ANNALES DE L'I. H. P., SECTION C

V. BANGERT

On minimal laminations of the torus

Annales de l'I. H. P., section C, tome 6, nº 2 (1989), p. 95-138
<http://www.numdam.org/item?id=AIHPC_1989__6_2_95_0>

© Gauthier-Villars, 1989, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section C » (http://www.elsevier.com/locate/anihpc) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 6, n° 2, 1989, p. 95-138. *Analyse non linéaire*

On minimal laminations of the torus

by

V. BANGERT

Mathematisches Institut, Universität Bern,
Sidlerstr. 5, CH 3012 Bern, Switzerland

ABSTRACT. — We investigate functions $u: \mathbf{R}^n \to \mathbf{R}$ which minimize a variational integral $\int F(x, u(x), u_x(x))\,dx$ where F: $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is periodic in $(x, u) \in \mathbf{R}^{n+1}$ and uniformly convex in $u_x \in \mathbf{R}^n$. We restrict attention to non-selfintersecting minimizers u, i. e. we require that the hypersurface graph $(u) \subseteq \mathbf{R}^{n+1}$ does not have selfintersections when projected into $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$. For such u there exists a "rotation vector" $\alpha = \alpha(u) \in \mathbf{R}^n$ such that $u(x) - \alpha \cdot x$ is bounded. Our main result determines the structure of the set $\mathcal{M}_\alpha$ of non-selfintersecting minimizers with fixed commensurable rotation vector α. The $u \in \mathcal{M}_\alpha$ are classified by secondary invariants. The projected graphs of the $u \in \mathcal{M}_\alpha$ with certain types of secondary invariants form foliations or laminations (i. e. foliations with gaps) of $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$.

Key words : $\mathbf{Z}^n$-periodic variational problems, minimizing solutions, laminations.

RÉSUMÉ. — On étudie des fonctions $u: \mathbf{R}^n \to \mathbf{R}$ qui minimisent une intégrale variationelle $\int F(x, u(x), u_x(x))\,dx$ où F: $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ est périodique en $(x, u) \in \mathbf{R}^{n+1}$ et uniformément convexe en $u_x \in \mathbf{R}^n$. On s'intéresse aux u pour lesquelles la projection du graphe de u dans $T^{n+1} = \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ est l'image d'une immersion injective. Cette condition implique que $u(x) - \alpha \cdot x$ soit bornée pour un $\alpha = \alpha(u) \in \mathbf{R}^n$. Cet α est appelé

Classification A.M.S. : 58 E 15, 58 F 18, 49 A 22.

vecteur de rotation de u. Le résultat principal détermine la structure de l'ensemble $\mathcal{M}_\alpha$ formé des minimales à vecteur de rotation α commensurable. Les $u \in \mathcal{M}_\alpha$ sont classifiées par des invariants secondaires. Après projection dans T^{n+1} les graphes des $u \in \mathcal{M}_\alpha$ à invariants secondaires d'un certain type forment soit un feuilletage de T^{n+1} soit une «lamination» (=feuilletage à creux) de T^{n+1}.

1. INTRODUCTION

A minimal solution of a variational problem with integrand $F: \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a C^1-function $u: \mathbf{R}^n \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^n} (F(x, (u+\varphi)(x), (u+\varphi)_x(x)) - F(x, u(x), u_x(x)))\, dx \geqq 0$$

for every C^1-function $\varphi: \mathbf{R}^n \to \mathbf{R}$ with compact support. Our main hypothesis is that F be $\mathbf{Z}$-periodic in the first $n+1$ variables so that F can be considered a map on $T^{n+1} \times \mathbf{R}^n$ where T^{n+1} denotes the torus $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$. We will study the minimal solutions without selfintersections of such problems. The condition "u without selfintersections" means that the hypersurface graph$(u) \subseteqq \mathbf{R}^{n+1}$ does not have nontrivial selfintersections when projected into T^{n+1}. So we look at those minimal solutions whose graphs could possibly occur as leaves of foliations or laminations (i.e. foliations with gaps) of T^{n+1}. In this study which has been initiated by J. Moser [12], *cf.* also [3] and [4], the standard case is the Dirichlet integrand $F_0(x, u(x), u_x(x)) = \frac{1}{2}|u_x(x)|^2$. The F_0-minimal solutions are the harmonic functions and the ones without selfintersections are the affine functions $u(x) = \alpha \cdot x + u_0$ where $\alpha \in \mathbf{R}^n$, $u_0 \in \mathbf{R}$. It is a purely topological fact that every minimal solution u without selfintersections determines a "rotation vector" or "average slope" $\alpha \in \mathbf{R}^n$ such that $u(x) - \alpha \cdot x$ is bounded uniformly for all $x \in \mathbf{R}^n$. We denote by $\mathcal{M}_\alpha = \mathcal{M}_\alpha(F)$ the set of nonselfintersecting F-minimal solutions with fixed rotation vector α. Under appropriate conditions on F we know from [12] that $\mathcal{M}_\alpha \neq \varnothing$ for all $\alpha \in \mathbf{R}^n$ and [3] shows that the graphs of functions in $\mathcal{M}_\alpha$ give rise to a lamination —possibly a foliation—on T^{n+1} if $\bar{\alpha} = (-\alpha, 1) = (-\alpha_1, \ldots, -\alpha_n, 1)$ is rationally independent. Under our assumptions this simply means that $\mathcal{M}_\alpha$ is totally ordered, i.e. if $u, v \in \mathcal{M}_\alpha$ then either $u = v$ or $u < v$ or $u > v$ everywhere on $\mathbf{R}^n$. This paper is devoted to the study of $\mathcal{M}_\alpha$ for rationally dependent $\bar{\alpha}=(-\alpha, 1)$. For such α the structure of $\mathcal{M}_\alpha$ can be much more complicated: $\mathcal{M}_\alpha$ can contain uncountably many laminations (possibly foliations) such that different laminations contain intersecting leaves. The appeal of this theory is based on the fact that we can analyze this complicated structure in detail under general hypotheses and by purely qualitative methods. It should be noted that for $\alpha \in \mathbf{Q}^n$ and $n>1$ the complicated situation is generic, i.e. occurs for most integrands F. On the other hand Moser shows in [13] that the "foliation" $\mathcal{M}_\alpha(\mathrm{F}_0)=\{u(x)=\alpha\cdot x+u_0 \mid u_0 \in \mathbf{R}\}$ survives (up to conjugation) small perturbations of F_0 provided α satisfies certain Diophantine inequalities (which, however, do not imply that $\bar{\alpha}$ is rationally independent). But large perturbations of F_0 can also destroy such foliations, *see* [4]. So, for rationally dependent $\bar{\alpha}=(-\alpha, 1)$ and $n>1$ the complications analyzed in this paper can be considered typical. The precise notions which enter into this analysis are developed in the text, in particular in Sect. 3. Here we present an informal summary of our results:

Before we start to describe the general situation we recall the case $n=1$, *cf.* [6], Sect. 3. There we have the following possibilities for the behaviour of a non-selfintersecting minimal solution u: $\mathbf{R} \to \mathbf{R}$.

If u has rational rotation number $\alpha=p/q$:

(*a*) periodic: graph(u) is invariant under translation by $(p, q) \in \mathbf{Z}^2$, i.e. $u(x-p)+q=u(x)$.

(*b*) heteroclinic: There exist periodic u^- and u^+ in $\mathcal{M}_\alpha$ such that $\lim_{x \to \pm\infty} (u(x)-u^{\pm}(x))=0$.

If u has irrational rotation number α:

(*c*) quasiperiodic: The $\mathbf{Z}^2$-translates of graph(u) form a dense subset of $\mathbf{R}^2$. The closure of the set of $\mathbf{Z}^2$-translates is a foliation whose leaves are the graphs of functions in $\mathcal{M}_\alpha$.

(*d*) generalized quasiperiodic: The closure of the set of $\mathbf{Z}^2$- translates of graph(u) defines a lamination. The set of points in which the leaves intersect the vertical coordinate axis contains and generically equals a Cantor set.

If the dimension n is arbitrary a function $u \in \mathcal{M}_\alpha$ can exhibit different types of behaviour in different directions of $\mathbf{R}^n$, e.g. u can be "periodic" on lines with direction e_1, "generalized quasiperiodic" on lines with direction e_2, etc.

Only if $\bar{\alpha} \in \mathbf{R}^{n+1}$ is rationally independent we necessarily have "quasiperiodic" or "generalized quasiperiodic" behaviour in every direction. In Sects. 3-5 we characterize the various possibilities for the behaviour of $u \in \mathcal{M}_\alpha$ with rationally dependent $\bar{\alpha}=(-\alpha, 1)$ by secondary invariants. These investigations are primarily topological. In Sect. 6 we use the minimality condition to show that the graphs of functions in $\mathcal{M}_\alpha$ with the same secondary invariants do not intersect. In Sect. 7 we settle the question of existence of minimal solutions $u \in \mathcal{M}_\alpha$ with prescribed secondary invariants. In particular we prove the existence of secondary laminations in the gaps between the functions in $\mathcal{M}_\alpha$ with maximal periodicity. Moreover we present examples for which these secondary laminations do not have gaps and hence give rise to secondary foliations. In Sect. 8 we mention two open problems and some partial results related to one of them.

This theory is based on the work of Morse [11] and Hedlund [7] on geodesics on surfaces which—in different contexts and with different motivation—was rediscovered and largely extended by Aubry/Le Daeron [1] and Mather, *cf.* [9] and [10]. In addition to the methods developed in [3] we use ideas by Morse and Aubry/Le Daeron. So our main tools are the compactness property of the set of minimal solutions derived by Moser [12], a $\mathbf{Z}^{n+1}$-action on the set of minimal solutions and a maximum principle for elliptic equations.

Finally we explain two concepts which we use and which might otherwise cause confusion. Slightly abusing terminology we will say that a subset $\mathcal{N} \subseteqq C^0(\mathbf{R}^n)$ is a foliation of a connected open set $W \subseteqq \mathbf{R}^{n+1}$ if

(1.1) $$\operatorname{graph}(u) \cap \operatorname{graph}(v) = \varnothing \quad \text{for all } u \neq v \text{ in } \mathcal{N},$$

and:

(1.2) $$\bigcup_{u \in \mathcal{N}} \operatorname{graph}(u) = W.$$

Very often we will encounter sets $\mathcal{N} \subseteqq C^0(\mathbf{R}^n)$ such that the graphs of the functions in $\mathcal{N}$ only form a "foliation with gaps". Thurston [14], p. 373, uses the term "lamination" in this context: $\mathcal{N} \subseteqq C^0(\mathbf{R}^n)$ is a lamination of $W \subseteqq \mathbf{R}^{n+1}$ if in addition to (1.1) we have

(1.3) $$\bigcup_{u \in \mathcal{N}} \operatorname{graph}(u) \quad \text{is closed in } W.$$

Actually, in order to exclude trivial cases we will only call $\mathcal{N}$ a lamination if $\mathcal{N}$ satisfies the following additional condition:

(1.4) The set $\{u(0) \mid u \in \mathcal{N}\} \subseteqq \mathbf{R}$ contains a Cantor set.

From topological dynamics we need the notion of a minimal set of a group action $\Gamma \times X \to X$ on a topological space X: A minimal set is a smallest Γ-invariant nonempty closed subset of X. Following Birkhoff [5], Chapter VII, Sect. 7, elements of a minimal set will be called (Γ-) recurrent. Unfortunately we cannot avoid to use the word "minimal" also in the completely different context of "minimal solutions". The laminations mentioned above will arise as minimal sets of the action of $\mathbf{Z}^{n+1}$ (or some subgroup thereof) on the set of minimal solutions.

2. THE VARIATIONAL PROBLEM

In this section we describe the setting of the problem and present some of Moser's [12] results which are fundamental for our investigations. We adopt the notation used in [12], [3] and [4].

The coordinates of a point in $\mathbf{R}^{2n+1}$ will be denoted by

$$(x, x_{n+1}, p)=(\bar{x}, p)$$

where

$$x\in\mathbf{R}^n, \qquad x_{n+1}\in\mathbf{R}, \qquad \bar{x}=(x, x_{n+1})\in\mathbf{R}^{n+1} \qquad \text{and} \qquad p\in\mathbf{R}^n.$$

We denote by $\mathrm{B}(x, r)\subseteqq\mathbf{R}^n$ the euclidean ball with raduis r and center $x\in\mathbf{R}^n$. The integrand of the variational problem is a function $\mathrm{F}: \mathbf{R}^{2n+1}\to\mathbf{R}$ with the following properties, *cf.* [12], (3.1).

(F_1) $\mathrm{F}\in\mathrm{C}^{2,\varepsilon}(\mathbf{R}^{2n+1})$ for some $\varepsilon>0$.

(F_2) F is $\mathbf{Z}^{n+1}$-periodic in $\bar{x}$, *i. e.*

$$\mathrm{F}(\bar{x}+\bar{k}, p)=\mathrm{F}(\bar{x}, p) \quad \text{for all } \bar{k}\in\mathbf{Z}^{n+1}$$

(F_3) $\delta|\xi|^2\leqq\sum_{\mu,\nu=1}^{n}\mathrm{F}_{p_\mu p_\nu}(\bar{x}, p)\xi_\mu\xi_\nu\leqq\delta^{-1}|\xi|^2$ for some $\delta\in(0, 1)$.

(F_4) and
$$|\mathrm{F}_{\bar{x}p}(\bar{x}, p)|\leqq c(1+|p|)$$
$$|\mathrm{F}_{\bar{x}\bar{x}}(\bar{x}, p)|\leqq c(1+|p|^2) \quad \text{for some } c>0.$$

If $\Omega\subseteqq\mathbf{R}^n$ is measurable and u is in the Sobolev space $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ we abbreviate

$$\mathrm{I}(u, \Omega)=\int_\Omega \mathrm{F}(x, u(x), u_x(x))\,dx$$

provided the integral exists as an extended real number.

Example. — The standard case is the Dirichlet integrand $\mathrm{F}_0(\bar{x}, p)=\frac{1}{2}|p|^2$. Then $\mathrm{I}(u, \Omega)$ is the Dirichlet integral of u over Ω.

If $\Omega\subseteqq\mathbf{R}^n$ is open let $\mathrm{W}^{1,2}_{\mathrm{comp}}(\Omega)$ denote the set of all $\varphi\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ with compact support.

(2.1) DEFINITION. — A function $u\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ is a minimal solution of the variational problem (briefly: u is minimal) if for all $\varphi\in\mathrm{W}^{1,2}_{\mathrm{comp}}(\mathbf{R}^n)$

$$\mathrm{I}(u+\varphi, \operatorname{supp}(\varphi))\geqq\mathrm{I}(u, \operatorname{supp}(\varphi))$$

If $\Omega\subseteqq\mathbf{R}^n$ is open we say that $u\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$ is minimal in Ω if this inequality is true for all $\varphi\in\mathrm{W}^{1,2}_{\mathrm{comp}}(\Omega)$.

Example. — For the Dirichlet integrand minimality is equivalent to harmonicity.

The regularity theory for minima u of our variational problem is quite delicate, cf. the remarks and references in [12]:

If $F \in C^{l,\varepsilon}(\mathbf{R}^{2n+1})$ and $l \geqq 2$ then $u \in C^{l,\varepsilon}(\Omega)$. One of our main tools is the following maximum principle:

(2.2) LEMMA. — *Suppose u and v are minimal in the connected open set $\Omega \subseteqq \mathbf{R}^n$.*

If $u \leqq v$ then either $u = v$ or $u < v$.

A detailed proof of (2.2) is given in [12], Sect. 4.

We consider the following $\mathbf{Z}^{n+1}$-action T on $C^0(\mathbf{R}^n)$:

$$(2.3) \quad \text{if} \quad \begin{gathered} (T_{\bar{k}} u)(x) = u(x-k) + k_{n+1} \\ \bar{k} = (k, k_{n+1}) \in \mathbf{Z}^{n+1} \quad \text{and} \quad u \in C^0(\mathbf{R}^n). \end{gathered}$$

This action corresponds to translation of graph (u):

$\operatorname{graph}(T_{\bar{k}} u) = \operatorname{graph}(u) + \bar{k}$. According to the $\mathbf{Z}^{n+1}$-periodicity (F_2) of F the action T maps minimal solutions to minimal solutions.

We partially order $C^0(\mathbf{R}^n)$ by defining $u < v$ if and only if $u(x) < v(x)$ for all $x \in \mathbf{R}^n$. Note that T preserves this order.

(2.4) DEFINITION. — A function $u \in C^0(\mathbf{R}^n)$ is said not to have selfintersections if the T-orbit of u is totally ordered, i. e. if for all $\bar{k} \in \mathbf{Z}^{n+1}$ we have $T_{\bar{k}} u < u$ or $T_{\bar{k}} u = u$ or $T_{\bar{k}} u > u$.

Geometrically this condition means that any two translates of graph (u) by integer vectors are either disjoint or identical.

In other words the projection of graph (u) into $T^{n+1} = \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ does not have nontrivial selfintersections. If $u \in C^0(\mathbf{R}^n)$ does not have selfintersections there exists a unique $\alpha \in \mathbf{R}^n$, "the rotation vector of u", such that $|u(x) - \alpha \cdot x|$ is bounded, *cf.* [12], Sect. 2 or [3], Sect. 4 or Sect. 3 of this paper.

(2.5) NOTATION. — We let $\mathcal{M} = \mathcal{M}(F)$ denote the set of minimal solutions without selfintersection. $\mathcal{M}$ decomposes into the disjoint union $\mathcal{M} = \bigcup_{\alpha \in \mathbf{R}^n} \mathcal{M}_\alpha$ where $\mathcal{M}_\alpha$ consists of the $u \in \mathcal{M}$ with rotation vector α.

An important property of $\mathcal{M}$ and of the $\mathcal{M}_\alpha$ is their T-invariance.

Example. — If F_0 is the Dirichlet integrand then

$$\mathcal{M}_\alpha = \{u \mid u(x) = \alpha \cdot x + u_0,\ u_0 \in \mathbf{R}\}.$$

This is a consequence of Liouville's Theorem. According to [12], Theorem 2.3, this is true more generally if F only depends on p.

Next we state Moser's estimates which are basic for everything to come, *cf.* [12], Theorems 2.1 and 3.1.

THEOREM. — *There exist constants* c_1, γ_1 *such that for all* $u \in \mathcal{M}_\alpha$:

(2.6) $$|u(x+y)-u(x)-\alpha\cdot y| \leqq c_1 \sqrt{1+|\alpha|^2}$$

(2.7) $$|u_x|_{C^\varepsilon} \leqq \gamma_1.$$

Here c_1 *only depends on* F *while* γ_1 *depends on* F *and* $|\alpha|$.

In particular, all $u \in \mathcal{M}_\alpha$ are Lipschitz with constant γ_1. As a simple consequence of (2.7) we obtain:

(2.8) COROLLARY. — *Every sequence* u_i *with* $u_i \in \mathcal{M}_{\alpha_i}$ *and both* $|u_i(0)|$ *and* $|\alpha_i|$ *bounded contains a subsequence which is* C^1*-convergent on compact sets to some* $u \in \mathcal{M}$.

In particular, the sets $\mathcal{M}$ and $\mathcal{M}_\alpha$, $\alpha \in \mathbf{R}^n$, are closed with respect to C^1-convergence on compact sets.

3. SECONDARY INVARIANTS FOR GRAPHS WITHOUT SELFINTERSECTIONS

According to [3], Sect. 4 the rotation vector $\alpha \in \mathbf{R}^n$ of a nonselfintersecting $u \in C^0(\mathbf{R}^n)$ is characterized by the following property (where $\bar{\alpha} := (-\alpha, 1) \in \mathbf{R}^{n+1}$):

$$\text{If } \bar{k} \in \mathbf{Z}^{n+1} \quad \text{and} \quad \bar{k}\cdot\alpha > 0 \quad \text{then } T_{\bar{k}} u > u.$$

However, if $\bar{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$ and $\bar{k}\cdot\bar{\alpha} = 0$ there are examples with $T_{\bar{k}} u > u$ or $T_{\bar{k}} u = u$ or $T_{\bar{k}} u < u$. In this section we define secondary invariants of u which completely determine which of the above possibilities holds for any given $\bar{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$ with $\bar{k}\cdot\bar{\alpha} = 0$. Of course, this is nontrivial only if $\bar{\alpha}$ is rationally dependent, i.e. if there exist $\bar{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$ with $\bar{k}\cdot\bar{\alpha} = 0$.

There is a close relation between nonselfintersecting $u \in C^0(\mathbf{R}^n)$ and orbits of n commuting circle homeomorphisms, i.e. orbits of a $\mathbf{Z}^n$-action on S^1 by homeomorphisms. The invariants are also defined for such an orbit and they describe how the orbit converges to the minimal set of the action (if $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$). If $u \in \mathcal{M}_\alpha$ these invariants allow us to investigate the action of T on the closure of the orbit $\{T_{\bar{k}} u \mid \bar{k} \in \mathbf{Z}^{n+1}\}$. The results in this section generalize those of [3], Sect. 4. They are fundamental for the uniqueness and existence results in Sects. 6 and 7.

If $u \in C^0(\mathbf{R}^n)$ does not have selfintersections we define

$$\Gamma^+(u) = \Gamma^+ = \{\bar{k} \in \mathbf{Z}^{n+1} \mid T_{\bar{k}} u \geqq u\}.$$

Then $\Gamma^+ \subset \mathbf{Z}^{n+1}$ is a semigroup and $\Gamma^+ \cup (-\Gamma^+) = \mathbf{Z}^{n+1}$. The following lemma shows that such semigroups are always the intersection of $\mathbf{Z}^{n+1}$ with some halfspace:

(3.1) LEMMA. — *Let* V *be a finite-dimensional euclidean vector space and*

$\Gamma \subseteqq V$ *a lattice, i. e.* Γ *is a discrete sub-group of* $(V, +)$ *with* $rk(\Gamma) = \dim(V)$. *Suppose* $\Gamma^+ \subseteqq \Gamma$ *is a semigroup such that* $\Gamma^+ \cup (-\Gamma^+) = \Gamma$. *Then either* $\Gamma^+ = \Gamma$ *or there exists a unique unit vector* $a \in V$ *such that* $\{k \in \Gamma \mid k \cdot a > 0\} \subseteqq \Gamma^+ \subseteqq \{k \in \Gamma \mid k \cdot a \geqq 0\}$.

Proof. — This is an invariant version of [3], Lemma (4.1). The proof given there applies also in the present situation if the following little remark is added:

$$\text{If } \Gamma^+ \subset \Gamma \quad \text{and} \quad \Gamma^+ \cup (-\Gamma^+) = \Gamma \quad \text{then } rk(\Gamma^+ \cap \Gamma^-) < rk(\Gamma).$$

In the present form the lemma can be applied iteratively. For Γ and Γ^+ as above consider $\Gamma_2 = \{k \in \Gamma \mid k \cdot a = 0\}$ and $V_2 = \operatorname{span}(\Gamma_2) \subseteqq \langle a \rangle^\perp$ with the induced scalar product. For $\Gamma_2^+ = \Gamma_2 \cap \Gamma^+$ we have $\Gamma_2^+ \cup (-\Gamma_2^+) = \Gamma_2$ so that we can apply (3.1) to V_2 and Γ_2, Γ_2^+.

Inductively we obtain:

(3.2) LEMMA. — *Let* V, Γ *and* Γ^+ *satisfy the hypotheses of* (3.1) *and suppose* $\Gamma^+ \subset \Gamma$. *Then there exist an integer* t, $1 \leqq t \leqq \dim V$, *and unit vectors* $a_1, \ldots, a_t$ *with* $a_s \in \operatorname{span}(\Gamma \cap \langle a_1, \ldots, a_{s-1} \rangle^\perp)$ $(1 < s \leqq t)$ *which are uniquely determined by the following properties:*

(*a*) $k \in \Gamma^+ \setminus (-\Gamma^+)$ *if and only if there exists* $1 \leqq s \leqq t$ *such that* $k \in \Gamma \cap \langle a_1, \ldots, a_{s-1} \rangle^\perp$ *and* $k \cdot a_s > 0$.

(*b*) $\Gamma^+ \cap (-\Gamma^+) = \Gamma \cap \langle a_1, \ldots, a_t \rangle^\perp$.

Here we denote by $\operatorname{span}(\Gamma)$ *the smallest linear subspace of* V *containing* Γ *and by* $\langle a_1, \ldots, a_t \rangle^\perp$ *the orthogonal complement of the linear subspace generated by* $a_1, \ldots, a_t$.

Now suppose $u \in C^0(\mathbf{R}^n)$ does not have selfintersections. We set $\bar{\Gamma}^+ = \bar{\Gamma}^+(u) = \{\bar{k} \in \mathbf{Z}^{n+1} \mid T_{\bar{k}} u \geqq u\}$. From Lemma (3.2) we obtain an integer $t = t(u)$, $1 \leqq t \leqq n+1$, and pairwise orthogonal unit vectors $\bar{a}_1 = \bar{a}_1(u), \ldots, \bar{a}_t = \bar{a}_t(u)$ in $\mathbf{R}^{n+1}$.

Note that $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$ according to [3], Lemma (4.1).

(3.3) DEFINITION. — The integer $t = t(u)$ and the vectors $\bar{a}_1(u), \ldots, \bar{a}_t(u)$ are the invariants of a nonselfintersecting $u \in C^0(\mathbf{R}^n)$.

In addition to being pairwise orthogonal the $\bar{a}_1(u), \ldots, \bar{a}_t(u)$ have the following property which is a direct consequence of their definition. If we set

$$(3.4) \quad \bar{\Gamma}_s = \bar{\Gamma}_s(u) = \mathbf{Z}^{n+1} \cap (\langle \bar{a}_1(u), \ldots, \bar{a}_{s-1}(u) \rangle^\perp) \qquad (2 \leqq s \leqq t+1)$$

then

$$(3.5) \qquad \bar{a}_s(u) \in \operatorname{span}(\bar{\Gamma}_s(u)) \qquad (2 \leqq s \leqq t).$$

For notational convenience we set $\bar{\Gamma}_1(u)=\mathbf{Z}^{n+1}$ so that (3.5) holds also for $s=1$. Note that Moser [12], p. 253, calls subgroups $\bar{\Gamma}\subseteq\mathbf{Z}^{n+1}$ maximal if $\bar{\Gamma}=\mathbf{Z}^{n+1}\cap\operatorname{span}(\bar{\Gamma})$. In this sense the $\bar{\Gamma}_s(u)$ are maximal. According to (3.2) the invariants t, $\bar{a}_1, \ldots, \bar{a}_t$ are characterized by the following properties:

(3.6) $T_{\bar{k}}u>u$ if and only if there exists $1\leqq s\leqq t$

such that $\bar{k}\in\bar{\Gamma}_s$ and $\bar{k}\cdot\bar{a}_s>0$.

(3.7) $T_{\bar{k}}u=u$ if and only if $\bar{k}\in\bar{\Gamma}_{t+1}$.

Remarks:

1. Invariance under the T-action: For all $\bar{k}\in\mathbf{Z}^{n+1}$ we have $t(T_{\bar{k}}u)=t(u)$, $\bar{a}_1(T_{\bar{k}}u)=\bar{a}_1(u), \ldots, \bar{a}_t(T_{\bar{k}}u)=\bar{a}_t(u)$.

2. If $rk(\Gamma_{t(u)+1}(u))=l$ then the projection of graph(u) into $T^{n+1}=\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ is an immersed submanifold of type $T^l\times\mathbf{R}^{n-l}$.

3. If $\bar{a}_1$ is rationally independent, i.e. if $\bar{\Gamma}_2=\{\bar{0}\}$, then $t=1$.

4. In [12], Sect. 2, or [3], Sect. 4, the "rotation vector" or "average slope" $\alpha\in\mathbf{R}^n$ of a nonselfintersecting $u\in C^0(\mathbf{R}^n)$ is defined. According to [3], Lemma (4.1), the rotation vector α and the invariant $\bar{a}_1$ are related by $\bar{a}_1=|\bar{\alpha}|^{-1}\bar{\alpha}$ where $\bar{\alpha}=(-\alpha, 1)$.

If this relation is fulfilled we will say that the invariant $\bar{a}_1$ and the rotation vector α correspond to each other.

For completeness we add a proof that (3.6) implies the boundedness of $|u(x)-\alpha\cdot x|$. This shows that $\bar{a}_1$ is a "mean unit normal" to graph(u).

(3.8) LEMMA. — *Suppose $u\in C^0(\mathbf{R}^n)$ has no selfintersections and $\bar{a}_1=|\bar{\alpha}|^{-1}\bar{\alpha}$ with $\bar{\alpha}=(-\alpha, 1)$. Then*

$$\sup\{|u(x)-\alpha\cdot x|\,|\,x\in\mathbf{R}^n\}\leqq\max\{|u(x)-\alpha\cdot x|\,|\,x\in[0,1]^n\}+1.$$

Proof. — Given $x\in\mathbf{R}^n$ choose $k\in\mathbf{Z}^n$ such that $x-k\in[0,1]^n$. If $u(x)-\alpha\cdot x>u(x-k)-\alpha\cdot(x-k)$ choose $k_{n+1}\in\mathbf{Z}$ such that $\bar{k}=(k, k_{n+1})$ satisfies $0<\bar{\alpha}\cdot\bar{k}\leqq 1$. Then (3.6) implies

$$(T_{\bar{k}}u)(x)=u(x-k)+k_{n+1}>u(x).$$

Hence

$$u(x)-\alpha\cdot x<u(x-k)-\alpha\cdot(x-k)+\bar{\alpha}\cdot\bar{k}\leqq u(x-k)-\alpha\cdot(x-k)+1.$$

Since $x-k\in[0,1]^n$ this proves our claim. The case

$$u(x)-\alpha\cdot x<u(x-k)-\alpha\cdot(x-k)$$

is analogous.

The invariants t, $\bar{a}_1, \ldots, \bar{a}_t$ can be similarly defined for an orbit of an action A: $\mathbf{Z}^n\times\mathbf{R}\to\mathbf{R}$ of $\mathbf{Z}^n$ on $\mathbf{R}$ by "circle maps", i.e.

$A(k, .)=A_k : \mathbf{R} \to \mathbf{R}$ is continuous, strictly increasing and $A_k(s+1)=A_k(s)+1$ for all $s \in \mathbf{R}$. The orbit of A through $u_0 \in \mathbf{R}$ is the "sequence" $(u_j)_{j \in \mathbf{Z}^n}$ defined by $u_j = A_j(u_0)$. Now such an orbit has a similar "no selfintersection property" as the restriction of a nonselfintersecting $u \in C^0(\mathbf{R}^n)$ to $\mathbf{Z}^n$: For $\bar{k} \in \mathbf{Z}^{n+1}$ we define

$$\tilde{T}_{\bar{k}}((u_j)_{j \in \mathbf{Z}^n}) = (v_j)_{j \in \mathbf{Z}^n} \quad \text{with} \quad v_j = u_{j-k} + k_{n+1}.$$

Then for all $\bar{k} \in \mathbf{Z}^{n+1}$ either

$$\tilde{T}_{\bar{k}}((u_j)) > (u_j) \quad \text{or} \quad \tilde{T}_{\bar{k}}((u_j)) = (u_j) \quad \text{or} \quad T_{\bar{k}}((u_j)) < (u_j).$$

So we can use Lemma (3.2) to obtain invariants $t, \bar{a}_1, \ldots, \bar{a}_t$ for such an orbit $(u_j)_{j \in \mathbf{Z}^n}$ which coincide with $t(u), \bar{a}_1(u), \ldots, \bar{a}_t(u)$ if $u_j = u(j)$ for a nonselfintersecting $u \in C^0(\mathbf{R}^n)$. Note that $\bar{a}_1 = |\bar{\alpha}|^{-1} \bar{\alpha}$ where $\bar{\alpha} = (-\alpha, 1)$ and $\alpha \in \mathbf{R}^n$ is the rotation vector of A, i. e. the components α_i of α are the rotation numbers of the circle maps $A(e_i, .)$.

So $\bar{a}_1$ only depends on A while for $s > 1$ the $\bar{a}_s$ depend on the choice of a particular orbit $(u_j)_{j \in \mathbf{Z}^n}$ of A.

These invariants determine how the orbit $(u_j)_{j \in \mathbf{Z}^n}$ converges to more and more periodic orbits of A. For minimal solutions $u \in \mathcal{M}_\alpha$ these things are treated in Sect. 4.

We will give a special name to those systems $\bar{a}_1, \ldots, \bar{a}_t$ of unit vectors which can arise as invariants of a nonselfintersecting $u \in C^0(\mathbf{R}^n)$:

(3.9) DEFINITION. — A system $(\bar{a}_1, \ldots, \bar{a}_t)$ of unit vectors in $\mathbf{R}^{n+1}$ is admissible if $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$ and if for $2 \leqq s \leqq t$:

$$\bar{a}_s \in \operatorname{span}(\bar{\Gamma}_s) \quad \text{where} \quad \bar{\Gamma}_s = \bar{\Gamma}_s(\bar{a}_1, \ldots, \bar{a}_t) = \mathbf{Z}^{n+1} \cap \langle \bar{a}_1, \ldots, \bar{a}_{s-1} \rangle^{\perp}.$$

If $(\bar{a}_1, \ldots, \bar{a}_t)$ is admissible and $1 \leqq s < t$ then $(\bar{a}_1, \ldots, \bar{a}_s)$ is admissible.

It is probably an elementary exercise to construct a nonselfintersecting $u \in C^0(\mathbf{R}^n)$ with invariants $t(u) = t$, $\bar{a}_1(u) = \bar{a}_1, \ldots, \bar{a}_t(u) = \bar{a}_t$ for a given admissible system $\bar{a}_1, \ldots, \bar{a}_t$. However we do not have to face this problem since the existence results for minimal solutions in Sect. 7 prove the existence of such u.

Now we investigate continuity properties of the $\bar{a}_s(u)$ as a function of the nonselfintersecting $u \in C^0(\mathbf{R}^n)$. This is easy in the case $s = 1$:

(3.10) LEMMA. — *Suppose the sequence $u_i \in C^0(\mathbf{R}^n)$ converges pointwise to $u \in C^0(\mathbf{R}^n)$ and u and the u_i do not have selfintersections. Then we have* $\lim \bar{a}_1(u_i) = \bar{a}_1(u)$.

Remark. — In [12], Lemma (3.4), a weaker statement is proved for minimal solutions.

Proof. — We argue by contradiction and assume that the sequence $\bar{a}_1(u_i)$ has a point of accumulation $\bar{a}\in S^n$ such that $\bar{a}\neq\bar{a}_1(u)$. Since $\{|\bar{k}|^{-1}\bar{k}\,|\,\bar{k}\in\mathbf{Z}^{n+1}\setminus\{0\}\}$ is dense in S^n there exists $\bar{k}\in\mathbf{Z}^{n+1}$ such that $\bar{k}\cdot\bar{a}_1(u)>0$ while $\bar{k}\cdot\bar{a}<0$. Hence $T_{\bar{k}}u>u$ while $T_{\bar{k}}u_i<u_i$ for infinitely many $i\in\mathbf{N}$. This contradicts the pointwise convergence of u_i to u.

For $s>1$ the situation gets much more complicated. This has to do with the fact that e. g. $\mathbf{Z}^{n+1}\cap\langle\bar{a}_1\rangle^{\perp}$ does not depend continuously on $\bar{a}_1$. So we will only treat a special case. The following lemma characterizes $\bar{a}_s$ by a slightly weaker property than (3.6).

(3.11) Lemma. — *Let $u\in C^0(\mathbf{R}^n)$ not have selfintersections. Suppose for some $s\leqq t(u)$ there exists a unit vector $\bar{a}\in\operatorname{span}(\bar{\Gamma}_s(u))$ such that $\bar{k}\in\bar{\Gamma}_s(u)$ and $\bar{k}\cdot\bar{a}>0$ imply $T_{\bar{k}}u\geqq u$. Then $\bar{a}=\bar{a}_s(u)$.*

Proof. — If $\bar{a}\neq\bar{a}_s(u)$ there exists $\bar{k}\in\bar{\Gamma}_s(u)$ such that $\bar{k}\cdot\bar{a}_s(u)<0<\bar{k}\cdot\bar{a}$. Now our hypothesis implies $T_{\bar{k}}u\geqq u$ while the definition of $\bar{a}_s(u)$ gives $T_{\bar{k}}u<u$. Hence we must have $\bar{a}=\bar{a}_s(u)$.

(3.12) Corollary. — *Suppose the sequence $u_i\in C^0(\mathbf{R}^n)$ converges pointwise to $u\in C^0(\mathbf{R}^n)$ and u and the u_i do not have selfintersections. If $t(u_i)=t$ and $\bar{a}_s(u_i)=\bar{a}_s$ for all $1\leqq s\leqq t$ and all $i\in\mathbf{N}$ then $t(u)\leqq t$ and $\bar{a}_s(u)=\bar{a}_s$ for all $1\leqq s\leqq t(u)$.*

Proof: According to (3.10) we have $\bar{a}_1(u)=\bar{a}_1$. Moreover $T_{\bar{k}}u\geqq u$ if $\bar{k}\in\mathbf{Z}^{n+1}\cap(\langle\bar{a}_1,\ \ldots,\ \bar{a}_{s-1}\rangle^{\perp})$ and $\bar{k}\cdot\bar{a}_s>0$ for some $1\leqq s\leqq t$. By induction (3.11) implies $\bar{a}_s(u)=\bar{a}_s$ for $1\leqq s\leqq\min\{t,\ t(u)\}$. Finally we have $t(u)\leqq t$ since $T_{\bar{k}}u=u$ for all $\bar{k}\in\mathbf{Z}^{n+1}\cap(\langle\bar{a}_1,\ \ldots,\ \bar{a}_t\rangle^{\perp})$.

4. APPLICATIONS TO THE SET OF MINIMAL SOLUTIONS

In this section we return to our original problem and consider the set $\mathscr{M}=\bigcup_{\alpha\in\mathbf{R}^n}\mathscr{M}_\alpha$ of minimal solutions without selfintersections with respect to some integrand F satisfying $(F_1)-(F_4)$. If $\bar{\alpha}=(-\alpha,\ 1)$ is rationally dependent we can use the invariants defined in the previous section to subdivide $\mathscr{M}_\alpha$ into T-invariant subsets $\mathscr{M}(\bar{a}_1,\ \ldots,\ \bar{a}_t)$, *see* (4.1) below.

We will study how a given $u\in\mathscr{M}(\bar{a}_1,\ \ldots,\ \bar{a}_t)$ converges to elements u^-, $u^+\in\mathscr{M}(\bar{a}_1,\ \ldots,\ \bar{a}_{t-1})$ with $u^-<u<u^+$. Since the graphs of u^- and u^+ are invariant under $\bar{\Gamma}_t$ they are more periodic than graph(u) which is only

invariant under $\bar{\Gamma}_{t+1}=\bar{\Gamma}_t \cap \langle \bar{a}_t \rangle^{\perp}$. Our arguments are basically topological and minimality is only used in form of the compactness properties (2.6)-(2.8) of $\mathscr{M}_\alpha$.

(4.1) DEFINITION. — For every admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ we define

$$\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)=\{ u \in \mathscr{M} \mid t(u)=t \text{ and } \bar{a}_s(u)=\bar{a}_s \text{ for } 1 \leqq s \leqq t \}.$$

Note that $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t) \cap \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_s)=\varnothing$ if $t \neq s$.

First we clarify the relation of $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ to the sets $\mathscr{M}_\alpha$, $\mathscr{M}_\alpha^{\text{rec}}$, $\mathscr{M}_\alpha^{\text{per}}$ defined in [3], Sects. 3 and 4. Recall that $\bar{a}_1(u)$ and the corresponding rotation vector α of u are related by $\bar{a}_1=|\bar{\alpha}|^{-1}\bar{\alpha}$ where $\bar{\alpha}=(-\alpha, 1)$. Hence for this $\alpha \in \mathbf{R}^n$:

$$\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t) \subseteqq \mathscr{M}_\alpha.$$

Lemma (4.6) in [3] says precisely that for $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ the unique minimal set $\mathscr{M}_\alpha^{\text{rec}}$ of the action T on $\mathscr{M}_\alpha$ is contained in $\mathscr{M}(\bar{a}_1)$. For generic integrand F one will have $\mathscr{M}_\alpha^{rec}=\mathscr{M}(\bar{a}_1)$. If $\alpha \in \mathbf{Q}^n$, $\bar{a}_1=|\bar{\alpha}|^{-1}\bar{\alpha}$, then $\mathscr{M}_\alpha^{\text{per}}=\mathscr{M}(\bar{a}_1)$ by definition.

For a given admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ the sets $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_s)$, $1 \leqq s \leqq t$, consist of minimal solutions whose graphs are invariant under $\bar{\Gamma}_s$. So this periodicity increases when s decreases and $\mathscr{M}(\bar{a}_1)$ consists of those $u \in \mathscr{M}_\alpha$ whose graph has maximal periodicity. Note that (3.10) and (3.12) imply that $\mathscr{M}(\bar{a}_1) \cup \mathscr{M}(\bar{a}_1, \bar{a}_2) \cup \ldots \cup \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is closed in $\mathscr{M}$.

Next we investigate the action of

$$\bar{\Gamma}_t=\bar{\Gamma}_t(\bar{a}_1, \ldots, \bar{a}_t)=\mathbf{Z}^{n+1} \cap (\langle \bar{a}_1, \ldots, \bar{a}_{t-1} \rangle^{\perp})$$

on $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$. Recall that on $\mathscr{M}$ we use the topology of C^1-convergence on compact sets.

(4.2) PROPOSITION. — *Suppose* $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *and* $t>1$. *Then there exist* u^- *and* u^+ *in* $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ *with the following properties:*

(*a*) *If* $\bar{k}_i \in \bar{\Gamma}_t$ *and* $\lim\limits_{i \to \infty} \bar{k}_i \cdot \bar{a}_t = \pm \infty$ *then* $\lim \mathrm{T}_{\bar{k}_i} u = u^{\pm}$.

(*b*) $u^- < u < u^+$ *and* $\mathrm{T}_{\bar{k}} u^- \geqq u^+$ *if* $\bar{k} \in \bar{\Gamma}_s$ *and* $\bar{k} \cdot \bar{a}_s > 0$ *for some* $1 \leqq s < t$.

(*c*) *If* $\lim\limits_{i \to \infty} \mathrm{T}_{\bar{k}_i} u = v$ *for a sequence* $\bar{k}_i \in \bar{\Gamma}_t$ *such that* $\bar{k}_i \cdot \bar{a}_t$ *is bounded then* $v \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$.

Proof: If $\bar{k}_1, \bar{k}_2 \in \mathbf{Z}^{n+1}$ satisfy $\bar{k}_1 \cdot \bar{a}_1 < 0 < \bar{k}_2 \cdot \bar{a}_1$ then we have

$$\mathrm{T}_{\bar{k}_1} u < \mathrm{T}_{\bar{k}} u < \mathrm{T}_{\bar{k}_2} u$$

for all $\bar{k} \in \bar{\Gamma}_t$. This follows from (3.6) applied to $\bar{k}_1 - \bar{k}$ and $\bar{k}_2 - \bar{k}$. Now Moser's compactness result (2.8) implies that the $\bar{\Gamma}_t$-orbit $\{ \mathrm{T}_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t \}$ is precompact. Using (3.6) again we see that every sequence $\mathrm{T}_{\bar{k}_i} u$ with $\bar{k}_i \in \bar{\Gamma}_t$ and $\lim \bar{k}_i \cdot \bar{a}_t = \infty$ converges to the same limit, called $u^+ \in \mathscr{M}$. Similarly we obtain $u^- \in \mathcal{M}$. Next we prove (*b*):

If $\bar{k} \in \bar{\Gamma}_s$ and $\bar{k} \cdot \bar{a}_s > 0$ for some $1 \leqq s < t$ then $\bar{k} - 2\bar{k}_i \in \bar{\Gamma}_s$ and $(\bar{k} - 2\bar{k}_i) \cdot \bar{a}_s > 0$ for every sequence $\bar{k}_i$ in $\bar{\Gamma}_t$ with $\lim \bar{k}_i \cdot \bar{a}_t = \infty$. Hence $T_{\bar{k}-\bar{k}_i} u > T_{\bar{k}_i} u$ for all $i \in \mathbf{N}$. In the limit this implies $T_{\bar{k}} u^- \geqq u^+$, and, in particular, $T_{\bar{k}} u^+ > u^+$ and $T_{\bar{k}} u^- > u^-$. So, in order to prove that $u^\pm \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ it is left to show $T_{\bar{k}} u^\pm = u^\pm$ for all $\bar{k} \in \bar{\Gamma}_t$: If $\bar{k} \in \bar{\Gamma}_t$ and if $\lim \bar{k}_i \cdot \bar{a}_t = \infty$ then $\lim (\bar{k} + \bar{k}_i) \cdot \bar{a}_t = \infty$. Hence

$$T_{\bar{k}} u^+ = \lim T_{(\bar{k} + \bar{k}_i)} u = u^+,$$

and similarly for u^-. To prove (*c*) note that there exist $\bar{k}_1, \bar{k}_2 \in \bar{\Gamma}_t$ such that $T_{\bar{k}_1} u < v < T_{\bar{k}_2} u$. Together with (*a*) and (*b*) this implies $T_{\bar{k}} v > v$ if and only if $T_{\bar{k}} u > u$. Hence $v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$.

(4.3) NOTATION. — For $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ we fix the notation u^- and u^+ for the elements of $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ defined by: $u^- = \lim T_{\bar{k}_i} u$ for every sequence $\bar{k}_i$ in $\bar{\Gamma}_t$ with $\lim \bar{k}_i \cdot \bar{a}_t = -\infty$, $u^+ = \lim T_{\bar{k}_i} u$ for every sequence $\bar{k}_i$ in $\bar{\Gamma}_t$ with $\lim \bar{k}_i \cdot \bar{a}_t = \infty$.

Now we investigate what the convergence in (4.2) (*a*) means for the functions u and $u^\pm$ themselves. Let $\rho : \mathbf{R}^{n+1} \to \mathbf{R}^n$ denote the projection $\bar{x} = (x, x_{n+1}) \to x$. For an admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ and $2 \leqq s \leqq t$ we introduce the following

(4.4) NOTATION. — We set $\Gamma_s = \rho(\bar{\Gamma}_s) = \rho(\mathbf{Z}^{n+1} \cap \langle \bar{a}_1, \ldots, \bar{a}_{s-1} \rangle^\perp)$ and $V_s = \operatorname{span}(\Gamma_s) = \rho(\operatorname{span} \bar{\Gamma}_s)$. We let b_s denote the unique vector in V_s such that

$$b_s \cdot k = \bar{a}_s \cdot \bar{k}$$

for all $\bar{k} = (k, k_{n+1}) \in \bar{\Gamma}_s$.

Remarks. — 1. Since $\bar{\Gamma}_s \subseteqq \langle \bar{a}_1 \rangle^\perp$ and $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$ the projection ρ is injective on span $(\bar{\Gamma}_s)$ and $\dim V_s = rk(\Gamma_s) = rk\ (\bar{\Gamma}_s)$.

2. One can calculate b_s as the orthogonal projection of $a_s + (a_s)_{n+1} \alpha$ to V_s where $\bar{a}_s = (a_s, (a_s)_{n+1})$. We have $b_s \neq 0$ since $\bar{a}_s \in \operatorname{span}(\bar{\Gamma}_s)$ and $\bar{a}_s \neq 0$.

3. Since $\bar{\Gamma}_{s+1} \subseteqq \bar{\Gamma}_s$ and $\bar{\Gamma}_s \subseteqq \langle \bar{a}_1, \ldots, \bar{a}_{s-1} \rangle^\perp$ we have $V_{s+1} \subseteqq V_s$ and $V_s \subseteqq \langle b_2, \ldots, b_{s-1} \rangle^\perp$.

The following lemma is basic for many arguments in this and the following sections.

(4.5) LEMMA. — *Let $(\bar{a}_1, \ldots, \bar{a}_{t-1})$ be admissible, let* E *be a measurable fundamental domain for V_t/Γ_t and let π_t denote the orthogonal projection $\pi_t : \mathbf{R}^n \to V_t$. Suppose $u, v \in C^0(\mathbf{R}^n)$ have the following property: If $1 \leqq s < t$ and $\bar{k} \in \bar{\Gamma}_s$ and $\bar{k} \cdot \bar{a}_s > 0$ then $T_{\bar{k}} u \geqq v$.*

Then

$$\int_{\pi_t^{-1}(E)} (v - u)(x)\, dx \leqq 1.$$

Remark. — If $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ then the hypothesis of (4.5) is fulfilled for u^- and u^+.

Proof. — Set $W = \{(x, x_{n+1}) \in \mathbf{R}^{n+1} \mid \pi_t(x) \in E \text{ and } u(x) < x_{n+1} < v(x)\}$. We claim that $(W + \bar{k}) \cap W = \varnothing$ for all $\bar{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$. Assume to the contrary that $\bar{x} = (x, x_{n+1}) \in W$ and $\bar{x} + \bar{k} \in W$ for some $\bar{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$. In particular $\pi_t(x) \in E$ and $\pi_t(x) + \pi_t(k) \in E$. This implies $\bar{k} \notin \bar{\Gamma}_t$, since otherwise $k = \pi_t(k) \in \Gamma_t \setminus \{0\}$. But for $\bar{k} \notin \bar{\Gamma}_t$ our hypothesis implies either $T_{-\bar{k}} u \geqq v$ or $T_{-\bar{k}} v \leqq u$, i. e.

$$u(x+k) \geqq v(x) + k_{n+1} \qquad \text{or} \qquad v(x+k) \leqq u(x) + k_{n+1}.$$

From $u(x) < x_{n+1} < v(x)$ we obtain

$$u(x+k) > x_{n+1} + k_{n+1} \qquad \text{or} \qquad v(x+k) < x_{n+1} + k_{n+1}.$$

Hence $\bar{x} + \bar{k} \notin W$ which contradicts our assumption. So the projection $\pi : \mathbf{R}^{n+1} \to T^{n+1} = \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ is injective on W. This implies our claim:

$$\int_{\pi_t^{-1}(E)} (v-u)(x)\, dx \leqq \mathrm{vol}_{n+1}(W) = \mathrm{vol}_{n+1}(\pi(W)) \leqq 1.$$

(4.6) COROLLARY. — *Suppose $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $t > 1$. For all $\varepsilon > 0$ there exists $C > 0$ such that $\mathrm{dist}(x, V_t) > C$ implies $(u^+ - u^-)(x) < \varepsilon$.*

Proof. — Let E be a measurable fundamental domain for V_t/Γ_t. Since u^+ and u^- are in $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ their difference $u^+ - u^-$ is Γ_t-periodic. Hence it suffices to prove our claim for $x = x_1 + x_2$ where $x_1 \in E$ and $x_2 \in (V_t)^\perp$, $|x_2| > C$. According to the remark following Lemma (4.5) we can apply (4.5) to u^- and u^+. Hence:

$$\int_{\pi_t^{-1}(E)} (u^+ - u^-)(x)\, dx \leqq 1.$$

By Moser's estimate (2.7) u^- and u^+ are Lipschitz. Hence the preceding inequality proves our claim.

Thus, if $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ then u^-, u^+ and u all converge to each other in directions not contained in V_t. The following proposition investigates the situation for the remaining directions.

(4.7) PROPOSITION. — *Suppose $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$. For every $\varepsilon > 0$ there exists $C > 0$ such that $u^+(x) - u(x) < \varepsilon$ if $x \cdot b_t < -C$ and $u(x) - u^-(x) < \varepsilon$ if $x \cdot b_t > C$. For every $D > 0$ there exists $\delta > 0$ such that $u^+(x) - u(x) > \delta$ and $u(x) - u^-(x) > \delta$ if $x \in V_t$ and $|x \cdot b_t| < D$.*

Proof. — We decompose $x = x_1 + x_2$ where $x_1 \in V_t$ and $x_2 \in (V_t)^\perp$. According to (4.6) it suffices to treat the case $|x_2| < C_1$ for some $C_1 = C_1(\varepsilon) > 0$. We argue by contradiction and assume that there exists a sequence $x^i = x_1^i + x_2^i$ in $\mathbf{R}^n$ such that

$$|x_2^i| < C_1, \ \lim (x^i \cdot b_t) = \infty \qquad \text{and} \qquad u(x^i) - u^-(x^i) \geqq \varepsilon.$$

Since $V_t = \mathrm{span}(\Gamma_t)$ we can find a sequence y_1^i in a bounded fundamental domain of V_t/Γ_t and a sequence $\bar{k}^i = (k^i, k_{n+1}^i) \in \bar{\Gamma}_t$ such that $x_1^i = y_1^i + k^i$. Then

$$\lim (\bar{k}^i \cdot \bar{a}_t) = \lim (k^i \cdot b_t) = \lim (x_1^i \cdot b_t) = \lim (x^i \cdot b_t) = \infty$$

so that $T_{(-\bar{k}^i)} u$ converges to u^- uniformly on compact sets. Since the points $y_1^i + x_2^i$ are contained in a compact set and $T_{(-\bar{k}^i)} u^- = u^-$ we conclude that

$$u(x^i) - u^-(x^i) = (T_{(-\bar{k}^i)} u)(y_1^i + x_2^i) - u^-(y_1^i + x_2^i)$$

converges to zero. This contradicts our assumption $u(x^i) - u^-(x^i) \geqq \varepsilon$.

The second assertion is proved similarly: One uses that every sequence $T_{\bar{k}_i} u$ with $\bar{k}_i \in \bar{\Gamma}_t$ and $\bar{k}_i \cdot \bar{a}_t$ bounded contains a convergent subsequence with limit v satisfying $u^- < v < u^+$.

If $u \in \mathcal{M}_\alpha$, $v \in \mathcal{M}_\beta$ and $\alpha \neq \beta$ then we can conclude from (2.6) that there exists $x \in \mathbf{R}^n$ with $u(x) = v(x)$. For the secondary invariants we obtain:

(4.8) COROLLARY. — *Suppose* $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$,

$$v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1}, \bar{a}_t'), \quad t > 1 \qquad \textit{and} \qquad \bar{a}_t \neq \bar{a}_t'.$$

If $u^+ = v^+$ *and* $u^- = v^-$ *there exist points* $x \in \mathbf{R}^n$ *with* $u(x) = v(x)$.

Remark. — Theorems (6.6) and (6.13) show that either $u^+ = v^+$, $u^- = v^-$ or $u^+ \leqq v^-$ or $v^+ \leqq u^-$. Hence, if our assumption $u^+ = v^+$, $u^- = v^-$ is not satisfied then $u < v$ or $v < u$. Moreover, Theorem (6.6) shows that $u^- = v^-$ implies $u^+ = v^+$ and vice versa. A generalization of (4.8) will be discussed at the end of Sect. 6.

Proof. — Let $\bar{\Gamma}_t = \bar{\Gamma}_t(\bar{a}_1, \ldots, \bar{a}_{t-1})$, $V_t = \mathrm{span}(\rho(\bar{\Gamma}_t))$ and let b_t resp. b_t' be the vectors in V_t such that $b_t \cdot k = \bar{a}_t \cdot \bar{k}$ resp. $b_t' \cdot k = \bar{a}_t' \cdot \bar{k}$ for all $\bar{k} \in \bar{\Gamma}_t$. First suppose that b_t and b_t' are linearly independent. Then there exist lines $\{x_0 + tv \mid t \in \mathbf{R}\} \subseteq V_t$ with $v \cdot b_t = 0$ and $v \cdot b_t' \neq 0$. According to Proposition (4.7) every such line contains a point x with $u(x) = v(x)$. Next we consider the case that b_t and b_t' are linearly dependent. Since $\bar{a}_t$ is uniquely determined by the set $\{\bar{k} \in \bar{\Gamma}_t \mid \bar{k} \cdot \bar{a}_t > 0\}$ and since this set is uniquely determined by the direction of b_t the vectors b_t and b_t' can be linearly dependent only if $\bar{a}_t = -\bar{a}_t'$ in which case $b_t = -b_t'$.

Hence Proposition (4.7) implies that every line in V_t with direction b_t contains a point x with $u(x) = v(x)$.

If $t>2$ we can iterate the construction of u^- and u^+ to obtain $(u^-)^- < (u^-)^+$ and $(u^+)^- < (u^+)^+$ in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-2})$. It is a simple consequence of (4.2) (*a*) and (*b*) that $(u^-)^- = (u^+)^-$ and $(u^-)^+ = (u^+)^+$. If $t>3$ one can continue the iteration and consider $((u^-)^-)^-$, etc. Since the proofs in Sects. 6 and 7 are by induction we will not need these higher iterates. However, they may be useful to clarify the geometric meaning of the invariants $\bar{a}_2, \ldots, \bar{a}_t$ and the corresponding $b_2, \ldots, b_t$. The proofs of the following statements are left to the reader.

According to (4.2) (*a*) we have

$$u^+ = \sup\{\mathrm{T}_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t\} \quad \text{and} \quad u^- = \inf\{\mathrm{T}_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t\}.$$

This leads us to define (for $1 \leqq s < t$)

$$u_s^+ = \sup\{\mathrm{T}_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_{s+1}\}, \qquad u_s^- = \inf\{\mathrm{T}_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_{s+1}\}.$$

Then we have $u_s^\pm \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_s)$ and $u_{t-1}^\pm = u^\pm$, $u_{t-2}^+ = (u^+)^+$, etc.

In analogy to (4.2) (*a*) and (*b*) we obtain:

(4.9) If $\bar{k}_i \in \bar{\Gamma}_{s+1}$ and $\lim \bar{k}_i \cdot \bar{a}_{s+1} = \pm\infty$ then $\lim \mathrm{T}_{\bar{k}_i} u = u_s^\pm$.

(4.10) If $1 \leqq \sigma \leqq s$ and $\bar{k} \in \bar{\Gamma}_\sigma$, $\bar{k} \cdot \bar{a}_\sigma > 0$ then $\mathrm{T}_{\bar{k}}(u_s^-) \geqq u_s^+$.

From (4.9) and (4.10) we can derive the following analogue of (4.7):

For every $\varepsilon > 0$ there exists $\mathrm{C} > 0$ such that $u_s^+(x) - u(x) < \varepsilon$ if $x \cdot b_{s+1} < -\mathrm{C}$ and $u(x) - u_s^-(x) < \varepsilon$ if $x \cdot b_{s+1} > \mathrm{C}$.

So we obtain the following picture: There exists a finite sequence

$$u_1^- < u_2^- < \ldots < u_{t-1}^- = u^- < u < u^+ = u_{t-1}^+ < \ldots < u_1^+$$

where $u_s^\pm \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_s)$ satisfies (4.9) and (4.10). In particular, graph$(u_s^\pm)$ is invariant under $\bar{\Gamma}_{s+1}$, i.e. the periodicity of graph $(u_s^\pm)$ decreases monotonically with s. The original u and all $u_\sigma^\pm$ for $\sigma > s$ converge to $u_s^\pm$ on halflines which are not orthogonal to b_s or do not lie in V_s.

5. SECONDARY LAMINATIONS

In this section we study the action of $\bar{\Gamma}_t = \mathbf{Z}^{n+1} \cap (\langle \bar{a}_1, \ldots, \bar{a}_{t-1} \rangle^\perp)$ on an arbitrary $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$, $t>1$.

From the preceding paragraph we know that the $\bar{\Gamma}_t$-orbit of u has u^+ as its supremum and u^- as its infimum.

If $rk(\bar{\Gamma}_{t+1}) = rk(\bar{\Gamma}_t) - 1$ then the $\bar{\Gamma}_t$-orbit of u is discrete.

If $rk(\bar{\Gamma}_{t+1}) < rk(\bar{\Gamma}_t) - 1$ then this orbit is not discrete and we study its closure $\mathcal{M}_t(u)$. The graphs of functions in $\mathcal{M}_t(u)$ laminate (or even foliate) the set between the graphs of u^- and u^+. We will call this a "secondary lamination" since for irrational α this lamination is contained in a gap of the lamination $\{\operatorname{graph}(v) \mid v \in \mathcal{M}_\alpha^{\text{rec}}\}$.

Actually we may have a finite hierarchy of laminations, one laminating the gaps of the next and so on. However, all except the one given by $\mathcal{M}_\alpha^{\text{rec}}$ will be called secondary. Obviously the union of all such laminations derived from u and its translates is the closure of the $\mathbf{Z}^{n+1}$-orbit of u and a lamination itself. However, the point is that $\mathcal{M}_t(u)$ may very well form a lamination even if the corresponding rotation vector α is rational so that $\mathcal{M}(\bar{a}_1) = \mathcal{M}_\alpha^{\text{per}}$ will in general be discrete.

We start by treating the simple case $rk(\bar{\Gamma}_{t+1}) = rk(\bar{\Gamma}_t) - 1$:

(5.1) LEMMA. — *If $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $rk(\bar{\Gamma}_{t+1}) = rk(\bar{\Gamma}_t) - 1$ then the $\bar{\Gamma}_t$-orbit of u is discrete.*

Proof. — Since $h : \bar{\Gamma}_t \to \mathbf{R}$, $h(\bar{k}) = \bar{k} \cdot \bar{a}_t$, is a homomorphism with kernel $\bar{\Gamma}_{t+1}$ its image $\{\bar{k} \cdot \bar{a}_t \mid \bar{k} \in \bar{\Gamma}_t\}$ is a finitely generated subgroup of $(\mathbf{R}, +)$ of rank 1, hence discrete. Now our claim follows easily from the facts that

$$T_{\bar{k}} u = u \qquad \text{if} \quad \bar{k} \in \bar{\Gamma}_{t+1} = \bar{\Gamma}_t \cap \langle a_t \rangle^\perp$$

and that, by (4.2) (*b*),

$$\lim T_{\bar{k}_i} u = u^\pm \qquad \text{if} \quad \lim \bar{k}_i \cdot \bar{a}_t = \pm\infty.$$

In analogy to [3], (4.3) we define:

(5.2) DEFINITION. — Suppose $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. We say that u can be approximated from below, resp. from above, if

$$u = \sup\{T_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t, \ \bar{k} \cdot \bar{a}_t < 0\}$$

resp. if

$$u = \inf\{T_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t, \ \bar{k} \cdot a_t > 0\}.$$

The set of all $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ which can be approximated from above or from below is denoted by $\mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$.

Remarks. — 1. We have $u \in \mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ if and only if $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and u is not an isolated point in its $\bar{\Gamma}_t$-orbit.

2. If $rk(\bar{\Gamma}_{t+1}) = rk(\bar{\Gamma}_t) - 1$ then $\mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t) = \varnothing$.

3. $\mathcal{M}^{\text{rec}}(\bar{a}_1) = \mathcal{M}_\alpha^{\text{rec}}$ if α is the rotation vector corresponding to $\bar{a}_1$.

NOTATION. — For $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ we let $\mathcal{M}_t(u)$ denote the closure of the orbit $\{T_{\bar{k}} u \mid \bar{k} \in \bar{\Gamma}_t\}$ inside the set $\{v \in \mathcal{M} \mid u^- < v < u^+\}$.

We set $\mathcal{M}_t^{\text{rec}}(u) = \mathcal{M}_t(u) \cap \mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$.

So $\mathcal{M}_t(u)$ consists of the limits of sequences $T_{\bar{k}_i} u$ where $\bar{k}_i \in \bar{\Gamma}_t$ and $\bar{k}_i \cdot \bar{a}_t$ remains bounded.

(5.3) LEMMA. — *Suppose $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ can be approximated from below, $v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$, $u^- < v < u^+$ and $\mathcal{M}_t(u) \cup \mathcal{M}_t(v)$ is totally ordered.*

Then

$$u=\sup\{T_{\bar{k}}v \mid \bar{k}\in\bar{\Gamma}_t \text{ and } T_{\bar{k}}v<u\}.$$

Remark. — A similar statement holds if u can be approximated from above.

Statement and proof of (5.3) are analogous to [3], (4.4).

(5.4) LEMMA. — *Suppose $u\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $rk(\bar{\Gamma}_{t+1})<rk(\bar{\Gamma}_t)-1$. Then $\mathcal{M}_t(u)\subseteqq\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $\mathcal{M}_t^{\mathrm{rec}}(u)\neq\emptyset$. We have $v\in\mathcal{M}_t^{\mathrm{rec}}(u)$ if and only if $u^-<v<u^+$ and v is a point of accumulation of $\mathcal{M}_t(u)$. $\mathcal{M}_t^{\mathrm{rec}}(u)$ is the unique minimal set of the action of $\bar{\Gamma}_t$ on $\mathcal{M}_t(u)$.*

Proof. — Suppose $v=\lim T_{\bar{k}_i}u$ where $\bar{k}_i\in\bar{\Gamma}_t$ and $\bar{k}_i\cdot\bar{a}_t$ is bounded, i.e. $u^-<v<u^+$. We want to show that $v\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. According to (3.12) we have $t(v)\leqq t$ and $\bar{a}_s(v)=\bar{a}_s$ for $1\leqq s\leqq t(v)$. On the other hand $u^-<v<u^+$ and (4.2) (*b*) imply $t(v)\geqq t-1$. So it suffices to prove that $t(v)\geqq t$ and $\bar{a}_t(v)=\bar{a}_t$. So suppose $\bar{k}\in\bar{\Gamma}_t(v)=\bar{\Gamma}_t(\bar{a}_1, \ldots, \bar{a}_{t-1})$ is given and $\bar{k}\cdot\bar{a}_t>0$. Since $v<u^+$ and $\lim_{n\to\infty} T_{n\bar{k}}u=u^+$ there exists $n\in\mathbf{N}$ such that $T_{n\bar{k}}u>v$. We may also assume that $n\bar{k}\cdot\bar{a}_t>\bar{k}_i\cdot\bar{a}_t$ for all $i\in\mathbf{N}$. Then we obtain

$$T_{2n\bar{k}}v=\lim_{i\to\infty} T_{(2n\bar{k}+\bar{k}_i)}u\geqq T_{n\bar{k}}u>v.$$

Since $2n\bar{k}\in\bar{\Gamma}_t(v)$ this implies $t(v)\geqq t$. Moreover we obtain $T_{\bar{k}}v>v$ for all $\bar{k}\in\bar{\Gamma}_t(v)$ with $\bar{k}\cdot\bar{a}_t>0$. Hence $\bar{a}_t(v)=\bar{a}_t$.

Next we show that $\mathcal{M}_t^{\mathrm{rec}}(u)\neq\emptyset$. Since $rk(\bar{\Gamma}_{t+1})<rk(\bar{\Gamma}_t)-1$ and $\bar{\Gamma}_{t+1}=\bar{\Gamma}_t\cap\langle a_t\rangle^{\perp}$ the subgroup $\{\bar{k}\cdot\bar{a}_t \mid \bar{k}\in\bar{\Gamma}_t\}$ of $(\mathbf{R}, +)$ has rank $\geqq 2$ so that it cannot be discrete. Hence there exists a sequence $\bar{k}_i\in\bar{\Gamma}_t$ such that $\bar{k}_i\cdot\bar{a}_t\neq 0$ and $\lim\bar{k}_i\cdot\bar{a}_t=0$. According to (2.8) the sequence $T_{\bar{k}_i}u$ has a limit point v, $u^-<v<u^+$, which is a point of accumulation of $\mathcal{M}_t(u)$. So, in order to prove that $\mathcal{M}_t^{\mathrm{rec}}(u)\neq\emptyset$ it suffices to prove that an arbitrary point of accumulation v of $\mathcal{M}_t(u)$ with $u^-<v<u^+$ belongs to $\mathcal{M}_t^{\mathrm{rec}}(u)$: We treat the case that $v=\lim T_{\bar{k}_i}u$ and $T_{\bar{k}_i}u<v$ for all $i\in\mathbf{N}$.

We set $\tilde{v}=\sup\{T_{\bar{k}}v \mid \bar{k}\in\bar{\Gamma}_t \text{ and } \bar{k}\cdot\bar{a}_t<0\}$. If $\tilde{v}<v$ there exist $\bar{k}, \bar{h}\in\bar{\Gamma}_t$ such that $\tilde{v}<T_{\bar{k}}v<T_{\bar{h}}u<v$, in particular $(\bar{h}-\bar{k})\cdot\bar{a}_t>0$. Hence $\tilde{v}<T_{(\bar{h}-\bar{k})}\tilde{v}<u$ and this is easily seen to contradict the definition of $\tilde{v}$. Hence $\tilde{v}=v$, i.e. v can be approximated from below. Conversely if $v\in\mathcal{M}_t^{\mathrm{rec}}(u)$ then v is a point of accumulation of $\mathcal{M}_t(u)$ by (5.3). Similarly the last claim of (5.4) follows from (5.3).

Next we prove that for $u\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $rk(\bar{\Gamma}_{t+1})<rk(\bar{\Gamma}_t)-1$ the set $\mathcal{M}_t^{\mathrm{rec}}(u)$ gives rise to a lamination of

$$W=\{\bar{x}=(x, x_{n+1}) \mid u^-(x)<x_{n+1}<u^+(x)\}.$$

Note that $\mathcal{M}_t^{\mathrm{rec}}(u)$ is totally ordered so that (1.1) is satisfied. Using Lemma (4.2) (*c*) we easily see that $\bigcup\{\mathrm{graph}(v) \mid v\in\mathcal{M}_t^{\mathrm{rec}}(u)\}$ is closed in W. So we only have to prove:

(5.5) PROPOSITION. — *Suppose* $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *and* $rk(\bar{\Gamma}_{t+1}) < rk(\bar{\Gamma}_t) - 1$. *Then the map* $\mathrm{H} : \mathcal{M}_t^{\mathrm{rec}}(u) \to (u^-(0), u^+(0)) \subseteqq \mathbf{R}$, $\mathrm{H}(v) := v(0)$, *is a homeomorphism either onto* $(u^-(0), u^+(0))$ *or onto a Cantor set in* $(u^-(0), u^+(0))$.

Proof. — Obviously H is a homeomorphism onto its image Im(H) and $\mathrm{Im}(\mathrm{H}) \neq \varnothing$ by (5.4). From (2.8) and (5.4) one easily concludes that $\mathrm{Im}(\mathrm{H}) \cup \{u^-(0), u^+(0)\} \subseteqq \mathbf{R}$ is compact. In particular Im(H) is closed in $(u^-(0), u^+(0))$. According to the definition of $\mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ we do not have isolated points in Im(H). So it remains to prove that $\mathrm{Im}(\mathrm{H}) = (u^-(0), u^+(0))$ if Im(H) contains a nonempty open interval I. Since Im(H) is closed all we have to show is that Im(H) is open. So let $v(0) \in \mathrm{Im}(\mathrm{H})$ be an arbitrary point. On Im(H) we have the induced $\bar{\Gamma}_t$-action $\tilde{\mathrm{T}}$, $\tilde{\mathrm{T}}_{\bar{k}}(v(0)) := (\mathrm{T}_{\bar{k}} v)(0) = v(-k) + k_{n+1}$. It follows from (5.3) that the $\tilde{\mathrm{T}}$-orbit of $v(0)$ is dense in Im(H). In particular, there exists $\bar{k} \in \bar{\Gamma}_t$ such that $\tilde{\mathrm{T}}_{\bar{k}}(v(0)) \in \mathrm{I}$. Hence $v(0)$ is contained in the open interval $\tilde{\mathrm{T}}_{-\bar{k}}(\mathrm{I}) \subseteqq \mathrm{Im}(\mathrm{H})$.

We are still considering $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ with $rk(\bar{\Gamma}_{t+1}) < rk(\bar{\Gamma}_t) - 1$. Next we will show that the action of $\bar{\Gamma}_t$ on $\mathcal{M}_t(u)$ is semiconjugate to the action of $\bar{\Gamma}_t$ on $\mathbf{R}$ defined by $s \in \mathbf{R} \to s + \bar{k} \cdot \bar{a}_t$.

(5.6) PROPOSITION. — *Let* $h : \mathcal{M}_t(u) \to \mathbf{R}$ *be defined by*

$$h(v) := \inf\{\bar{k} \cdot \bar{a}_t \mid \bar{k} \in \bar{\Gamma}_t \text{ and } \mathrm{T}_{\bar{k}} u \geqq v\}.$$

Then $h(v) = \sup\{\bar{k} \cdot \bar{a}_t \mid k \in \bar{\Gamma}_t \textit{ and } \mathrm{T}_{\bar{k}} u \leqq v\}$, *h is continuous, nondecreasing and satisfies for all* $\bar{k} \in \bar{\Gamma}_t$, $v \in \mathcal{M}_t(u)$:

$$h(\mathrm{T}_{\bar{k}} v) = h(v) + \bar{k} \cdot \bar{a}_t.$$

If v can be approximated from below, resp. from above, then

$$w < v \;\Rightarrow\; h(w) < h(v), \qquad \textit{resp.} \quad w > v \;\Rightarrow\; h(w) > h(v).$$

Proof. — While $h(v) \geqq \sup\{\bar{k} \cdot \bar{a}_t \mid \bar{k} \in \bar{\Gamma}_t \text{ and } \mathrm{T}_{\bar{k}} u \leqq v\}$ follows directly from the definitions the reversed inequality can easily be proved using the fact that $\{\bar{k} \cdot \bar{a}_t \mid \bar{k} \in \bar{\Gamma}_t\}$ is dense in $\mathbf{R}$. Using this density again the definition of h shows that h is continuous from the left while our first claim shows that h is continuous from the right as well.

Obviously h is nondecreasing and

$$h(\mathrm{T}_{\bar{k}} v) = h(v) + \bar{k} \cdot \bar{a}_t \qquad \text{for} \quad \bar{k} \in \bar{\Gamma}_t, \quad v \in \mathcal{M}_t(u).$$

Finally suppose $v \in \mathcal{M}_t(u)$ can be approximated from below and $w < v$ for some $w \in \mathcal{M}_t(u)$. According to (5.3) there exist $\bar{k}, \bar{h} \in \bar{\Gamma}_t$ such that

$$w < \mathrm{T}_{\bar{h}} u < \mathrm{T}_{\bar{k}} u < v.$$

This implies $h(w) \leqq \bar{h} \cdot \bar{a}_t < \bar{k} \cdot \bar{a}_t \leqq h(v)$.

The following corollary is an analogue of [3], (4.7) and will be an important tool in Sect. 6.

(5.7) COROLLARY. — *Suppose $u, v \in \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ can both be approximated from below. Let sequences $\bar{k}_i$, $\bar{h}_i$ in $\bar{\Gamma}_t$ be given such that*

$$\lim \mathrm{T}_{\bar{k}_i} u = \tilde{u}, \qquad \lim \mathrm{T}_{\bar{k}_i} v = \tilde{v}, \qquad \mathrm{T}_{\bar{h}_i} \tilde{u} < u \quad \textit{and} \quad \lim \mathrm{T}_{\bar{h}_i} \tilde{u} = u.$$

Then we have: $\lim \mathrm{T}_{\bar{h}_i} \tilde{v} = v$.

Proof. — The key observation is that the actions of $\bar{\Gamma}_t$ on $\mathcal{M}_t(u)$ and $\mathcal{M}_t(v)$ are both semi-conjugate to the same action of $\bar{\Gamma}_t$ on $\mathbf{R}$. Let $h_u:\mathcal{M}_t(u) \to \mathbf{R}$ resp. $h_v:\mathcal{M}_t(v) \to \mathbf{R}$ be the functions defined in (5.6). Our hypotheses $\lim \mathrm{T}_{\bar{k}_i} u = \tilde{u}$, $\lim \mathrm{T}_{\bar{k}_i} v = \tilde{v}$ imply $h_u(\tilde{u}) = \lim \bar{k}_i \cdot \bar{a}_t = h_v(\tilde{v})$.

On the other hand we have $0 = h_u(u) = \lim h_u(\mathrm{T}_{\bar{h}_i} \tilde{u}) = h_u(\tilde{u}) + \lim \bar{h}_i \cdot \bar{a}_t$, hence $\lim \bar{h}_i \cdot \bar{a}_t = -\lim \bar{k}_i \cdot \bar{a}_t = -h_v(\tilde{v})$. Since u can be approximated from below the last statement in (5.6) shows that $\mathrm{T}_{\bar{h}_i} \tilde{u} < u$ implies

$$h_u(\mathrm{T}_{\bar{h}_i} \tilde{u}) = h_u(\tilde{u}) + \bar{h}_i \cdot \bar{a}_t < h_u(u) = 0.$$

So

$$\bar{h}_i \cdot \bar{a}_t < h_u(\tilde{u}) = h_v(\tilde{v}),$$

hence

$$h_v(\mathrm{T}_{\bar{h}_i} \tilde{v}) = h_v(\tilde{v}) + \bar{h}_i \cdot \bar{a}_t < 0.$$

Since h_v is nondecreasing this implies $\mathrm{T}_{\bar{h}_i} \tilde{v} < v$. Now v can be approximated from below and

$$\mathrm{T}_{\bar{h}_i} \tilde{v} < v \qquad \text{and} \qquad \lim h_v(\mathrm{T}_{\bar{h}_i} \tilde{v}) = h_v(\tilde{v}) + \lim \bar{h}_i \cdot \bar{a}_t = 0;$$

using the last statement in (5.6) again we see that this implies $\lim \mathrm{T}_{\bar{h}_i} \tilde{v} = v$.

For completeness we recall what happens in the case $t = 1$, i. e. $u \in \mathcal{M}(\bar{a}_1)$, $\bar{\Gamma}_t = \mathbf{Z}^{n+1}$, which was excluded up to now.

If α is rational we have $\mathcal{M}(\bar{a}_1) = \mathcal{M}_\alpha^{\mathrm{per}}$ and the $\mathbf{Z}^{n+1}$-orbit of u is discrete. If α is irrational then the set of accumulation points of the $\mathbf{Z}^{n+1}$-orbit of u is the unique minimal set $\mathcal{M}_\alpha^{\mathrm{rec}}$ of the $\mathbf{Z}^{n+1}$-action T on $\mathcal{M}_\alpha$.

This follows from [3], (5.2) and some simple additional arguments.

As we remarked earlier for generic F one will have $\mathcal{M}(\bar{a}_1) = \mathcal{M}_\alpha^{\mathrm{rec}}$ so that u itself belongs to the minimal set $\mathcal{M}_\alpha^{\mathrm{rec}}$.

6. UNIQUENESS RESULTS

In this section our main goal is to prove that for every admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ the set $\mathcal{M}(\bar{a}_1) \cup \ldots \cup \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered. This can be thought of as a uniqueness result since it tells us that for every

$\bar{x}=(x, x_{n+1})\in\mathbf{R}^{n+1}$ there is at most one

$$u\in\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) \qquad \text{such that } u(x)=x_{n+1}.$$

However, there is a different way to look upon this as a uniqueness result: It shows that for every $u\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ the set $\mathcal{M}_t^{\mathrm{rec}}(u)$ which gives rise to the secondary laminations considered in Sect. 5 does not depend on u and equals the set of $v\in\mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ with $u^-<v<u^+$. The results and the proofs in this section are generalizations of those given in [3]. The existence results in the next section will rely on the uniqueness results obtained here.

We introduce the following abbreviations for functions $u\in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$, $\varphi\in W^{1,2}_{\mathrm{comp}}(\mathbf{R}^n)$ and measurable sets $\Omega\subseteq\mathbf{R}^n$:

$$I(u, \Omega)=\int_\Omega F(x, u, u_x)\,dx$$

provided this integral exists as an extended real number.

$$\Delta(u, \varphi, \Omega)=\int_\Omega (F(x, u+\varphi, u_x+\varphi_x)-F(x, u, u_x))\,dx$$
$$D(u, \Omega)=\sup\{-\Delta(u, \psi, \Omega)\mid \psi\in W^{1,2}_{\mathrm{comp}}(\Omega)\}\geqq 0.$$

So $\Delta(u, \varphi, \Omega)$ is the amount by which the integral for $u+\varphi$ over Ω exceeds the integral for u over Ω. $D(u, \Omega)$ tells us by how much we can reduce the integral for u over Ω by variations of u compactly supported in Ω. In particular, u is minimal in Ω if and only if

$$\Delta(u, \varphi, \Omega)\geqq 0 \qquad \text{for all } \varphi\in W^{1,2}_{\mathrm{comp}}(\Omega)$$

or if and only if $D(u, \Omega)=0$. Obviously $I(u, \Omega)$, $\Delta(u, \varphi, \Omega)$ and $D(u, \Omega)$ are additive in Ω.

In all the proofs in this section the essential point is to show that the graphs of two minimals $u\neq v\in\mathcal{M}_\alpha$ with certain properties cannot intersect. In order to motivate the estimates which we will derive we start with a lemma which represents the final step in most proofs in this section. Roughly, the idea is as follows: If $u\neq v$ are minimal and $u(x_0)=v(x_0)$ then—by the maximum principle (2.2)—$\max(u, v)$ and $\min(u, v)$ are not minimal. We will prove that the amount

$$D(\max(u, v), B(x_0, R)), \qquad D(\min(u, v), B(x_0, R))$$

by which we can reduce the integrals

$$I(\max(u, v), B(x_0, R)), \qquad I(\min(u, v), B(x_0, R))$$

by variations supported in $B(x_0, R)$ has to be smaller than $\text{const.}\cdot\int_{\partial B(x_0, R)}|u-v|\,d\sigma_R$ where $d\sigma_R$ is the volume element of $\partial B(x_0, R)$ induced from the euclidean metric. Hence, in order to show that the assumption $u(x_0)=v(x_0)$ leads to a contradiction we will have to prove that $D(\max(u, v), B(x_0, R))$ and $D(\min(u, v), B(x_0, R))$ grow faster with $R \to \infty$ than $\int_{\partial B(x_0, R)} |u-v|\, d\sigma_R$.

(6.1) LEMMA. — *Suppose $u, v \in \mathcal{M}_\alpha$ and there exists $j \geqq 0$ such that*

$$\liminf_{R \to \infty} (R^{-j} D(\max(u, v), B(0, R))) > 0$$

and

$$\liminf_{R \to \infty} (R^{-j} D(\min(u, v), B(0, R))) > 0.$$

Then we have

$$\liminf_{R \to \infty} \left(R^{-j-1} \int_{B(0, R)} |u-v|(x)\, dx\right) > 0.$$

Proof. — Assume to the contrary that

$$\liminf_{R \to \infty} \left(R^{-j-1} \int_{B(0, R)} |u-v|(x)\, dx\right) = 0.$$

From this we are going to derive that not both u and v can be minimal. Integration in polar coordinates shows that there exists a sequence $R_i \geqq 1$ diverging to ∞ such that

$$\lim_{i \to \infty} \left(R_i^{-j} \int_{\partial B(0, R_i)} |u-v|\, d\sigma_{R_i}\right) = 0.$$

Now we apply [3], Lemma (6.9), to $w_2 = \max(u, v)$ and $w_1 = u$.

Moser's estimates (2.6) and (2.7) show that the hypotheses for this lemma are satisfied. Hence there exists a constant $\tilde{A} = \tilde{A}(F, \alpha)$ and functions $\varphi_i \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ such that $u + \varphi_i \,|\, B(0, R_i) = \max(u, v) \,|\, B(0, R_i)$ and

$$\Delta(u, \varphi_i, \mathbf{R}^n \setminus B(0, R_i)) \leqq \tilde{A} \int_{\partial B(0, R_i)} (\max(u, v) - u)\, d\sigma_{R_i}.$$

Using our assumption and the fact that $\max(u, v) - u \leqq |u-v|$ we obtain for every $\varepsilon > 0$ an integer $i(\varepsilon)$ such that

$$\Delta(u, \varphi_i, \mathbf{R}^n \setminus B(0, R_i)) \leqq \varepsilon R_i^j \tag{6.2}$$

for all $i \geqq i(\varepsilon)$. Similarly we obtain $\bar{\varphi}_i \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ such that

$$v+\bar{\varphi}_i \,|\, B(0, R_i)=\min(u, v) \,|\, B(0, R_i)$$

and

$$\Delta(v, \bar{\varphi}_i, \mathbf{R}^n \setminus B(0, R_i)) \leqq \varepsilon R_i^j \tag{6.3}$$

for all $i \geqq i(\varepsilon)$.

On the other hand, since $u+\varphi_i \,|\, B(0, R_i)=\max(u, v) \,|\, B(0, R_i)$ our first hypothesis implies that there exists $\delta>0$ such that for sufficiently large i there are $\psi_i \in W^{1,2}_{\text{comp}}(B(0, R_i))$ such that

$$I(u+\varphi_i+\psi_i, B(0, R_i))<I(u+\varphi_i, B(0, R_i))-\delta R_i^j. \tag{6.4}$$

Similarly we obtain $\bar{\psi}_i \in W^{1,2}_{\text{comp}}(B(0, R_i))$ such that

$$I(v+\bar{\varphi}_i+\bar{\psi}_i, B(0, R_i))<I(v+\bar{\varphi}_i, B(0, R_i))-\delta R_i^j. \tag{6.5}$$

Now we note that quite generally

$$I(\max(u, v), \Omega)+I(\min(u, v), \Omega)=I(u, \Omega)+I(v, \Omega).$$

Since

$$\begin{aligned} u+\varphi_i \,|\, B(0, R_i)&=\max(u, v) \,|\, B(0, R_i), \\ v+\bar{\varphi}_i \,|\, B(0, R_i)&=\min(u, v) \,|\, B(0, R_i) \end{aligned}$$

we obtain by adding (6.4) and (6.5):

$$\Delta(u, \varphi_i+\psi_i, B(0, R_i))+\Delta(v, \bar{\varphi}_i+\bar{\psi}_i, B(0, R_i))<-2\delta R_i^j$$

Since $(\operatorname{supp}(\psi_i) \cup \operatorname{supp}(\bar{\psi}_i)) \subseteqq B(0, R_i)$ the inequalities (6.2) and (6.3) imply

$$\Delta(u, \varphi_i+\psi_i, \mathbf{R}^n \setminus B(0, R_i))+\Delta(v, \bar{\varphi}_i+\bar{\psi}_i, \mathbf{R}^n \setminus B(0, R_i)) \leqq 2\varepsilon R_i^j$$

for $i \geqq i(\varepsilon)$. Adding these last inequalities we obtain

$$\Delta(u, \varphi_i+\psi_i, \mathbf{R}^n)+\Delta(v, \bar{\varphi}_i+\bar{\psi}_i, \mathbf{R}^n)<0$$

provided $\varepsilon>0$ is chosen smaller than δ and $i \geqq i(\varepsilon)$.

Hence, not both u and v can be minimal which contradicts our hypothesis.

Our first uniqueness result relies on an idea of M. Morse [11] which was rediscovered by Aubry-Le Daeron [1] in their setting.

(6.6) THEOREM. — *Suppose $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ and $t>1$. Then there does not exist $v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ such that $u^-<v<u^+$.*

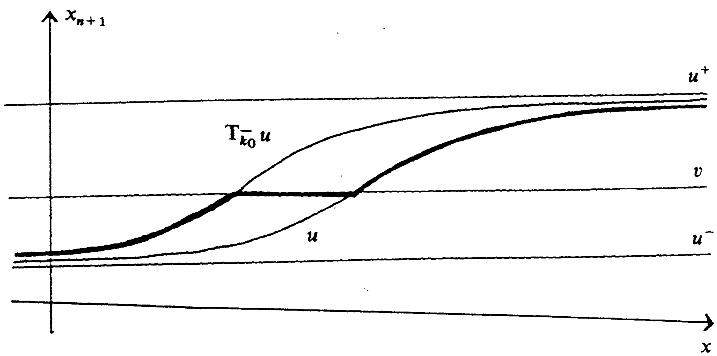
Remark. — If $\mathcal{N}$ is an ordered set we say that two elements $u_1<u_2$ of $\mathcal{N}$ are neighboring elements in $\mathcal{N}$ if there does not exist $u_3 \in \mathcal{N}$ such that $u_1<u_3<u_2$. So (6.6) says that u^- and u^+ are neighbors in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$.

Proof: We assume that there exists $v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ such that $u^- < v < u^+$ and we will prove that this contradicts the minimality of u. We choose $\bar{k}_0 \in \bar{\Gamma}_t$ with $\bar{k}_0 \cdot \bar{a}_t > 0$ and define

$$w = \max(\min(v, T_{\bar{k}_0} u), \max(v, u)),$$

i. e.

$$\begin{aligned} w(x) &= v(x) \quad \text{if} \quad u(x) \leqq v(x) \leqq (T_{\bar{k}_0} u)(x) \\ w(x) &= u(x) \quad \text{if} \quad u(x) \geqq v(x) \\ w(x) &= (T_{\bar{k}_0} u)(x) \quad \text{if} \quad (T_{\bar{k}_0} u)(x) \leqq v(x) \end{aligned}$$



Graph of w (fat line), case $n=1$.

In particular we have $w \geqq u$. The proof is based on the following two estimates where $j := rk(\bar{\Gamma}_t)$:

(6.7) $\displaystyle\int_{B(0,R)} (w-u)\,dx \leqq CR^{j-1}$ for some constant $C>0$.

(6.8) $D(w, B(0, R)) > \delta R^{j-1}$ for some $\delta > 0$ and all $R > R_0$.

Proof of (6.6) *assuming* (6.7) *and* (6.8). — This is a variation of the proof of Lemma (6.1). As before we apply [3], Lemma (6.9), to the functions $w \geqq u$ and obtain a constant $\tilde{A} = \tilde{A}(F, \alpha)$ such that for all $R \geqq 1$ there exist $\varphi_R \in W^{1,2}_{\text{comp}}(\mathbf{R}^n)$ with $u + \varphi_R \mid B(0, R) = w \mid B(0, R)$ and

$$\Delta(u, \varphi_R, \mathbf{R}^n \setminus B(0, R)) \leqq \tilde{A} \int_{\partial B(0, R)} (w-u)\,d\sigma_R.$$

According to (6.7) we can find $R > \max\{R_0, \tilde{A}C\delta^{-1}\}$ such that

$$\int_{\partial B(0, R)} (w-u)\,d\sigma_R \leqq CR^{j-2}.$$

Hence for this $\varphi=\varphi_R$ we have

$$\Delta(u,\ \varphi,\ \mathbf{R}^n\setminus B(0,\ R))\leqq \tilde{A}\, C R^{j-2}.$$

On the other hand

$$u+\varphi\,|\,B(0,\ R)=w\,|\,B(0,\ R) \qquad \text{and} \qquad D(w,\ B(0,\ R))>\delta R^{j-1}$$

imply that there exists $\psi\in W^{1,2}_{\text{comp}}(B(0,\ R))$ such that

$$\Delta(u,\ \varphi+\psi,\ B(0,\ R))<-\delta R^{j-1}.$$

We add the preceding inequalities, use

$$\varphi+\psi\,|\,\mathbf{R}^n\setminus B(0,\ R)=\varphi\,|\,\mathbf{R}^n\setminus B(0,\ R)$$

and obtain

$$\Delta(u,\ \varphi+\psi,\ \mathbf{R}^n)<(\tilde{A}C-\delta R)\,R^{j-2}<0.$$

This contradicts the minimality of u.

Proof of (6.7). — This is a variation of the proof of Lemma (4.5). Set $\Gamma_t^\perp=\{\bar{k}\in\mathbf{Z}^{n+1}\,|\,\bar{k}\cdot\bar{h}=0$ for all $\bar{h}\in\Gamma_t\}$ and $\tilde{\Gamma}=\Gamma_t^\perp+\mathbf{Z}\,\bar{k}_0$.

Then $rk(\tilde{\Gamma})=rk(\Gamma_t^\perp)+1=n+2-j$. The important property of $\tilde{\Gamma}$ is that for every $\bar{k}\in\tilde{\Gamma}\setminus\{0\}$ we have either $T_{\bar{k}}u\geqq w$ or $T_{\bar{k}}w\leqq u$: If $\bar{k}=\bar{k}'+n\bar{k}_0$ and $\bar{k}'\in\Gamma_t^\perp\setminus\{0\}$ then there exists $1\leqq s<t$ such that $\bar{k}'\in\Gamma_s$ and $\bar{k}'\cdot\bar{a}_s\neq 0$, say $\bar{k}'\cdot\bar{a}_s>0$.

Since $\bar{k}_0\in\Gamma_t$ we have $\bar{k}\in\Gamma_s$ and $\bar{k}\cdot\bar{a}_s>0$, hence $T_{\bar{k}}u^-\geqq u^+$ by (4.2) (*b*). Since $u^-<u\leqq w<u^+$ this implies $T_{\bar{k}}u>w$. If $\bar{k}=n\bar{k}_0$ and e. g. $n>0$ then $T_{\bar{k}}u\geqq T_{\bar{k}_0}u\geqq w$.

Hence the set

$$W=\{(x,\ x_{n+1})\,|\,u(x)<x_{n+1}<w(x)\}$$

has one-to-one projection into the cylinder

$$\mathbf{R}^{n+1}/\tilde{\Gamma}\simeq T^{n+2-j}\times\mathbf{R}^{j-1}.$$

Hence there exists $\tilde{C}>0$ such that

$$\operatorname{vol}(W\cap B(\bar{0},\ R))\leqq\tilde{C}\,R^{j-1}$$

Since $u,\ v\in\mathcal{M}_\alpha$ the set W is contained between two hyperplanes with normal $(-\alpha,\ 1)$. So there exists $C\geqq\tilde{C}$ such that

$$\int_{B(0,\ R)}(w-u)\,dx=\operatorname{vol}(W\cap(B(0,\ R)\times\mathbf{R}))\leqq C\,R^{j-1}.$$

Proof of (6.8). — Here we rely on Proposition (4.7). Since u^+-v and $v-u^-$ are positive and periodic with respect to Γ_t there exists $\varepsilon>0$ such that

$$u^+(x)-v(x)\geqq\varepsilon \qquad \text{and} \qquad v(x)-u^-(x)\geqq\varepsilon$$

for all $x \in V_t = \operatorname{span}(\Gamma_t) \subseteq \mathbf{R}^n$. Recall from (4.7) that there exists $C>0$ such that $u^+(x) - u(x) < \varepsilon$ if $x \cdot b_t < -C$ and $u(x) - u^-(x) < \varepsilon$ if $x \cdot b_t > C$.

Hence $u(x) > v(x)$ if $x \in V_t$ and $x \cdot b_t < -C$ while $u(x) < v(x)$ if $x \in V_t$ and $x \cdot b_t > C$. In particular, every line in V_t with direction b_t intersects the set $\{x \in V_t \mid u(x) = v(x)\}$.

Finally we will prove that there exist $\varepsilon > 0$, $r > 0$ such that

$$(6.9) \qquad D(w, B(x, r)) \geqq \varepsilon$$

for every $x \in V_t$ with $u(x) = v(x)$. This will complete our proof since the preceding remark shows that the number of disjoint balls $B(x, r) \subseteq B(0, R)$ with $x \in V_t$ and $u(x) = v(x)$, grows like R^{j-1} since $\dim V_t - 1 = j - 1$. To prove (6.9) we argue by contradiction and assume that there exist sequences $x_i \in V_t$, $r_i \to \infty$ such that

$$u(x_i) = v(x_i) \quad \text{and} \quad \lim D(w, B(x_i, r_i)) = 0.$$

Note that $u(x_i) = v(x_i)$ implies

$$(T_{\bar{k}_0} u)(x_i + k_0) = (T_{\bar{k}_0} v)(x_i + k_0) = v(x_i + k_0).$$

Choose $\bar{k}_i \in \bar{\Gamma}_t$ such that $y_i = x_i + k_i$ are bounded for all $i \in \mathbf{N}$.

We may assume that $\lim y_i = y$ and $\lim T_{\bar{k}_i} u = \tilde{u}$ exist.

Since $u(x_i) = v(x_i)$ we have $|x_i \cdot b_t| \leqq C$ so that $k_i \cdot b_t = \bar{k}_i \cdot \bar{a}_t$ is bounded. According to (4.2) (*c*) this implies that $\tilde{u} \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$, in particular $T_{\bar{k}_0} \tilde{u} > \tilde{u}$. Using $T_{\bar{k}_i} v = v$ we easily see that

$$\tilde{w} := \lim T_{\bar{k}_i} w = \max(\min(v, T_{\bar{k}_0} u), \max(v, \tilde{u})).$$

Now we use our assumption $\lim D(w, B(x_i, r_i)) = 0$ which—according to [3], Lemma (6.5) — implies $D(\tilde{w}, \mathbf{R}^n) = 0$, i. e. $\tilde{w}$ is minimal.

On the other hand we have

$$\tilde{u} \leqq \tilde{w} \leqq T_{\bar{k}_0} \tilde{u} \quad \text{and} \quad \tilde{u}(y) = v(y) = \tilde{w}(y),$$
$$(T_{\bar{k}_0} \tilde{u})(y + k_0) = v(y + k_0) = \tilde{w}(y + k_0).$$

Hence the maximum principle (2.2) implies $\tilde{u} = \tilde{w} = T_{\bar{k}_0} \tilde{u}$ which contradicts $\tilde{u} < T_{\bar{k}_0} \tilde{u}$.

This completes the proof of (6.9).

The following obvious consequence of (6.6) will be useful in the inductive proof that $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered.

(6.10) COROLLARY. — *Suppose* $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ *is totally ordered. If* $u \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *and* $v \in \mathscr{M}(\bar{a}_1, \ldots, \bar{a}_{t-1}, \bar{a}_t')$ *and if there exists* $x_0 \in \mathbf{R}^n$ *with*

$$u^-(x_0) \leqq v(x_0) \leqq u^+(x_0) \quad \textit{then} \quad u^- = v^- \quad \textit{and} \quad u^+ = v^+.$$

The first step in the proof that $\mathscr{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered is to show that $\mathscr{M}(\bar{a}_1)$ is totally ordered. As we mentioned earlier we have $\mathscr{M}(\bar{a}_1) = \mathscr{M}_\alpha^{\text{per}}$ if $\alpha \in \mathbf{Q}^n$ is the rotation vector corresponding to $\bar{a}_1$. In this case $\mathcal{M}(\bar{a}_1)=\mathcal{M}_\alpha^{\text{per}}$ is totally ordered according to [12], Theorem (5.2). Moreover, for $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ and generic F we will have $\mathcal{M}(\bar{a}_1)=\mathcal{M}_\alpha^{\text{rec}}$ which is totally ordered by [3], Theorem (5.1). Here we treat the remaining case.

(6.11) LEMMA. — *For all $\bar{a}_1 \in S^n$ with $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$ the set $\mathcal{M}(\bar{a}_1)$ is totally ordered.*

Proof. — We assume that $u \neq v \in \mathcal{M}(\bar{a}_1)$, $u(x_0)=v(x_0)$ for some $x_0 \in \mathbf{R}^n$ and that $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$. By the maximum principle (2.2) $\max(u, v)$ and $\min(u, v)$ are not minimal. Hence there exist $\varepsilon > 0$, $r > 0$ such that

$$D(\max(u, v), B(x_0, r)) \geqq \varepsilon \quad \text{and} \quad D(\min(u, v), \ B(x_0, r)) \geqq \varepsilon.$$

Since

$$T_{\bar{k}}(\max(u, v)) = \max(u, v), \qquad T_{\bar{k}}(\min(u, v)) = \min(u, v)$$

for all $\bar{k} \in \bar{\Gamma}_2$ we obtain with $j = rk(\bar{\Gamma}_2)$:

$$\liminf_{R \to \infty} (R^{-j} D(\max(u, v), B(0, R))) > 0$$

and

$$\liminf_{R \to \infty} (R^{-j} D(\min(u, v), B(0, R))) > 0.$$

So Lemma (6.1) implies:

$$\liminf_{R \to \infty} \left(R^{-j-1} \int_{B(0, R)} |u-v|(x)\, dx \right) > 0. \tag{6.12}$$

On the other hand we note that for every element $w \in \mathcal{M}_\alpha$ the set $\{w\} \cup \mathcal{M}_\alpha^{\text{rec}}$ is totally ordered since $\mathcal{M}_\alpha^{\text{rec}}$ is contained in the closure of the $\mathbf{Z}^{n+1}$-orbit of w, *cf.* [3], Corollary (5.2). In particular we have $u \notin \mathcal{M}_\alpha^{\text{rec}}$, $v \notin \mathcal{M}_\alpha^{\text{rec}}$.

Set $u_1 = \inf\{w \in \mathcal{M}_\alpha^{\text{rec}} \mid w > u\}$, $u_2 = \sup\{w \in \mathcal{M}_\alpha^{\text{rec}} \mid w < u\}$.

Then $u_1 < u < u_2$ and $T_{\bar{k}} u_1 \geqq u_2$ for all $\bar{k} \in \mathbf{Z}^{n+1}$ with $\bar{k} \cdot \bar{a}_1 > 0$.

Since $u_1, u_2 \in \mathcal{M}_\alpha^{\text{rec}}$ and $u_1(x_0) < v(x_0) < u_2(x_0)$ we have $u_1 < v < u_2$ as well. From (4.5) we obtain

$$\int_{\pi_2^{-1}(E)} (u_2 - u_1)(x)\, dx \leqq 1$$

where $\pi_2 : \mathbf{R}^n \to V_2 = \operatorname{span}(\Gamma_2)$ is the orthogonal projection and E is a measurable fundamental domain for the action of Γ_2 on V_2. Since $u_1 < u < u_2$ and $u_1 < v < u_2$ we conclude that there exists $C > 0$ such that

$$\int_{B(0, R)} |u - v|\, dx \leqq C R^j$$

where $j = \dim V_2 = rk(\bar{\Gamma}_2)$. This contradicts (6.12).

Now we state the central theorem of this section:

(6.13) THEOREM. — *For every admissible system* $(\bar{a}_1, \ldots, \bar{a}_t)$ *the set* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *is totally ordered.*

We will prove (6.13) by induction on t the case $t=1$ being treated in (6.11). Our first step is a result which is an analogue of [3], Theorem (5.1). Its claim is nontrivial only if $rk(\bar{\Gamma}_{t+1})<rk(\bar{\Gamma}_t)-1$. We will use the results (5.3)-(5.7) on $\mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$.

(6.14) LEMMA. — *Suppose* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ *is totally ordered and* $t>1$. *Then* $\mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ *is totally ordered.*

Proof. — We argue by contradiction and assume that there exist $u \neq v \in \mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ such that $u(x_0)=v(x_0)$ for some $x_0 \in \mathbf{R}^n$. We will show that this contradicts the minimality of u or v. We may assume that both u and v can be approximated from below: According to Lemma (5.3) there exist $u_i, v_i \in \mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ such that $u=\lim u_i$, $v=\lim v_i$ and such that u_i and v_i can be approximated from below; now the maximum principle (2.2) implies that for sufficiently large $i \in \mathbf{N}$ we have $u_i \neq v_i$ and there exists $x_i \in \mathbf{R}^n$ with $u_i(x_i)=v_i(x_i)$.

We set $j=rk(\bar{\Gamma}_t)$. We will prove the following estimates

$$(6.15)\qquad \begin{aligned} &\liminf_{R\to\infty} (R^{-j+1} D(\max(u, v), B(0, R)))>0 \\ &\liminf_{R\to\infty} (R^{-j+1} D(\min(u, v), B(0, R)))>0 \end{aligned}$$

and

$$(6.16)\qquad \lim_{R\to\infty}\left(R^{-j}\int_{B(0,R)} |u-v|\,dx\right)=0.$$

Since (6.15) and (6.16) contradict Lemma (6.1) this will complete the proof of (6.14).

Proof of (6.15). — We only consider $\max(u, v)$; $\min(u, v)$ can be treated similarly. Since $W:=V_t \cap \langle b_t \rangle^{\perp}$ has dimension $j-1$ it suffices to prove that there exist $r>0$, $\varepsilon>0$ such that

$$(6.17)\qquad D(\max(u, v), B(x, r)) \geqq \varepsilon$$

for all $x \in W$. Assume to the contrary that there exist sequences $x_i \in W$, $r_i \to \infty$ such that

$$\lim D(\max(u, v), B(x_i, r_i))=0.$$

Choose $\bar{k}_i \in \bar{\Gamma}_t$ such that the sequence $y_i=x_i+k_i$ is bounded. We may assume that $\lim T_{\bar{k}_i} u=\tilde{u}$ and $\lim T_{\bar{k}_i} v=\tilde{v}$ exist. Since

$$D(\max(u, v), B(x_i, r_i))=D(\max(T_{\bar{k}_i} u, T_{\bar{k}_i} v), B(y_i, r_i))$$

and the y_i remain bounded and $\lim r_i=\infty$ we can use [3], Lemma (6.5), to conclude that $\max(\tilde{u}, \tilde{v})=\lim(\max(T_{\bar{k}_i} u, T_{\bar{k}_i} v))$ is minimal. Hence the maximum principle (2.2) implies $\tilde{u}<\tilde{v}$ or $\tilde{u}=\tilde{v}$ or $\tilde{u}>\tilde{v}$. On the other hand the sequence $\bar{k}_i \cdot \bar{a}_t=k_i \cdot b_t=y_i \cdot b_t$ is bounded so that $\tilde{u}, \tilde{v} \in \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ by Lemma (5.4). Since $\mathcal{M}_t(\tilde{u}) \subseteqq \mathcal{M}_t(u)$ and since u can be approximated from below we can apply Lemma (5.3) and obtain a sequence $\bar{h}_i \in \bar{\Gamma}_t$ such that $T_{\bar{h}_i} \tilde{u}<u$ and $\lim T_{\bar{h}_i} \tilde{u}=u$. But now Corollary (5.7) implies that $\lim T_{\bar{h}_i} \tilde{v}=v$. Since $\tilde{u}>\tilde{v}$ or $\tilde{u}=\tilde{v}$ or $\tilde{u}<\tilde{v}$ we obtain $u>v$ or $u=v$ or $u<v$ and this contradicts our assumption on u and v.

Proof of (6.16). — Since our claim depends neither on the norm defining $B(0, R)$ nor on the normalization of dx we may assume that Γ_t equals $\mathbf{Z}^j \times \{0\} \subseteqq \mathbf{Z}^n$ and $V_t=\mathbf{R}^j \times \{0\}$.

We denote $x=(y, z) \in \mathbf{R}^n$ where $y \in \mathbf{R}^j$, $z \in \mathbf{R}^{n-j}$. From Corollary (6.10) we know that $u^-=v^-$ and $u^+=v^+$, hence $|u-v|<u^+-u^-$.

The idea for the proof is as follows: Applying (4.5) to u^- and u^+ we reduce our claim to the case where we only integrate over those $(y, z) \in B(0, R)$ with $|z| \leqq R_0$ for some $R_0>0$ not depending on R. Hence $R^{-j} \int_{B(0, R)} |u-v| \, dx$ is bounded above. In order to prove that the limit for $R \rightarrow \infty$ is zero we use the convergence of u and v to u^- and u^+ expressed in Proposition (4.7).

Lemma (4.5) implies

$$\int_{E \times \mathbf{R}^{n-j}} (u^+-u^-) \, dx \leqq 1$$

where $E=[0, 1)^j$. Hence, for every $\varepsilon>0$ there exists $R_0>0$ such that

$$\int_{|y|<R} \left(\int_{|z|>R_0} (u^+-u^-)(y, z) \, dz \right) dy \leqq \varepsilon (R+1)^j.$$

To see this we choose R_0 such that

$$\int_{y \in E} \left(\int_{|z|>R_0} (u^+-u^-)(y, z) \, dz \right) dy \leqq \varepsilon 2^{-j}$$

and observe that $(u^+-u^-)(y, z)$ is $\mathbf{Z}^j$-periodic in y and that the ball $\{y \in \mathbf{R}^j \mid |y|<R\}$ can be covered by $(2[R]+2)^j$ copies of E. Hence it remains to prove that

$$\lim_{R \rightarrow \infty} \left(R^{-j} \int_{|y|<R} \left(\int_{|z|<R_0} |u-v|(y, z) \, dz \right) dy \right)=0 \tag{6.18}$$

for every $R_0>0$. So let $R_0>0$, $\delta>0$ be given and set $\eta=(\omega_j\omega_{n-j}R_0^{n-j})^{-1}\delta$ where ω_i denotes the volume of the i-dimensional unit ball.

From Proposition (4.7) we know that we can find $C>0$ such that for all $x\in\mathbf{R}^n$ with $|x\cdot b_t|>C$ we have $|u-v|(x)<\eta$.

So, if $x=(y,z)$ and $|z|\leqq R_0$ and $|(y,0)\cdot b_t|>C+R_0|b_t|$ then $|u-v|(x)<\eta$. Hence we can estimate the integral

$$\int_{|y|\leqq R}\left(\int_{|z|\leqq R_0}|u-v|(y,z)\,dz\right)dy$$

by the sum of the integral of $|u-v|$ over the set

$$\{x=(y,z)\mid\ |y|<R \text{ and } |(y,0)\cdot b_t|<C+R_0|b_t|,\ |z|<R_0\}$$

and the integral

$$\int_{|y|<R}\left(\int_{|z|<R_0}\eta\,dz\right)dy=\eta\omega_j R^j\omega_{n-j}R_0^{n-j}=\delta R^j.$$

Since $|u-v|$ is bounded the first integral can be estimated above by DR^{j-1} where $D>0$ depends on C, R_0, $|b_t|$. Hence we obtain for all $R>0$:

$$\int_{|y|<R}\left(\int_{|z|<R_0}|u-v|(y,z)\,dz\right)dy\leqq DR^{j-1}+\delta R^j.$$

This proves (6.18) and thus completes the proofs of (6.16) and (6.14).

Our next step in the proof that $\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_t)$ is totally ordered is to show:

(6.19) LEMMA. *If $\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_{t-1})$ is totally ordered and $rk(\bar{\Gamma}_{t+1})=rk(\bar{\Gamma}_t)-1$ then $\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_t)$ is totally ordered.*

Proof. — Contrary to our claim we assume that there exist $u\neq v\in\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_t)$ such that $u(x_0)=v(x_0)$ for some $x_0\in\mathbf{R}^n$. Since $u-v$ is Γ_{t+1}-periodic we obtain

$$\liminf_{R\to\infty}(R^{-j+1}D(\max(u,v),B(0,R)))>0$$

and

$$\liminf_{R\to\infty}(R^{-j+1}D(\min(u,v),B(0,R)))>0$$

where $j=rk(\bar{\Gamma}_t)=rk(\bar{\Gamma}_{t+1})+1$. On the other hand Lemma (6.10) implies $u^-=v^-$, $u^+=v^+$, hence $|u-v|<u^+-u^-$. According to Lemma (4.5) we

have

$$\int_{\pi_t^{-1}(E)} (u^+ - u^-)(x)\,dx \leqq 1$$

where $\pi_t: \mathbf{R}^n \to V_t$ is the orthogonal projection and E a fundamental domain for V_t/Γ_t. Recall that $\dim(V_t) = rk(\bar{\Gamma}_t) = j$.

Proposition (4.7) applies to u and v so that for every $\varepsilon > 0$ there exists $C > 0$ such that $|u - v|(x) < \varepsilon$ if $|x \cdot b_t| > C$. Hence the same arguments used to prove (6.16) allow us to conclude

$$\lim_{R \to \infty} \left(R^{-j} \int_{B(0,\,R)} |u - v|(x)\,dx \right) = 0.$$

This contradicts Lemma (6.1) and completes the proof of (6.19).

The following lemma is the final step in the inductive proof of Theorem (6.13):

(6.20) LEMMA. — *If* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ *is totally ordered and* $rk(\bar{\Gamma}_{t+1}) < rk(\bar{\Gamma}_t) - 1$ *then* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *is totally ordered.*

Proof. — Again we assume that there exist $u \neq v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ such that $u(x_0) = v(x_0)$ for some $x_0 \in \mathbf{R}^n$. So $u^- = v^-$, $u^+ = v^+$ by Lemma (6.10). Lemma (6.14) says that $\mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered. According to Lemma (5.3) this implies that

$$\mathcal{M}_t^{\mathrm{rec}}(u) = \mathcal{M}_t^{\mathrm{rec}}(v) = \{ w \mid w \in \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t), u^- < w < u^+ \}.$$

In particular $\{u, v\} \cap \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t) = \varnothing$ and

$$\{u\} \cup \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t), \qquad \{v\} \cup \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$$

are totally ordered. We set

$$u_1 = \sup \{ w \in \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t) \mid w < u \}$$
$$u_2 = \inf \{ w \in \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t) \mid w > u \}.$$

Then $u_1 < u < u_2$ and $T_{\bar{k}} u_1 \geqq u_2$ for all $\bar{k} \in \bar{\Gamma}_t$ with $\bar{k} \cdot \bar{a}_t > 0$.

Moreover $u(x_0) = v(x_0)$ implies $u_1 < v < u_2$ since $\{v\} \cup \mathcal{M}^{\mathrm{rec}}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered.

Now we use Lemma (4.5) to conclude

$$\limsup_{R \to \infty} \left(R^{-j} \int_{B(0,\,R)} (u_2 - u_1)(x)\,dx \right) < \infty$$

where $j = rk(\bar{\Gamma}_{t+1})$. On the other hand the $\bar{\Gamma}_{t+1}$-periodicity of $u - v$ implies

$$\liminf_{R \to \infty} (R^{-j} D(\max(u, v), B(0, R))) > 0$$

and

$$\liminf_{R \to \infty} (R^{-j} D(\min(u, v), B(0, R))) > 0.$$

Since $|u-v| < u_2 - u_1$ these inequalities contradict Lemma (6.1).

This proves Lemma (6.20) and completes the proof of Theorem (6.13).

As a byproduct of the proof of Lemma (6.20) we obtain the following final form of Lemma (5.4):

(6.21) COROLLARY. — *Suppose* $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *and*

$$rk(\bar{\Gamma}_{t+1}) < rk(\bar{\Gamma}_t) - 1.$$

Then the set $\{v \in \mathcal{M}^{\text{rec}}(\bar{a}_1, \ldots, \bar{a}_t) \mid u^- < v < u^+\}$ *is the unique minimal set of the action of* $\bar{\Gamma}_t$ *on* $\{v \in \mathcal{M}(a_1, \ldots, a_t) \mid u^- < v < u^+\}$.

This corollary is a generalization of [3], Corollary (5.2).

It says that a secondary lamination between neighboring elements $u^- < u^+$ in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ is uniquely determined by $\bar{a}_t$.

(6.22) THEOREM. — *Suppose* $(\bar{a}_1, \ldots, \bar{a}_t)$ *is an admissible system. Then* $\mathcal{M}(\bar{a}_1) \cup \mathcal{M}(\bar{a}_1, \bar{a}_2) \cup \ldots \cup \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *is a totally ordered, closed, T-invariant set of minimal solutions.*

Proof. — Closedness follows from Corollary (3.12) while T-invariance is trivial. Iterated application of (4.2), (6.6) and (6.13) proves that $\mathcal{M}(\bar{a}_1) \cup \ldots \cup \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered.

Finally we make a few remarks on the converse of (6.22):

Under what conditions do different minimal solutions

$$u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t), \qquad v \in \mathcal{M}(\bar{a}'_1, \ldots, \bar{a}'_\tau)$$

coincide at some point?

According to Theorem (6.22) a necessary condition is that for some smallest $1 \leqq s \leqq \min(t, \tau)$ we have $\bar{a}_s \neq \bar{a}'_s$. If $s=1$, i. e. $\bar{a}_1 \neq \bar{a}'_1$, let $\alpha \neq \alpha'$ denote the corresponding rotation vectors. In this case (2.6) implies that the set $S(u, v) = \{x \in \mathbf{R}^n \mid u(x) = v(x)\}$ separates two halfspaces

$$\{x \in \mathbf{R}^n \mid x \cdot (\alpha - \alpha') < C_1\} \qquad \text{and} \qquad \{x \in \mathbf{R}^n \mid x \cdot (\alpha - \alpha') > C_2\}$$

where $C_2 > C_1$. If $s > 1$ the graphs of u and v may be disjoint:

If $\text{graph}(u) \cap \text{graph}(v) \neq \varnothing$ then $\text{graph}(u) \cap \text{graph}(T_{\bar{k}} v) = \varnothing$ for all $\bar{k} \in \mathbf{Z}^{n+1}$ with $\bar{k} \cdot \bar{a}_1 \neq 0$. In the case $s > 1$ a necessary and sufficient condition is that $u_{s-1}^- = v_{s-1}^-$ where $u_{s-1}^-, v_{s-1}^- \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{s-1})$ are defined at the end of Sect. 4 by applying the operation $u \to u^-$ an appropriate number of times.

Without this condition it is not even generally true that u and some translate $T_{\bar{k}} v$ coincide at some point: For non-generic integrands F it can happen that

$$u_{s-1}^{-} < u_{s-1}^{+} \leqq v_{s-1}^{-} < v_{s-1}^{+}$$

and that for all $\bar{k} \notin \bar{\Gamma}_s(\bar{a}_1, \ldots, \bar{a}_{s-1})$ we have either $T_{\bar{k}} u_{s-1}^{-} \geqq v_{s-1}^{+}$ or $T_{\bar{k}} v_{s-1}^{+} \leqq u_{s-1}^{-}$. The proofs for these statements follow from Lemma (4.9) and Theorem (6.6); they are left to the reader.

7. EXISTENCE RESULTS

The uniqueness results of the preceding section are meaningful only if there are objects to which they apply.

The purpose of this section is to discuss the existence of minimal solutions in the sets $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. We recall Moser's basic existence results: In our notation Corollary (5.5) in [12] states that $\mathcal{M}(\bar{a}_1) \neq \emptyset$ if the rotation vector α corresponding to $\bar{a}_1$ is rational. Theorem (5.6) in [12] proves that $\mathcal{M}_\alpha \neq \emptyset$ for all $\alpha \in \mathbf{R}^n$. From Moser's compactness theorem (2.8) one can easily deduce that the $\mathbf{Z}^{n+1}$-action T on $\mathcal{M}_\alpha$ has a minimal set. According to [3], Lemma (4.6) such a minimal set is contained in $\mathcal{M}(\bar{a}_1)$, in particular $\mathcal{M}(\bar{a}_1) \neq \emptyset$ also if $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$. So Moser's results show that $\mathcal{M}(\bar{a}_1) \neq \emptyset$ for all admissible $\bar{a}_1$, i. e. for all $\bar{a}_1 \in S^n$ with $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$.

It should be underlined that we do not have to use any hard analysis in order to obtain new minimal solutions. We simply use the compactness property (2.8) together with the $\mathbf{Z}^{n+1}$-action and the uniqueness results from Sect. 6. This is a well-known and useful technique for this type of problem.

The following theorem is the main result of this section. We recall that $u_1 < u_2$ are neighboring elements in

$$\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) \text{ if } u_1, u_2 \in \mathcal{M}(\bar{a}_1, \ldots, a_t)$$

and there does not exist $u_3 \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ with $u_1 < u_3 < u_2$.

(7.1) THEOREM. — *Let $(\bar{a}_1, \ldots, \bar{a}_t)$ be an admissible system with $t \geqq 2$. If $u_1 < u_2$ are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ there exists $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ such that $u_1 < u < u_2$ and, consequently, $u^- = u_1$, $u^+ = u_2$.*

We will say that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ has gaps if there exists a pair of neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. So Theorem (7.1) shows that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) \neq \emptyset$ if $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ has gaps.

Conversely if $u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ then $u^- < u^+$ are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ according to Theorem (6.6), i. e. $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ has gaps.

Historical remark: In the case $n=1$ Theorem (7.1) says:
If $u_1<u_2$ are neighboring elements in

$$\mathcal{M}(\bar{a}_1)=\mathcal{M}_\alpha^{\text{per}} \quad \text{with} \quad \alpha\in\mathbf{Q},\ \bar{a}_1=(1+|\alpha|^2)^{-1/2}(-\alpha,1)$$

and if $\bar{a}_2=(1+|\alpha|^2)^{-1/2}(1,\alpha)$ there exists $u\in\mathcal{M}(\bar{a}_1,\bar{a}_2)$ with $u^-=u_1$, $u^+=u_2$. Since $b_2=\sqrt{1+\alpha^2}$ Proposition (4.7) implies $\lim_{x\to\infty}(u-u_1)(x)=0$, $\lim_{x\to-\infty}(u_2-u)(x)=0$.

If one looks at the situation on the torus $T^2=\mathbf{R}^2/\mathbf{Z}^2$ then $x\to(x,u_1(x))$, $x\to(x,u_2(x))$ parametrize homologous simple closed curves $\mathscr{C}_1$ and $\mathscr{C}_2$ (generically we will have $\mathscr{C}_1=\mathscr{C}_2$) and $x\to(x,u(x))$ is a curve without selfintersections which is α-asymptotic to $\mathscr{C}_2$, ω-asymptotic to $\mathscr{C}_1$ and which does not intersect $\mathscr{C}_1\cup\mathscr{C}_2$. If we replace $\bar{a}_2$ by $-\bar{a}_2$ we obtain $v\in\mathcal{M}(\bar{a}_1,-\bar{a}_2)$ with $v^-=u_1$, $v^+=u_2$ and hence

$$\lim_{x\to\infty}(u_2-v)(x)=0=\lim_{x\to-\infty}(v-u_1)(x).$$

This case $n=1$ has been treated in [6], Theorem 3.5.

In the (parametric) case of geodesics on surfaces such pairs of heteroclinic (resp. homoclinic if $\mathscr{C}_1=\mathscr{C}_2$) minimal geodesics connecting freely homotopic minimal closed geodesics were first found by M. Morse [11]. Analogous results were obtained by Aubry-Le Daeron [1], Appendix 6, for a model in solid state physics. *See* [2] for the relation between these topics.

Proof of (7.1). — We argue by contradiction and assume that t is the smallest integer ≥ 2 such that there exist an admissible system $(\bar{a}_1,\ldots,\bar{a}_t)$ and neighboring elements $u_1<u_2$ in $\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_{t-1})$ for which our claim does not hold.

For $\varepsilon>0$ we set

$$\bar{a}(\varepsilon)=(1+\varepsilon^2)^{-1/2}(\bar{a}_{t-1}+\varepsilon\bar{a}_t).$$

Since

$$\bar{a}_{t-1},\ \bar{a}_t\in\operatorname{span}(\mathbf{Z}^{n+1}\cap\langle\bar{a}_1,\ldots,\bar{a}_{t-2}\rangle^\perp)$$

the system $(\bar{a}_1,\ldots,\bar{a}_{t-2},\bar{a}(\varepsilon))$ is admissible. In the case $t=2$ we have to take ε smaller than some ε_0 so that $\bar{a}(\varepsilon)\cdot\bar{e}_{n+1}>0$.

The crucial step is to prove that there exists $\delta>0$ with the following property:

(7.2) For all $\varepsilon\in(0,\varepsilon_0)$ there exist $v_\varepsilon\in\mathcal{M}(\bar{a}_1,\ldots,\bar{a}_{t-2},\bar{a}(\varepsilon))$ and x_ε in a bounded subset of $\mathbf{R}^n$ such that

$$u_1(x_\varepsilon)+\delta<v_\varepsilon(x_\varepsilon)<u_2(x_\varepsilon)-\delta.$$

Proof of (7.2) *in the case* $t=2$. — Choose $\tilde{v}_\varepsilon \in \mathcal{M}(\bar{a}_\varepsilon)$ and $\bar{k}=(k, k_{n+1}) \in \bar{\Gamma}_2$ such that $\bar{k}\cdot\bar{a}_2>0$. We will find a translate $v_\varepsilon = \mathrm{T}_{n_\varepsilon \bar{k}} \tilde{v}_\varepsilon$ where $n_\varepsilon \in \mathbf{Z}$ and some $x_\varepsilon \in \mathbf{R}^n$ with $|x_\varepsilon| \leqq |k|$ such that (7.2) is satisfied.

We choose $\delta>0$ such that $2\delta<(u_2-u_1)(tk)$ for all $t\in\mathbf{R}$.

This is possible since $t \to (u_2-u_1)(tk)$ is positive and periodic.

We have $\bar{k}\cdot\bar{a}_\varepsilon = \varepsilon(1+\varepsilon^2)^{-1/2}\bar{k}\cdot\bar{a}_2>0$ and $\bar{k}\cdot\bar{a}_1=0$.

Since the graphs of u_1 and u_2 lie within finite distance from a hyperplane with normal $\bar{a}_1$ and since graph$(\tilde{v}_\varepsilon)$ lies within finite distance from a hyperplane with normal $\bar{a}_\varepsilon$ we see that

$$\lim_{t\to\pm\infty}\frac{u_1(tk)}{t}=\lim_{t\to\pm\infty}\frac{u_2(tk)}{t}\neq\lim_{t\to\pm\infty}\frac{\tilde{v}_\varepsilon(tk)}{t}.$$

Hence there exists $t_\varepsilon\in\mathbf{R}$ such that

$$u_1(t_\varepsilon k)+\delta<\tilde{v}_\varepsilon(t_\varepsilon k)<u_2(t_\varepsilon k)-\delta.$$

If we set $n_\varepsilon=-[t_\varepsilon]\in\mathbf{Z}$ and $x_\varepsilon=(t_\varepsilon+n_\varepsilon)k$ we obtain $|x_\varepsilon|\leqq|k|$. Now $\mathrm{T}_{\bar{k}}u_1=u_1$ and $\mathrm{T}_{\bar{k}}u_2=u_2$ imply

$$u_1(x_\varepsilon)+\delta<(\mathrm{T}_{n_\varepsilon\bar{k}}\tilde{v}_\varepsilon)(x_\varepsilon)<u_2(x_\varepsilon)-\delta.$$

Hence $v_\varepsilon := \mathrm{T}_{n_\varepsilon\bar{k}}\tilde{v}_\varepsilon$ and x_ε have the properties required in (7.2).

Proof of (7.2) *in the case* $t>2$. — This follows along the same lines. Since $u_1<u_2$ are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ we have $u_1^-=u_2^-$ and $u_1^+=u_2^+$, *cf.* (4.2) (*a*). In particular, u_1^- and u_2^+ are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-2})$.

By the choice of t there exists $\tilde{v}_\varepsilon\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-2}, \bar{a}_\varepsilon)$ with $(\tilde{v}_\varepsilon)^-=u_1^-$ and $(\tilde{v}_\varepsilon)^+=u_2^+$. Choose

$$\bar{k}\in\bar{\Gamma}_t=\mathbf{Z}^{n+1}\cap\langle\bar{a}_1, \ldots, \bar{a}_{t-1}\rangle^\perp \quad \text{with} \quad \bar{k}\cdot\bar{a}_t>0 \quad \text{and} \quad \delta>0$$

such that $2\delta<(u_2-u_1)(tk)$ for all $t\in\mathbf{R}$. Then we have $\bar{k}\cdot\bar{a}_\varepsilon>0$ and, consequently,

$$\lim_{n\to-\infty}\mathrm{T}_{n\bar{k}}\tilde{v}_\varepsilon=(\tilde{v}_\varepsilon)^-<u_1 \quad\text{and}\quad \lim_{n\to\infty}\mathrm{T}_{n\bar{k}}\tilde{v}_\varepsilon>u_2.$$

Hence there exists $m_\varepsilon\in\mathbf{N}$ such that

$$\tilde{v}_\varepsilon(nk)>u_2(nk) \quad \text{for} \quad n<-m_\varepsilon$$

and

$$\tilde{v}_\varepsilon(nk)<u_1(nk) \quad \text{for} \quad n>m_\varepsilon.$$

By continuity we obtain $t_\varepsilon\in\mathbf{R}$ such that

$$u_1(t_\varepsilon k)+\delta<\tilde{v}_\varepsilon(t_\varepsilon k)<u_2(t_\varepsilon k)-\delta.$$

As before we set $n_\varepsilon=-[t_\varepsilon]\in\mathbf{Z}$. Then $v_\varepsilon=\mathrm{T}_{n_\varepsilon\bar{k}}\tilde{v}_\varepsilon$ and $x_\varepsilon=(t_\varepsilon+n_\varepsilon)k$ have the properties required in (7.2).

Finally we show that (7.2) implies our claim. Using (2.8) we see that there is a sequence $\varepsilon_i>0$ with $\lim \varepsilon_i=0$ such that for $v_i=v_{\varepsilon_i}$ and $x_i=x_{\varepsilon_i}$ we have limits $\lim_{i\to\infty} v_i=v$, $\lim_{i\to\infty} x_i=x_0$. Then (7.2) implies

$$u_1(x_0)<v(x_0)<u_2(x_0).$$

So—according to Theorem (6.6)—all we have to do is to show that $v\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. If $t>2$ we have

$$t(v)\geqq t-1 \qquad \text{and} \qquad \bar{a}_1(v)=\bar{a}_1, \ldots, \bar{a}_{t-2}(v)=\bar{a}_{t-2}$$

since

$$u_1^-<v<u_2^+ \qquad \text{and} \qquad u_1^- \text{ and } u_2^+$$

are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-2})$, *cf.* (4.2) (*b*) and (6.6). Moreover, if

$$\bar{k}\in\bar{\Gamma}_{t-1} \qquad \text{and} \qquad \bar{k}\cdot\bar{a}_{t-1}>0$$

then $\bar{k}\cdot\bar{a}(\varepsilon)>0$ for small enough $\varepsilon>0$.

Hence we have $T_{\bar{k}}v_i>v_i$ for almost all $i\in\mathbf{N}$ and consequently $T_{\bar{k}}v\geqq v$. According to Lemma (3.11) this implies $\bar{a}_{t-1}(v)=\bar{a}_{t-1}$.

If $t=2$ we have $\bar{a}_1(v)=\lim_{\varepsilon\to 0}\bar{a}(\varepsilon)=\bar{a}_1$ since $\bar{a}_1(u)$ depends on u continuously, *cf.* Lemma (3.10). Next we show that $t(v)\geqq t$ and $\bar{a}_t(v)=\bar{a}_t$: Since $u_1(x_0)<v(x_0)<u_2(x_0)$ and u_1, u_2 are neighboring elements we cannot have $v\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$, hence $t(v)\geqq t$. If $\bar{k}\in\bar{\Gamma}_t$ and $\bar{k}\cdot\bar{a}_t>0$ then $\bar{k}\cdot\bar{a}(\varepsilon)>0$ for all $\varepsilon\in(0, \varepsilon_0)$. As above this implies $\bar{a}_t(v)=\bar{a}_t$. Finally $T_{\bar{k}}v_i=v_i$ for all $\bar{k}\in\bar{\Gamma}_{t+1}$ and $i\in\mathbf{N}$ implies $T_{\bar{k}}v=v$ for all $\bar{k}\in\bar{\Gamma}_{t+1}$, hence $t(v)=t$ and $v\in\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. This completes the proof of Theorem (7.1).

The following corollary is not as obvious as it may sound.

(7.3) COROLLARY. — *Suppose* $(\bar{a}_1, \ldots, \bar{a}_t)$ *is admissible. The graphs of functions in* $\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ foliate $\mathbf{R}^{n+1}$ *if and only if* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *does not have gaps.*

Proof. — Suppose first that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ does not have gaps. Since $\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is totally ordered we only have to show that for every $x_0\in\mathbf{R}^n$ the image Im(H) of the map

$$H:\ \mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)\to\mathbf{R}, \qquad H(u)=u(x_0)$$

is all of $\mathbf{R}$. Since $\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is closed so is Im(H). If $\mathrm{Im}(H)\neq\mathbf{R}$ we can find $u(x_0)<v(x_0)$ in Im(H) such that $(u(x_0), v(x_0))\cap\mathrm{Im}(H)=\varnothing$.

This implies

$$\bar{\Gamma}^+(u)=\bar{\Gamma}^+(v) \qquad \text{where} \quad \bar{\Gamma}^+(w)=\{\bar{k}\in\mathbf{Z}^{n+1}\,|\,T_{\bar{k}}w\geqq w\}:$$

If $T_{\bar{k}} u < u$ while $T_{\bar{k}} v \geqq v$ then $u < T_{-\bar{k}} u < T_{-\bar{k}} v \leqq v$ so that

$$(T_{-\bar{k}} u)(x_0) \in (u(x_0), v(x_0)) \cap \operatorname{Im}(H).$$

Now $\bar{\Gamma}^+(u) = \bar{\Gamma}^+(v)$ shows that u and v have the same invariants. Hence there exists $1 \leqq s \leqq t$ such that u and v are neighboring elements in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_s)$. For $s = t$ this contradicts our hypothesis while for $s < t$ Theorem (7.1) provides $w \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{s+1})$ with $w(x_0) \in (u(x_0), v(x_0))$ which contradicts the choice of u and v.

The converse is almost obvious: Note that for neighboring elements $u_1 < u_2 \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ we can have $v \in \mathcal{M}$ with $u_1 < v < u_2$ only if $t(v) > t$.

With the help of (7.3) we can state Theorem (7.1) in the following final form:

(7.4) COROLLARY. — *Let $(\bar{a}_1, \ldots, \bar{a}_t)$ be admissible. Then $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) = \varnothing$ if and only if there exists $1 \leqq s < t$ such that $\mathcal{M}(\bar{a}_1) \cup \ldots \cup \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_s)$ gives rise to a foliation. Put differently we have $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) \neq \varnothing$ if and only if $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_s)$ has gaps for all $1 \leqq s < t$.*

One can call an admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ complete if $\mathbf{Z}^{n+1} \cap \langle \bar{a}_1, \ldots, \bar{a}_t \rangle^{\perp} = \{0\}$. Using Corollary (4.8) it is easy to show the following: If an admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$ is complete then $\mathcal{M}(\bar{a}_1) \cup \ldots \cup \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ is a totally ordered T-invariant closed subset of $\mathcal{M}$ which is maximal with these properties.

Finally we briefly discuss the interesting question which of the various possibilities for foliations and laminations with gaps can actually occur. The simplest case is that F only depends on p. Then

$$\mathcal{M} = \{ u(x) = \alpha \cdot x + u_0 \mid \alpha \in \mathbf{R}^n,\ u_0 \in \mathbf{R} \}$$

so that $\mathcal{M}(\bar{a}_1)$ defines an affine foliation for all admissible $\bar{a}_1$. This was proved by Moser [12], Theorem (2.3), and it also follows easily from (6.13), *cf.* the proof of (7.7) below.

According to J. Moser [12], Theorem (8.1), such affine minimal foliations are stable under small perturbations of the integrand F if the rotation vector α satisfies certain Diophantine conditions. On the other hand, a sufficiently large perturbation of the Dirichlet integrand $F(\bar{x}, p) = \frac{1}{2} |p|^2$ can destroy all $\mathbf{Z}^{n+1}$-invariant minimal foliations with rotation vector α satisfying $|\alpha| \leqq A$ for some preassigned A. This was proved with some labor in [4]. Using this result we will prove:

(7.5) THEOREM. — *Let $\{0\} \neq \bar{\Gamma} \subseteq \mathbf{Z}^{n+1}$ be a subgroup with $\bar{\Gamma} \cap \mathbf{Z} . \bar{e}_{n+1} = \{0\}$. For every $\delta > 0$ there exist integrands F satisfying (F_1)-(F_4) such that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ has gaps if $\bar{a}_1 \cdot \bar{e}_{n+1} > \delta$ and if*

$\bar{\Gamma}_{t+1}:=(\mathbf{Z}^{n+1}\cap\langle\bar{a}_1, \ldots, \bar{a}_t\rangle^{\perp})\supseteqq\bar{\Gamma}$ *while* $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ *does not have gaps if* $\bar{\Gamma}_{t+1}$ *does not contain* $\bar{\Gamma}$.

Remark. — In particular, if $(\bar{a}_1, \ldots, \bar{a}_t)$ is admissible we can choose $\bar{\Gamma}=\mathbf{Z}^{n+1}\cap\langle\bar{a}_1, \ldots, \bar{a}_{t-1}\rangle^{\perp}$. Then we obtain integrands F such that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ has gaps while $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ does not have gaps. Hence $\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ gives rise to a foliation which is not conjugate to a foliation by affine functions. The integrands F we construct are very special since they do not depend on $\bar{x}$ in some direction. In particular, $\mathcal{M}(\bar{a}_1)\cup\ldots\cup\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ defines a foliation for every complete admissible system $(\bar{a}_1, \ldots, \bar{a}_t)$. However, if we just want to retain the property that for some fixed $(\bar{a}_1, \ldots, \bar{a}_t)$ there are gaps in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ but not in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$ we can arbitrarily perturb F outside some set which is compact in $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}\times\mathbf{R}^n$ and thereby destroy most of these foliations.

Before we start with the proof of (7.5) we present the type of integrands F that we will use and we prove a lemma about minimal solutions of such F. Let us first assume that $\bar{\Gamma}\subseteqq\mathbf{Z}^n\times\{0\}$, say $\operatorname{span}(\bar{\Gamma})=\{0\}\times\mathbf{R}^{n-j}\times\{0\}$. Then a corresponding F can be easily described: We denote

$$x=(y, z)\in\mathbf{R}^j\times\mathbf{R}^{n-j} \qquad \text{and} \qquad p=(q, r)\in\mathbf{R}^j\times\mathbf{R}^{n-j}.$$

We choose an integrand G: $\mathbf{R}^j\times\mathbf{R}\times\mathbf{R}^j\to\mathbf{R}$ according to [4] (satisfying (F_1)-(F_4) with j replacing n) and set

$$(7.6) \qquad F(x, u, p)=G(y, u, q)+|r|^2 \qquad (x=(y, z), p=(q, r))$$

We can take G to be of the type $G(\bar{y}, q)=(1+\lambda f(\bar{y}))|q|^2$ where λ is a large constant and $f:\mathbf{R}^{j+1}\to[0, 1]$ is $\mathbf{Z}^{j+1}$-periodic with $f(1/2, \ldots, 1/2)=1$ and $f\equiv 0$ in a neighborhood of the boundary of the cube $[0, 1]^{j+1}$.

(7.7) LEMMA. — *Suppose* F *and* G *are related by* (7.6) *and* $\bar{\Gamma}\subseteqq\mathbf{Z}^n\times\{0\}$ *generates* $\{0\}\times\mathbf{R}^{n-j}\times\{0\}$. *If* $u\in\mathcal{M}(F)$ *and* $T_{\bar{k}}u=u$ *for all* $\bar{k}\in\bar{\Gamma}$ *then there exists* $\tilde{u}\in\mathcal{M}(G)$ *such that* $u(y, z)=\tilde{u}(y)$.

Proof. — Suppose $u\in\mathcal{M}(F)$ and $(k, 0)\in\bar{\Gamma}$. It follows easily from (7.6) that $u_s(x)=u(x-sk)$ is in $\mathcal{M}(F)$ for all $s\in\mathbf{R}$.

Moreover u_s has the same invariants as u. Hence (6.13) implies that for every s either $u_s=u$ or $u_s>u$ or $u_s<u$.

If $s=\frac{p}{q}\in\mathbf{Q}$ and $u_s\neq u$, say $u_s>u$, then $u_p>u_{(q-1)s}>\ldots>u$ which contradicts $T_{p\bar{k}}u=u$. Hence $u_s=u$ for all $s\in\mathbf{Q}$ and by continuity this implies

$u(x)=u(x-sk)$ for all $s\in\mathbf{R}$. Since the set of all $k\in\mathbf{Z}^n$ with $\bar{k}\in\bar{\Gamma}$ generates $\{0\}\times\mathbf{R}^{n-j}$ we obtain $u(y, z)=\tilde{u}(y)$.

Finally we have to show that $\tilde{u}\in\mathcal{M}(G)$. Obviously $\tilde{u}$ does not have selfintersections. For the minimality of $\tilde{u}$ we have to prove

$$I_G(\tilde{u}+\tilde{\varphi}, \operatorname{supp}(\tilde{\varphi}))-I_G(\tilde{u}, \operatorname{supp}(\tilde{\varphi}))=:\eta\geqq 0$$

for every $\tilde{\varphi}\in W^{1,2}_{\mathrm{comp}}(\mathbf{R}^j)$. Extend $\tilde{\varphi}$ to $\varphi_R\in W^{1,2}_{\mathrm{comp}}(\mathbf{R}^n)$ by

$$\varphi_R(y, z)=\lambda_R(z)\,\varphi(y)$$

where

$$\lambda_R(z)=\begin{cases} 1 & \text{if } |z|\leqq R-1 \\ R-|z| & \text{if } R-1<|z|<R \\ 0 & \text{if } |z|\geqq R. \end{cases}$$

A simple estimate shows that

$$I_F(u+\varphi_R, \operatorname{supp}(\varphi_R))-I_F(u, \operatorname{supp}(\varphi_R))\leqq\eta\omega_{n-j}R^{n-j}+C(R^{n-j-1}+1)$$

where $C>0$ does not depend on R and ω_{n-j} is the volume of the $(n-j)$-dimensional unit ball. If $\eta<0$ then for sufficiently large $R>0$ this inequality contradicts the minimality of u. This proves that $\tilde{u}$ is minimal.

Proof of (7.5). — Suppose first that $\operatorname{span}(\bar{\Gamma})=\{0\}\times\mathbf{R}^{n-j}\times\{0\}$.

We define F by (7.6) where G is so chosen that there are no $\mathbf{Z}^{j+1}$-invariant G-minimal foliations of $\mathbf{R}^{j+1}$ with rotation vector $\tilde{\alpha}$ satisfying $|\tilde{\alpha}|^2\leqq\delta^{-1}$. Now Lemma (7.7) shows that $\mathcal{M}_F(\bar{a}_1)\cup\ldots\cup\mathcal{M}_F(\bar{a}_1, \ldots, \bar{a}_t)$ cannot define a foliation if $\bar{\Gamma}_{t+1}(\bar{a}_1, \ldots, \bar{a}_t)\supseteqq\bar{\Gamma}$ and $\bar{a}_1\cdot\bar{e}_{n+1}>\delta$. Hence $\mathcal{M}_F(\bar{a}_1, \ldots, \bar{a}_t)$ has gaps, *cf.* Corollary (7.3). Conversely suppose that

$$\bar{\Gamma}\nsubseteq\bar{\Gamma}_{t+1}(\bar{a}_1, \ldots, \bar{a}_t) \qquad\text{and}\qquad u\in\mathcal{M}_F(\bar{a}_1. \ldots, \bar{a}_t).$$

Then there exists $\bar{k}=(k, 0)\in\bar{\Gamma}\setminus\bar{\Gamma}_{t+1}$. As in (7.7) we see that

$$u_s\in\mathcal{M}_F(\bar{a}_1, \ldots, \bar{a}_t) \qquad\text{where}\quad u_s(x)=u(x-sk)$$

and that $s\to u_s$ is monotonic. Since $T_{\bar{k}}u\neq u$ we even have strict monotonicity. Hence $\mathcal{M}_F(\bar{a}_1, \ldots, \bar{a}_t)$ does not contain pairs of neighboring elements, i.e. $\mathcal{M}_F(\bar{a}_1, \ldots, \bar{a}_t)$ does not have gaps.

Finally we have to remove the condition that

$$\operatorname{span}(\bar{\Gamma})=\{0\}\times\mathbf{R}^{n-j}\times\{0\}.$$

Now, if

$$\bar{\Gamma}\subseteqq\mathbf{R}^n\times\{0\} \qquad\text{and}\qquad rk(\bar{\Gamma})=n-j$$

then this condition is satisfied after changing coordinates in $\mathbf{R}^n$ by a suitable element of SL $(n, \mathbf{Z})$.

If $\bar{\Gamma} \not\subseteq \mathbf{R}^n \times \{0\}$ we can write

$$\operatorname{span}(\bar{\Gamma}) = \{(x, \alpha \cdot x) \mid x \in \operatorname{span}(\Gamma) \subseteqq \mathbf{R}^n\}$$

for a uniquely determined rational vector $\alpha \in \operatorname{span}(\Gamma)$, say $l\alpha \in \mathbf{Z}^n$ for some integer $l > 0$. Now we choose an integrand F which solves our problem for the subgroup $\Gamma \times \{0\} \subseteqq \mathbf{Z}^n \times \{0\}$.

We may assume that F has period l^{-1} in u. Then

$$\tilde{F}(x, u, p) = F(x, u - \alpha \cdot x, p - \alpha)$$

has the required properties for the group $\bar{\Gamma}$.

8. OPEN PROBLEMS

We present two open problems which are closely related to our results. The first question is in the spirit of Moser's Theorem (8.1) from [12] which extends KAM-Theory to the context considered here: a minimal foliation

$$\mathcal{M}_\alpha = \mathcal{M}_\alpha(F_0) = \{u \mid u(x) = \alpha \cdot x + u_0,\ u_0 \in \mathbf{R}\}$$

is stable (up to conjugation) under small perturbations of the integrand F_0 if its rotation vector α satisfies certain Diophantine inequalities. Now Theorem (7.5) provides examples of integrands F with secondary foliations: There exist admissible systems $(\bar{a}_1, \ldots, \bar{a}_t)$ such that $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ has gaps which are filled by foliations in $\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t)$. If $rk(\bar{\Gamma}_t) \geqq 2$ then $\bar{a}_t$ can determine an irrational direction in $\operatorname{span}(\bar{\Gamma}_t)$ and one can ask for an analogue to Moser's theorem in this case:

Is there a stability result for secondary foliations?

To motivate our second problem we return to the Dirichlet integrand $F_0(\bar{x}, p) = \frac{1}{2}|p|^2$ whose minimals are the harmonic functions. In the 1-dimensional case $n=1$ every harmonic function is affine. This generalizes to arbitrary $\mathbf{Z}^2$-periodic integrands: In the case $n=1$ every minimal solution does not have selfintersections. For $n>1$, however, not every harmonic function is affine. Accordingly, if one generalizes the Dirichlet integrand F_0 to $\mathbf{Z}^{n+1}$-periodic integrands F the natural class of functions corresponding to the affine functions is not characterized by F-minimality alone, one has to impose an additional condition. The topological condition "no selfintersections" used by Moser [12] fulfills this purpose and it is particularly natural in the context of foliations on a torus. Liouville's

Theorem on the growth of harmonic functions, however, would lead one to different and more analytic conditions: One might assume that the minimal solutions u satisfy $\sup|u(x)-\alpha\cdot x|<\infty$ for some $\alpha\in\mathbf{R}^n$ or, even weaker, $\sup|u_x|<\infty$. According to (2.6) and (2.7) every minimal solution without selfintersections has these properties. So the question is if one can deduce the property "no selfintersections" from these conditions (and minimality).

We present two partial results in this direction. Here minimality is always meant with respect to some integrand F satisfying (F_1)-(F_4).

(8.1) THEOREM. — *Suppose u is minimal and* $\sup|u_x|<\infty$. *Then there exists a sequence $\bar{k}_i\in\mathbf{Z}^{n+1}$ such that $T_{\bar{k}_i}u$ converges to a minimal solution without self-intersections.*

Remark. — We will also prove: If u is an element of a minimal set of the $\mathbf{Z}^{n+1}$-action on the set of all minimal solutions then u does not have selfintersections.

Proof. — As a consequence of Theorem (6.5) in Chapter 4 of Ladyzhenskaya/Ural'tseva's book [8] we have: If u_i is a sequence of minimal solutions such that $\sup|(u_i)_x|$ and $|u_i(0)|$ are bounded then u_i contains a subsequence which converges to a minimal solution in the C^1-topology on compact sets. This compactness property implies the existence of a minimal set of the $\mathbf{Z}^{n+1}$-action on the closure of the orbit $\{T_{\bar{k}}u\,|\,\bar{k}\in\mathbf{Z}^{n+1}\}$.

So it suffices to show that every element v of such a minimal set does not have selfintersections. Assume to the contrary that $v(x_0)=(T_{\bar{k}}v)(x_0)$ for some $\bar{k}\in\mathbf{Z}^{n+1}$, $x_0\in\mathbf{R}^{n+1}$ and that $v\neq T_{\bar{k}}v$. Since v belongs to a minimal set we obtain:

If w is a limit of $\mathbf{Z}^{n+1}$-translates of v then $w\neq T_{\bar{k}}w$ and there exists

$$y_0\in\mathbf{R}^n \qquad \text{with} \quad w(y_0)=(T_{\bar{k}}w)(y_0).$$

This implies:

There exist $\varepsilon>0$, $r>0$ such that for all $x\in\mathbf{R}^n$:

$$\begin{aligned} &D(\max(v,\ T_{\bar{k}}v),\ B(x,r))\geqq\varepsilon \\ &D(\min(v,\ T_{\bar{k}}v,\ B(x,r))\geqq\varepsilon \end{aligned} \tag{8.2}$$

On the other hand we have

(8.3) v and $T_{\bar{k}}v$ are uniformly Lipschitz; in particular $\sup|v-T_{\bar{k}}v|<\infty$.

The arguments used in the proof of Lemma (6.1) show that (8.2) and (8.3) contradict the minimality of v.

If we assume the stronger hypothesis that $|u(x)-\alpha\cdot x|$ is bounded for some $\alpha\in\mathbf{R}^n$ we obtain:

(8.4) THEOREM. — *Suppose u is minimal, $|u(x)-\alpha\cdot x|$ is bounded and $\bar{\alpha}=(-\alpha, 1)$ is rationally independent. Then u does not have selfintersections, i. e. $u\in\mathcal{M}_\alpha$.*

Proof. — As in the proof of [12], Theorem (3.1), one can use Theorem (5.2) in [8], Chapter 4 to conclude that u is uniformly Lipschitz. Assume that there exists $\bar{k}\in\mathbf{Z}^{n+1}$ and $x_0\in\mathbf{R}^n$ such that $u(x_0)=(\mathrm{T}_{\bar{k}}u)(x_0)$ while $u\neq\mathrm{T}_{\bar{k}}u$. Since $\bar{\alpha}$ is rationally independent we may assume that $\bar{k}\cdot\bar{\alpha}>0$.

The maximum principle (2.2) implies that the set $\mathrm{W}=\{x\,|\,(\mathrm{T}_{\bar{k}}u)(x)<u(x)\}\subseteq\mathbf{R}^n$ is not empty. We will prove the following two statements:

(8.5) There exist $\varepsilon>0$, $r>0$ such that for all $x\in\mathrm{W}$:

$$\mathrm{D}(\max(u, \mathrm{T}_{\bar{k}}u), \mathrm{B}(x, r))\geqq\varepsilon \quad \text{and} \quad \mathrm{D}(\min(u, \mathrm{T}_{\bar{k}}u), \mathrm{B}(x, r))\geqq\varepsilon.$$

(8.6) Let $\mathrm{N}(\mathrm{R})$ denote the maximal number of points in $\mathrm{W}\cap\mathrm{B}(0, \mathrm{R})$ with pairwise distance $\geqq 2r$. For every $\mathrm{C}>0$ there exists $\mathrm{R}\geqq\mathrm{C}$ such that

$$\mathrm{N}(\mathrm{R})\geqq\mathrm{C}\int_{\partial\mathrm{B}(0, \mathrm{R})\cap\mathrm{W}}(u-\mathrm{T}_{\bar{k}}u)\,d\sigma_{\mathrm{R}}.$$

Note that

$$\int_{\partial\mathrm{B}(0, \mathrm{R})\cap\mathrm{W}}(u-\mathrm{T}_{\bar{k}}u)\,d\sigma_{\mathrm{R}}=\int_{\partial\mathrm{B}(0, \mathrm{R})}(\max(u, \mathrm{T}_{\bar{k}}u)-\mathrm{T}_{\bar{k}}u)\,d\sigma_{\mathrm{R}}$$
$$=\int_{\partial\mathrm{B}(0, \mathrm{R})}|\min(u, \mathrm{T}_{\bar{k}}u)-u|\,d\sigma_{\mathrm{R}}.$$

Using the uniform Lipschitz continuity of u and $\mathrm{T}_{\bar{k}}u$ we can argue as in Lemma (6.1) and show that (8.5) and (8.6) contradict the minimality of u.

Proof of (8.5). — Assume to the contrary that there exists a sequence $x_i\in\mathrm{W}$ such that

$$\lim_{i\to\infty}\mathrm{D}(\max(u, \mathrm{T}_{\bar{k}}u), \mathrm{B}(x_i, i))=0.$$

Choose $\bar{k}_i\in\mathbf{Z}^{n+1}$ such that $y_i=x_i+k_i\in[0, 1]^n$ and such that $(\mathrm{T}_{\bar{k}_i}u)(0)$ is bounded. As a consequence of Theorem (6.5) in [8], Chapter 4 a subsequence of $\mathrm{T}_{\bar{k}_i}u$ converges to a minimal solution v. As in the proof of (6.15) we conclude from

$$\lim_{i\to\infty}\mathrm{D}(\max(u, \mathrm{T}_{\bar{k}}u), \mathrm{B}(x_i, i))=0 \quad \text{that } \max(v, \mathrm{T}_{\bar{k}}v)$$

is minimal.

For every limit point y of the sequence $y_i \in [0, 1]^n$ we have $(T_{\bar{k}} v)(y) \leqq v(y)$. Hence the maximum principle implies $T_{\bar{k}} v \leqq v$. On the other hand v is a limit of translates of u so that $|v(x) - \alpha \cdot x|$ is bounded. Since $\bar{k} \cdot \bar{\alpha} > 0$ this implies that $T_{m\bar{k}} v > v$ for sufficiently large $m \in \mathbf{N}$. This contradicts $T_{\bar{k}} v \leqq v$ and concludes the proof of (8.5).

Proof of (8.6). — First of all we have the trivial estimate

$$N(R) \geqq (\mathrm{vol}_n(B(0, 4r)))^{-1} \cdot \mathrm{vol}_n(W \cap B(0, R)) \tag{8.7}$$

where vol_n denotes n-dimensional Lebesgue measure. On the other hand integration in polar coordinates gives:

$$\mathrm{vol}_n(W \cap B(0, R)) = \int_0^R \sigma_\rho(W \cap \partial B(0, \rho))\, d\rho$$

where σ_ρ is the measure on the sphere $\partial B(0, \rho)$ induced from the euclidean structure of $\mathbf{R}^n$. Since $\mathrm{vol}_n(W \cap B(0, R))$ does not grow exponentially in R the quotient

$$\mathrm{vol}_n(W \cap B(0, R)) \cdot (\sigma_R(W \cap \partial B(0, R)))^{-1} \tag{8.8}$$

cannot remain bounded for $R \to \infty$. Since $|u - T_{\bar{k}} u|$ is bounded the inequalities (8.7) and (8.8) imply (8.6).

In view of Theorem (8.4) one might be inclined to believe that it is only a small step to prove that u minimal and $\sup |u(x) - \alpha \cdot x| < \infty$ imply $u \in \mathcal{M}_\alpha$ for an arbitrary $\alpha \in \mathbf{R}^n$. However, as we saw in the preceding sections the case "$\bar{\alpha}$ rationally dependent" may be much more complicated than the case "$\bar{\alpha}$ rationally independent".

So we ask:

Do there exist minimal solutions u with selfintersections such that $|u(x) - \alpha \cdot x|$ is bounded for some $\alpha \in \mathbf{R}^n$?

ACKNOWLEDGEMENT

This paper was completed while the author enjoyed the hospitality of the Nonlinear Systems Laboratory at the Mathematics Institute of the University of Warwick.

REFERENCES

[1] S. Aubry and P. Y. Le Daeron, The discrete Frenkel-Kontorova Model and Its Extensions I. Exact Results for the Ground States, *Physica*, 8 D, 1983, pp. 381-422.

[2] V. Bangert, Mather Sets for Twist Maps and Geodesics on Tori, Dynamics Reported, Vol. 1, U. Kirchgraber and H. O. Walther éds., pp. 1-56. Stuttgart-Chichester, B. G. Teubner-John Wiley, 1988.

[3] V. BANGERT, A Uniqueness Theorem for $\mathbf{Z}^n$-periodic Variational Problems, *Comment. Math. Helv.*, Vol. 62, 1987, pp. 511-531.
[4] V. BANGERT, The Existence of Gaps in Minimal Foliations. *Aequationes Math.*, Vol. 34, 1987, pp. 153-166.
[5] G. D. BIRKHOFF, *Dynamical Systems*, Amer. Math. Soc. Colloq. Publ., Vol. IX, New York, *Amer. Math. Soc.*, 1927.
[6] J. DENZLER, Mather Sets for Plane Hamiltonian Systems, *Z. Angew. Math. Phys.* (*ZAMP*), Vol. 38, 1987, pp. 791-812.
[7] G. A. HEDLUND, Geodesics on a Two-dimensional Riemannian Manifold with Periodic Coefficients, *Ann. of Math.*, Vol. 33, 1932, pp. 719-739.
[8] O. A. LADYZHENSKAYA and N. N. URAL'TSEVA, *Linear and Quasilinear Elliptic Equations*, New York-London, Academic Press, 1968.
[9] J. N. MATHER, Existence of quasi-periodic Orbits for Twist Homeomorphisms of the Annulus, *Topology*, Vol. 21, 1982, pp. 457-467.
[10] J. N. MATHER, More Denjoy Minimal Sets for Area Preserving Diffeomorphisms, *Comment. Math. Helv.*, Vol. 60, 1985, pp. 508-557.
[11] M. MORSE, A Fundamental Class of Geodesics on Any Closed Surface of Genus Greater than One, *Trans. Amer. Math. Soc.*, Vol. 26, 1924, pp. 25-60.
[12] J. MOSER, Minimal Solutions of Variational Problems on a Torus, *Ann. Inst. Henri-Poincaré (Analyse non linéaire)*, Vol. 3, 1986, pp. 229-272.
[13] J. MOSER, A Stability Theorem for Minimal Foliations on a Torus, *Ergod. Th. Dynam. Sys.*, Vol. 8, 1988, pp. 251-281.
[14] W. P. THURSTON, Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry, *Bull. (N.S.) Amer. Math. Soc.*, Vol. 6, 1982, pp. 357-381.

(*Manuscrit reçu le* 14 *mars* 1988.)