^{\vec{\gamma}})$, $1 \leqq p \leqq \infty$, let $l=(l_1, l_2, \ldots, l_m)$ be a nonnegative integer m-vector, such that

$$\kappa = 1 - \frac{1}{\gamma_1} \sum_{j=1}^{m} l_j > 0.$$

Then

$$f^{(l)} := \frac{\partial^{|l|}}{\partial_1^{l_1} \dots \partial x_m^{l_m}} f(t, x) \in B_p^{\varkappa\vec{\gamma}}(H_p^{\varkappa\vec{\gamma}})$$

and

$$\| f^{(l)} \|_{B_p^{\varkappa\vec{\gamma}}} \leqq c \| f \|_{B_p^{\vec{\gamma}}}$$

where c does not depend on f. The same holds for the space H.

(3) (Trace theorem) $B_p^{\vec{\gamma}}(\mathbf{R}^{1+m}) \subsetneq B_p^{\varkappa\gamma_1}(\mathbf{R}^m)$ if $\varkappa = 1 - \frac{1}{\gamma_0 p} > 0$.

(4) $B_p^{\vec{\gamma}}(\mathbf{R}^{1+m}) \subsetneq B_q^{\varkappa\vec{\gamma}}(\mathbf{R}^{1+m})$, if $1 \leqq p < q \leqq \infty$, and if

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{m}{\gamma_1} + \frac{1}{\gamma_0}\right) > 0.$$

The following inequalities are obtained by the above relations.

PROPOSITION 2.1. — *Let* $Q_T = [0, T] \times M$, *where* M *is a two dimensional compact Riemann manifold.* $\forall f \in W_p^{(1, 2)}(Q_T)$, $4 < p < \infty$, we have a constant $C_\alpha > 0$ *such that*

$$\sup_{t \in [0, T]} \| f(t, .) \|_{C^{1+\alpha}(M)} \leqq C_\alpha \| f \|_{W_p^{1, 2}(Q_T)}$$

where $\alpha = 1 - \frac{4}{p}$.

Proof: We have

$$W_p^{1,2}(Q_T) \subsetneq B_p^{1,2}(Q_T) \quad [\text{by}\,(1)]$$
$$\subsetneq B_p^{2(1-1/p)}(\{t\} \times M), \qquad \forall t \in [0, T] \quad (\text{by the trace theorem})$$
$$\subsetneq B_\infty^{2\varkappa(1-1/p)}(\{t\} \times M) \quad [\text{by}\,(4)]$$

if $\varkappa = 1 - \frac{1}{p-1} > 0$, *i. e.* $p > 2$.

However, if $p > 4$, then $2\varkappa\left(1 - \frac{1}{p}\right) = 2 - \frac{4}{p}$. Let $\alpha = 1 - \frac{4}{p}$, again by (1), we have

$$B_\infty^{2\varkappa(1-1/p)}(\{t\} \times M) \subsetneq H_\infty^{1+\alpha}(\{t\} \times M).$$

PROPOSITION 2.2. — $\forall f \in W_p^{1,2}(Q_T)$, $p > 4$, $\nabla_x f \in C^{\gamma, 2\gamma}(Q_T)$ *for* $\gamma = \frac{1}{2}\left(1 - \frac{4}{p}\right)$ *and* $\exists\, C_p > 0$ *such that*

$$\| \nabla_x f \|_{C^{\gamma, 2\gamma}(Q_T)} \leqq C_p \| f \|_{W_p^{1, 2}(Q_T)}.$$

Proof. — According to the embedding theorem and (1), we have

$$f \in W_p^{1,2} \subsetneq B_p^{1,2},$$

and

$$\nabla_x f \in B_p^{1/2, 1}.$$

Apply (4) and (1),

$$\nabla_x f \in B_\infty^{\kappa(1/2, 1)} \hookrightarrow H_\infty^{\kappa(1/2, 1)}$$

where $\kappa = 1 - \frac{4}{p} > 0$; let $\gamma = \frac{1}{2}\kappa$, we obtain the desired conclusion.

Next we turn out to study the basic properties of the heat flow.

PROPOSITION 2.3. — *Suppose that* $f \in C^{1+(\gamma/2), 2+\gamma}(Q_T, N)$, *for some* $\gamma > 0$ *satisfies the evolution equation* (E). *Then*

$$\frac{d}{dt} E(f(t, .)) \leqq 0, \qquad \forall t \in [0, T]$$

and that the equality occurs at some point t_0, *iff* $f(t_0, .)$ *is a harmonic map.*

Proof. — By Green's formula,

$$\begin{aligned} \frac{d}{dt} E(f(t, .)) &= \int_M \langle \nabla f(t, .), \nabla \partial_t f(t, .) \rangle dVg \\ &= - \int_M \langle \underline{\Delta} f(t, .), \partial_t f(t, .) \rangle dVg \\ &\quad + \int_{\partial M} g^{ij} h_{\alpha\beta} \partial_t f^\beta(t, .) \; f^\alpha_{,i} . n_j dSg \\ &= - \int_M |\underline{\Delta} f(t, .)|^2 dVg \end{aligned} \tag{2.1}$$

where dSg is the line element on ∂M, and $\{n_j | j = 1, 2\}$ is the normal vector.

PROPOSITION 2.4. — *Under the same assumptions as Proposition* 2.3,

$$\int_{Q_T} |\partial_t f(t, .)|^2 dt\, dVg \leqq E(\varphi).$$

Proof. — According to the above equality (2.1),

$$\begin{aligned} \int_M |\partial_t f(t, .)|^2 dVg &= \int_M |\underline{\Delta} f(t, .)|^2 dVg \\ &= - \frac{d}{dt} E(f(t, .)). \end{aligned}$$

After integration, we have

$$\int_{Q_T} |\partial_t f(t,.)|^2 \, dt \, dV g = E(\varphi) - \lim_{t \to T-0} E(f(t,.)).$$

PROPOSITION 2.5. — *Under the assumptions of proposition* 2.3, *there exists a sequence* $t_i \nearrow T$, *and a map* $\tilde{u} \in W_2^1(M, N)$ *such that* $f(t_i, .) \rightharpoonup \tilde{u}$ *weakly in* $W_2^1(M, N)$, *with*

$$\underline{\lim}_{i \to \infty} E(f(t,.)) \geqq E(\tilde{u}).$$

Furthermore, if $T = \infty$, then $\partial_t f(t_i, .) \to 0$ *in the strong* $L^2(M, N)$ *topology.*

Proof. — The first conclusion follows from the weak compactness of bounded sets of $W_2^1(M, N)$, in conjunction with proposition 2.3.

As to the second conclusion, we observe that the integral

$$\int_0^\infty \left(\int_M |\partial_t f(t,.)|^2 \, dV g \right) dt$$

is convergent, there must be a sequence $t_i \nearrow +\infty$ such that

$$\int_M |\partial_t f(t_i, .)|^2 \, dV g \to 0.$$

This is what we need.

Finally, in order to write down the evolution equation (E) in coordinate form, we embed N into a suitable Euclidean space $\mathbf{R}^k$, according to Eells-Sampson [ES1]. We do not take the ordinary Euclidean metric on $\mathbf{R}^k$. Let T be a tubular neighborhood of N, extend the metric on N smoothly to a metric on **T** such that there is an isometry $i: \mathbf{T} \to \mathbf{T}$ on the tubular neighborhood, having precisely N for its fixed point set. Let B be a large ball in the Euclidean metric of $\mathbf{R}^k$, containing **T**, we extend the metric on T smoothly to all of $\mathbf{R}^k$ so as to equal the Euclidean metric outside of B. R. Hamilton [H1] proved that

(1) If $u: M \to N \subseteqq B$ with the above metric on B, then

$$\underline{\Delta}_B u = \underline{\Delta}_N u,$$

where the subscripts of $\underline{\Delta}$ denote $\underline{\Delta}$ under the corresponding metrics.

(2) If $f \in C^{1+\gamma/2, 2+\gamma}(Q_T, B)$ satisfies the evolution equation

$$(E_B) \qquad \begin{aligned} \partial_t f &= \underline{\Delta}_B f \quad \text{in } \mathring{Q}_T \\ f(0, x) &= \varphi, \qquad \forall x \in M \\ f(t,.)|_{\partial M} &= \chi(.), \qquad \forall t \in [0, T] \end{aligned}$$

and if $\varphi \in C_\chi^{2+\gamma}(\bar{M}, N)$ and $\chi \in C^{2+\gamma}(\partial M, N)$, then $f \in C^{1+\gamma/2, 2+\gamma}(Q_T, N)$, and it satisfies

$$\partial_t f = \underline{\Delta}_N f \quad \text{in } \mathring{Q}_T.$$

In this sense, we shall only study the evolution equation (E) in the flat target space (with the nonflat metric).

III. THE LOCAL EXISTENCE

The first step towards the global existence is to prove the local existence, *i. e.* the existence of the heat flow in short time.

THEOREM 3.1. — *There exists,* $\varepsilon > 0$ *and a unique* $f \in W_p^{1,2} \cap C^{1+(\gamma/2), 2+\gamma}(\mathring{Q}_\varepsilon, N)$, $p > \dfrac{4}{1-\gamma}$, *which satisfies the equation* (E) *in* $\mathring{Q}_\varepsilon$.

Proof. — First, we define a nonlinear map as follows.

$$A: \quad W_p^{1,2}(Q_T, \mathbf{R}^k) \to L^p(Q_T, \mathbf{R}^k) \times B_p^{2(1-1/p)}(M, \mathbf{R}^k) \times B_p^{1-(1/2p), 2-(1/p)}(S_T, \mathbf{R}^k),$$
$$f \mapsto (\partial_t f - \underline{\Delta}_B f, f(0, .), f(t, x)|_{S_T}).$$

Since the derivative of A at the function φ reads as $dA(\varphi)$:

$$v \mapsto (\partial_t - \Delta_M) v - 2\Gamma(\varphi)(\nabla\varphi, \nabla v) - \partial\Gamma(\varphi).v(\nabla\varphi, \nabla\varphi), v(0, .), v|_{S_T}.$$

The associated linear parabolic system

$$\partial_t v = \Delta_M v + 2\Gamma(\varphi)(\nabla\varphi, \nabla v) + \partial\Gamma(\varphi).v(\nabla\varphi, \nabla\varphi) + g,$$
$$v(0, .) = 0,$$
$$v(t, .)|_{\partial M} = 0, \qquad \forall t \in [0, T]$$

possesses a unique solution in $W_p^{1,2}(Q_T, \mathbf{R}^k)$, $p > 4$, for each $g \in L^p(Q_T)$, (*cf.* [LSU1] and [W1]). And then, $dA(\varphi)$ is an isomorphism. On the other hand

$$A(\varphi) = (g_0, \varphi, \chi)$$

where $g_0 = \underline{\Delta}\varphi \in C^\gamma(\bar{M}, N) \subset L^p(Q_T, N)$, we may find $\varepsilon > 0$ small enough so that the function

$$g_\varepsilon(t, x) = \begin{cases} 0 & 0 \leqq t \leqq \varepsilon \\ g_0 & \varepsilon < t < T \end{cases}$$

is in a small $L^p(Q_T)$ neighborhood of g_0, so that the inverse function theorem applies. Therefore, we obtain a solution $f \in W_p^{1,2}(Q_T)$ satisfying (*cf.* [LSU1] and [W1])

$$\partial_t f = \underline{\Delta}_B f + g_\varepsilon, \qquad \forall (t, x) \in \mathring{Q}_T$$
$$f(0, x) = \varphi(x)$$
$$f(t, .)|_{\partial M} = \chi.$$

In particular, $f|_{Q_\varepsilon}$ is a solution of $(E)_B$ in Q_ε. According to the remarks at the end of the last section, $f|_{Q_\varepsilon}$ is in $W_p^{1,2}(Q_T, N)$ and solves the evolution equation (E).

Second, we prove the regularity. According to Proposition 2.2, $\nabla_x f \in C^{\gamma', 2\gamma'}(\mathring{Q}_\varepsilon)$, for $\gamma' = \frac{1}{2}\left(1 - \frac{4}{p}\right)$, and since $C^{\gamma', 2\gamma'}(\mathring{Q}_\varepsilon)$ is an algebra, we see

$$\Gamma(f)(\nabla f, \nabla f) \in C^{\gamma', 2\gamma'}(\mathring{Q}_\varepsilon).$$

Applying the Schauder estimate again, it follows $f \in C^{1+(\gamma/2), 2+\gamma}(\mathring{Q}_\varepsilon, N)$.

In the following, we denote $[0, \omega)$, $\omega > 0$, the maximal solvable interval, on which $f \in C^{1+(\gamma/2), 2+\gamma}(\mathring{Q}_T, N)$ for any $T < \omega$.

Remark. — Basically, this result is already known from Hamilton [H1].

IV. THE MAIN ESTIMATES

Sacks-Uhlenbeck [SU] establish a L^p local estimate for the perturbed harmonic maps, by which a C^1 convergence, except at finitely many points, was proved. In this section, we establish an analogy for the heat flow.

LEMMA 4.1. — *Suppose that* $1 < p, q < \infty$, *and that* $t \mapsto g(t, .) \in L^p(M)$ *for a. e.* $t \in [0, T]$. *Assume that*

$$\int_0^T \|g(t, .)\|_{L^p(M)}^q \, dt < \infty,$$

and

$$\begin{aligned} \partial_t f - \Delta_M f = g \quad &\text{in } Q_T \\ f(0, .) = 0 \quad &\text{on } M \\ f(t, .)|_{\partial M} = 0 & \end{aligned} \tag{4.1}$$

then we have a constant $C = C_{p,q}$ *such that*

$$\int_0^T \|f(t, .)\|_{W_p^2(M)}^q \, dt \leqq C \int_0^T \|g(t, .)\|_{L^p(M)}^q \, dt.$$

Proof. — The linear equation (4.1) is considered as an evolution equation associated with the analytic semigroup $T(t)$, which generator Δ_M is sectorial on the space $L^p(M)$ with domain (*see* for instance A. Friedman [F1])

$$D(\Delta_M) = \{u \in W_p^2(M) \mid u|_{\partial M} = 0\}.$$

Thus the solution $f(t, .)$ is expressed by the semigroup:

$$f(t, .) = \int_0^t T(t - \tau) \, g(\tau, .) \, d\tau.$$

For the sectorial operator, there is a $\delta > 0$ such that

$$\|\Delta_M f(t, .)\|_{L^p(M)} \leqq \int_0^t \|\Delta_M T(t-\tau)\|_{B(L^p)} \|g(\tau, .)\|_{L^p} d\tau$$
$$\leqq C_\delta \int_0^t \frac{e^{-\delta(t-\tau)}}{t-\tau} \|g(t, .)\|_{L^p} d\tau.$$

The L^q estimate for the singular integrals (Hilbert transform) provides the following inequality (*see* for instance E. M. Stein [St1])

$$\int_0^T \|\Delta_M f(t, .)\|_{L^p}^q dt \leqq C_q \int_0^T \|g(t, .)\|_{L^p}^q dt.$$

Applying the L^p estimate for elliptic operators due to Agmon-Douglise-Nirenberg, we obtain the desired inequality

$$\int_0^T \|f(t, .)\|_{W_p^2(M)}^q dt \leqq C \int_0^T \|g(t, .)\|_{L^p}^q dt.$$

LEMMA 4.2. — *There exists a positive number* $\varepsilon_0 > 0$ *such that for a solution of the system* (E) *in a domain* $[0, T] \times D$, *where* $D = B_\rho(x_0) \cap \bar{M}$, *for some* $x_0 \in \bar{M}$, *and* $\rho > 0$, *if*

$$\sup_{t \in [t_0, t_1]} \int_D |\nabla f(t, .)|^2 dVg < \varepsilon_0$$

for some $t_0, t_1 \in (0, T)$, *then for any* $\rho' \in (0, \rho)$, *and* $(t_0', t_1') \subset (t_0, t_1)$, *we have some* $\alpha > 0$ *and a constant* C *depending on* ε_0, α, ρ', ρ, *and* t_0, t_1, t_0', t_1' *only such that*

$$\sup_{t \in [t_0', t_1']} \|f(t, .)\|_{C^{1+\alpha}(D')} \leqq C \left[1 + \|\chi\|_{C^{2+\gamma}(\partial M \cap D)} + \left(\int_{[t_0, t_1] \times D} |\nabla f|^{2p} dt\, dVg \right)^{1/p} \right]$$

for $p > 4$, where $D' = B_{\rho'}(x_0) \cap \bar{M}$.

Proof. — Define a cutoff function $\varphi_1 \in C^\infty(Q_T)$, satisfying

$$0 \leqq \varphi_1 \leqq 1$$

and

$$\varphi_1(t, x) = \begin{cases} 1 & (t, x) \in [t_0', t_1'] \times D' \\ 0 & (t, x) \notin [t_0, t_1] \times D \end{cases}$$

Let $F = \varphi_1 . f$, then

$$\partial_t F - \Delta_M F = \Gamma(f)(\nabla F, \nabla f) - \Gamma(f)(f \nabla \varphi_1, \nabla f) - 2 \nabla f . \nabla \varphi_1 + f(\partial_t - \Delta_M) \varphi_1$$

$$F(t_0, .)=0$$
$$F(t, .)|_{\partial M}=\varphi_1 \cdot \chi.$$

According to Proposition 2.1, and linear L^p theory, we have $\alpha=1-4/p>0$ such that

$$\begin{aligned}\sup_{t\in[t_0', t_1']} \|f(t, .)\|_{C^{1+\alpha}(D')} &\leqq \sup_{t\in[t_0, t_1]} \|F(t, .)\|_{C^{1+\alpha}(D)} \leqq C_\alpha \|F\|_{W_p^{1,2}([t_0, t_1]\times D)} \\ &\leqq C[1+\|\chi\|_{C^{2+\gamma}(\partial M\cap D)}+\|\nabla f\|_{L^p([t_0, t_1]\times D)} \\ &\qquad +\|\nabla F . \nabla f\|_{L^p([t_0, t_1]\times D)}]. \quad (4.2)\end{aligned}$$

However, provided by the Sobolev imbedding theorem together with Lemma 4.1, let $p_1=2p/(p+1)$, we have

$$\int_{t_0}^{t_1} \|\nabla F\|_{L^{2p}(D)}^{2p} dt \leqq \int_{t_0}^{t_1} \|F\|_{W_{p_1}^2(D)}^{2p} dt \leqq C[1+\|\chi\|_{C^{2+\gamma}(\partial M\cap D)}^{2p} + \int_{t_0}^{t_1} \|\nabla f\|_{L^{p_1}(D)}^{2p} dt + \int_{t_0}^{t_1} \|\nabla F . \nabla f\|_{L^{p_1}(D)}^{2p} dt]. \quad (4.3)$$

Applying the Hölder inequality,

$$\int_{t_0}^{t_1} \|\nabla F . \nabla f\|_{L^{p_1}(D)}^{2p} dt \leqq \int_{t_0}^{t_1} \|\nabla f\|_{L^2(D)}^{2p} \|\nabla\|_{L^{2p}(D)}^{2p} dt \leqq \varepsilon_0^p \int_{t_0}^{t_1}\int_D |\nabla F(t, x)|^{2p} dV_g dt. \quad (4.4)$$

For sufficiently small $\varepsilon_0>0$, we put the two inequalitites (4.3), (4.4) together and obtain

$$\int_{t_0}^{t_1} \|\nabla F(t, .)\|_{L^{2p}(D)}^{2p} dt \leqq C_{\varepsilon_0}\left[1+\|\chi\|_{C^{2+\gamma}(\partial M\cap D)}^{2p} + \int_{t_0}^{t_1} \|\nabla f(t, .)\|_{L^{p_1}(D)}^{2p} dt\right]. \quad (4.5)$$

Again by the Hölder inequality,

$$\begin{aligned}\|\nabla F . \nabla f\|_{L^p([t_0, t_1]\times D)} &\leqq \|\nabla F\|_{L^{2p}} . \|\nabla f\|_{L^{2p}} \\ &\leqq C_{\varepsilon_0}[1+\|\chi\|_{C^{2+\gamma}(\partial M\cap D)}+\|\nabla f\|_{L^{2p}([t_0, t_1]\times D)}] \|\nabla f\|_{L^{2p}([t_0, t_1]\times D)}.\end{aligned}$$

Return to (4.2), we have

$$\sup_{t\in[t_0', t_1']} \|f(t, .)\|_{C^{1+\alpha}(D')} \leqq C[1+\|\chi\|_{C^{2+\gamma}(\partial M\cap D)}+\|\nabla f\|_{L^{2p}([t_0, t_1]\times D)}^2].$$

LEMMA 4.3. — *Let* $\omega>0$ *be finite or infinite. Assume that* $\forall T<\omega$, $f\in W_p^{1,2}(Q_T, N)$, $p>4$, *is a solution of* (E). *If there is a relatively open* *set* $D \subset M$ *and a sequence of intervals* $I_j \subset [0, \omega)$ *with* $\text{mes}(I_j) \geqq \delta > 0$ *such that*

$$\sup_{t \in I_j} \int_D |\nabla f(t, .)|^2 dVg < \varepsilon_0.$$

Then for any open subset $D' \subset\subset D$, *there is a sequence* $\{t_j\}$ *such that* $t_j \in I_j$, $f(t_j, .)$ *is* $C^1(\bar{D}', N)$ *convergent to some* $\tilde{u} \in W_2^1(M, N)$.

Proof. — Since

$$\int_M |\nabla f(t, .)|^2 dVg \leqq E(\varphi),$$

the family of maps $\{f(t_j, .) \mid j=1, 2, \ldots\}$ is weakly compact in $W_2^1(M, \mathbb{R}^k)$, so that there is a subsequence $\{t'_j\}$ along which $f(t'_j, .) \to \tilde{u}$ in $W_2^1(M, \mathbb{R}^k)$ weakly.

Starting from (4.5) with $p=2$, we obtain a constant, which depends on ε_0, χ and δ, dominating the norms $\|\nabla F\|_{L^4(I_j \times D)}\ \forall_j$. Applying (4.2), so is $\|F\|_{W_2^{1,2}(I_j \times D)}$. Then, the Sobolev embedding theorem implies the boundedness of $\|\nabla F\|_{L^{2p}(I_j \times D)}\ \forall p > 4$.

Thus, we have

$$\|f(t_j, .)\|_{C^{1+\alpha}(D')} \leqq \text{Const.}, \qquad \forall t_j \in I_j,$$

provided by Lemma 4.2. This implies a subsequence $\{t'_j\}$ such that $f(t'_j, .)$ C^1 converges to $\tilde{u}$.

THEOREM 4.1. — *Suppose that* $f \in W_p^{1,2}(Q_T, N)$, $\forall T < \omega$, *is a solution of* (E), *where* $p > 4$, *then there is a sequence* $t_j \to T-0$ *and a finite number of points* $\{x_1, \ldots, x_l\} \subset M$ *such that*

$$f(t_j, .) \to \tilde{u}(.) \quad \text{in } C^{1+\alpha'}(M \setminus \{x_1, \ldots, x_l\}, N)$$

for some $\tilde{u} \in W_2^1(M, N)$, *and* $0 < \alpha' < \alpha = 1 - 4/p$.

Proof. — According to a covering theorem due to Besicovitch, there is an open covering of M consisting of disks $\{B_r(y_i) \mid i=1, \ldots, p\}$ such that

(*a*) $M \subset \bigcup_{i=1}^{p} B_{r/2}(y_i)$.

(*b*) $\forall x \in M$, there exist at most h disks $B_r(y_i)$ covering x, where h is independent of r.

Then

$$\sum_i \int_{B_r(y_i)} |\nabla f(t, .)|^2 dVg \leqq h E(f(t, .)) \leqq h E(\varphi).$$

Hence $\forall t$, $\exists$ at most $l=\left[\dfrac{2hE(\varphi)}{\varepsilon_0}\right]+1$ disks $B_r(y_i)$, $i=1, 2, \ldots, l$, on which

$$\int_{B_r(y_i)} |\nabla f(t,\,.)|^2 dVg \geqq \frac{\varepsilon_0}{2}.$$

Fixing l such disks, there is a sequence $t_j \uparrow \omega - 0$ such that

$$\int_{B_r(y_i)} |\nabla f(t_j,\,.)|^2 dVg < \frac{\varepsilon_0}{2}, \qquad \forall i > l.$$

Now we apply lemma 3.6 in Struwe [S1], which assures a uniform bound $\delta > 0$ such that

$$\sup_{|t-t_j| \leqq \delta} \int_{B\,3r/4\,(y_i)} |\nabla f(t,\,.)|^2 dVg < \varepsilon_0. \qquad \forall i > l.$$

We apply Lemma 4.3 to these remaining disks, there is a sequence $t_j \uparrow \omega - 0$ such that $f(t_j,\,.)$ is $C^{1+\alpha'}$ convergent on $M \diagdown \bigcup_{i=1}^{l} B_{r/2}(y_i)$. Let $\gamma = 2^{-k}$, $k=1, 2, \ldots$, by the diagonal process, there is a subsequence, we still denote it by $\{t_j\}$, so that $f(t_j,\,.)\,C^{1+\alpha'}$-converges on $M \diagdown \{x_1, \ldots, x_l\}$, because the upper bound of the number of exceptional disks is independent of r.

V. GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

We prove the main theorem in this section. Actually, conclusion (3) follows from conclusion (2) directly.

Proof. — We cover M by small balls $\bigcup_{i=1}^{p} B_{r/2}(x_i)$ such that $C_0^2 \operatorname{mes}(B_r(x_i)) < \varepsilon_0$. According to Lemma 4.2,

$$\sup_{t \in [k,\, k+1]} \| f(t,\,.) \|_{C^{1+\alpha}(B_{r/2}(x_i))} \leqq C[1 + \| \chi \|_{C^{2+\alpha}(\partial M \cap B_r(x_i))} + 2C_0^2],$$

$k=1, 2, \ldots, i=1, \ldots, p$, which implies

$$\sup_{t \in [1,\, \infty]} \| f(t,\,.) \|_{C^{1+\alpha}(M)} \leqq C_1, \quad \text{a constant.}$$

On the other hand, according to Proposition 2.5, there is a sequence $t_j \nearrow +\infty$ such that

$$\partial_t f(t_j,\,.) \to 0 \quad \text{in } L^2(M, \mathbf{R}^k).$$

We apply Theorem 4.1 to the sequence $f(t_j, .)$, because

$$\int_{B_r(x_i)} |\nabla f(t_j, x)|^2 dVg < \varepsilon_0.$$

It follows that $f(t_j, .) \to \tilde{u}(.) C^{1+\alpha'}(\bar{M}, N)$. Thus

$$\int_M [g(\nabla\tilde{u}, \nabla\varphi_1) + \Gamma(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u})\varphi_1] dVg = 0,$$

$\forall \varphi_1 \in C_0^\infty(M, N)$. Apply the elliptic regularity theorem again; we conclude $\tilde{u} \in C^{2+\gamma}(M, N)$, and

$$\Delta\tilde{u} = 0$$
$$\tilde{u}|_{\partial M} = \chi.$$

Now we turn to the study of global existence.

From the local existence theorem, we get the maximal existence interval $[0, \omega)$, where ω is finite or infinite. $\forall T \in [0, \omega)$, let

$$\theta_T = \max_{(t, x) \in Q_T} |\nabla f(t, x)|.$$

It is easily seen that the function $T \mapsto \theta_T$ is monotone nondecreasing.

LEMMA 5.1. — *Suppose that* θ_T *is not bounded, then*

$$E(\varphi) \geqq m + b.$$

Proof. — We may find sequences $T_k \nearrow \omega$ and $a_k \in \bar{M}$ such that

$$|\nabla f(T_k, a_k)| = \max_{x \in \bar{M}} |\nabla f(T_k, x)| = \theta_{T_k}$$

$k = 1, 2, \ldots$ In the sequel, we write θ_{T_k} simply as θ_k.

Neglecting subsequences, we may only consider the following two possibilities:

(1) $\theta_k \operatorname{dist}(a_k, \partial M) \to +\infty$

(2) $\theta_k \operatorname{dist}(a_k, \partial M) \to \rho < +\infty$

in both cases, we may assume $a_k \to a \in \bar{M}$.

Take a local chart U of a. Let

$$D_k = \left\{ y \in \mathbf{R}^2 \mid a_k + \frac{y}{\theta_k} \in U \right\},$$

and

$$I_k = [-\theta_k^2 T_k, \theta_k^2 (\omega - T_k)).$$

Define a function on $I_k \times \mathbf{D}_k$ as follows:

$$v_k(\tau, y) = f\left(T_k + \frac{\tau}{\theta_k^2}, a_k + \frac{y}{\theta_k}\right).$$

$k=1, 2, \ldots$ Then we see

$$\partial_\tau v_k = \underline{\Delta}\, v_k, \tag{5.1}$$

and

$$\max_{(\tau, y) \in I_k \times D_k} |\nabla_y v_k(\tau, y)| \leqq 1, \qquad k=1, 2, \ldots \tag{5.2}$$

Let

$$h_k(\tau) = \int_{D_k} |\partial_\tau v_k(\tau, y)|^2 \, dy.$$

Then $h_k(\tau) \geqq 0$, and $\forall e > 0$

$$\int_{-e}^{0} h_k(\tau)\, d\tau \leqq \int_{T_k - e/\theta_k^2}^{T_k} dt \int_M |\partial_t f(t, x)|^2 \, dV g$$
$$= E\left(f\left(T_k - \frac{e}{\theta_k^2}, \cdot\right)\right) - E(f(T_k, \cdot)) \to 0$$

as $k \to \infty$.

Thus, neglecting a subsequence, we may assume

$$h_k(\tau) \to 0 \quad \text{a.e. } \tau \in [-e, 0]$$

i.e. for almost all $\tau \in [-e, 0]$,

$$\int_{D_k} |\partial_\tau v_k(\tau, y)|^2 \, dy \to 0. \tag{5.3}$$

In case (1), $a_k \in \mathring{M}$, and $\mathbf{D}_k \to \mathbf{R}^2$ in the sense that $\forall R > 0$, $\exists k_0 > 0$, the ball B_R centered at θ in $\mathbf{R}^2$ is included in $\mathbf{D}_k$ for $k \geqq k_0$.

On one hand by (5.3)

$$\partial_\tau v_k(\tau^*, y) \to 0, \qquad L^2(B_R, \mathbf{R}^k), \quad \forall R > 0. \tag{5.4}$$

for almost all $\tau^* \in [-e, 0]$.

On the other hand, by Lemma 4.2, we have

$$\sup_{\tau \in [-e, 0]} \| v_k(\tau, \cdot) \|_{C^{1+\alpha}(B_R)} \leqq C[1 + (e 4 \pi R^2)^{1/p}]. \tag{5.5}$$

This implies a subsequence, where we do not change the subscripts, so that

$$v_k(\tau^*, y) \to \tilde{v}(y), \; C^{1+\alpha'}(\mathbf{R}^2) \quad \text{locally}$$

for some $\tau^* \in [-e, 0]$ (actually in a countable dense subset of $[-e, 0]$). We conclude that

$$\underline{\Delta}\, \tilde{v} = 0 \quad \text{in } \mathbf{R}^2.$$

According to the removable singularity theorem due to Sacks-Uhlenbeck, $\tilde{v}$ is extendible to a harmonic map $\bar{v}: S^2 \to N$.

We are going to show that $\bar{v}$ is nonconstant. Indeed,

$$|\nabla_y v_k(0, \theta)| = \frac{1}{\theta_k} |\nabla_x f(T_k, a_k)| = 1$$

since v_k satisfies (5.1) on $I_k \times D_k$ with the condition (5.2). The Schauder estimate applies to obtain an estimate:

$$\| v_k(\tau, y) \|_{C^{1+(\gamma/2), 2+\gamma}([-e, 0] \times (B_\delta(0) \cap D_k))} \leqq C \{1 + [e\pi(2\delta)^2]^{1/p}\} \quad (5.6)$$

for some $\delta > 0$ small depending on U. The right hand side of the inequality is a constant independent of k. According to the embedding theorem (2) mentioned in section II. (Actually this is due to Bernstein-Montel and Nikol'ski.)

$$\| \nabla_y v_k(\tau, y) \|_{C^{(1+\gamma)/2, 1+\gamma}([-e, 0] \times (B_\delta(\theta) \cap D_k))} \leqq C_1$$

where C_1 is a constant independent of k. Hence

$$|\nabla_y v_k(\tau^*, \theta) - \nabla_y v_k(0, \theta)| < C_1 \tau^*.$$

We may choose $\tau^* > 0$ small enough so that

$$|\nabla v_k(\tau^*, \theta)| > \frac{1}{2}. \quad (5.7)$$

It proves that $\bar{v}$ is nonconstant.

Let $T'_k = T_k + \frac{\tau^*}{\theta_k^2}$, since $T'_k \to \omega$, Theorem 4.1 suggests that for a subsequence T'_k, $|\nabla f(T'_k, .)|$ blows up at most finitely many points $\{x_1, \ldots, x_l\}$, which includes the limit set of $\{a_k\}$.

We choose $\delta > 0$ small enough,

$$E(f(T'_k, .)) = \int_M |\nabla f(T'_k, x)|^2 dVg$$
$$= \int_{M \setminus \bigcup_{j=1}^{l} B_\delta(x_j)} + \sum_{j=1}^{l} \int_{B_\delta(x_j)} |\nabla f(T'_k, x)|^2 dVg.$$

Since

$$f(T'_k, .) \to \tilde{u}(.), \qquad C^{1+\alpha'}(M \setminus \bigcup_{j=1}^{l} B_\delta(x_j), \mathbf{R}^k)$$

and there exists at least one j_0 such that $a = x_{j_0}$. We have

$$\lim_{k \to \infty} \int_{M \setminus \bigcup_{j=1}^{l} B_\delta(x_j)} |\nabla f(T'_k, x)|^2 dVg = \int_{M \setminus \bigcup_{j=1}^{l} B_\delta(x_j)} |\nabla \tilde{u}(x)|^2 dVg,$$

and

$$\int_{B_\delta(x_{j_0})} |\nabla f(T'_k, x)|^2 dV g \geqq \int_{B_{\theta_k \delta/2}(\theta)} |\nabla v_k(\tau^*, y)|^2 dy$$

for k large. First let $k \to \infty$, by definition

$$\lim_{k \to \infty} \int_{B_\delta(x_{j_0})} |\nabla f(T'_k, x)|^2 dV g \geqq b,$$

and then because $\delta > 0$ is arbitrary,

$$E(\varphi) \geqq \lim_{k \to \infty} \int_M |\nabla f(T'_k, x)|^2 dV g \geqq \int_M |\nabla \tilde{u}(x)|^2 dV g + b \geqq m + b. \quad (5.8)$$

This is the desired conclusion.

In case (2), $a \in \partial M \cap U$. We choose a suitable coordinate (y_1, y_2) in $\mathbf{R}^2$, such that the y_2-axis is parallel to the tangent at a of ∂M, and the y_1-axis points to the interior of U. Thus $\mathbf{D}_k$ tends to the half plane $\mathbf{R}^2_+ = \{(y_1, y_2) \mid y_1 > -\rho\}$, and for each point on the boundary: $y_1 = -\rho$.

$$a_k + \frac{y}{\theta_k} \to a.$$

As in the proof of (5.5), now we have $\forall R > 0$,

$$\sup_{\tau \in [-e, 0]} \| v_k(\tau, \, .) \|_{C^{1+\alpha}(B_R \cap D_k)} \leqq C \Big[1 + (e 4 \pi R^2)^{1/p} + \Big\| \chi\Big(a_k + \frac{y}{\theta_k}\Big) \Big\|_{C^{2+\gamma}(\partial D_k \cap B_R)} \Big]$$

since on the right hand side, there is a constant control independent of k. We find a function $\tilde{v}^*$ on $\mathbf{R}^2_+$ and a subsequence $v_k(\tau^*, \, .)$ such that

$$v_k(\tau^*, y) \to \tilde{v}^*(y) \; C^{1+\alpha'}(\mathbf{R}^2_+)$$

and then

$$\underline{\Delta} \tilde{v}^* = 0 \quad \text{in } \mathbf{R}^2_+,$$
$$\tilde{v}^* |_{\partial \mathbf{R}^2_+} = \chi(a).$$

On one hand, similar to the proofs of (5.6) and (5.7), we see that $\tilde{v}^*$ is nonconstant, on the other hand, let us define a complex function

$$\eta(z) = h(\tilde{v}^*_z, \tilde{v}^*_{\bar{z}})$$

where h is the Riemannian metric on N, and

$$\tilde{v}_z^* = \frac{1}{2}(\partial_{y_1} - i\,\partial_{y_2})\,\tilde{v}^*,$$
$$z = y_1 + i y_2.$$

So

$$\eta(z) = h(\tilde{v}_{y_1}^*, \tilde{v}_{y_1}^*) - h(\tilde{v}_{y_2}^*, \tilde{v}_{y_2}^*) - 2\,ih(\tilde{v}_{y_1}^*, \tilde{v}_{y_2}^*).$$

The harmonics of $\tilde{v}^*$ implies the analyticity of the function η. The boundary condition on $\tilde{v}$ implies that the function η can be analytically extended to the whole complex plane. From the condition

$$\eta(-\rho + i y_2) = 0$$

we conclude that $\eta(z) \equiv 0$, and hence $\tilde{v}^*$ is a constant map. This is a contradiction. Therefore Lemma 5.1 is proved.

We continue the proof of our main theorem.

By the assumption $E(\varphi) < m + b$, and Lemma 5.1, we conclude

$$\sup_{(t,\,x) \in [0,\,\omega) \times M} |\nabla f(t, x)| \leqq c_0. \tag{5.9}$$

Thus the norm $\| f \|_{W_p^{1,2}(Q_\omega)}$, $p > 4$, and then the norm $\| f \|_{C^{1+(\gamma/2),\,2+\gamma}(\mathring{Q}_\omega)}$ are bounded if $\omega < \infty$. So the evolution equation is extendible beyond the interval of $\omega < \infty$. This contradicts the maximality of ω. Therefore ω must be infinite, *i.e.* the global solution exists: At the same time, (5.9) is the conclusion (2).

In the following, we assume $\pi_2(N) = 0$. We shall improve the conclusion of Lemma 5.1 to the following:

$$E(\varphi) \geqq m_{\mathbf{E}} + b$$

where $\mathbf{E}$ is the homotopy class of φ, and

$$m_{\mathbf{E}} = \inf\{E(u) \mid u \in \mathbf{E}\}.$$

Only the inequality (5.8) should be fixed. It is known that $f(T'_k, .) \to \tilde{u}(.)$ in $C^{1+\alpha'}(M \setminus \{x_1, \ldots, x_l\}, \mathbf{R}^k)$. We only want to show $\tilde{u} \in \mathbf{E}$. One may choose suitable subsequences $\delta_k \downarrow 0$, $T''_k \to \omega$ such that $B_{\delta_k}(x_i) \cap B_{\delta_k}(x_j) = \varnothing$, if $i \neq j$, and the joining maps

$$\hat{f}_k(x) = \begin{cases} f(T''_k, x), & \forall\, x \notin \bigcup_{i=1}^{l} B_{\delta_k}(x_i), \\ \exp_{\tilde{u}(x_i)}\left(\eta\left(\dfrac{|x - x_i|}{\delta_k}\right)\exp_{\tilde{u}(x_i)}^{-1} f(T''_k, x)\right), & \forall\, x \in B_{\delta_k}(x_i), \end{cases}$$

converge to $\tilde{u}$ in $C(M, N)$, where $\eta \in C^\infty(\mathbf{R}^1)$ satisfies

$$\eta(r) = \begin{cases} 1 & r \geqq 1 \\ 0 & r \leqq \frac{1}{2}, \end{cases}$$

and exp is the exponential map. Since $\pi_2(N) = 0$, and

$$\hat{f}_k \big|_{\partial B_\delta(x_i)} = f(T_k'', .) \big|_{\partial B_\delta(x_i)}$$

we see that $\hat{f}_k$ remains in the same homotopy class **E**, And from $\hat{f}_k \to \tilde{u} C(M, N)$, we conclude $\tilde{u} \in \mathbf{E}$. ■

COROLLARY 1. — Now follows directly from the improved inequality.

As to the *proof of Corollary* 2, first, we have a minimum $\tilde{u}$ of $E(u)$, *i. e.*

$$E(\tilde{u}) = m.$$

Obviously, this is a harmonic map. Second, by the "principle of adding spheres" due to Wente, *cf.* Jost [J2], one can find a map v homotopically different from $\tilde{u}$, such that

$$E(v) < E(\tilde{u}) + b = m + b.$$

Another harmonic map is obtained by the main theorem in the homotopy class $[v]$.

Remark. — It is well known that there is no harmonic map from $P^2 \to S^2$ (and also $T^2 \to S^2$) of degree ± 1, *cf.* Eells-Wood [EW1] and Eells-Lemaire [EL1]. Also neither nonconstant harmonic map from the unit disc D to S^2 with constant boundary conditions, *cf.* Lemaire [L1].

The heat flow initiated from any map of these homotopy classes blows up at some time (either finite or infinite).

Finally, we prove the continuous dependence of the initial value, *i. e.* conclusion (4).

In the following, let us denote f by f_φ to indicate its initial value $f(0, .) = \varphi$.

First, we prove that the flow $\varphi \mapsto f_\varphi$ is locally uniformly bounded, *i. e.*

$$\forall \varphi_0 \in E_c = \{u \in C_\chi^{2+\gamma}(\bar{M}, N) \mid E(u) \leqq c\},$$

where $c < m + b$ [in case $\pi_2(N) = 0$, $c < m_E + b$], there exist $\delta > 0$ and $c_1 > 0$ such that

$$\sup_{(t, x) \in [0, \infty) \times \bar{M}} |\nabla f_\varphi(t, x)| \leqq C_1, \qquad \forall \varphi \in B_\delta(\varphi_0), \tag{5.9'}$$

where B_δ is the δ-ball on the Banach manifold $C_\chi^{2+\gamma}(\bar{M}, N)$.

The proof is quite similar to the proof for Lemma 5.1. If the conclusion is not true, we have.

$$\varphi_k \to \varphi_0 \quad \text{in } C_\chi^{2+\gamma}(\bar{M}, N),$$

and $\exists \{T_k\}_1^\infty$ and $\{a_k \in \bar{M} \mid k=1, 2, \ldots\}$ such that

$$\theta_k = \sup_{(t,\, x) \in [0,\, T_k] \times \bar{M}} |\nabla f_{\varphi_k}(t, x)| = |\nabla f_{\varphi_k}(T_k, a_k)| \to \infty.$$

Define a sequence of flows v_k as above. Follow the above proof step by step. Similarly, one shows that $\exists T'_k \nearrow +\infty$ such that

$$\lim_{k \to \infty} E(f_{\varphi_k}(T'_k, .)) \geqq m+b \qquad [\text{or } m_E+b \text{ if } \pi_2(N)=0].$$

However,

$$E(\varphi_0) = \lim_{k \to \infty} E(\varphi_k) \geqq \lim_{k \to \infty} E(f_{\varphi_k}(T'_k, .)).$$

This contradicts the assumption $\varphi_0 \in E_c$.

Next, we prove the C-norm continuous dependence, *i. e.* as $\varphi_k \to \varphi_0\, C_\chi^{2+\gamma}(\bar{M}, N)$, $\sup_{t \in [0,\, T]} \| f_{\varphi_k}(t, .) - f_{\varphi_0}(t, .)\|_{L^\infty(M,\, N)} \to 0$, $\forall T > 0$.

Let σ be a smooth function defined on $N \times N$,

$$\sigma(y_1, y_2) = \frac{1}{2} \operatorname{dist}(y_1, y_2)^2,$$

and let $\rho(t, x) = \sigma(f_\varphi(t, x), f_{\varphi_0}(t, x))$, $\forall \varphi \in B_\delta(\varphi_0)$. Since $|\nabla f_\varphi(t, x)|$ is bounded, according to the computations in Hamilton [H1], p. 105-107. We have a constant $C_2 > 0$ such that

$$\partial_t \rho \leqq \Delta \rho + C_2\, \rho.$$

Thus, by the maximum principle, for each $T > 0$,

$$\operatorname{dist}(f_\varphi(t, x), f_{\varphi_0}(t, x)) \leqq e^{C_2 T} \operatorname{dist}(\varphi, \varphi_0), \qquad \forall t \in [0, T].$$

Third, we prove the following estimate

$$\| f_\varphi(t, x) - f_{\varphi_0}(t, x)\|_{C^{1+(\gamma/2),\, 2+\gamma}([0,\, T] \times \bar{M},\, N)} \leqq C_3 \| \varphi - \varphi_0 \|_{C^{2+\gamma}(\bar{M},\, N)},$$

where C_3 is a constant, $\forall \varphi \in B_\delta(\varphi_0)$. Let $f = f_\varphi - f_{\varphi_0}$, we write

$$\begin{aligned} \Delta(\Gamma(f)(\nabla f, \nabla f)) &= \Gamma(f_\varphi)(\nabla f_\varphi, \nabla f_\varphi) - \Gamma(f_{\varphi_0})(\nabla f_{\varphi_0}, \nabla f_{\varphi_0}) \\ &= (\Gamma(f_\varphi) - \Gamma(f_{\varphi_0}))(\nabla f_\varphi, \nabla f_\varphi) \\ &\quad + \Gamma(f_{\varphi_0})(\nabla f, \nabla f_\varphi) + \Gamma(f_{\varphi_0})(\nabla f_{\varphi_0}, \nabla f). \end{aligned}$$

Thus

$$\left.\begin{aligned} &\partial_t f = \Delta_M f + \Delta(\Gamma(f)(\nabla f, \nabla f)), \\ &f(0, .) = \varphi - \varphi_0, \\ &f|_{[0,\, \infty) \times \partial M} = 0 \end{aligned}\right\} \tag{5.10}$$

$\forall T > 0$, we apply the L^p-estimates to the equation (5.10)

$$\begin{aligned} \| f \|_{W_p^{1,\, 2}(Q_T)} \leqq C[&\| \varphi - \varphi_0 \|_{C^{2+\gamma}} \\ &+ C_1^2 C_3 \| f_\varphi - f_{\varphi_0} \|_{L^\infty(Q_T)} (T \operatorname{mes}(M))^{1/p} + 2 C_1 C_3 \| \nabla f \|_{L^p(Q_T)}] \end{aligned}$$

for $p > 4$, which implies (according to Proposition 2.2 and an interpolation inequality)

$$\| f \|_{W_p^{1,2}(Q_T)} \leqq C_4 \| \varphi - \varphi_0 \|_{C^{2+\gamma}(\bar{M}, N)}.$$

Again, by a bootstrap iteration, we obtain the desired conclusion.

VI. MINIMAX PRINCIPLE

In order to use the heat flow as deformations, the conclusion (4) of Theorem 1.1 provides the continuity of f_φ with respect to the initial data φ. Unfortunately, the flow $t \mapsto f_\varphi(t, .)$ is not known to be continuous at $t=0$ on the Banach manifold $C_\chi^{2+\gamma}(\bar{M}, N)$ under the strong topology! We overcome this difficulty by introducing a weaker topology W_p^2 with $1 - \frac{4}{p} > \gamma$. A new problem is that the manifold is incomplete under the weaker topology.

For technical simplicity, we assume $\chi \in C^\infty(\partial M, N)$, the flow $f_\varphi(t, .)$ is smooth for $t > 0$, if $E(\varphi) < b+m$, according to Lemma 4.2. Lemma 5.1 and the regularity theory. [When $\pi_2(N)=0$, $E(\varphi) < b+m_E$.]

We study the energy function

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dV g$$

on the manifold $C_\chi^{2+\gamma}(\bar{M}, N)$ with W_p^2 topology. For any $a \geqq 0$, denote

$$E_a = \{ u \in C_\chi^{2+\gamma}(\bar{M}, N) \mid E(u) \leqq a \}$$

as the level set, and denote

$$K = \{ u \in C_\chi^{2+\gamma}(\bar{M}, N) \mid \underline{\Delta} u = 0 \}$$

as the critical set. We write $K_a = K \cap E^{-1}(a)$.

In case $K \cap E^{-1}[a, d] = \varnothing$, where $0 \leqq a < d < m+b$ [or $m_E + b$ if $\pi_2(N)=0$], let us define a deformation retract as follows: $\forall \varphi \in E_d \setminus E_a$, we know from the conclusion (1) that $f_\varphi(t, .)$ exists globally, and from the conclusion (3),

$$\lim_{t \to +\infty} E(f_\varphi(t, .)) < a,$$

therefore $\exists T = T_\varphi > 0$ such that

$$E(f_\varphi(T_\varphi, .)) = a.$$

We point out that the function $\varphi \mapsto T_\varphi$ is continuous. Indeed, the function $E(f_\varphi(t, .))$ is continuously differentiable with respect to t,

$$\partial_t E(f_\varphi(t, .))|_{t=T(\varphi)} = -\int_M \langle \partial_t f(t, .), \Delta f(t, .)\rangle dVg|_{t=T(\varphi)}$$
$$= -\int_M |\partial_t f(t, .)|^2 dVg|_{t=T(\varphi)} < 0.$$

The implicit function theorem is applied to assure the continuous dependence. We define

$$\eta(\tau, \varphi) = \begin{cases} \varphi & \text{if } \varphi \in E_a \\ f_\varphi(T_\varphi \tau, .) & \text{if } \varphi \in E_d \setminus E_a \end{cases} \tag{6.1}$$

as a deformation: $[0, 1] \times E_d \to E_d$. It is a strong deformation retract, satisfying:

$$\eta(0, .) = \mathrm{id}_{E_d}, \qquad \eta(t, .)|_{E_a} = \mathrm{id}_{E_a}$$
$$\eta(1, E_d) \subset E_a, \qquad E(\eta(t, \varphi)) \leqq E(\varphi), \qquad \forall (\tau, \varphi) \in [0, 1] \times E_d.$$

We emphasize that η is a deformation under the W_p^2 topology, but not the $C_\chi^{2+\gamma}$-strong topology. (The W_p^2-continuous dependence of φ is verified in the same way.)

We need an approximation lemma.

LEMMA 6.1. — *Suppose that* Q *is a compact manifold. Assume that* $l: Q \to W_p^2(M, N)_\chi$ *is a continuous function, with* $l(\partial Q) \subset C_\chi^{2+\gamma}(\bar{M}, N)$ *then for any* $\varepsilon > 0$ *there exists* $\tilde{l}: Q \to C_\chi^{2+\gamma}(\bar{M}, N)$, *which is continuous under* W_p^2-topology, *and satisfies*

$$\tilde{l}|_{\partial Q} = l|_{\partial Q}$$

and

$$\mathrm{dist}_{W_p^2}(\tilde{l}(q), l(q)) < \varepsilon, \qquad \forall q \in Q.$$

Proof. — Since Q is paracompact, and since l is continuous, there exists an open covering $\{U_\alpha \mid \alpha \in \Lambda\}$, and an associated partition of unity

$$\{\rho_\alpha \in C(Q, W_{p,\chi}^2) \mid \mathrm{supp}\, \rho_\alpha \subset\subset U_\alpha, \alpha \in \Lambda\}$$

such that

$$\mathrm{OSC}_{U_\alpha}(l) := \sup\{\mathrm{dist}_{W_p^2}(l(q), l(q')) \mid q, q' \in U_\alpha\} < \varepsilon.$$

For any $\alpha \in \Lambda$, we choose $a_\alpha \in C_\chi^{2+\gamma}(\bar{M}, N)$ such that

$$\mathrm{dist}_{W_p^2}(l(q), a_\alpha) < \varepsilon, \qquad \forall q \in U_\alpha.$$

For those $\alpha \in \Lambda$ with $U_\alpha \cap \partial Q \neq \varnothing$, we choose suitable coordinates $q = (y_1, \ldots, y_s)$ in U_α such that ∂Q is the hyperplane $y_s = 0$, and U_α is in

$y_s \geqq 0$. Define

$$b_\alpha(q) = l(y_1, \ldots, y_{s-1}, 0)(1-y_s) + a_\alpha y_s.$$

We see again

$$\text{dist}_{W_p^2}(l(q), b_\alpha(q)) < \varepsilon, \qquad \forall q \in U_\alpha.$$

We may assume $\varepsilon > 0$ is so small that the tabular neighborhood T of N, which possesses a smooth projection π onto N, includes the ε-neighborhood of N. The projection π extends to a smooth projection $\tilde{\pi}$ from $W_p^2(M, T)_\chi$ onto $W_p^2(M, N)_\chi$ and from $C_\chi^{2(1+\gamma)}(\bar{M}, N)$ to $C_\chi^{2(1+\gamma)}(\bar{M}, N)$. Now let us define

$$\tilde{l}(q) = \tilde{\pi}\Big(\sum_{U_\alpha \cap \partial Q \neq \varnothing} b_\alpha(q)\rho_\alpha(q) + \sum_{U_\alpha \cap \partial Q = \varnothing} a_\alpha \rho_\alpha(q)\Big),$$

it satisfies the requirement, $\tilde{l}: Q \to C_\chi^{2(1+\gamma)}(\bar{M}, N)$ and

$$\text{dist}_{W_p^2}(\tilde{l}(q), l(q)) < \varepsilon.$$

Noticing that the critical set K is closed, so is its image E(K). The following Minimax Principle holds.

THEOREM 6.2. — *Suppose that $\pi_2(N) = 0$, and let* **E** *be a component of* $C_\chi^{2(1+\gamma)}(\bar{M}, N)$.
Let Q *be a compact manifold, and*

$$\Gamma = \{l \in C(Q, \hat{E}) \mid l(\partial Q) \subset C_\chi^{2(1+\gamma)}(\bar{M}, N) \cap E_e \text{ and } l|_{\partial Q} \text{ is fixed}\}$$

where e is a constant less than $m_{\mathbf{E}} + b$, and $\hat{\mathbf{E}}$ *is the closure of* **E** *under the* W_p^2 *topology, endowed with the* W_p^2 topology. Define

$$c = \inf_{l \in \Gamma} \sup_{q \in Q} E.l(q).$$

If $e < c < b + m_{\mathbf{E}}$, then c is a critical value of **E**.

Proof. — Since E(K) is closed, if c is not a critical value, $\exists\, \varepsilon_0 > 0$ such that $K \cap E^{-1}[c-\varepsilon_0, c+\varepsilon_0] = \varnothing$, and $e < c-\varepsilon_0 < c+\varepsilon_0 < b+m_{\mathbf{E}}$. By definition, $\exists\, l \in \Gamma$ such that

$$E(l(q)) < c + \frac{\varepsilon_0}{2}, \qquad \forall q \in Q.$$

According to Lemma 6.1, $\exists\, \tilde{l}_k: Q \to C_\chi^{2(1+\gamma)}(\bar{M}, N)$ with $\sup_{q \in Q} \text{dist}_{W_p^2}(\tilde{l}_k(q), l(q)) \to 0$ as $k \to \infty$. Choose a suitable k_0 such that

$$E(\tilde{l}_{k_0}(q)) < c + \varepsilon_0.$$

We apply the deformation η on the set $E_{c+\varepsilon_0}$ with $a = c - \varepsilon_0$, then $q \mapsto \eta(1, \tilde{l}_{k_0}(q))$ is well defined, and is continuous under the W_p^2-topology. But $E(\eta(1, \tilde{l}_{k_0}(q))) \leqq c - \varepsilon_0$. This contradicts the definition of c.

Remark 6.1. — In case $\partial Q=\emptyset$, $\Gamma=C(Q, \hat{E})$. In particular, $Q=S^k$ for some k, Γ contains the k-homotopy class.

Remark 6.2. — In case $Q=[0, 1]$, this is the mountain pass theorem. And in the case $Q=B^k$ for some k, this is the high link version of the mountain pass theorem.

VII. LJUSTERNIK SCHNIRELMANN THEORY

We continue our study of the manifold $C_\chi^{2+\gamma}(\bar{M}, N)$ endowed with a weaker W_p^2-topology, where $1-\frac{4}{p}>\gamma$. In order to extend the Ljusternik Schnirelmann multiplicity theorem, we shall prove a stronger deformation lemma, and study some properties of the category in different topologies.

In the following, we always assume $\pi_2(N)=0$.

LEMMA 7.1. — *Suppose that* $\hat{\gamma}>\gamma$ *and that* $\chi\in C^{2+\hat{\gamma}}(\partial M, N)$. *If* E *is a component of* $C_\chi^{2+\gamma}(\bar{M}, N)$, *and if* $c<m_E+b$, *then the critical set* K_c *is compact in* $C_\chi^{2+\gamma}(\bar{M}, N)$ *under the strong topology.*

Proof. — First we prove

$$\|\nabla u(x)\|_{C(\bar{M},\, N)}\leqq a \text{ const.},\qquad \forall u\in K_c. \tag{7.1}$$

If not, $\exists\, u_k\in K_c$, and $a_k\in M$ such that

$$\underline{\Delta} u_k=0$$
$$\theta_k=\|\nabla u_k(x)\|_{C(\bar{M},\, N)}=|\nabla u_k(a_k)|\to\infty.$$

Let

$$v_k(y)=u_k\left(a_k+\frac{y}{\theta_k}\right),$$

which is defined similarly to that in section V. We prove similarly by the blowup technique, that

$$\varliminf_{k\to\infty} E(u_k)\geqq b+m_E.$$

This is a contradiction.

From (7.1), we obtain from the Schauder estimate

$$\|u\|_{C^{2+\hat{\gamma}}(\bar{M},\, N)}\leqq C, \text{ a constant.}$$

The compactness follows directly.

Now we study the category of K_c under different topologies. In the following, we denote $X=W_p^2(M, N)_\chi$ and $Y=C_\chi^{2+\gamma}(\bar{M}, N)$ with the weaker W_p^2-topology. The set K_c is compact in both spaces. We denote

$\text{cat}_X(K_c)$ and $\text{cat}_Y(K_c)$ the categories in different spaces. We shall prove the following.

LEMMA 7.2. — $\text{cat}_X(K_c)=\text{cat}_Y(K_c)$. *And there is a closed neighborhood* U *of* K_c *in* Y *such that*

$$\text{cat}_X(K_c)=\text{cat}_Y(U).$$

Proof. — Since $\text{cat}_X(K_c)=r$ is finite, by definition, $\exists$ closed contractible subsetse $F_1, \ldots, F_r \subset X$ such that

$$K_c \subset \bigcup_{i=1}^{r} F_i.$$

Also since $K_c \subset Y$, the sets $F_i \cap Y$, $i=1, \ldots, r$ are nonempty closed subsets in Y (under the weaker topology!) so we have

$$\text{cat}_Y(K_c) \leqq \text{cat}_X(K_c).$$

On the other hand, let $s=\text{cat}_Y(K_c)$, $\exists$ closed contractible subsets $G_1, \ldots, G_s \subset Y$. Let $\bar{G}_i \subset X$ be the closure of G_i in X, then $\bar{G}_i$ is contractible. Hence

$$\text{cat}_X(K_c)=\text{cat}_Y(K_c).$$

We prove the second part of the lemma, since F_i is contractible and X is complete, by the continuity extension theorem. $\exists\, \varphi_i$: $[0, 1]\times X \to X$ such that

$$\varphi_i(t, .)=\text{id},$$
$$\varphi_i(1, F_i)=p_i \in X.$$

Now for any $\varepsilon > 0$, we find an approximation map similar to what was done in Lemma 6.1.

$$\tilde{\varphi}_i: \quad [0, 1]\times Y \to Y$$

satisfying

$$\tilde{\varphi}_i(0, .)=\text{id}_Y$$
$$\tilde{\varphi}_i(1, F_i \cap Y)=q_i \in Y$$

and

$$\text{dist}_Y(\varphi_i(t, y), \tilde{\varphi}_i(t, y)) < \varepsilon, \qquad \forall (t, y) \in [0, 1]\times Y.$$

For $q_i \in Y$, there is a closed contractible neighborhood $V_i \subset Y$, let

$$U_i=\tilde{\varphi}_i^{-1}(1, V_i),$$

then $U_i \subset Y$ is a closed contractible neighborhood of $F_i \cap Y$. Define $U=\bigcup_{i=1}^{r} U_i$, which is the desired closed neighborhood of K_c with

$$\text{cat}_Y(U) \leqq r=\text{cat}_X(K_c).$$

However from $K_c \subset U$, we have

$$\mathrm{cat}_Y(K_c) \leqq \mathrm{cat}_Y(U).$$

Applying the first conclusion, we obtain

$$\mathrm{cat}_X(K_c) = \mathrm{cat}_Y(U).$$

Now we are going to prove a stronger deformation theorem.

First, according to Lemma 7.2, $\exists$ closed neighborhood U of K_c, and $\exists\, \delta > 0$ such that

$$U_\delta = \{u \in C_\chi^{2+\gamma}(\bar{M}, N) \mid \mathrm{dist}_{W_p^2}(u, K_c) \leqq \delta\} \subset U.$$

Define a C^1 function on $W_p^2(M, N)_\chi$ satisfying $0 \leqq \gamma \leqq 1$, and

$$\gamma(u) = \begin{cases} 1 & u \notin U_{\delta/4} \\ 0 & u \in U_{\delta/8}, \end{cases}$$

with $\mathrm{supp}\,\gamma$ = the complementary of $U_{\delta/8}$. We shall study the solution of the following equation:

$$\left.\begin{aligned} \partial_t f(t, .) &= \gamma(f(t, .))\,\underline{\Delta} f(t, .), \\ f(0, .) &= \varphi, \qquad f(t, .) \in \mathbf{E}. \end{aligned}\right\} \tag{7.2}$$

This is not a differential equation, because the coefficient $\gamma(f(t, .))$ depends on $f(t, .)$ globally. The difference between this equation and the evolution equation (E) only occurs in the neighborhood $U_{\delta/4}$. So first we shall forcus our attention on $U_{\delta/4}$.

LEMMA 7.3. — *Suppose that* X, Y *are Banach spaces and that* $A \in \mathscr{L}(X, Y)$ *is invertible. Let* $b \in X^*$, $e \in Y$. *Then the operator* $A_1 = A - \langle b, . \rangle e$ *is invertible if and only if*

$$\langle b, A^{-1} e \rangle \neq 1.$$

Proof. — $\ker(A_1) = \{x \in X \mid A x = \langle b, x \rangle e\}$.

If $\theta \neq x_0 \in \ker(A_1)$, then $\langle b, x_0 \rangle \neq 0$ because A is invertible. Since we have

$$\langle b, x_0 \rangle (\langle b, A^{-1} e \rangle - 1) = 0$$

it follows $\langle b, A^{-1} e \rangle = 1$.

On the contrary, if $\langle b, A^{-1} e \rangle = 1$, then for any $\lambda \in \mathbf{R}^1$, $x = \lambda A^{-1} e \in \ker(A_1)$.

LEMMA 7.4. — $\forall\, \varphi \in U_{\delta/4} \setminus \bar{U}_{\delta/8}$, *the equation* (7.2) *is locally solvable.*

Proof. — We linearize the equation at φ, and get the following equation.

$$\partial_t v = \gamma(\varphi)(\Delta_M v + 2\Gamma(\varphi)(\nabla\varphi, \nabla v) + \partial\Gamma(\varphi).v(\nabla\varphi, \nabla\varphi)) + \langle \gamma'(\varphi), v \rangle \underline{\Delta}\varphi. \tag{7.3}$$

One wants to show that there exist $\varepsilon > 0$ such that if $v \in W_2^{1,2}(Q_\varepsilon)$ satisfying

$$v(0, .) = 0,$$
$$v(t, .)|_{\partial M} = 0$$

then $v \equiv 0$. In fact, let

$$A = \partial_t - \gamma(\varphi)(\Delta_M + 2\Gamma(\varphi)(\nabla\varphi, \nabla .) + \partial\Gamma(\varphi).(\nabla\varphi, \nabla\varphi)).$$

Since

$$\| A^{-1} \underline{\Delta}\varphi \|_{W_p^{1,2}(Q_T)} \leqq C \| \underline{\Delta}\varphi \|_{L^p(Q_T)} = C\left(\int_0^T \| \underline{\Delta}\varphi \|_{L^p(M)}^p \, dt\right)^{1/p},$$

therefore if $T > 0$ small enough, such that $|\langle \gamma'(\varphi), A^{-1}\underline{\Delta}\varphi \rangle| < 1$. According to Lemma 7.3, the linearized equation (7.3) possesses only the null solution. The implicit function theorem is applied to show that (7.2) is locally solvable. Furthermore, $f(t, x) \in C_\chi^{1+(\gamma/2), 2+\gamma}$, according to regularity theory.

LEMMA 7.5. — *Suppose that $\hat{\gamma} > \gamma$ and that $\chi \in C^{2+\hat{\gamma}}(\partial M, N)$. $\forall \varphi \in E_c$, with $c < m_E + b$, the equation* (7.2) *is globally solvable.*

Proof. — We only want to prove the global existence of the flow emanating from $\varphi \in U_{\delta/4} \setminus \bar{U}_{\delta/8}$. According to Lemma 7.4, one may assume that the flow $f_\varphi(t, .)$ has a maximal existence interval $[0, \omega)$. Since $f_\varphi(t, .) \in U_{\delta/2}$, $\nabla f_\varphi(t, .) \in B_p^{1/2, 1} \hookrightarrow L^q$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{4}$. After a bootstrap iteration, again we conclude

$$\| f_\varphi \|_{C_\chi^{1+(\gamma/2), 2+\gamma}(\mathring{Q}_\omega)} \leqq C_\delta, \qquad a \text{ const.}$$

Moreover, by an argument of interior estimate,

$$\| f \|_{C_\chi^{1+(\hat{\gamma}/2), 2+\hat{\gamma}}(Q_\omega \quad Q_{\omega/2})} \leqq C_{\delta, \omega}, \qquad a \text{ const.}$$

Therefore for any sequence $\{t_j \nearrow \omega\}$, $f_\varphi(t_j, .)$ is subconvergent in $C^{2+\gamma}$. However, $\forall t_1, t_2 \in (0, \omega)$,

$$\begin{aligned} \| f_\varphi(t_1, .) - f_\varphi(t_2, .) \|_{L^2} &\leqq \int_{t_1}^{t_2} \| \partial_t f \|_{L^2} \, dt \\ &\leqq \int_{t_1}^{t_2} \| \underline{\Delta} f(t, .) \|_{L^2(M, R^k)} \, dt \\ &\leqq c \int_{t_1}^{t_2} \| f(t, .) \|_{W_2^2(M, N)} \, dt \\ &\leqq C |t_1 - t_2|. \end{aligned}$$

If $\omega < +\infty$, we see that $f_\varphi(t, .) \to$ some u in $L^2(M, N)$. This shows that $f_\varphi(t, .) \to u(.)\, C_\chi^{2+\gamma}(\bar{M}, N)$ as $t \to \omega$. Thus either the flow $t \mapsto f_\varphi(t, .)$ is extendible beyond ω, *i. e.* $\omega = +\infty$, or $f_\varphi(\omega, .) \in \partial(U_{\delta/4} \setminus \bar{U}_{\delta/8})$. In the latter case, since the equation becomes PDE (E) or $\partial_t f = 0$, the global existence is evidently true.

LEMMA 7.6. — *For $\delta > 0$ defined above, $\exists \varepsilon_0 = \varepsilon_0(\delta)$ such that $\| \underline{\Delta} u \|_{L^2(M, N)} \geqq \varepsilon_0$ for all u of the form $f_\varphi(t, .)$, $t \geqq 0$ satisfying* $\mathrm{dist}_{W_p^2}(u, K) \geqq \delta/4$, *and* $E(\varphi) < m_E + b$.

Proof. — If not, $\exists \{u_k\} \subset \mathring{E}_{m_E + b}$, $\exists \delta_0 > 0$ satisfying $\| \underline{\Delta} u_k \|_{L^2} \to 0$, and $\mathrm{dist}_{W_p^2}(u_k, K) \geqq \delta_0$. One may prove that $\| \nabla u_k \|_{L^\infty}$ is bounded, for otherwise, a blowup technique can be applied as in Lemma 7.1, which contradicts the energy condition. Therefore $\| u_k \|_{C_\chi^{2+\gamma}(\bar{M}, N)}$ is bounded provided by the L^p-estimate and the Schauder estimate. Hence $\{u_k\}$ subconverges to a map $\tilde{u}$ in the W_p^2 topology, which implies $\underline{\Delta} \tilde{u} = 0$. This contradicts the assumption.

Now we are ready to prove the following

THEOREM 7.1. — *For any closed neighborhood $U \subset C_\chi^{2+\gamma}$ of K_c in the W_p^2-topology, $\exists \varepsilon > 0$ and a W_p^2-continuous deformation $\eta: [0, 1] \times E_{c+\varepsilon} \to E_{c+\varepsilon}$ satisfying*

$$\eta(0, .) = \mathrm{id}_{E_{c+\varepsilon}}$$
$$\eta(1, E_{c+\varepsilon} \setminus U) \subset E_{c-\varepsilon}.$$

Proof. — We define the global flow as the solution of the equation (7.2). According to Lemma 7.5, it exists for each φ with $E(\varphi) < m_\delta + b$.

We want to find $\varepsilon > 0$ small enough, and a finite $T > 0$ such that $f(T, E_{c+\varepsilon} \setminus U) \subset E_{c-\varepsilon}$.

In fact, for any $\varphi \in E_{c+\varepsilon} \setminus U_\delta$, we want to show that, if $\varepsilon > 0$ is small, the flow $f_\varphi(t, .)$ will never enter into $U_{\delta/4}$.

If not, $\exists \varphi \in E_{c+\varepsilon} \setminus U_\delta$, $\exists t_2 > t_1 \geqq 0$ such that

$$\mathrm{dist}_{W_p^2}(f_\varphi(t_1, .), K_c) = \delta,$$
$$\mathrm{dist}_{W_p^2}(f_\varphi(t_2, .), K_c) = \frac{\delta}{2}$$

and

$$\frac{\delta}{2} < \mathrm{dist}_{W_p^2}(f_\varphi(t, .), K_c) < \delta, \qquad \forall t \in (t_1, t_2).$$

Since it is known that

$$\| f_\varphi \|_{C^{1+(\gamma/2),\, 2+\gamma}} \leqq C_\delta, \qquad \forall t \in [t_1, t_2] \tag{7.4}$$

we have

$$\| f_\varphi(t_1, .) - f_\varphi(t_2, .) \|_{W_p^2} \leqq C_\delta \, | t_1 - t_2 |^{\gamma/2}$$

provided by the embedding theorem. Therefore

$$|t_1-t_2| \geqq \left(\frac{\delta}{2\,C_\delta}\right)^{2/\gamma}.$$

On the other hand, (7.4) implies

$$\|\Delta f_\varphi(t, .)\|_{L^2} \geqq \varepsilon_0, \qquad \forall t\in[t_1, t_2]$$

according to Lemma 7.6. The inequalities

$$\begin{aligned} &E(f(t_2, .))-E(f(t_1, .)) \\ &= \int_{t_1}^{t_2} \|\partial_t f_\varphi(t, .)\|_{L^2}^2\, dt \\ &= \int_{t_1}^{t_2} \|\Delta f_\varphi(t, .)\|_{L^2}^2\, dt \\ &\geqq \varepsilon_0^2 (t_2-t_1) \geqq \varepsilon_0^2 \left(\frac{\delta}{2\,C_\delta}\right)^{2/\gamma} \end{aligned}$$

provide a upper bound for ε which prevents the flow entering $U_{\delta/4}$ before it arrives at $E_{c-\varepsilon}$. So we take $\varepsilon < \min\left\{\frac{1}{2}\varepsilon_0^2\left(\frac{\delta}{2\,C_\delta}\right)^{2/\gamma}, m_E+b-c\right\}$. As to these $\varphi\in E_{c+\varepsilon}\setminus U_\delta$, by Lemma 7.6, we have $\varepsilon_1 > 0$ such that

$$\|\Delta f_\varphi(t, .)\|_{L^2} \geqq \varepsilon_1, \qquad \forall t > 0.$$

Again, we have an estimate of the arriving time

$$T_\varphi \leqq \frac{2\varepsilon}{\varepsilon_1^2}.$$

Let us define

$$\eta(t, \varphi)=f_\varphi\left(\frac{2\varepsilon}{\varepsilon_1^2} t, .\right).$$

It satisfies all desired properties in the theorem [the continuous dependence on φ is proved similarly, *cf.* Conclusion (4) of paragraphe 5].

Proof of Theorem 1.2. — Let **F** denote the family of closed subsets of $C_\chi^{2+\gamma}(\bar{M}, N)$ endowed with the W_p^2 topology, and let

$$\mathbf{F}_k=\{A\in \mathbf{F} \mid \operatorname{cat}(A) \geqq k\}$$

$k=1, 2, \ldots$ Define

$$c_k= \inf_{A\in \mathbf{F}_k} \sup_{u\in A} E(u), \qquad k=1, 2, \ldots$$

We assume that

$$c=c_{k+1}=\ldots=c_{k+r} < m_E+b.$$

By Lemma 7.2, K_c possesses a closed neighborhod U in $C^{2+\gamma}$, with

$$\operatorname{cat}(U)=\operatorname{cat}_{W_p^2}(K_c).$$

By definition, $\forall \varepsilon > 0$, $\exists F_\varepsilon \in \mathbf{F}_{k+r}$ such that

$$E(u) < c+\varepsilon, \qquad \forall u \in F_\varepsilon.$$

According to Theorem 7.1, let $\eta=\eta(1, .)$, we have

$$\eta(F_\varepsilon \setminus U) \subset E_{c-\varepsilon}.$$

Thus

$$\begin{aligned} k+r \leqq \operatorname{cat}(F_\varepsilon) &\leqq \operatorname{cat}(F_\varepsilon \setminus \mathring{U})+\operatorname{cat}(U) \\ &\leqq \operatorname{cat}(E_{c-\varepsilon})+\operatorname{cat}_{W_p^2}(K_c) \\ &\leqq k+\operatorname{cat}_{W_p^2}(K_c). \end{aligned}$$

i. e.

$$\operatorname{cat}_{W_p^2}(K_c) \geqq r.$$

A proof of Corollary 3: Now we shall prove that for each homotopy class **E** in $C_\chi^{2+\gamma}(\bar{M}, N)$ and for some $d < m_{\mathbf{E}}+b$, $\operatorname{cat}(E_d \cap \mathbf{E}) \geqq 2$.

We only wish to construct an essential map

$$\sigma \in C(S^{n-2}, \mathbf{E})$$

with the property

$$\sup_{s \in S^{n-2}} E(\sigma(s)) \leqq d.$$

The existence of such a map σ was constructed by Benci-Coron [BeC1], only slight modifications are needed. In particular, we choose a local chart U outside of which, σ is defined to be $\tilde{u}$, the minimizer, and inside U we choose a small disk $B_\delta(z_0)$ on which $\sigma(s)(z)$: $S^{n-2} \times B_\delta(z_0) \to S^n$ is a homeomorphism. The map σ is connected smoothly, and a careful construction makes

$$E(\sigma(s)) < m_{\mathbf{E}}+b, \qquad \forall s \in S^{n-2}.$$

The condition $\pi_2(S^n)=0$ again guarantees $\sigma(s) \in \mathbf{E}$, $\forall s \in S^{n-2}$.

It is not difficult to show that σ: $S^{n-2} \times M \to S^n$ is essential, because outside the disk $B_\delta(z_0)$, $\sigma(s)(z)=\tilde{u}(z)$ for $z \in M \setminus B_\delta(z_0)$, which is contractible, and inside the disk σ: $S^{n-2} \times B_\delta(z_0) \to S^n$ is a homeomorphism. Therefore σ is homotopic to a homemorphism of S^n, it must be essential. Therefore

$$\operatorname{cat}(E_d \cap \mathbf{E}) \geqq 2.$$

The Ljusternik-Schnirelman category theory is applied to obtain at least two distinct harmonic maps.

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(*Manuscrit received May 26th*, 1988)
(*accepted December 15th*, 1988.)