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A general mountain pass principle for locating and classifying critical points

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ABSTRACT. — A general "Mountain Pass" principle that extends the theorem of Ambrosetti-Rabinowitz and which gives more information about the location of critical points, is established. This theorem also covers the problem of the "limiting case", i. e. when "the separating mountain range has zero altitude". It is also shown how this principle yields localized versions of recent results of Hofer and Pucci-Serrin concerning the structure of the critical set.

Key words : Mountain pass, Ambrosetti-Rabinowitz theorem, saddle point.

RÉSUMÉ. — On démontre une extension du théorème de col d'Ambrosetti-Rabinowitz dans laquelle une information auxiliaire sur la position du point critique est établie. On en déduit, d'une part, des résultats nouveaux, notamment le cas « limite » et d'autre part des démonstrations simples de résultats récents de Hofer et Pucci-Serrin sur la structure de l'ensemble critique.

I. INTRODUCTION AND MAIN RESULTS

The mountain pass theorem of Ambrosetti-Rabinowitz [1] is a useful tool for establishing the existence of critical points for non-linear functionals on infinite dimensional spaces and consequently for finding solutions to some non-linear differential equations via variational methods. For a survey, *see* Nirenberg [6] and Rabinowitz [10]. In this theorem, one considers on a Banach space X, a real valued C^1-function φ that verifies a compactness condition of Palais-Smale type and which also satisfies the following condition:

There exists a sphere S_R centred at 0 with radius $R>0$ such that $b=\inf\{\varphi(x);\ x\in S_R\}>\max\{\varphi(0),\ \varphi(e)\}=a$ where e is a point outside the ball B_R (i. e. $\|e\|>R$).

The classical theorem of Ambrosetti-Rabinowitz gives then the existence of a critical point x_0 (*i. e.* $\varphi'(x_0)=0$) different from 0 and e and with critical value $c\geqq b>a$ (*i. e.* $\varphi(x_0)=c$). Moreover, c is given by the formula $c=\inf_{g\in\Gamma}\max_t \varphi(g(t))$ where Γ is the space of all continuous paths joining 0 to e.

In this case, the critical value occurs, because 0 and e are low points on either side of the "mountain range" S_R so that between 0 and e there must be a lowest critical point or "mountain pass". In [7] it is asked whether the conclusion of the theorem remains true if the "mountain range" separating 0 and e is assumed to be of "zero altitude" (*i. e.* if $c=b=a$), and whether in this case the "pass" itself can be chosen to be on the mountain range (*i. e.* $\|x_0\|=R$).

In this paper we formulate a more general principle which besides giving the existence of critical points, provides some information about their location. It will contain the theorem of Ambrosetti-Rabinowitz and will give a positive answer to the problem of the "limiting case" mentioned above. Moreover, this principle can also be used to give simple proofs for stronger versions of some of the recent results of Hofer [3], [4] and Pucci-Serrin ([7], [8], [9]) about the structure of the set of critical points in the mountain pass theorem. The proof is based on Ekeland's variational principle and is a refinement of his proof of the classical case.

DEFINITION (0). — A closed subset H of a Banach space X is said to separate two points u and v in X if u and v belong to disjoint connected components of $X\setminus H$.

We denote by Γ_u^v the set of all continuous paths joining u and v; that is:

$$\Gamma_u^v=\{g\in C([0,\ 1];\ X);\ g(0)=u \text{ and } g(1)=v\}$$

where $C([0, 1]; X)$ is the space of all X-valued continuous functions on $[0, 1]$. The distance of a point x in X to a set F will be denoted $\operatorname{dist}(x, F) = \inf\{\|x-y\|;\ y \in F\}$.

THEOREM (1). — *Let $\varphi: X \to \mathbf{R}$ be a continuous and Gâteaux-differentiable function on a Banach space X such that $\varphi': X \to X^*$ is continuous from the norm topology of X to the weak*-topology of X^*. Take two points u and v in X and consider the number*

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} \varphi(g(t))$$

where $\Gamma = \Gamma_u^v$ is the set of all continuous paths joining u and v. Suppose F is a closed subset of X such that $F \cap \{x \in X;\ \varphi(x) \geq c\}$ separates u and v, then:

There exists a sequence $(x_n)_n$ in X verifying the following:

(i) $\lim_n \operatorname{dist}(x_n, F) = 0$

(ii) $\lim_n \varphi(x_n) = c$

(iii) $\lim_n \|\varphi'(x_n)\| = 0.$

We now define a Palais-Smale type condition that will insure the existence of a cluster point for the sequence $(x_n)_n$ obtained in Theorem (1) above and hence the existence of a critical point for φ.

DEFINITION (2). — Assume $\varphi: X \to \mathbf{R}$ is Gâteaux-differentiable on a Banach space X. Let F be a subset of X and let c be a real number. We shall say that φ *verifies the Palais-Smale condition around F at the level c* (in short $(PS)_{F, c}$) if every sequence $(x_n)_n$ in X verifying $\lim_n \operatorname{dist}(x_n, F) = 0$, *lim* $\lim_n \varphi(x_n) = c$ and $\lim_n \|\varphi'(x_n)\| = 0$, has a convergent subsequence.

Note that the *classical Palais-Smale condition (PS) on* φ corresponds to the case where it verifies $(PS)_{F, c}$ for any F in X and any $c \in \mathbf{R}$, while the *weak Palais-Smale condition* (WPS) on φ means that the latter verifies $(PS)_{F, c}$ for any c in $\mathbf{R}$ and all bounded sets F. Other (PS) conditions depending on the level c have also been considered (*see* for instance [2]).

The following is now immediate:

THEOREM (1. bis). — *Let X, φ, $\{u, v\}$, c and F be as in Theorem* (1). *Assume φ verifies $(PS)_{F, c}$, then:*

There exists a critical point φ on F with critical value c.

In the theorem of Ambrosetti-Rabinowitz [1], one assumes that $c > \max\{\varphi(u), \varphi(v)\} = a$, but this means that $\{x \in X;\ \varphi(x) \geq c\}$ separates u and v. Hence Theorem (1. bis) applies with $F = X$.

The limiting case described above, corresponds to the situation where u and v are on different sides of a sphere S on which φ is larger or equal to c. Hence, Theorem (1. *bis*) is applicable with $F = S = S \cap \{ x \in X;\ \varphi(x) \geqq c \}$.

Our next application deals with "localizations" of the results of Hofer ([3], [4]) and Pucci-Serrin ([7], [8], [9]) concerning the structure of the critical set in the Mountain Pass theorem. We recall the following notions: Let X, φ and c be as in Theorem (1). Denote by

$$K_c = \{ x \in X;\ \varphi(x) = c, \varphi'(x) = 0 \}$$
$$M_c = \{ x \in K_c;\ x \text{ is a local minimum for } \varphi \}$$
$$P_c = \{ x \in K_c;\ x \text{ is a local maximum for } \varphi \}$$

$S_c = \{ x \in K_c;\ x$ is a saddle point for φ *i.e.* in each neighbourhood of x there exist two points y and z such that $\varphi(y) < \varphi(x) < \varphi(z) \}$

Following Hofer [3] we say that a point x in K_c is of *mountain-pass type* if for any neighbourhood N of x the set $\{ y \in N;\ \varphi(y) < c \}$ is non-empty and not path-connected.

THEOREM (1. ter). — *Let X, φ, $\{u, v\}$, c and F be as in Theorem* (1). *Assume φ verifies* (PS). *Then:*

(*a*) *Either* $F \cap \bar{M}_c \neq \varnothing$ *or* $F \cap K_c$ *contains a critical point of mountain-pass type.*

Moreover, if $F \cap P_c$ *contains no compact set that separates u and v* (*which always holds if X is infinite dimensional*), *we also have:*

(*b*) *Either* $F \cap M_c \neq \varnothing$ *or* $F \cap K_c$ *contains a saddle point.*

and

(*c*) *Either* $F \cap \bar{M}_c \neq \varnothing$ *or* $F \cap K_c$ *contains a saddle point of mountain-pass type.*

Note that if $c > a$, then the set F = Boundary of $\{ x \in X;\ \varphi(x) \geqq c \}$ separates u and v. Since $F \cap M_c$ is necessarily empty, the above theorem applied to this particular set F gives the following known results.

COROLLARY (3). — *Let X, φ, $\{u, v\}$ and c be as in Theorem* (1). *Assume that φ verifies* (*PS*) *and that* $c > a = max \{ \varphi(u), \varphi(v) \}$. *Then:*

(*a*) (Höfer [4]) *Either* $\bar{M}_c \setminus M_c \neq \varnothing$ or K_c *contains a point of mountain-pass type.*

Moreover, if X is infinite dimensional, we have:

(*b*) (Pucci-Serrin [9]) K_c *contains a saddle point*

and

(*c*) (Pucci-Serrin [8]) *Either* $\bar{M}_c \setminus M_c \neq \varnothing$ or K_c *contains a saddle point of mountain-pass type.*

Notice that we have used for (*b*) and (*c*) the fact that a compact subset of an infinite dimensional Banach space cannot separate two points in its complement. More precise statements than those in Corollary (3) can be deduced from Theorem (1. *Ter*) whenever an additional "constraint set"

F is involved. The details are left to the interested reader. It will be interesting to know whether the above method can be used to obtain the other results of Pucci-Serrin [8] concerning the structure of the critical set.

We would like to thank I. Ekeland for bringing to our attention the reference [11] after a first version of this paper was written and also for his invaluable help in preparing this revised version. In that paper ([11]) another proof of the "limiting case" which uses the deformation Lemma is given.

Extensions of the above results to the case of "higher dimensional links" will be investigated in a forthcoming paper [12].

II. PROOFS

For the convenience of the reader we start by recalling the statement of Ekeland's variational principle ([2] Corollary 5.3.2).

LEMMA 4. — *Let* (Γ, d) *be a complete metric space, and* $I : \Gamma \to \mathbf{R} \cup \{+\infty\}$ *a bounded below lower semi-continuous function on* Γ. *Let* $\varepsilon > 0$ *and* g *in* Γ *be such that* $I(g) \leqq \inf_{\Gamma} I + \varepsilon^2$. *Then there exists* $\hat{g}$ *in* Γ *such that:*

(1) $I(\hat{g}) \leqq I(g)$

(2) $d(\hat{g}, g) \leqq \varepsilon$

(3) $I(f) \geqq I(\hat{g}) - \varepsilon d(\hat{g}, f)$ *for all* f *in* Γ.

Proof of Theorem (1). — Let $\tilde{F} = F \cap \{x \in X;\ \varphi(x) \geqq c\}$. Since $\tilde{F}$ separates u and v and since X is locally connected, we can find two disjoint open sets U and V such that $X \setminus \tilde{F} = U \cup V$ and $u \in U$ while $v \in V$. Fix ε so that $0 < \varepsilon < \frac{1}{2} \min(1, \operatorname{dist}(u, \tilde{F}), \operatorname{dist}(v, \tilde{F}))$. We shall prove the existence of a point x_ε in X such that:

(i) $c \leqq \varphi(x_\varepsilon) \leqq c + \frac{5}{4}\varepsilon^2$.

(ii) $\operatorname{dist}(x_\varepsilon, \tilde{F}) \leqq 3\varepsilon/2$.

(iii) $\|\varphi'(x_\varepsilon)\| \leqq 3\varepsilon/2$.

To do that, let g be a function in $C([0, 1], X)$ such that $g(0) = u$, $g(1) = v$ and

$$\operatorname{Max}\{\varphi(g(t));\ 0 \leqq t \leqq 1\} < c + \varepsilon^2/4 \tag{1}$$

Define two numbers a and b with $0 < a < b < 1$ by:

$$a = \sup\{t \in [0, 1];\ g(t) \in U \text{ and } \operatorname{dist}(g(t), \tilde{F}) \geqq \varepsilon\}$$
$$b = \inf\{t \in [a, 1];\ g(t) \in V \text{ and } \operatorname{dist}(g(t), \tilde{F}) \geqq \varepsilon\},$$

so that necessarily $\operatorname{dist}(g(t), \tilde{F}) < \varepsilon$ whenever $a < t < b$.

Consider the space:

$$\Gamma = \Gamma(a, b) = \{ k \in C([a, b]; X);\ k(a) = g(a) \text{ and } k(b) = g(b) \}$$

equipped with the uniform distance:

$$\| k_1 - k_2 \| = \max \{ \| k_1(t) - k_2(t) \|;\ a \leqq t \leqq b \}$$

Set $\psi(x) = \max\{0, \varepsilon^2 - \varepsilon \operatorname{dist}(x, \tilde{F})\}$
and define a function $I : \Gamma(a, b) \to \mathbf{R}$ by

$$I(k) = \operatorname{Max} \{ \varphi(k(t)) + \psi(k(t));\ a \leqq t \leqq b \}$$

Note that for any k in $\Gamma(a, b)$ we have that $k([a, b]) \cap \tilde{F} \neq \varnothing$, since $k(a) \in U$, $k(b) \in V$ and $X \setminus \tilde{F} = U \cup V$. It follows that for any such k in Γ:

$$I(k) \geqq \operatorname{Max} \{ \varphi(k(t)) + \psi(k(t));\ a \leqq t \leqq b \text{ and } k(t) \in \tilde{F} \} \geqq c + \varepsilon^2$$

so that

$$\inf_{\Gamma} I \geqq c + \varepsilon^2. \tag{2}$$

On the other hand, let $\tilde{g}$ be the restriction of g to $[a, b]$. We have:

$$I(\tilde{g}) \leqq \operatorname{Max} \{ \varphi(g(t)) + \psi(g(t));\ 0 \leqq t \leqq 1 \} \leqq \left(c + \frac{\varepsilon^2}{4} \right) + \varepsilon^2. \tag{3}$$

The function I is bounded below and lower semi-continuous on the complete metric space $\Gamma(a, b)$ and the function $\tilde{g}$ is a point in $\Gamma(a, b)$ such that $I(\tilde{g}) \leqq \operatorname{Inf}_{\Gamma} I + \varepsilon^2/4$. We can now apply Lemma (4) to find a path $\hat{g}$ in $\Gamma(a, b)$ such that:

$$I(\hat{g}) \leqq I(\tilde{g}) \tag{4}$$

$$\| \hat{g} - \tilde{g} \| \leqq \varepsilon/2 \tag{5}$$

$$I(f) \geqq I(\hat{g}) - (\varepsilon/2) \| f - \hat{g} \| \quad \text{for all } f \text{ in } \Gamma(a, b). \tag{6}$$

Let now M be the subset of $[a, b]$ consisting of all points where $(\varphi + \psi) \circ \hat{g}$ attains its maximum on $[a, b]$. We first prove the following *claim*: There exists $t_0 \in M$ such that $\| \varphi'(\hat{g}(t_0)) \| \leqq 3\,\varepsilon/2$.

Indeed, first note that (6) gives for any h in $C([a, b], X)$ with $h(a) = h(b) = 0$

$$-(\varepsilon/2) \| h \| \leqq \liminf_{\lambda \downarrow 0} \frac{1}{\lambda} [I(\hat{g} + \lambda h) - I(\hat{g})]$$

Using the definition of φ' and the fact that, for $t \in [a, b]$ and $\lambda > 0$, we have $\psi(\hat{g}(t) + \lambda h(t)) \leqq \psi(\hat{g}(t)) + \lambda \operatorname{Lip}(\psi) \| h(t) \|$, it follows that the last quantity is dominated by

$$\leqq \liminf_{\lambda \downarrow 0} \frac{1}{\lambda}\{ \max_{t \in [a, b]} ((\varphi+\psi)(\hat{g}(t))+\lambda<\varphi'(\hat{g}(t)), h(t)> \} - \max_{t \in [a, b]} (\varphi+\psi)(\hat{g}(t))]+\|h\| \operatorname{Lip}(\psi)$$

Hence:

$$-(\varepsilon+\varepsilon/2)\|h\| \leqq \liminf_{\lambda \downarrow 0} \frac{1}{\lambda}[\mathrm{N}(\alpha+\lambda\beta)-\mathrm{N}(\alpha)] \tag{7}$$

where $\alpha=(\varphi+\psi)\circ\hat{g}$, $\beta=\langle \varphi'(\hat{g}), h \rangle$ and N is the (continuous convex) function on the space $C[a, b]$ which associates to any continuous $\gamma:[a, b]\rightarrow \mathbf{R}$ its maximal value $\mathrm{N}(\gamma)=\sup_{t \in [a, b]} \gamma(t)$.

Now we can proceed as in the proof of Theorem 5.5.5 of [2]. Consider the subdifferential $\partial\mathrm{N}(\gamma)$ of N at the point γ and recall that

$\partial\mathrm{N}(\gamma)=\{\mu;\ \mu$ Radon probability measure supported in $\mathrm{M}(\gamma)\}$ where $\mathrm{M}(\gamma)=\{t\in[a, b];\ \gamma(t)=\mathrm{N}(\gamma)\}$.

First we show that:

$$\mathrm{M}\cap\{a, b\}=\varnothing \tag{8}$$

Indeed by combining (2) and (1) we get:

$$\begin{aligned} \mathrm{I}(\hat{g}) \geqq c+\varepsilon^2 &\geqq 3\varepsilon^2/4+\max\{\varphi(g(t));\ 0\leqq t\leqq 1\} \\ &\geqq 3\varepsilon^2/4+\max\{\varphi(\gamma(a)),\ \varphi(g(b))\} \\ &=3\varepsilon^2/4+\max\{(\varphi+\psi)(g(a)),\ (\varphi+\psi)(g(b))\}. \end{aligned}$$

This clearly implies (8).

Back to (7), we get:

$$\begin{aligned} -(3\varepsilon/2)\|h\| &\leqq \liminf_{\lambda \downarrow 0}[\mathrm{N}(\alpha+\lambda\beta)-\mathrm{N}(\alpha)] \\ &\leqq \max\{\langle \beta, \mu \rangle;\ \mu\in\partial\mathrm{N}(\alpha)\} \\ &=\max\left\{\int\langle \varphi'(\hat{g}), h\rangle d\mu;\ \mu\in\partial\mathrm{N}(\alpha)\right\}. \end{aligned}$$

By a standard minimax theorem ([2], Th. 6.2.7) we have

$$\begin{aligned} -3\varepsilon/2 &\leqq \inf_h \max_\mu \left\{\int\langle \varphi'(\hat{g}), h\rangle;\ \mu\in\partial\mathrm{N}(\alpha),\ \|h\|\leqq 1,\ h(a)=0=h(b)\right\} \\ &=\max_\mu \inf_h \left\{\int\langle \varphi'(\hat{g}), h\rangle d\mu;\ \mu\in\partial\mathrm{N}(\alpha),\ \|h\|\leqq 1,\ h(a)=0=h(b)\right\} \\ &=\max_\mu\left\{-\int\|\varphi'(\hat{g})\|d\mu;\ \mu\in\partial\mathrm{N}(\alpha)\right\} \\ &=-\min\{\|\varphi'(\hat{g}(t))\|;\ t\in\mathrm{M}((\varphi+\psi)\circ\hat{g})\end{aligned}$$

It follows that there exists $t_0 \in \mathrm{M}$ such that $\| \varphi'(\hat{g}(t_0)) \| \leqq 3\varepsilon/2$.
and the claim is proved.

It remains to show that the point $x_\varepsilon = \hat{g}(t_0)$ satisfies (i) and (ii). For (i) combine (2), (3) (4) and the claim to get:

$$c + \varepsilon^2 \leqq \inf_{\Gamma} \mathrm{I} \leqq \varphi(\hat{g}(t_0)) + \psi(\hat{g}(t_0)) = \mathrm{I}(\hat{g}) \leqq \mathrm{I}(\tilde{g}) \leqq c + 5\varepsilon^2/4.$$

Since $0 \leqq \psi \leqq \varepsilon^2$ we obtain $c \leqq \varphi(x_\varepsilon) \leqq c + \frac{5}{4}\varepsilon^2$. For (ii) it is enough to notice that (8) implies $a < t_0 < b$, hence $\operatorname{dist}(\tilde{g}(t_0), \tilde{\mathrm{F}}) = \operatorname{dist}(g(t_0), \tilde{\mathrm{F}}) \leqq \varepsilon$. This combined with (5) gives that $\operatorname{dist}(x_\varepsilon, \tilde{\mathrm{F}}) = \operatorname{dist}(\hat{g}(t_0, \tilde{\mathrm{F}}) \leqq 3\varepsilon/2$. ■

Proof of Theorem (1. *Ter*) (*i*). — Suppose $\mathrm{F} \cap \mathrm{K}_c$ contains no critical point of mountain-pass type. Let $\tilde{\mathrm{F}} = \mathrm{F} \cap \{\varphi \geqq c\}$ and let U be a component of $\mathrm{X} \setminus \tilde{\mathrm{F}}$ containing u. The hypothesis implies that v does not belong to U. Let $\mathrm{G} = \{x;\ \varphi(x) < c\}$. We claim that

(★) there exist finitely many components of G, say $\mathrm{C}_1, \ldots, \mathrm{C}_p$ and $\varepsilon_1 > 0$ such that

$\mathrm{G} \cap \{x;\ d(x, \tilde{\mathrm{F}} \cap \mathrm{K}_c) < \varepsilon_1\} \subset \mathrm{C}_1 \cup \mathrm{C}_2 \ldots \cup \mathrm{C}_p$.

Indeed, otherwise we could find a sequence x_i in $\tilde{\mathrm{F}} \cap \mathrm{K}_c$ and a sequence $(\mathrm{C}_i)_i$ of different components of G such that $\operatorname{dist}(x_i, \mathrm{C}_i) \to 0$. But then any limit point of the sequence x_i would be a critical point for φ of mountain-pass type belonging to $\tilde{\mathrm{F}} \subset \mathrm{F}$, thus contradicting our initial assumption. Hence (★) is verified.

Let now $\mathrm{M}_i = \mathrm{F} \cap \mathrm{K}_c \cap \bar{\mathrm{C}}_i$. Since any point of $\mathrm{M}_i \cap (\bigcup_{j \neq i} \bar{\mathrm{C}}_j)$ would be a critical point of Mountain-pass type, we may find ε_2 in $(0, \varepsilon_1)$ such that $\{x;\ d(x, \mathrm{M}_i) < \varepsilon_2\} \cap (\bigcup_{j \neq i} \bar{\mathrm{C}}_j) = \varnothing$. We can also assume that

$$\varepsilon_2 < \min(\operatorname{dist}(u, \tilde{\mathrm{F}}), \operatorname{dist}(v, \tilde{\mathrm{F}})).$$

Let N_1 be the set of all i in $\{1, \ldots, p\}$ such that $\mathrm{C}_i \subset \mathrm{U}$ and let N_2 be $\{1, \ldots, p\} \setminus \mathrm{N}_1$. For every ε in $(0, \varepsilon_2)$ the set

$$\mathrm{H}(\varepsilon) = [\mathrm{U} \cup \bigcup_{i \in \mathrm{N}_1} \{x;\ d(x, \mathrm{M}_i) < \varepsilon\}] \setminus \bigcup_{j \in \mathrm{N}_2} \overline{\{x;\ d(x, \mathrm{M}_j) < \varepsilon\}}$$

is an open subset of X containing u while $v \notin \overline{\mathrm{H}(\varepsilon)}$. Hence the boundary $\mathrm{F}(\varepsilon) = \partial \mathrm{H}(\varepsilon)$ is a closed set separating u and v. Since $\varepsilon < \varepsilon_2$,

$$\bigcup_{i \in \mathrm{N}_1} \mathrm{C}_i \subset \mathrm{H}(\varepsilon) \quad \text{and} \quad \bigcup_{i \in \mathrm{N}_2} \mathrm{C}_i \subset \mathrm{X} \setminus \mathrm{H}(\varepsilon),$$

thus $\mathrm{F}(\varepsilon) \cap \mathrm{G} = \mathrm{F}(\varepsilon) \cap \left(\bigcup_{i=1}^{p} \mathrm{C}_i\right) = \varnothing$. Consequently $\varphi \geqq c$ on $\mathrm{F}(\varepsilon)$. Since $\mathrm{F}(\varepsilon)$ separates u and v, we may use Theorem (1. *bis*) to find a critical point x_ε in $\mathrm{F}(\varepsilon) \cap \mathrm{K}_c$.

We claim that for each $\varepsilon>0$, x_ε is a local minimum for φ. Indeed, otherwise there is $i\in\{1, \ldots, p\}$ such that $x_\varepsilon\in\bar{C}_i$. If $i\in N_1$ we would conclude that $x_\varepsilon\in M_i\subset H(\varepsilon)$ and, since $H(\varepsilon)$ is open, $x_\varepsilon\notin\partial H(\varepsilon)=F(\varepsilon)$ which is a contradiction. On the other hand, if $i\in N_2$, we would get $x_\varepsilon\in M_i\subset X\setminus\overline{H(\varepsilon)}$ and again that $x_\varepsilon\notin F_\varepsilon$; another contradiction.

Finally observe that $\operatorname{dist}(x_\varepsilon, \tilde{F})\to 0$ and since K_c is compact, this shows that $\tilde{F}\cap\bar{M}_c\neq\varnothing$.

For the rest we shall need the following standard topological result.

Lemma (5). — *Assume* F *is a closed subset of* X *separating two points* u *and* v. *Then there exists a closed connected subset* $\hat{F}$ *of* F *separating* u *and* v *such that* $\hat{F}=\partial U=\partial V$ *where* U, V *are components of* $X\setminus\hat{F}$ *containing* u *and* v *respectively.*

Proof. — *See* Kuratowski ([Ku], ch VIII, § 57, III, theorem 1) and ([Ku], ch VI, § 49, V, theorems 1 and 3).

Proof of Theorem (1. *Ter*) (*ii*). — Suppose that $F\cap M_c=\varnothing=F\cap S_c$. Let $\tilde{F}=F\cap\{\varphi\geqq c\}$. Use Lemma (5) to find a closed connected subset $\hat{F}\subset\tilde{F}$ that also separates u and v. Note that $\hat{F}\cap K_c=\hat{F}\cap P_c$. Since $\hat{F}\cap P_c$ is open in $\hat{F}$, $\hat{F}\cap P_c$ is a compact set clopen in $\hat{F}$. Since $\hat{F}$ is connected, either $\hat{F}\cap P_c=\varnothing$ or $\hat{F}\cap P_c=\hat{F}$. The first case is impossible since $\hat{F}\cap P_c=\hat{F}\cap K_c\neq\varnothing$ according to Theorem (1. *bis*). Hence $\hat{F}\subset P_c$ and the corollary is proved, since then $F\cap P_c$ will contain the compact set $\hat{F}\cap K_c=\hat{F}\cap P_c=\hat{F}$ which separates u and v.

Proof of Theorem (1. *Ter*) (*iii*). — Let $\tilde{F}=F\cap\{\varphi\geqq c\}$. Since $\tilde{F}$ is a closed set separating u and v, we can use Lemma (5) to get a closed connected subset $\hat{F}\subset\tilde{F}$ separating u and v such that $\hat{F}=\partial U=\partial V$ where U and V are two components of $X\setminus\tilde{F}$ containing u and v respectively. Assume $F\cap\bar{M}_c=\varnothing$. The set $K=\hat{F}\cap P_c$ is an open subset relative to $\hat{F}$. If K is not closed then any x in $\bar{K}\setminus K$ is a saddle point since $\hat{F}\cap M_c=\varnothing$. Moreover, if H is any open neighbourhood of x not intersecting M_c and such that $\varphi\leqq c$ on H, then both sets $U\cap H$ and $V\cap H$ meet the set $\{\varphi<c\}$. This shows that x is a point of Mountain pass type.

Assume now that K is closed. Then it is a clopen set in the connected space $\hat{F}$. Hence either $K=\hat{F}$ or $K=\varnothing$. In the first case $\hat{F}$ is then contained in P_c and since it separates u and v, we get a contradiction. In the second case, $\hat{F}\cap K_c$ contains a point of mountain-pass type by part (i). Moreover, such a point is necessarily a saddle point since $\hat{F}\cap M_c=\varnothing$. This clearly finishes the proof.

Remark (6). — Note that in the proof of Theorem (1. *Ter*) (ii) we only used that φ verifies $(PS)_{F,c}$, while for (i) and (iii) the proof requires that φ verifies $(PS)_{F_\varepsilon,c}$ for some $\varepsilon>0$ where F_ε is the ε-neighbourhood of F, *i.e.* $F_\varepsilon=\{x\in X;\ \operatorname{dist}(x, F)<\varepsilon\}$.

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