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A new approach for the analysis of Vortex Methods in two and three dimensions

by

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ABSTRACT. — A new approach for the analysis of Vortex Methods is described. This analysis which is based on the notion of weak solution for the convection deformation vorticity form of the Euler equations leads to convergence results in the case of Vortex In Cell methods as well as for three dimensional Vortex Methods.

Key words : Vortex methods, error estimates, linear first order hyperbolic equations, weak solutions.

RÉSUMÉ. — On écrit une nouvelle approche permettant une analyse simplifiée des méthodes de vortex. Cette analyse utilise la notion de solution faible des équations de convection déformation du tourbillon et aboutit à des résultats de convergence en particulier dans le cas des

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méthodes de Vortex In Cell ainsi que pour des méthodes de vortex tridimensionnelles.

I. INTRODUCTION

This paper is devoted to a new mathematical analysis of Vortex Methods. In fact these methods, which are commonly implemented for the simulation of incompressible flows at high Reynolds numbers, have been given a large amount of theoretical work, since the first work of O. Hald [11]. The most recent contributions deal with some important extensions of these methods in two and three dimensions, such as Filament Vortex Methods ([9], [17]), Vortex In Cell Methods [6], viscous flows ([7], [8], [14]) among many others (*see* also [1]).

The main purpose of this paper is to provide a very simple understanding of the convergence of these methods with a wide range of applications. Let us briefly describe the idea on which this work is based. Consider for instance the three-dimensional Euler equations in velocity-vorticity formulation:

$$\frac{\partial \omega}{\partial t} + \nabla . (u \otimes \omega) - (\omega . \nabla) u = 0 \qquad \text{(E. 1)}$$

$$\omega(.,0) = \omega_0$$

$$\operatorname{curl} u = \omega$$

$$\operatorname{div} u = 0 \qquad \text{(E. 2)}$$

$$|u| \to 0 \quad \text{at infinity.}$$

A very simple way to define a particle approximation of this problem is to consider an approximation of the initial condition ω_0 by a linear combination of Dirac measures $\omega_0 \simeq \sum \alpha_j \delta(x - x_j)$, and then to define the evolution of the positions and the weights of the particles in a way that is consistent with (E. 1). Beale and Majda [4] first suggested modifying the weights by methods using finite-difference approximations of the stretching term $(\omega . \nabla) u$. Later Greengard [9] improved this method by observing that only derivatives of the velocity along the vorticity are needed and proposed an analysis of a Filament Vortex type method.

In fact it is possible to define a way to compute the functions $\alpha_j(t)$ by using directly the interactions of particles. Such methods have been implemented since 1977 by Rehbach. The mathematical basis of this approach is that, given a differentiable velocity field, there exists an explicit measure solution of (E. 1) when ω_0 is a Dirac mass; it seems that this property was clearly written for the first time in [16].

Starting from this remark, since the method is based on the explicit solution of the equation satisfied by the vorticity, it seems natural to look for estimates of the vorticity rather than the velocity. Therefore we have to work in distribution spaces whose choice is made to:

1. give back a satisfactory control of the velocity in order to ensure stability in the nonlinear terms;

2. express properties of optimal accuracy for the approximation of continuous functions by Dirac measures.

These properties are actually easily proved to be shared by $\mathbf{W}^{-m, p}$ type spaces; in particular the first property is then related to the Calderon's theorem, while the second one is related to quadrature estimates. Furthermore it turns out that this analysis is largely independant of the precise way the velocity is computed from the vorticity. This means that, for instance, from this point of view Vortex In Cell methods are naturally connected to methods which use calculation of the velocity based on integral methods.

Finally we emphasize that, since the essential tool that is required for the analysis is that the approximate vorticity must be an explicit solution of the original equation, methods like Contour Dynamics Methods and Filament Vortex Methods are obviously good candidates for this analysis. Let us point out that this analysis seems also promising when dealing with boundary terms or time discretization.

An outline of this paper is as follows. In section 2 we give some very simple preliminary results concerning distributions and stability properties of advection equations in distribution spaces. In section 3, we analyze the standard two-dimensional Vortex Method and in section 4 we give a proof of the convergence of Vortex In Cell methods that is largely independent of the author's previous work [6]. Finally in section 5 we focus on the three-dimensional grid free point Vortex method. For a different analysis of this method we refer to [2].

II. PRELIMINARY RESULTS

In all this section $n=2$ or 3. We introduce, for $m \geqq 0$ and $p \in [1, +\infty]$, the Sobolev spaces

$$W^{m,p}(\mathbb{R}^n) = \{f, \partial^\alpha f \in L^p(\mathbb{R}^n), |\alpha| \leqq m\}$$

and, for $p \in]1, +\infty[$, $W^{-m,p}(\mathbb{R}^n)$ the dual of $W^{m,p^*}(\mathbb{R}^n)$, with $1/p + 1/p^* = 1$. We denote by $\|.\|_{m,p}$ and $\|.\|_{-m,p}$ the corresponding norms, and by $|.|_{m,p}$ the usual semi-norm of $W^{m,p}(\mathbb{R}^n)$:

$$\|f\|_{m,p} = (\sum_{0 \leqq |\alpha| \leqq m} \|\partial^\alpha f\|_{0,p}^p)^{1/p}$$

$$|f|_{m,p} = (\sum_{|\alpha| = m} \|\partial^\alpha f\|_{0,p}^p)^{1/p}.$$

In addition $\vartheta_{m,p}$ will be the space of all distributions $T \in W^{-m,p}(\mathbb{R}^n)$ such that there exists a constant C independent of φ in $W^{m,p^*}(\mathbb{R}^n)$ satisfying:

$$|\langle T, \varphi \rangle| \leqq C(\sum_{1 \leqq |\alpha| \leqq m} \|\partial^\alpha \varphi\|_{0,p^*}^{p^*})^{1/p^*}.$$

For such T we also set

$$[T]_{-m,p} = \sup \{ |\langle T, \varphi \rangle| / (\sum_{1 \leqq |\alpha| \leqq m} \|\partial^\alpha \varphi\|_{0,p^*}^{p^*})^{1/p^*}; \varphi \in W^{m,p^*}(\mathbb{R}^n) \}.$$

Observe that $[T]_{-m,p} \geqq \|T\|_{-m,p}$ if $T \in \vartheta_{m,p}$.

Let us prove the following result:

LEMMA 2.1. – *Let r, m be two integers with $r \leqq m$; let (p_k) be a family of integers whose conjugates are denoted by p_k^*. Assume that the distribution* **T** *satisfies:*

$$|\langle T, \varphi \rangle| \leqq \sum_{r \leqq k \leqq m} C_k |\varphi|_{k, p_k^*}, \qquad C_k > 0, \tag{2.1}$$

for all φ C^∞ with compact support.

Then we can write:

$$T = \sum_{r \leqq |\alpha| \leqq m} \partial^\alpha g_\alpha$$

with

$$\| g_\alpha \|_{0, p_k} \leqq C_k, \qquad r \leqq |\alpha| = k \leqq m.$$

Proof. — For the sake of simplicity we only give the proof corresponding to the case $p_k \equiv p$; the general case follows with minor modifications. We classically begin by viewing **T** as a linear continuous form on the subspace $\mathbf{H} \subset (\mathbf{L}^{p^*}(\mathbb{R}^n))^N$ made by the n-uples

$$(\partial^\alpha \varphi)_{r \leqq |\alpha| \leqq m}, \qquad \varphi \in \mathbf{W}^{m, p^*}(\mathbb{R}^n).$$

We then provide $(\mathbf{L}^{p^*}(\mathbb{R}^n))^N$ with the following norm:

$$(f_\alpha)_{r \leqq |\alpha| \leqq m} \to \sum_{r \leqq |\alpha| \leqq m} C_\alpha \| f_\alpha \|_{0, p^*}.$$

By (2.1), we know that the norm of **T** is less than 1 on **H**. So that we may extend **T** on the whole space $(\mathbf{L}^{p^*})^N$, with a norm $\leqq 1$. Let $\underline{\mathbf{T}}$ the resulting linear functional and $(g_\alpha)_{r \leqq |\alpha| \leqq m} \in (\mathbf{L}^p(\mathbb{R}^n))^N$ such that

$$\langle \underline{\mathbf{T}}, f \rangle = \sum_{r \leqq |\alpha| \leqq m} \int f_\alpha g_\alpha \, dx.$$

We first get classically

$$\mathbf{T} = \sum_{r \leqq |\alpha| \leqq m} \partial^\alpha g_\alpha.$$

Since $\| \underline{\mathbf{T}} \| \leqq 1$ our choice of the norm in $(\mathbf{L}^p(\mathbb{R}^n))^N$ leads to:

$$\sum_{r \leqq |\alpha| \leqq m} \int f_\alpha g_\alpha \, dx \leqq \sum_{r \leqq |\alpha| \leqq m} C_\alpha \| f_\alpha \|_{0, p^*}$$

for all $(f_\alpha) \in (\mathbf{L}^{p^*}(\mathbb{R}^n))^N$. This proves that $\| g_\alpha \|_{0, p} \leqq C_\alpha$. ■

The above result proves in particular that the elements of $\vartheta_{m, p}$ are precisely the distributions **T** which can be written

$$\mathbf{T} = \sum_{1 \leqq |\alpha| \leqq m} \partial^\alpha g_\alpha.$$

In this case, observe also that if in the above proof we choose the following norm in $(\mathbf{L}^p(\mathbb{R}^n))^N$

$$(f_\alpha)_{1 \leqq |\alpha| \leqq m} \to \Big(\sum_{1 \leqq |\alpha| \leqq m} \| f_\alpha \|_{0, p^*}^{p^*} \Big)^{1/p^*}$$

we easily obtain

$$[\mathbf{T}]_{-m,p}=\min\{(\sum_\alpha \|g_\alpha\|_{0,p}^p)^{1/p};\ \mathbf{T}=\sum_{1\leq|\alpha|\leq m}\partial^\alpha g_\alpha,\ g_\alpha\in\mathbf{L}^p(\mathbb{R}^n)\}$$

which is the analogue in $\vartheta_{m,p}$ of the classical result

$$\|\mathbf{T}\|_{-m,p}=\min\{(\sum_\alpha \|g_\alpha\|_{0,p}^p)^{1/p};\ \mathbf{T}=\sum_{0\leq|\alpha|\leq m}\partial^\alpha g_\alpha,\ g_\alpha\in\mathbf{L}^p(\mathbb{R}^n)\}.$$

In both cases we shall call a canonical decomposition of **T**, in $\mathbf{W}^{-m,p}(\mathbb{R}^n)$ or in $\vartheta_{m,p}$, depending on wether **T** belongs to $\mathbf{W}^{-m,p}(\mathbb{R}^n)$ or $\vartheta_{m,p}$, a (non unique) decomposition of **T** which realizes the above minima.

The following lemma deals with stability properties in $\mathbf{W}^{-m,p}(\mathbb{R}^n)$ for the linear hyperbolic equations of 1st order. To begin with we recall some standard properties about classical solutions of the following problem:

$$\begin{aligned}&\frac{\partial\xi}{\partial t}+\nabla.(v\xi)=\theta\\ &\xi(.,0)=\xi_0.\end{aligned}\tag{2.2}$$

We define the characteristics $X(s_1; x, s_2)$ related to v as the solutions of the system

$$\begin{aligned}&\frac{\partial X}{\partial s_1}=v(X, s_1)\\ &X(s_2; x, s_2)=x.\end{aligned}\tag{2.3}$$

and we denote by J the jacobian determinant of the transformation X:

$$J(t; x, s)=\det\left(\frac{\partial X}{\partial x}(t; x, s)\right).$$

If θ and ξ_0 are smooth enough the unique classical solution of (2.2) is given by

$$\xi(.,t)=J(0; x, t)\,\xi_0(X(0; x, t))+\int_0^t J(s; x, t)\,\theta(X(s; x, t), s)\,ds.\tag{2.4}$$

It is easily seen from (2.3) and (2.4) that if

$$\xi_0\in\mathbf{W}^{-m,p}(\mathbb{R}^n),\qquad \theta\in\mathbf{L}^1(0,\tau;\mathbf{W}^{-m,p}(\mathbb{R}^n)),\qquad v\in\mathbf{L}^\infty(0,\tau;\mathbf{W}^{m,\infty}(\mathbb{R}^n))^n$$

then

$$\xi \in \mathbf{L}^\infty(0, \tau; \mathbf{W}^{m,p}(\mathbb{R}^n))^n.$$

We also recall the following property which is a straigthforward consequence of (2.3):

$$\text{if } v \in \mathbf{L}^\infty(0, \tau; \mathbf{W}^{m,\infty}(\mathbb{R}^n))^n \text{ then } \mathbf{X} \in \mathbf{L}^\infty([0, \tau]^2; \mathbf{W}^{m,\infty}(\mathbb{R}^n))^n.$$

We state now the

LEMMA 2.2. — *Let $m \geqq 1$, $p \in]1, +\infty[$, and $\tau > 0$. Let v be a vector valued function in $\mathbf{L}^\infty(0, \tau; (\mathbf{W}^{m,\infty}(\mathbb{R}^n))^n)$; given ξ_0 in $\mathbf{W}^{-m,p}(\mathbb{R}^n)$ and θ in $\mathbf{L}^1(0, \tau; \mathbf{W}^{-m,p}(\mathbb{R}^n))$, the problem (2.2) has a unique solution ξ in $\mathbf{L}^\infty(0, \tau; \mathbf{W}^{-m,p}(\mathbb{R}^n))$ and there exists a constant* C *only depending on τ and v such that*

$$\| \xi(\,.\,, t) \|_{-m,p} \leqq C \left\{ \| \xi_0 \|_{-m,p} + \int_0^t \| \theta(\,.\,, s) \|_{-m,p}\, ds \right\}. \quad t \in [0, \tau]. \tag{2.5}$$

Moreover, if ξ_0 and $\theta(\,.\,, t)$ lie in $\vartheta_{m,p}$ for all time, so does $\xi(\,.\,, t)$, and the above estimate remains valid by changing $\| \quad \|_{-m,p}$ into $[\quad]_{-m,p}$.

Proof. — To begin with, let us recall in what sense we speak of weak solutions; we denote by $\mathbf{L}$ the differential operator such that:

$$\mathbf{L}\varphi = \frac{\partial \varphi}{\partial t} + \nabla.(v\varphi)$$

and $\mathbf{L}^*$ its formal adjoint:

$$\mathbf{L}^*\varphi = -\left(\frac{\partial \varphi}{\partial t} + (v.\nabla)\varphi \right).$$

A distribution ξ will be called a weak solution in $\mathbf{L}^\infty(0, \tau; \mathbf{W}^{-m,p}(\mathbb{R}^n))$ of (2.2) if

$$\int_0^\tau \langle \xi(\,.\,, t), \mathbf{L}^*\varphi(\,.\,, t) \rangle dt = \int_0^\tau \langle \theta(\,.\,, t), \varphi(\,.\,, t) \rangle dt + \langle \xi_0, \varphi(\,.\,, 0) \rangle \tag{2.6}$$

for all function φ such that

$$\left.\begin{array}{c}\varphi \in \mathbf{C}^0(0, \tau; \mathbf{W}^{m, p^*}(\mathbb{R}^n)) \\ \varphi(., \tau)=0 \\ \mathbf{L}^* \varphi \in \mathbf{L}^1(0, \tau; \mathbf{W}^{m, p^*}(\mathbb{R}^n)).\end{array}\right. \tag{2.7}$$

Let us now prove the uniqueness of such a weak solution of (2.2). If ξ_1 and ξ_2 are two solutions of (2.6), then $\xi_1-\xi_2=\xi$ is solution of the homogeneous problem, i.e. (2.6) with $\theta=0$ and $\xi_0=0$. Given ψ in $\mathbf{L}^1(0, \tau; \mathbf{W}^{m, p^*}(\mathbb{R}^n))$, let φ be the classical solution of

$$\left.\begin{array}{c}\mathbf{L}^* \varphi=\psi \\ \varphi(., \tau)=0.\end{array}\right\} \tag{2.8}$$

Using the characteristics defined in (2.3), the solution of (2.8) can be written in the form:

$$\varphi(x, t)=-\int_t^\tau \psi(\mathbf{X}(s; x, t), s)\, ds$$

from where it is easily seen that φ fullfils the requirements (2.7). Hence we get from (2.6) with $\theta=0$ and $\xi_0=0$:

$$\int_0^\tau \langle \xi(., t), \psi(., t)\rangle\, dt=0, \qquad \forall \psi \in \mathbf{L}^1(0, \tau; \mathbf{W}^{m, p^*}(\mathbb{R}^n)).$$

Taking in particular $\psi(., t)=\lambda(t)\psi$, $\lambda \in \mathbf{L}^1(0, \tau)$, $\psi \in \mathbf{W}^{m, p^*}(\mathbb{R}^n)$ gives

$$\int_0^\tau \lambda(t)\langle \xi(., t), \psi\rangle\, dt=0, \qquad \forall \lambda \in \mathbf{L}^1(0, \tau), \quad \forall \psi \in \mathbf{W}^{m, p^*}(\mathbb{R}^n)$$

and therefore $\xi \equiv 0$.

In order to prove (2.3) it will be convenient to use a canonical decomposition of θ and ξ in $\mathbf{W}^{-m, p}(\mathbb{R}^n)$:

$$\theta=\sum_{0 \leq |\alpha| \leq m} \partial^\alpha \theta_\alpha, \qquad \xi_0=\sum_{0 \leq |\alpha| \leq m} \partial^\alpha \xi_{0, \alpha}.$$

Using the fact that

$$-\mathbf{L}^* \varphi(\mathbf{X}(t; y, s), t)=\frac{d}{dt} \varphi(\mathbf{X}(t; y, s), t)$$

it is an easy matter to check that the distribution ξ defined by

$$\langle \xi(.,t), \varphi \rangle = \sum_{\alpha} (-1)^{|\alpha|} \left\{ \int \xi_{0,\alpha}(y)\, \partial_y^\alpha \varphi(X(t;y,0))\, dy \right. \\ \left. + \int_0^t ds \int \theta_\alpha(y,s)\, \partial_y^\alpha \varphi(X(t;y,s))\, dy \right\} \quad (2.9)$$

is a weak solution of (2.2) in the sense of (2.6), (2.7).

Now, by the smoothness of X, which is a consequence of the smoothness of v, (2.9) leads to

$$|\langle \xi(.,t), \varphi \rangle| \leq C \left\{ \left(\|\xi_{0,0}\|_{0,p} + \int_0^t \|\theta_0(.,s)\|_{0,p} ds \right) \right. \\ \times \|\varphi\|_{0,p^*} + \sum_{1 \leq |\alpha| \leq m} \left(\|\xi_{0,\alpha}\|_{0,p} + \int_0^t \|\theta_\alpha(.,s)\|_{0,p} ds \right) \\ \left. \times \sum_{1 \leq |\alpha| \leq m} \|\partial^\alpha \varphi\|_{0,p^*} \right\} \quad (2.10)$$

Therefore we get on the one hand:

$$\|\xi(.,t)\|_{-m,p} \leq C \left\{ \|\xi_0\|_{-m,p} + \int_0^t \|\theta(.,s)\|_{-m,p} ds \right\},$$

which is the desired result (2.4).

On the other hand if ξ_0 and θ are in $\vartheta_{m,p}$, then for a canonical decomposition of ξ_0 and θ in $\vartheta_{m,p}$ we have $\xi_{0,0} = \theta_0 = 0$. Therefore by (2.10) we get

$$|\langle \xi(.,t), \varphi \rangle| \leq C \left([\xi_0]_{-m,p} + \int_0^t [\theta(.,s)]_{-m,p} ds \right) . \left(\sum_{1 \leq |\alpha| \leq m} \|\partial^\alpha \varphi\|_{0,p^*} \right)^{1/p^*}$$

which implies that $\xi(.,t) \in \vartheta_{m,p}$ and

$$[\xi(.,t)]_{-m,p} \leq C \left\{ [\xi_0]_{-m,p} + \int_0^t [\theta(.,s)]_{-m,p} ds \right\}. \quad \blacksquare$$

We shall end this section with a result concerning quadrature formulas in $\mathbb{R}^n$.

LEMMA 2.3. — *Let $h>0$ and let x_j be the points in $\mathbb{R}^n$ defined by*

$$x_j=(j_1 h, \ldots, j_n h), \qquad (j_1, \ldots, j_n)\in {}^n;$$

then we have for any g in $\mathbf{W}^{m,1}(\mathbb{R}^n)$ *with* $m \geqq n$

$$\left| \int g(x)\,ds - h^n \sum_j g(x_j) \right| \leqq C\,|g|_{m,1}.$$

The proof of this result can be easily deduced from the arguments given in [16], theorem 3.1, and from the property that $\mathbf{W}^{m,1}(\mathbb{R}^n)$ is imbedded in the space of continuous functions as soon as $m \geqq n$.

III. VORTEX METHODS WITH EXPLICIT KERNELS IN TWO DIMENSIONS

In the two-dimensional case the Euler equations may be written in the form:

$$\frac{\partial \omega}{\partial t} + \nabla . (\operatorname{curl} \psi \omega) = 0 \tag{3.1}$$

$$\omega(.,0)=\omega_0 \tag{3.2}$$

$$\left.\begin{aligned} -\Delta\psi &= \omega \\ |\nabla \psi| \to 0, &\qquad |x| \to \infty \end{aligned}\right\} \tag{3.3}$$

with $x=(x_1, x_2)\in\mathbb{R}^2$, $t>0$.

We denote by $\mathbf{G}$ the elementary solution of (3.3) and we set

$$\mathbf{K}(x)=\operatorname{curl}\mathbf{G}=\frac{1}{2\pi |x|^2}(-x_2, x_1)$$

so that if ψ is the solution of (3.3) we have

$$u=\operatorname{curl}\psi=\mathbf{K} * \omega \tag{3.4}$$

Now let us introduce a particle approximation ω_0^h of ω_0:

$$\omega_0^h(x)=\sum_j \alpha_j \delta(x-x_j)$$

with

$$x_j=(j_1 h, j_2 h), \qquad j=(j_1, j_2)\in {}^2, \quad h>0$$
$$\alpha_j=h^2\,\omega_0(x_j).$$

We shall denote by B_j the square

$$[j_1 h-h/2,\, j_1 h+h/2]\times[j_2 h-h/2,\, j_2 h+h/2].$$

Let u^h be an approximate velocity field; the weak solution of

$$\begin{aligned}&\frac{\partial\omega^h}{\partial t}+\nabla.(u^h\omega^h)=0\\&\omega^h(.,0)=\omega_0^h\end{aligned}\tag{3.5}$$

is the measure

$$\omega^h(x,t)=\sum_j \alpha_j\,\delta(x-X_j^h(t))$$

where $(X_j^h)_j$ is obtained through the ordinary differential system

$$\begin{aligned}&\frac{\partial X_j^h}{\partial s}=u^h(X_j^h, s)\\&X_j^h(0)=x_j.\end{aligned}\tag{3.6}$$

In order to couple u^h and ω^h, we first need to regularize $\mathbf{K}$; thus let ζ be a function in $\mathbf{L}^\infty(\mathbb{R}^2)\cap\mathbf{L}^1(\mathbb{R}^2)$ and

$$\zeta_\varepsilon(x)=\varepsilon^{-2}\zeta(x/\varepsilon), \qquad \varepsilon>0.$$

We set

$$\mathbf{K}_\varepsilon=\mathbf{K}*\zeta_\varepsilon$$

and

$$u^h=\mathbf{K}_\varepsilon*\omega^h.\tag{3.7}$$

The approximation is finally defined by (3.5), (3.7).

Let us state our convergence result:

THEOREM 3.1. — *Assume that ω_0 is smooth enough. Assume also that the following conditions hold*

$$\zeta\in W^{m,\infty}(\mathbb{R}^2)\cap W^{m,1}(\mathbb{R}^2), \qquad \forall m>0,\tag{3.8}$$

$$\int \zeta(x)\,dx = 1$$

$$\int x^\alpha \zeta(x)\,dx = 0, \qquad 1 \leqq |\alpha| \leqq d-1, \tag{3.9}$$

$$\int |x|^d \zeta(x)\,dx < \infty$$

$$\exists\, C>0, \quad s>0, \qquad h \leqq C\varepsilon^{1+s} \tag{3.10}$$

Then there exists a constant C *such that for* h *and* ε *small enough*

$$\|(u-u^h)(.,t)\|_{0,p} \leqq C\varepsilon^d, \qquad p\in]2,+\infty], \quad t\in[0,\tau].$$

In fact it will be a consequence of the foregoing proof that convergence holds in a more general situation i. e. when $\zeta \in W^{m,\infty}(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^2)$, m finite. This would lead to error estimates of the kind of those found in [16], for instance, but with stability conditions more restrictive than (3.10) for small values of m.

Our arguments will use the following steps:

1. use (3.1), (3.2), (3.5) in order to derive estimates of $\omega-\omega^h$ in some $W^{-m,p}(\mathbb{R}^2)$ spaces;

2. then go back to the velocity by using the properties of the regularized kernel.

More precisely the needed properties of the regularized kernel are summarized in the following lemma:

LEMMA 3.2. — (i) *Let* $\mathbf{T}$ *be in* $L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$; *then*

$$\|\mathbf{K} * \mathbf{T}\|_{0,\infty} \leqq C(\|\mathbf{T}\|_{0,\infty} + \|\mathbf{T}\|_{0,1}). \tag{3.12}$$

(ii) *Let* ζ *be in* $L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ *and* $\mathbf{T}$ *be in* $L^q(\mathbb{R}^2)$, $q<2$; *then* $\mathbf{K}_\varepsilon * \mathbf{T}$ *is in* $L^p(\mathbb{R}^2)$ *for any* p *such that* $1/p \leqq 1/q - 1/2$ *and*

$$\|\mathbf{K}_\varepsilon * \mathbf{T}\|_{0,p} \leqq C\varepsilon^{-\alpha}\|\mathbf{T}\|_{0,q}, \qquad \alpha = 2(1/q-1/p)-1. \tag{3.13}$$

(iii) *Let* ζ *be in* $W^{m,\infty}(\mathbb{R}^2)\cap W^{m,1}(\mathbb{R}^2)$ *and* $\mathbf{T}$ *in* $\vartheta_{m,p}$, $m \geqq 1$, $1<p<+\infty$; *then*

$$\|\mathbf{K}_\varepsilon * \mathbf{T}\|_{0,p} \leqq C/\varepsilon^{m-1}\,[\mathbf{T}]_{-m,p}. \tag{3.14}$$

(iv) *Under the assumption* (3.9) *we get*

$$\|(\mathbf{K}-\mathbf{K}_\varepsilon)*\mathbf{T}\|_{0,p}\leqq C\,\varepsilon^d(\|\mathbf{T}\|_{d,1}+\|\mathbf{T}\|_{d,\infty}),$$

if $\mathbf{T}\in W^{d,\infty}(\mathbb{R}^2)\cap W^{d,1}(\mathbb{R}^2)$

$$\|(\mathbf{K}-\mathbf{K}_\varepsilon)*\mathbf{T}\|_{0,p}\leqq C\,\varepsilon^d\|\mathbf{T}\|_{d-1,p}, \tag{3.15}$$

if $\mathbf{T}\in W^{d-1,p}(\mathbb{R}^2)$, $1<p<\infty$.

Proof. — The proof of (3.12) is a classical matter and can be omitted here. To check (3.13) let us consider $p<2$ and $q>2$; we write

$$\mathbf{K}_\varepsilon*\mathbf{T}=\zeta_\varepsilon*\mathbf{K}*\mathbf{T}$$

and hence

$$\|\mathbf{K}_\varepsilon*\mathbf{T}\|_{0,q}\leqq\|\zeta_\varepsilon\|_{0,r}\|\mathbf{K}*\mathbf{T}\|_{0,q'},\qquad 1/r+1/q'=1+1/q. \tag{3.16}$$

On the one hand we have by the Sobolev inequalities (*see* [5])

$$\|\mathbf{K}*\mathbf{T}\|_{0,q'}\leqq C\|\mathbf{T}\|_{0,q'},\qquad 1/q'=1/p-1/2.$$

On the other hand an easy calculation shows that

$$\|\zeta_\varepsilon\|_{0,r}\leqq C\,\varepsilon^{-2+2/r}.$$

Therefore (3.16) gives

$$\|\mathbf{K}_\varepsilon*\mathbf{T}\|_{0,q}\leqq C\,\varepsilon^{-\alpha}\|\mathbf{T}\|_{0,p},\qquad \alpha=2(1-1/r)=2(1/p-1/q)-1.$$

Let us now prove (3.14). If $\mathbf{T}\in\vartheta_{m,p}$, $m\geqq 1$, $1<p<+\infty$, we write a canonical decomposition of $\mathbf{T}$ in $\vartheta_{m,p}$:

$$\mathbf{T}=\frac{\partial}{\partial x_1}\Big(\sum_{0\leqq|\alpha|\leqq m-1}\partial^\alpha g_\alpha^1\Big)+\frac{\partial}{\partial x_2}\Big(\sum_{0\leqq|\alpha|\leqq m-1}\partial^\alpha g_\alpha^2\Big)$$

with

$$\|g_\alpha^i\|_{0,p}\leqq C\,[\mathbf{T}]_{-m,p};$$

thus

$$\mathbf{K}_\varepsilon*\mathbf{T}=\frac{\partial\mathbf{K}}{\partial x_1}*\Big(\zeta_\varepsilon*\sum_{0\leqq|\alpha|\leqq m-1}\partial^\alpha g_\alpha^1\Big)+\frac{\partial\mathbf{K}}{\partial x_2}*\Big(\zeta_\varepsilon*\sum_{0\leqq|\alpha|\leqq m-1}\partial^\alpha g_\alpha^2\Big).$$

Since, by Calderon's theorem, the convolution operators $f \to \partial \mathbf{K} * f$ are continuous from $\mathbf{L}^p$ into $\mathbf{L}^p$ for $1<p<+\infty$ (*see* again [5]), we have:

$$\|\mathbf{K}_\varepsilon * \mathbf{T}\|_{0,p} \leqq C \sum_{0 \leqq |\alpha| \leqq m-1} \|\zeta_\varepsilon * \partial^\alpha g_\alpha\|_{0,p}$$

$$\leqq C \sum_{0 \leqq |\alpha| \leqq m-1} \|\partial^\alpha \zeta_\varepsilon\|_{0,1} \|g_\alpha\|_{0,p} \leqq C/\varepsilon^{m-1} [\mathbf{T}]_{-m,p}.$$

Finally the last assertion results from easy calculations for which we refer to [16] for instance. ■

We now come to the proof of the theorem itself. Since all the velocity fields introduced in this section are clearly divergence free, we shall equivalently use either conservative or non conservative forms of the nonlinear terms [such as $(u.\nabla)\omega = \nabla.(u\omega)$]. We shall consider separately the part of the error coming from the particle discretization and the one coming from the regularization of the kernel; we begin with the regularization error and for this purpose we introduce the following intermediate problem:

$$\begin{aligned} &\frac{\partial \omega_\varepsilon}{\partial t} + \nabla.(u_\varepsilon \omega_\varepsilon) = 0 \\ &\omega_\varepsilon(.,0) = \omega_0 \\ &u_\varepsilon = \mathbf{K}_\varepsilon * \omega_\varepsilon. \end{aligned} \tag{3.17}$$

Let us estimate $u - u_\varepsilon$:

LEMMA 3.3. — *Assume that* (3.9) *holds and that* $\omega_0 \in \mathbf{W}^{m,\infty}(\mathbb{R}^2) \cap \mathbf{W}^{m,1}(\mathbb{R}^2)$ *with* $m \geqq d$. *Then we have for some positive constant* C *and for all* $t \in [0, \tau]$:

$$\left.\begin{aligned} \|\omega(.,t)\|_{m,1} + \|\omega(.,t)\|_{m,\infty} \leqq C \\ \|\omega_\varepsilon(.,t)\|_{m,1} + \|\omega_\varepsilon(.,t)\|_{m,\infty} \leqq C \end{aligned}\right\} \tag{3.18}$$

$$\|(u-u_\varepsilon)(.,t)\|_{0,p} \leqq C\varepsilon^d, \qquad 2<p \leqq +\infty. \tag{3.19}$$

Proof. — We omit the proof of (3.18). In fact it is easily seen that it is enough to obtain the bound for $\varepsilon = 0$, which is precisely a well-known regularity result for the two-dimensional Euler equation (*see* [3] for instance). Let us focus on (3.19).

Using (3.17) on the one hand and (3.1), (3.2) on the other hand, we get:

$$\frac{\partial(\omega-\omega_\varepsilon)}{\partial t}+\nabla.(u_\varepsilon(\omega-\omega_\varepsilon))=\nabla.((u_\varepsilon-u)\,\omega)$$
$$\omega-\omega_\varepsilon(.,0)=0.$$

Since u and u_ε are divergence free, the above system can be rewritten as follows

$$\frac{\partial(\omega-\omega_\varepsilon)}{\partial t}+\nabla.(u_\varepsilon(\omega-\omega_\varepsilon))=((u_\varepsilon-u).\nabla)\,\omega \qquad (3.20)$$
$$\omega-\omega_\varepsilon(.,0)=0$$

whereas (3.4) and (3.17) give

$$u-u_\varepsilon=(\mathbf{K}-\mathbf{K}_\varepsilon)*\omega+\mathbf{K}_\varepsilon*(\omega-\omega_\varepsilon). \qquad (3.21)$$

Let us denote by X_ε the characteristic curves associated with the flow u_ε. Writing the solution of (3.20) as in (2.4) gives, since in the present case $\mathrm{J}\equiv 1$

$$(\omega-\omega_\varepsilon)(x,t)=\int_0^t((u-u_\varepsilon).\nabla\omega)(\mathrm{X}_\varepsilon(s;x,t),s)\,ds$$

and therefore

$$\|(\omega-\omega_\varepsilon)(.,t)\|_{0,1}\leqq \mathrm{C}\int_0^t\|((u-u_\varepsilon).\nabla\omega)(.,s)\|_{0,1}\,ds$$
$$\|(\omega-\omega_\varepsilon)(.,t)\|_{0,\infty}\leqq \mathrm{C}\int_0^t\|((u-u_\varepsilon).\nabla\omega)(.,s)\|_{0,\infty}\,ds. \qquad (3.22)$$

Next, by (3.15)

$$\|(\mathbf{K}-\mathbf{K}_\varepsilon)*\omega\|_{0,\infty}\leqq \mathrm{C}\,\varepsilon^d(\|\omega(.,t)\|_{0,1}+\|\omega(.,t)\|_{0,\infty})$$

which gives by (3.18)

$$\|(\mathbf{K}-\mathbf{K}_\varepsilon)*\omega\|_{0,\infty}\leqq \mathrm{C}\,\varepsilon^d. \qquad (3.23)$$

Then (3.12) yields

$$\|\mathbf{K}_\varepsilon*(\omega-\omega_\varepsilon)\|_{0,\infty}\leqq \mathrm{C}(\|\omega-\omega_\varepsilon\|_{0,\infty}+\|\omega-\omega_\varepsilon\|_{0,1}). \qquad (3.24)$$

Combining (3.21), (3.23), (3.24) and setting

$$y(t)=\|(\omega-\omega_\varepsilon)(.,t)\|_{0,\infty}+\|(\omega-\omega_\varepsilon)(.,t)\|_{0,1}$$

we obtain the inequality:

$$\|(u-u_\varepsilon)(.,t)\|_{0,\infty}\leqq C(\varepsilon^d+y(t)).$$

It remains now to insert the above estimate in (3.22) to get

$$y(t)\leqq C\left(\varepsilon^d+\int_0^t y(s)\,ds\right)$$

from which we obtain by Gronwall's lemma

$$y(t)\leqq C\,\varepsilon^d.$$

This gives the desired estimate for $u-u_\varepsilon$ and $p=\infty$. Finally let us consider the case p finite. By (3.13) we can write

$$\|\mathbf{K}_\varepsilon*(\omega-\omega_\varepsilon)\|_{0,p}\leqq C\|\omega-\omega_\varepsilon\|_{0,q}, \qquad 1/p=1/q-1/2.$$

Then we observe that, since $y(t)\leqq C\,\varepsilon^d$, we have also

$$\|\omega-\omega_\varepsilon\|_{0,q}\leqq C\,\varepsilon^d.$$

Next, by (3.15) we get

$$\|(\mathbf{K}_\varepsilon-\mathbf{K})*\omega\|_{0,p}\leqq C\,\varepsilon^d\|\omega\|_{1,p}.$$

Combining (3.21) and the above estimates gives the desired result. ■

Using (3.6), (3.7), and (3.17), we can derive the following system:

$$\frac{\partial(\omega_\varepsilon-\omega^h)}{\partial t}+\nabla(u_\varepsilon.(\omega_\varepsilon-\omega^h))=\nabla.((u_\varepsilon-u^h)(\omega_\varepsilon-\omega^h))-\nabla.((u_\varepsilon-u^h)\,\omega_\varepsilon)$$

$$\omega_\varepsilon-\omega^h(.,0)=\omega_0-\omega_0^h$$

$$u_\varepsilon-u^h=\mathbf{K}_\varepsilon*(\omega_\varepsilon-\omega^h).$$

Then we write

$$\omega_\varepsilon-\omega^h=\lambda^h+\mu^h$$

where λ^h and μ^h are respectively solutions of:

$$\begin{aligned} &\frac{\partial \mu^h}{\partial t} + \nabla . (u_\varepsilon \mu^h) = 0 \\ &\mu^h(.,0) = \omega_0 - \omega_0^h \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\frac{\partial \lambda^h}{\partial t} + \nabla . (u^h \lambda^h) = \nabla . ((u_\varepsilon - u^h)(\mu^h - \omega_\varepsilon)) \\ &\lambda^h(.,0) = 0. \end{aligned} \tag{3.26}$$

Let us first derive estimates of μ^h, which will prove the consistency of the method:

LEMMA 3.4. — *Let $m \geqq 2$. Assume $\omega_0 \in W^{m,\infty}(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^2)$; then we get:*

$$\| \mu^h(.,t) \|_{-m,p} \leqq C h^m, \qquad t \in [0,\tau], \quad p \in]1,+\infty[. \tag{3.27}$$

Moreover we can write for any p, $q \in]1,+\infty[$

$$\mu^h = \bar{\mu}^h + \mu'^h \tag{3.28}$$

where the distributions $\bar{\mu}^h \in L^q(\mathbb{R}^2)$ and $\mu'^h \in \vartheta_{m,p}$ satisfy

$$\| \bar{\mu}^h(.,t) \|_{0,q} \leqq C h^m, \qquad [\mu'^h(.,t)]_{-m,p} \leqq C h^m. \tag{3.29}$$

Proof. — Let φ be some test function C^∞ with compact support; we have

$$\langle \mu^h(.,0), \varphi \rangle = \sum_j \left(\int_{B_j} \omega_0 \varphi \, dx - h^2 \omega_0(x_j) \varphi(x_j) \right).$$

By lemma 2.3 we have therefore

$$|\langle \mu^h(.,0), \varphi \rangle| \leqq C h^m \| \omega_0 \varphi \|_{m,1} \leqq C h^m \| \omega_0 \|_{m,p} \| \varphi \|_{m,p^*}$$

which can be rewritten as

$$|\langle \mu^h(.,0), \varphi \rangle| \leqq C h^m (\| \omega_0 \|_{m,q} \| \varphi \|_{0,q^*} + \| \omega_0 \|_{m,p} \sum_{1 \leqq |\alpha| \leqq m-1} \| \partial^\alpha \varphi \|_{0,p^*})$$

By lemma 2.1 with $p_0 = q$, $p_k = p$ for $k \neq 0$, this implies that we can write

$$\mu^h(.,0) = \sum_{0 \leqq |\alpha| \leqq m} \xi_{0,\alpha}$$

where $\xi_{0,0} \in L^q(\mathbb{R}^2)$ and $\xi_{0,\alpha} \in \vartheta_{|\alpha|,p}$ and

$$\|\xi_{0,0}\|_{0,q} \leqq C h^m, \qquad [\xi_{0,\alpha}]_{-|\alpha|,p} \leqq C h^m, \quad |\alpha| \geqq 1.$$

Next consider the solution ξ_α of (3.25) with initial condition $\xi_{0,\alpha}$. By lemma 2.2 we have

$$\|\xi_0\|_{0,q} \leqq C \|\xi_{0,0}\|_{0,q} \leqq C h^m,$$
$$[\xi_\alpha]_{-|\alpha|,p} \leqq C [\xi_{0,\alpha}]_{-|\alpha|,p} \leqq C h^m, \qquad |\alpha| \geqq 1.$$

Since clearly $\mu^h = \sum_\alpha \xi_\alpha$, writing $\bar{\mu}^h = \xi_0$ and $\mu'^h = \sum_{1 \leqq |\alpha| \leqq m} \xi_\alpha$ gives (3.28) and (3.29). ■

We now turn to λ^h. We set

$$\rho^h = \nabla . ((u_\varepsilon - u^h)(\mu^h - \omega_\varepsilon)).$$

To begin with, we observe that $\omega_\varepsilon - \mu^h$ is solution of the following system:

$$\frac{\partial(\omega_\varepsilon - \mu^h)}{\partial t} + \nabla . (u_\varepsilon(\omega_\varepsilon - \mu^h)) = 0$$
$$(\omega_\varepsilon - \mu^h)(.,0) = \omega_0^h.$$

Therefore we can write

$$(\omega_\varepsilon - \mu^h)(x,t) = \sum_j \alpha_j \delta(x - X_{j,\varepsilon}(t))$$

where $X_{j,\varepsilon}(t) = X_\varepsilon(t; x_j, 0)$. This means that if φ is some test function C^∞ with compact support

$$\begin{aligned} \langle \rho^h(.,t), \varphi \rangle &= \langle \nabla . ((u_\varepsilon - u^h)(\omega_\varepsilon - \mu^h)(.,t)), \varphi \rangle \\ &= \sum_j \alpha_j ((u_\varepsilon - u^h) . \nabla \varphi)(X_{j,\varepsilon}(t)) \\ &= \langle (\rho_1^h + \rho_2^h)(.,t), \varphi \rangle \end{aligned}$$

where

$$\langle \rho_1^h(.,t), \varphi \rangle = \int \omega_0(y)(u_\varepsilon - u^h)(X_\varepsilon(t,y,0),t) . \nabla \varphi(X_\varepsilon(t;y,0))\, dy \quad (3.30)$$

and

$$\langle \rho_2^h(.,t), \varphi \rangle$$

$$=\sum_j \int_{B_j} \omega_0(y)(u_\varepsilon - u^h)(X_\varepsilon(t,y,0),t).\nabla\varphi(X_\varepsilon(t;y,0))\,dy$$

$$-\sum_j \alpha_j (u_\varepsilon - u^h)(X_{j,\varepsilon}(t)).\nabla\varphi(X_{j,\varepsilon}(t)). \quad (3.31)$$

Let us evaluate ρ_1^h, ρ_2^h.

LEMMA 3.5. — *Assume that* $\omega_0 \in W^{m,\infty}(\mathbb{R}^2) \cap W^{m,1}(\mathbb{R}^2)$, *with* $m \geqq 1$. *We get:*

(i) $\rho_1^h(.,t) \in \vartheta_{1,p}$ *and*

$$[\rho_1^h(.,t)]_{-1,p} \leqq C \| (u_\varepsilon - u^h)(.,t) \|_{0,p} \quad (3.32)$$

(ii) $\rho_2^h \in \vartheta_{3,p}$ *and the following decomposition holds* $\rho_2^h = \sum_{1 \leqq k \leqq 3} \rho_{2,k}^h$, *with* $\rho_{2,k}^h(.,t)) \in \vartheta_{k,p}$ *and*

$$[\rho_{2,k}^h(.,t)]_{-k,p} \leqq C h^2 \| (u_\varepsilon - u^h)(.,t) \|_{3-k,p}, \qquad 1 \leqq k \leqq 3. \quad (3.33)$$

Proof. — Using the change of variables $z = X_\varepsilon(t;y,0)$ whose jacobian determinant is one because u_ε is divergence free, we get from (3.30):

$$|\langle \rho_1^h(.,t), \varphi \rangle| \leqq \| \omega_0 \|_{0,\infty} \| u_\varepsilon - u^h \|_{0,p} \| \nabla\varphi \|_{0,p^*}$$

which implies that $\rho_1^h \in \vartheta_{1,p}$ and gives immediately (3.32).

Next, using lemma 2.3 we obtain the following estimate

$$|\langle \rho_2^h(.,t), \varphi \rangle| \leqq C h^2 | \omega_0 (u_\varepsilon - u^h)(X_\varepsilon(t,.,0),t).\nabla\varphi(X_\varepsilon(t;.,0)) |_{2,1}.$$

Since $\omega_0 \in W^{2,\infty}(\mathbb{R}^2) \cap W^{2,1}(\mathbb{R}^2)$ we have by lemma 3.3

$$\| \omega_\varepsilon(.,t) \|_{2,\infty} + \| \omega_\varepsilon(.,t) \|_{2,1} \leqq C.$$

Therefore

$$\| u_\varepsilon(.,t) \|_{2,\infty} \leqq C \qquad \text{and} \qquad \| X_\varepsilon(t;.,0) \|_{2,\infty} \leqq C.$$

Thus we obtain

$$|\langle \rho_2^h(.,t), \varphi \rangle| \leqq C h^2 \sum_{1 \leqq k \leqq 3} \| (u_\varepsilon - u^h)(.,t) \|_{3-k,p} \| \varphi \|_{k,p^*}.$$

Therefore, following lemma 2.1, we can write the desired decomposition of ρ_2^h and (3.33). ■

Denote now by λ_1^h, $\lambda_{2,k}^h$, $1 \leqq k \leqq 3$, the solutions of (3.26) with right hand sides ρ_1^h, $\rho_{2,k}^h$ respectively. Then we have

$$\lambda_2^h = \sum_{1 \leqq k \leqq 3} \lambda_{2,k}^h.$$

Let M a constant to be defined below. We define:

$$\tau_M^h = \sup \{ t \in [0, \tau]; \| u^h(.,t) \|_{3,\infty} \leqq M \}.$$

Using lemma 2.2 and 3.5 we obtain immediately the following result:

$$\lambda_1^h(.,t) \in \vartheta_{1,p}, \qquad \lambda_{2,k}^h(.,t) \in \vartheta_{k,p}, \quad 1 \leqq k \leqq 3$$

and there exists a constant C depending only on M and τ such that for $t \in [0, \tau_M^h]$:

$$[\lambda_1^h(.,t)]_{-1,p} \leqq C \int_0^t \| (u_\varepsilon - u^h)(.,s) \|_{0,p}\, ds \tag{3.34}$$

$$[\lambda_{2,k}^h(.,t)]_{-k,p} \leqq C h^2 \int_0^t \| (u_\varepsilon - u^h)(.,s) \|_{3-k,p}\, ds, \quad 1 \leqq k \leqq 3. \tag{3.35}$$

We are now able to present the

Proof of theorem 3.1. — Let us fix some p in $]2, +\infty[$ we set:

$$y_p(t) = \int_0^t ([\lambda_1^h(.,s)]_{-1,p} + \sum_{1 \leqq k \leqq 3} \varepsilon^{1-k} [\lambda_{2,k}^b(.,s)]_{-k,p})\, ds.$$

By (3.4) and (3.35) we have, for $0 \leqq t \leqq \tau_M^h$:

$$\frac{\partial}{\partial t} y_p(t) \leqq C \int_0^t (\| (u_\varepsilon - u^h)(.,s) \|_{0,p} + h^2 \sum_{1 \leqq k \leqq 3} \varepsilon^{1-k} \| (u_\varepsilon - u^h(.,s) \|_{3-k,p}). \tag{3.36}$$

Next we write

$$u_\varepsilon - u^h = \mathbf{K}_\varepsilon * \mu^h + \mathbf{K}_\varepsilon * \lambda^h = A_1 + A_2. \tag{3.37}$$

From (3.13), (3.14) and lemma 3.4 with q such that $1/q=1/p-1/2$ we get

$$\begin{aligned}\|A_1\|_{0,p} &\leqq C(\|\bar{\mu}^h\|_{0,q}+\varepsilon^{1-m}[\mu'^h]_{-m,p}), \qquad 1/q=1/p-1/2\\ &\leqq C\,h^m/\varepsilon^{m-1}.\end{aligned}$$

It is also possible to derive estimates of the derivatives of A_1, in the following way: we write $\mu^h=\sum \partial^\alpha f_\alpha$ and if r is an integer $\geqq 1$,

$$\begin{aligned}|A_1|_{r,p} &\leqq \sum_\alpha |\mathbf{K}_\varepsilon * \partial^\alpha f_\alpha|_{r,p}\\ &\leqq |\zeta_\varepsilon|_{r,1}\,|\mathbf{K}_\varepsilon * f_0|_{0,p} + \sum_{0\leqq|\alpha|\leqq m-1} |\zeta_\varepsilon|_{r+|\alpha|-1}\,|\mathbf{K} * f_\alpha|_{1,p}.\end{aligned}$$

By lemma 3.2 this yields

$$\begin{aligned}\|A_1\|_{r,p} &\leqq C\,\varepsilon^{-r}(\|f_0\|_{0,q}+\varepsilon^{1-m}\sum_\alpha \|f_\alpha\|_{0,p}), \qquad 1/q=1/p-1/2\\ &\leqq C\,h^m/\varepsilon^{m-1+r}.\end{aligned}$$

Using the same argument combined with the fact that λ^h lies in $\vartheta_{3,p}$, we obtain

$$\|A_2\|_{r,p} \leqq C\,\varepsilon^{-r}([\lambda_1^h]_{-1,p}+\sum_k \varepsilon^{1-k}[\lambda_{2,k}^h]_{-k,p}). \tag{3.39}$$

Combining (3.36), (3.37), (3.38) and (3.39) we conclude:

$$\frac{\partial}{\partial t} y_p(t) \leqq C\left(1+h^2\sum_{1\leqq k\leqq 3}\frac{\varepsilon^{1-k}}{\varepsilon^{3-k}}\right)\left(y_p(t)+\frac{h^m}{\varepsilon^{m-1}}\right), \qquad t\in[0,\tau_M^h].$$

Since by (3.10) h/ε is bounded, this yields

$$\frac{\partial}{\partial t} y_p(t) \leqq C\left(y_p(t)+\frac{h^m}{\varepsilon^{m-1}}\right), \qquad t\in[0,\tau_M^h]. \tag{3.40}$$

By Gronwall's lemma we conclude:

$$y_p(t)\leqq C\frac{h^m}{\varepsilon^{m-1}}, \qquad t\in[0,\tau_M^h].$$

Using (3.37) to (3.40) we then obtain

$$\|(u_\varepsilon-u^h)(.,t)\|_{r,p} \leqq C\frac{h^m}{\varepsilon^{m-1+r}}, \qquad t\in[0,\tau_M^h]. \tag{3.41}$$

To prove the estimate announced in the theorem with finite p we observe that, due to (3.10) we can choose m such that $h^m/\varepsilon^{m-1} \leqq \varepsilon^d$; thus (3.41) with $r=0$ and (3.18) gives

$$\|(u-u^h)(.,t)\|_{0,p} \leqq C\,\varepsilon^d, \qquad t\in[0,\tau_M^h], \quad 2<p<+\infty.$$

For $p=+\infty$ we use the following interpolation inequality

$$\|f\|_{0,\infty} \leqq C\,\|f\|_{0,p}^{s}\,\|f\|_{1,p}^{1-s}, \qquad s=2/p, \quad 2<p<+\infty$$

which combined with (3.39), $r=0$, 1, p large enough, gives the desired result.

Finally we have to prove that, for a suitable choice of the constant M, $\tau_M^h=\tau$. For that we start from (3.39) with $r=3$ and 4 we use the above interpolation argument and (3.18); we obtain that for ε and h small enough:

$$\|u^h(.,\tau_M^h)\|_{3,\infty} \leqq 2\,\|u_\varepsilon(.,\tau_M^h)\|_{3,\infty}$$

which is the desired result if, using (3.18), we choose

$$M=2\max_{0<t<\tau;\,\varepsilon>0}\|u_\varepsilon(.,t)\|_{3,\infty}.$$

This ends the proof of the theorem.

IV. VORTEX IN CELL METHODS IN TWO DIMENSIONS

In this section we plan to show that the approach developped in the previous section provides a good tool for the analysis of the convergence of Vortex In Cell methods. For a previous analysis based on different techniques we refer to [6].

In short, VIC methods consist in the following steps

1. Solve the system:

$$\frac{\partial\omega^h}{\partial t}+\nabla.(u^h\omega^h)=0$$

$$\omega^h(.,0)=\sum_j \alpha_j\,\delta(x-x_j).$$

2. Then compute u^h from ω^h by some coupled finite element particle scheme.

The advantage of such methods is that any Finite Element method on a uniform rectangular grid can be easily formulated in terms of Fourier Series, leading to Fast Poisson solvers which require only $O(\mathrm{N}\ \mathrm{Log}\ (\mathrm{N}))$ operations [to be compared with the $O(\mathrm{N}^2)$ operations for the naive way of computing interactions in section III].

We focus below on the case of P_1, homogeneous, Finite Element Methods, because optimal L^p estimates are known in this case for any p, but we believe that, since our techniques are rather general, they can work in many situations in which both the polynomial space and the artificial boundary condition are more accurately choosen.

IV. 1. Definition of the finite element-particle scheme

Let $\varepsilon>0$, $\mathrm{R}>0$; let ω be a bounded measure which will be typically a linear combination of Dirac measures. We introduce the characteristic function χ of the square $[-1/2, +1/2]^2$ and we set

$$\chi_\varepsilon(x)=\varepsilon^{-2}\chi(x/\varepsilon), \qquad \varepsilon>0, \quad x\in\mathbb{R}^2.$$

We also set $\Omega_\mathrm{R}=[-\mathrm{R}, +\mathrm{R}]^2$, $\Gamma_\mathrm{R}=\partial\Omega_\mathrm{R}$. If f is a scalar function we define the vector valued function:

$$\begin{aligned} \mathrm{curl}_\varepsilon f(x)=(2\varepsilon)^{-1}([f(x_1, x_2+\varepsilon)-f(x_1, x_2-\varepsilon)],& \\ [-f(x_1+\varepsilon, x_2)+f(x_1-\varepsilon, x_2)]) \quad & (4.1) \\ x=(x_1, x_2).& \end{aligned}$$

Next we consider the Dirichlet problem:

$$\left.\begin{aligned} -\Delta u_\mathrm{R}=\mathrm{curl}_\varepsilon(\omega * \chi_\varepsilon) \quad \text{on } \Omega_\mathrm{R} \\ u_\mathrm{R}=0 \quad \text{on } \Gamma_\mathrm{R}. \end{aligned}\right\} \qquad (4.2)$$

Let $\mathscr{T}_\varepsilon$ be a sequence of triangulations of Ω_R satisfying the usual regularity condition: there exist two constants c_1 and c_2 such that each triangle $\kappa\in\mathscr{T}_\varepsilon$ contains a circle of radius $c_1\varepsilon$ and is contained in a circle of radius $c_2\varepsilon$. We define Ξ_ε to be the space of continuous functions on Ω_R, piecewise linear on each $\kappa\in\mathscr{T}_\varepsilon$ and vanishing on Γ_R. Finally we denote by Π_ε the Ritz projection on Ξ_ε defined by:

$$\int_{\Omega_R} \nabla \Pi_\varepsilon(f) . \nabla g \, dx = \int_{\Omega_R} \nabla f . \nabla g \, dx, \qquad \forall g \in \Xi_\varepsilon. \tag{4.3}$$

We set:

$$\tilde{u}_R^\varepsilon = \left\{ \begin{array}{ll} \Pi_\varepsilon(u_R) & \text{on } \Omega_R \\ 0 & \text{elsewhere} \end{array} \right\} \tag{4.4}$$

and

$$u_R^\varepsilon = \tilde{u}_R^\varepsilon * \chi_\varepsilon. \tag{4.5}$$

We finally define the mapping $\mathscr{L}_R^\varepsilon$ by (4.2), (4.4), (4.5) and:

$$u_R^\varepsilon = \mathscr{L}_R^\varepsilon(\omega). \tag{4.6}$$

The approximation ω_0^h being defined as in section 3, the numerical method consists now in looking for (u^h, ω^h) such that

$$\begin{aligned} &\frac{\partial \omega^h}{\partial t} + \nabla . (u^h \omega^h) = 0 \\ &\omega^h(., 0) = \omega_0^h \\ &u^h = \mathscr{L}_R^\varepsilon(\omega^h). \end{aligned} \tag{4.7}$$

Let us observe that the velocity field as defined in (4.5) is not in general divergence free; this will not lead to additional difficulties in our analysis because it is essentially based on the conservative form (4.7). However it must be pointed out that VIC methods are usually formulated in terms of finite difference schemes for the stream function ψ (such that $u = \text{curl}\, \psi$). This point of view is developed in [6] and the links between the both approaches are briefly discussed below in section IV.4.

The advantage of the finite element approach leading to u^h is that it enables us to derive optimal stability properties for $\mathscr{L}_R^\varepsilon$. This is the purpose of the following section.

IV.2. Stability and consistency properties of $\mathscr{L}_R^\varepsilon$

To begin with, we recall some classical results concerning the Dirichlet problem and its corresponding Ritz projection.

In the sequel $\|.\|_{k,p,\mathrm{R}}$ will denote the norm of $\mathbf{W}^{k,p}(\Omega_\mathrm{R})$ whereas $|.|_{k,p,\mathrm{R}}$ will be the corresponding semi norm; in addition if s is some real number $\|.\|_{s,p,\Gamma_\mathrm{R}}$ will be the usual norm of $\mathbf{W}^{s,p}(\Gamma_\mathrm{R})$.

PROPOSITION 4.1. — (i) Δ *is an homeomorphism from* $\mathring{\mathbf{W}}^{1,p}(\Omega_\mathrm{R}) \cap \mathbf{W}^{k,p}(\Omega_\mathrm{R})$ *onto* $\mathbf{W}^{k-2,p}(\Omega_\mathrm{R})$ *for* $k=1,2$, $p\in]1,+\infty[$. *Moreover there exist constants* C *independent of* $\mathrm{R}\geqq \mathrm{R}_0$ *such that*

$$|v|_{k,p,\mathrm{R}} \leqq \mathrm{C}(\|\Delta v\|_{k-2,p,\mathrm{R}} + \|v\|_{k-1/p,p,\Gamma_\mathrm{R}}),\qquad k=1,2,\qquad p\in]1,+\infty[. \tag{4.8}$$

If $\Delta v \in \mathbf{W}^{-1,p}(\Omega_\mathrm{R})$ *has compact support in* $\Omega_{\mathrm{R}/2}$ *and* $v=0$ *on* Γ_R, *we also have*

$$\|v\|_{0,p,\mathrm{R}} \leqq \mathrm{C}\|\Delta v\|_{-2,p,\mathrm{R}} \tag{4.9}$$

(ii) *The Ritz projection* Π_ε *is stable on* $\mathbf{W}^{1,p}(\Omega_\mathrm{R})$ *and there exist constants* C *independant of* $\mathrm{R}\geqq\mathrm{R}_0$ *and* $\varepsilon\leqq\varepsilon_0$ *such that*

$$\left.\begin{aligned} \|\Pi_\varepsilon(v)\|_{0,p,\mathrm{R}} &\leqq \mathrm{C}(\|v\|_{0,p,\mathrm{R}} + \varepsilon|v|_{1,p,\mathrm{R}}), && p\in]1,+\infty[,\\ |\Pi_\varepsilon(v)|_{1,p,\mathrm{R}} &\leqq \mathrm{C}\|v\|_{1,p,\mathrm{R}}, && p\in]1,+\infty], \end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned} \|\Pi_\varepsilon(v)-v\|_{0,p,\mathrm{R}} &\leqq \mathrm{C}\varepsilon^2\|v\|_{2,p,\mathrm{R}}, && p\in]1,+\infty[,\\ |\Pi_\varepsilon(v)-v|_{1,p,\mathrm{R}} &\leqq \mathrm{C}\varepsilon\|v\|_{2,p,\mathrm{R}}, && p\in]1,+\infty]. \end{aligned}\right\} \tag{4.11}$$

Moreover we have

$$|\Pi_\varepsilon(v)-v|_{1,\infty,\mathrm{R}} \leqq \mathrm{C}\varepsilon^{1-2/p}\|v\|_{2,p,\mathrm{R}},\qquad p\in]1,+\infty[. \tag{4.12}$$

Proof. — For $k=1$, 2, our first assertion follows from classical tools concerning the Laplace problem in smooth domains. For these results we refer to [13] for $k=1$ and [10] for $k=2$. To derive (4.8) we use a scaling argument. This argument which is detailed in [6] consists in using the change of variables

$$x=\mathrm{R}\,\hat{x},\qquad x\in\Omega_\mathrm{R},\quad \hat{x}\in\Omega_1=\hat{\Omega}$$

and setting for any function f defined in Ω_R

$$\hat{f}(\hat{x})=\mathrm{R}^{-2}f(\mathrm{R}\,\hat{x}).$$

We easily check that

$$|\hat{f}|_{k,p,1}=\mathrm{R}^{k-2-2/p}|f|_{k,p,\mathrm{R}};\qquad |\hat{f}|_{k,p,\Gamma_1}=\mathrm{R}^{k-2-1/p}|f|_{k,p,\Gamma_\mathrm{R}}.$$

Then, using the results of [10] and [13] on $\hat{\Omega}$, we obtain

$$|\hat{v}|_{k,p,1} \leqq C(\|\Delta\hat{v}\|_{k-2,p,1} + \|\hat{v}\|_{k-1/p,p,\Gamma_R}),$$
$$k=1,2, \qquad p\in]1,+\infty[,$$

which implies

$$|v|_{k,p,1} \leqq C(\|\Delta v\|_{k-2,p,1} + \|v\|_{k-1/p,p,\Gamma_R}), \qquad k=1,2, \qquad p\in]1,+\infty[.$$

For (4.9) we use a duality argument. Let φ a test function in $L^q(\Omega_R)$, with $1/p+1/q=1$ and let w such that

$$\begin{cases} \Delta w=\varphi & \text{on} \quad \Omega_R \\ w=0 & \text{on} \quad \Gamma_R. \end{cases}$$

By (4.8) with $k=2$ we get $w\in W^{2,q}(\Omega_R)$ with

$$\|w\|_{2,q,R} \leqq C\|\varphi\|_{0,q,R}. \tag{4.13}$$

Then we write

$$\int_{\Omega_R} v\,\varphi\,dx = -\int_{\Omega_R} v\,\Delta w\,dx. \tag{4.14}$$

Since $\Delta v\in W^{-1,p}(\Omega_R)$ and $v=0$ on Γ_R, setting $f=\Delta v$ we can write by the Green's formula

$$-\int_{\Omega_R} v\,\Delta w\,dx = -\int_{\Omega_R} \nabla v\,\nabla w\,dx = \langle f, w\rangle_R \tag{4.15}$$

where $\langle .,. \rangle_R$ denotes the duality pairing between $\mathring{W}^{1,p}(\Omega_R)$ and $W^{-1,p}(\Omega_R)$.

Next we introduce the function $\bar{w}\in \mathring{W}^{2,q}(\Omega_R)$ such that:

$$\bar{w}=w \quad \text{on} \quad \Omega_{R/2}$$
$$\|\bar{w}\|_{2,q,R} \leqq C\|w\|_{2,q,R}.$$

Since $\operatorname{Supp} f\subset \Omega_{R/2}$, we thus have

$$|\langle f,w\rangle_R| = |\langle f,\bar{w}\rangle_R| \leqq \|f\|_{-2,p,R}\|\bar{w}\|_{2,q,R}$$
$$\leqq C\|f\|_{-2,p,R}\|w\|_{2,q,R}. \tag{4.16}$$

Combining (4.13) to (4.16) we obtain

$$\int_{\Omega_R} v\,\varphi\,dx \leqq C\,\|f\|_{-2,p,R}\,\|w\|_{2,q,R} \leqq C\,\|f\|_{-2,p,R}\,\|\varphi\|_{0,q,R},$$

and therefore

$$\|v\|_{0,p,R} \leqq C\,\|f\|_{-2,p,R}.$$

Finally to get (4.10) to (4.12) we note that

$$\hat{\Pi}_\varepsilon(v) = \Pi_{\varepsilon/R}(\hat{v}).$$

Then, let us apply in $\hat{\Omega}$ the L^p estimates given in [15]; we obtain for instance

$$\|\hat{\Pi}_\varepsilon(v)\|_{0,p,1} = \|\Pi_{\varepsilon/R}(\hat{v})\|_{0,p,1} \leqq C(\|\hat{v}\|_{0,p,1} + \varepsilon/R\,|\hat{v}|_{1,p,1}).$$

Therefore

$$R^{-2-2/p}\,\|\Pi_\varepsilon(v)\|_{0,p,R} \leqq C(R^{-2-2/p}\,\|v\|_{0,p,R} + (\varepsilon/R)\,R^{1-2-2/p}\,|v|_{1,p,R})$$

which yields the first estimate in (4.10). The other results derive in a similar way from the estimates given in [15]. ■

We sum up in the following lemma some remarks whose proof is straightforward and which will be useful in the sequel

LEMMA 4.2. — (i) *For $i=1$ or 2, $\partial\chi_\varepsilon/\partial x_i$ is a bounded measure with total mass less than C/ε.*

(ii) *One may rewrite* (4.1) *in the following way*

$$\operatorname{curl}_\varepsilon f = \varepsilon^{-1}\left(\int_{-\varepsilon}^{+\varepsilon} \frac{\partial f}{\partial x_2}(x_1, x_2+y_2)\,dy_2 - \int_{-\varepsilon}^{+\varepsilon} \frac{\partial f}{\partial x_1}(x_1+y_1, x_2)\,dy_1\right). \tag{4.17}$$

As a consequence of (4.17), it is easily seen that we have for any pair (k, p)

$$\|\operatorname{curl}_\varepsilon f\|_{k,p} \leqq \|\operatorname{curl} f\|_{k,p}. \tag{4.18}$$

We now sum up the stability properties of $\mathscr{L}_R^\varepsilon$ in the following lemma:

LEMMA 4.3. — *$\mathscr{L}_R^\varepsilon$ is a linear mapping which satisfies the following property: if ω is a distribution with compact support in $\Omega_{R/2}$, there exist* *positive constants independent of* $\varepsilon \leqq \varepsilon_0$ *and* $R \geqq R_0$ *such that the following estimates hold*

$$\| \mathscr{L}_R^\varepsilon(\omega) \|_{k,p} \leqq \begin{cases} C\varepsilon^{1-r-k} \| \omega \|_{-r,p}, \\ 0 \leqq k \leqq 2, \quad 1 \leqq r \leqq 2, \quad p \in]1, +\infty[\\ C\varepsilon^{-k} \| \omega \|_{0,q}, \\ 0 \leqq k \leqq 2, \quad p \in]2, +\infty[; \quad 1/q = 1/p + 1/2; \end{cases} \tag{4.19}$$

$$\left. \begin{array}{c} | \mathscr{L}_R^\varepsilon(\omega) |_{k,\infty} \leqq C\varepsilon^{s-r-k} \| \omega \|_{-r,p}, \\ 0 \leqq k \leqq 2, \quad 1 \leqq r \leqq 2, \quad p \in]2, +\infty[, \quad s = 1 - 2/p. \end{array} \right\} \tag{4.20}$$

Proof. — Let p be a finite number. We begin by deriving (4.19) for $k = 0, 1$. From (4.5), (4.6) we get:

$$\| \mathscr{L}_R^\varepsilon(\omega) \|_{k,p} \leqq \| \tilde{u}^\varepsilon \|_{k,p,R}, \qquad k = 0 \text{ or } 1. \tag{4.21}$$

Next (4.10) and (4.4) give

$$\left. \begin{array}{c} \| \tilde{u}_R^\varepsilon \|_{0,p,R} \leqq C(\| u_R \|_{0,p,R} + \varepsilon \| u_R \|_{1,p,R}) \\ | \tilde{u}_R^\varepsilon |_{1,p,R} \leqq C \| u_R \|_{1,p,R}. \end{array} \right\} \tag{4.22}$$

Combining (4.22), (4.8) and (4.9) and using the fact that $\tilde{u}_R^\varepsilon = 0$ outside of Ω_R give

$$\left. \begin{array}{c} \| \tilde{u}_R^\varepsilon \|_{0,p} \leqq C(\| \Delta u_R \|_{-2,p,R} + \varepsilon \| \Delta u_R \|_{-1,p,R}) \\ | \tilde{u}_R^\varepsilon |_{1,p} \leqq C \| \Delta u_R \|_{-1,p,R}. \end{array} \right\} \tag{4.23}$$

Since ω has its support contained in $\Omega_{R/2}$ the following inclusions clearly hold for ε small enough:

$$\operatorname{Supp} \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \subset \Omega_{R/2+2\varepsilon} \subset \Omega_{3R/4}.$$

Hence we can write

$$\| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-k,p,R} = \| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-k,p}, \qquad k = 1, 2,$$

and by virtue of (4.2), (4.8) and (4.9), (4.23) leads to:

$$\left. \begin{array}{c} \| \tilde{u}_R^\varepsilon \|_{0,p} \leqq C(\| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-2,p} + \varepsilon \| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-1,p} \\ | \tilde{u}_R^\varepsilon |_{1,p} \leqq C \| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-1,p}). \end{array} \right\} \tag{4.24}$$

Next, by (4.1) we have

$$\| \operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon) \|_{-1,p} \leqq C/\varepsilon \| \omega * \chi_\varepsilon \|_{-1,p} \leqq C/\varepsilon \| \omega \|_{-1,p}. \tag{4.25}$$

By (4.18) we have also

$$\|\operatorname{curl}_\varepsilon(\omega * \chi_\varepsilon)\|_{-2,p} \leqq \|\operatorname{curl}(\omega * \chi_\varepsilon)\|_{-2,p} \leqq \|\omega * \chi_\varepsilon\|_{-1,p} \leqq \|\omega\|_{-1,p}. \quad (4.26)$$

Combining (4.21), (4.22) and (4.24) we obtain (4.19) for $k=0$, 1 and $r=1$.

When $k=0$, 1 and $r=0$ we observe that, denoting by p^* and q^* the conjugate exponents of p and q, if $\varphi \in \mathbf{W}^{1,p^*}(\mathbb{R}^2)$ and $\omega \in \mathbf{L}^q(\mathbb{R}^2)$, we may write

$$|\langle \omega, \varphi \rangle| \leqq \|\omega\|_{0,q} \|\varphi\|_{0,q^*} \leqq C \|\omega\|_{0,q} \|\varphi\|_{1,p^*}$$

this last bound being a consequence of the already mentionned Sobolev imbedding

$$\|\varphi\|_{0,q^*} \leqq C \|\varphi\|_{1,p^*}, \qquad 1/p^* = 1/q^* + 1/2.$$

Thus we get

$$\|\omega\|_{-1,p} \leqq C \|\omega\|_{0,q}$$

and (4.21), (4.22), (4.24) leads to (4.19) for $k=0$, 1 and $r=0$.

To deal with the other cases, we make use of lemma 4.2, assertion (1); thus we can write

$$\|\omega * \chi_\varepsilon\|_{-1,p} \leqq C/\varepsilon \|\omega\|_{-2,p}$$

and also, by (4.5)

$$\|\mathscr{L}_R^\varepsilon(\omega)\|_{2,p} \leqq C/\varepsilon \|\tilde{u}_R^\varepsilon\|_{1,p}.$$

Combined with the above estimates, this yields (4.18) for $k=2$ or $r=2$.

Now let us derive (4.20). By the Gagliardo-Kohn-Nirenberg inequality, we first obtain:

$$\|\mathscr{L}_R^\varepsilon(\omega)\|_{0,\infty} \leqq C \|\mathscr{L}_R^\varepsilon(\omega)\|_{0,p}^s \|\mathscr{L}_R^\varepsilon(\omega)\|_{1,p}^{1-s}, \qquad p>2; \quad s=1-2/p$$

which by virtue of (4.19) gives (4.20) for $k=0$. For $k=1$, we come back to the definition of $\tilde{u}_R^\varepsilon$. We observe that, since on each $\kappa \in \mathscr{T}_\varepsilon \tilde{u}_R^\varepsilon$ is the linear function which interpolates the values of $\tilde{u}_R^\varepsilon$ at the vertices of κ, we have by the inverse inequality for quasi uniform mesh:

$$\|\tilde{u}_R^\varepsilon\|_{1,\infty,R} \leqq C/\varepsilon \|\tilde{u}_R^\varepsilon\|_{0,\infty,R}.$$

Using the Gagliardo-Kohn-Nirenberg inequality together with (4.23), (4.24) and (4.25), this implies immediately (4.20) for $k=1$.

Finally for $k=2$ we use again (4.17), and we obtain

$$\begin{aligned}\|\mathscr{L}_R^\varepsilon(\omega)\|_{2,\infty} &\leqq C\|\tilde{u}_R^\varepsilon * \chi_\varepsilon\|_{2,\infty,R}\\ &\leqq C/\varepsilon\|\tilde{u}_R^\varepsilon\|_{1,\infty,R}\end{aligned}$$

and thus (4.20) for $k=2$. ■

We now turn to the following consistency result:

LEMMA 4.4. – *Let* $\omega\in W^{3,1}(\mathbb{R}^2)\cap W^{3,\infty}(\mathbb{R}^2)$ *with support contained in* $\Omega_{R/2}$, *and* $u=K*\omega$. *There exist constants* C *such that*

$$\begin{aligned}&\|\mathscr{L}_R^\varepsilon(\omega)-u\|_{0,p}\leqq C(\varepsilon^2+R^{2/p-1}), &&2<p<+\infty\\ &|\mathscr{L}_R^\varepsilon(\omega)-u|_{1,p}\leqq C(\varepsilon+\varepsilon^{-1}R^{2/p-1}), &&2<p<+\infty \qquad (4.27)\\ &|\mathscr{L}_R^\varepsilon(\omega)-u|_{2,p}\leqq C(1+\varepsilon^{-2}R^{2/p-1}), &&2<p<+\infty,\end{aligned}$$

and

$$\left.\begin{aligned}\|\mathscr{L}_R^\varepsilon(\omega)-u\|_{1,\infty}&\leqq C(\varepsilon^{1-2/p}+\varepsilon^{-1}R^{2/p-1})\\ |\mathscr{L}_R^\varepsilon(\omega)-u|_{2,\infty}&\leqq C(\varepsilon^{-2/p}+\varepsilon^{-2}R^{2/p-1})\end{aligned}\right\}2<p<+\infty. \qquad (4.28)$$

Proof. – Denote by $\tilde{u}_R$ the function $\begin{cases}u_R \text{ on } \Omega_R\\ 0 \text{ elsewhere}\end{cases}$.

Following (4.4) and (4.5), we can write

$$\mathscr{L}_R^\varepsilon(\omega)-u=(\tilde{u}_R^\varepsilon-\tilde{u}_R)*\chi_\varepsilon+(\tilde{u}_R-u)*\chi_\varepsilon+(u*\chi_\varepsilon-u)=A_1+A_2+A_3.$$

First of all observe that, since $\omega\in W^{2,1}(\mathbb{R}^2)\cap W^{2,\infty}(\mathbb{R}^2)$ we have $u\in W^{2,p}(\mathbb{R}^2)$ for any p in $]2,+\infty]$ (by lemma 3.2 for instance); therefore we obtain easily (*see* [16] for a detailed proof)

$$\left.\begin{aligned}\|A_3\|_{0,p}+\varepsilon\|A_3\|_{1,p}+\varepsilon^2\|A_3\|_{2,p}\leqq C\|u\|_{2,p}\varepsilon^2\leqq C\varepsilon^2,\\ p\in]2,+\infty].\end{aligned}\right\} \qquad (4.29)$$

Next, in order to estimate A_1 we use (4.8) and (4.11) to get

$$\|\tilde{u}_R^\varepsilon-\tilde{u}_R\|_{0,p}\leqq C\varepsilon^2\|u_R\|_{2,p,R}\leqq C\varepsilon^2\|\operatorname{curl}_\varepsilon(\omega*\chi_\varepsilon)\|_{0,p,R}\leqq C\varepsilon^2\|\operatorname{curl}\omega\|_{0,p},$$
$$2<p<+\infty,$$

$$|\tilde{u}_R^\varepsilon-\tilde{u}_R|_{1,p}\leqq C\varepsilon\|u_R\|_{2,p,R}\leqq C\varepsilon\|\operatorname{curl}\omega\|_{0,p}, \qquad 2<p\leqq+\infty,$$

and therefore

$$\left.\begin{aligned} \| A_1 \|_{0,p} &\leqq C\varepsilon^2 (\| \omega \|_{1,1} + \| \omega \|_{1,\infty}), \qquad 1<p<+\infty. \\ | A_1 |_{1,p} &\leqq C\varepsilon (\| \omega \|_{1,1} + \| \omega \|_{1,\infty}), \qquad 1<p\leqq+\infty. \end{aligned}\right\} \tag{4.30}$$

We have also

$$\begin{aligned} | A_1 |_{2,p} &\leqq C/\varepsilon \, | \tilde{u}_R^\varepsilon - \tilde{u}_R |_{1,p} \\ &\leqq C(\| \omega \|_{1,1} + \| \omega \|_{1,\infty}), \qquad 2<p\leqq+\infty \end{aligned} \tag{4.31}$$

To estimate A_2 we observe that $u_R - u$ is solution of

$$\left.\begin{aligned} \Delta (u_R - u) &= \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \quad \text{on} \quad \Omega_R \\ u_R - u &= -u \quad \text{on} \quad \Gamma_R. \end{aligned}\right\} \tag{4.32}$$

Therefore, by virtue of proposition 4.1, assertion (i), we may write for $1<p<+\infty$.

$$\begin{aligned} | u_R - u |_{2,p,R} &\leqq C(\| \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \|_{0,p,R} + \| u \|_{2-1/p,p,\Gamma_R}), \\ | u_R - u |_{1,p,R} &\leqq C(\| \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \|_{-1,p,R} + \| u \|_{1-1/p,p,\Gamma_R}). \end{aligned} \tag{4.33}$$

Next, to get a bound for $\| u_R - u \|_{0,p}$ for $2<p<+\infty$ let us consider the function w such that

$$\left.\begin{aligned} \Delta w &= 0 \quad \text{on} \quad \Omega_R \\ w &= u \quad \text{on} \quad \Gamma_R. \end{aligned}\right\} \tag{4.34}$$

We first get by the maximum principle

$$\| w \|_{0,\infty,R} \leqq \| u \|_{0,\infty,\Gamma_R}$$

and therefore

$$\| w \|_{0,p,R} \leqq C \| u \|_{0,\infty,\Gamma_R} \cdot R^{2/p}. \tag{4.35}$$

Then we get from (4.32) and (4.34)

$$\begin{cases} \Delta (u_R - u + w) = \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \quad \text{on} \quad \Omega_R \\ u_R - u + w = 0 \quad \text{on} \quad \Gamma_R. \end{cases}$$

Since ω has its support in $\Omega_{R/2}$, $\operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega$ has its support in $\Omega_{3R/4}$ for ε small enough and therefore (4.9) allows us to write

$$\| u_R - u + w \|_{0,p,R} \leqq C \| \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \|_{-2,p,R}. \tag{4.36}$$

Then, combining (4.35) and (4.36) gives

$$\| u_R - u \|_{0, p, R} \leqq C(\| \operatorname{curl}_\varepsilon (\omega * \chi_\varepsilon) - \operatorname{curl} \omega \|_{-2, p, R} + R^{2/p} \| u \|_{0, \infty, \Gamma_R}). \quad (4.37)$$

In order to estimate the right hand side of (4.34) and (4.37) we first recall the following result concerning the decay of $u(x)$ for large x: if $u = \mathbf{K} * \omega$, where ω is compactly supported and smooth enough we get

$$|u(x)| + |\nabla u(x)| + |\nabla^2 u(x)| \leqq C(1 + |x|)^{-1}.$$

This implies in particular that for $2 < p < +\infty$

$$\| u \|_{W^{2, p}(\mathbb{R}^2/\Omega_R)} \leqq C R^{2/p - 1}.$$

Thus we have by a classical trace theorem and the now familiar scaling argument

$$\begin{aligned} \| u \|_{1 - 1/p, p, \Gamma_R} &\leqq C \| u \|_{W^{2, p}(\mathbb{R}^2/\Omega_R)} \\ &\leqq C R^{2/p - 1}, \qquad 2 < p < +\infty. \end{aligned} \quad (4.38)$$

Next, argueing as for the estimate of A_3, we obtain

$$\| \operatorname{curl} \omega * \chi_\varepsilon - \operatorname{curl} \omega \|_{0, p} = \| (\operatorname{curl} \omega) * \chi_\varepsilon - \operatorname{curl} \omega \|_{0, p} \leqq C \varepsilon^2 | \operatorname{curl} \omega |_{2, p}; \quad (4.39)$$

whereas using similar arguments and starting from (4.18) lead to

$$\| \operatorname{curl}_\varepsilon \omega * \chi_\varepsilon - \operatorname{curl} \omega * \chi_\varepsilon \|_{0, p} \leqq C \varepsilon^2 | \operatorname{curl} \omega |_{2, p}. \quad (4.40)$$

Since $\omega \in W^{3, 1}(\mathbb{R}^2) \cap W^{3, \infty}(\mathbb{R}^2)$, combining (4.39) and (4.40) yields

$$\| \operatorname{curl}_\varepsilon \omega * \chi_\varepsilon - \operatorname{curl} \omega \|_{0, p} \leqq C \varepsilon^2 \quad (4.41)$$

and thus (4.33), (4.37) and (4.41) give

$$\| u_R - u \|_{2, p, R} \leqq C(\varepsilon^2 + R^{2/p - 1}), \qquad 2 < p < +\infty.$$

Since $u_R - u = -u$ on $\mathbb{R}^2 - \Omega_R$ the above bound and the decay of u imply

$$\| \tilde{u}_R - u \|_{2, p} \leqq C(\varepsilon^2 + R^{2/p - 1}), \qquad 2 < p < +\infty,$$

which proves that

$$\| A_2 \|_{2,p} \leqq C(\varepsilon^2 + R^{2/p-1}), \qquad 2<p<+\infty. \tag{4.42}$$

It remains now to combine (4.29), (4.30), (4.31) and (4.38) to get (4.27).
For (4.28) we use (4.12) and obtain

$$\left. \begin{array}{c} \| A_1 \|_{1,\infty} \leqq \| \tilde{u}_R^\varepsilon - \tilde{u}_R \|_{1,\infty} \leqq C\varepsilon^{1-2/p} \| \tilde{u}_R \|_{2,p,R} \leqq C\varepsilon^{1-2/p}, \\ 2<p<+\infty. \end{array} \right\} \tag{4.43}$$

$$\| A_1 \|_{2,\infty} \leqq C\varepsilon^{-1} \| \tilde{u}_R^\varepsilon - \tilde{u}_R \|_{1,\infty} \leqq C\varepsilon^{-2/p}, \qquad 2<p<+\infty. \tag{4.44}$$

Next we can write, thanks to the Gagliardo-Kohn-Nirenberg inequality

$$\| u - \tilde{u}_R \|_{1,\infty} \leqq C \| u - \tilde{u}_R \|_{1,p}^s \| u - \tilde{u}_R \|_{2,p}^{1-s} \leqq C(\varepsilon^2 + R^{2/p-1})$$

and thus

$$\left. \begin{array}{c} \| A_2 \|_{1,\infty} \leqq \| u - \tilde{u}_R \|_{1,\infty} \leqq C(\varepsilon^2 + R^{2/p-1}) \\ \| A_2 \|_{2,\infty} \leqq C/\varepsilon \| u - \tilde{u}_R \|_{1,\infty} \leqq C(\varepsilon + \varepsilon^{-1} R^{2/p-1}). \end{array} \right\} \tag{4.45}$$

Combining (4.29), (4.43), (4.44) and (4.45) finally gives (4.28). ■

IV.3. Error estimates

We shall prove the following result

THEOREM 4.5. – *Let us assume that* $\omega_0 \in W^{2,1}(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ *with compact support. Let* u^h *and* ω^h *be defined by* (3.5) *and* (4.7). *We also assume that there exist constants positive constants* C *and* a *such that*

$$h \leqq C\varepsilon^{1+a}, \qquad R^{-1} \leqq C\varepsilon^{1+a}. \tag{4.46}$$

Then the following bounds hold for ε *small enough*

$$\| (u-u^h)(.,t) \|_{0,p} \leqq C[\varepsilon^2 + h^2/\varepsilon + R^{2/p-1}], \qquad p \in]2, +\infty[. \tag{4.47}$$

In order to apply the techniques introduced in the previous section, we first need some refinement of lemma 2.2. The lemma which follows indicates in what way the various constants involved in lemma 2.2 actually depend on v.

LEMMA 4.6. — *Assume the hypotheses of lemma* 2.2 *with* $m \leqq 2$. *Then we may write the solution of* (2.2) *in the following way*

$$\xi(.,t) = \sum_{0 \leqq |\alpha| \leqq m} \xi_\alpha(.,t) \tag{4.48}$$

where $\xi_\alpha \in \mathbf{W}^{-|\alpha|, p}(\mathbb{R}^2)$. *Moreover*

$$\|\xi_\alpha(.,t)\|_{-|\alpha|,p} \leqq C_{|\alpha|,m} \left\{ \|\xi_0\|_{-m,p} + \int_0^t \|\theta(.,s)\|_{-m,p} ds \right\}, \quad t \in [0, \tau]. \tag{4.49}$$

where the constants $C_{|\alpha|,m}$ *satisfy*

$$\begin{gathered} C_{0,m} \leqq C(\tau) \\ C_{1,2} \leqq C(\tau)\,[1 + \operatorname*{Max}_{t \leqq \tau} \|v(.,t)\|_{2,\infty} \exp(3 \operatorname*{Max}_{t \leqq \tau} \|v(.,t)\|_{1,\infty} \tau) \\ C_{2,2} + C_{1,1} \leqq C(\tau) \exp(\operatorname*{Max}_{t \leqq \tau} \|v(.,t)\|_{1,\infty} \tau). \end{gathered} \tag{4.50}$$

Furthermore, if ξ_0 *and* θ *have compact support in a ball of radius* R, *then* $\xi(.,t)$ *vanishes out of a ball of radius* R (t) *where*

$$R(t) = R + t \operatorname*{Max}_{t \leqq \tau} \|v(.,t)\|_{0,\infty}.$$

Proof. — Let us focus on the case $m = 2$, the other cases being treated in a similar way. The idea is to start from (2.9) and write in a precise way the terms of the form $\partial_y^\alpha \varphi(X(t; y, s))$. First we need to estimate $\partial^i X$, for $|i| = 1, 2$ in terms of the derivatives of u. All along this proof we shall use the following notation:

$$|v|_{k,\infty} = \operatorname*{Max}_{t \leqq \tau} |v(.\,t)|_{k,\infty}.$$

By differentiating (2.3) with respect to x, we obtain

$$\frac{d}{dt} |X(t;.,s)|_{1,\infty} \leqq |v|_{1,\infty} |X(t;.,s)|_{1,\infty}$$

$$|X(s;.,s)|_{1,\infty} = 1.$$

Integrating this differential inequality gives for $s, t \in [0, \tau]$

$$|X(t;.,s)|_{1,\infty} \leqq \exp(|v|_{1,\infty}(t-s)). \tag{4.51}$$

Differentiating once again (2.3) we obtain

$$\frac{d}{dt}|X(t;.,s)|_{2,\infty} \leqq |v|_{1,\infty}|X(t;.,s)|_{2,\infty} + |v|_{2,\infty}|X(t;.,s)|_{1,\infty}^2$$
$$|X(s;.,s)|_{2,\infty}=0.$$

which yields

$$|X(t;.,s)|_{2,\infty} \leqq \exp(|v|_{1,\infty}t)\int_s^t |v|_{2,\infty}|X(\theta;.,s)|_{1,\infty}^2 \exp(-|v|_{1,\infty}\theta)\,d\theta.$$

By (4.51) we get therefore

$$|X(t;.,s)|_{2,\infty} \leqq \tau \exp(2|v|_{1,\infty}\tau)|v|_{2,\infty}, \qquad s, t \in [0,\tau]. \tag{4.52}$$

Next developing (2.9) yields

$$\begin{aligned}|\langle \xi(.,t), \varphi\rangle| \leqq C\Big\{\Big(\|\xi_{0,0}\|_{0,p} + \int_0^t \|\theta_0(.,s)\|_{0,p}\,ds\Big)\\ \times\|\varphi\|_{0,p^*} + \sum_{|\alpha|=1}\Big(\|\xi_{0,\alpha}\|_{0,p} + \int_0^t \|\theta_\alpha(.,s)\|_{0,p}\,ds\Big)\\ \times|X(t;.,0)|_{1,\infty}|\varphi|_{1,p^*} + \sum_{|\alpha|=2}\Big(\|\xi_{0,\alpha}\|_{0,p} + \int^t \|\theta_\alpha(.,s)\|_{0,p}\,ds\Big)\\ \times[|X(t;.,0)|_{2,\infty}|\varphi|_{1,p^*} + |X(t;.,0)|_{1,\infty}^2|\varphi|_{2,p^*}]\Big\}.\end{aligned}$$

By virtue of (4.51) and (4.52) the agove estimate becomes

$$\begin{aligned}|\langle \xi(.,t), \varphi\rangle| \leqq C\Big\{\Big(\|\xi_{0,0}\|_{0,p} + \int_0^t \|\theta_0(.,s)\|_{0,p}\,ds\Big)\\ \times\|\varphi\|_{0,p^*} + \sum_{|\alpha|=1}\Big(\|\xi_{0,\alpha}\|_{0,p} + \int_0^t \|\theta_\alpha(.,s)\|_{0,p}\,ds\Big)\\ \times C_{1,2}|\varphi|_{1,p^*} + \sum_{|\alpha|=2}\Big(\|\xi_{0,\alpha}\|_{0,p} + \int^t \|\theta_\alpha(.,s)\|_{0,p}\,ds\Big)\\ \times[C_{1,2}|\varphi|_{1,p^*} + C_{2,2}|\varphi|_{2,p^*}]\Big\}.\end{aligned} \tag{4.53}$$

where the constants $C_{k,2}$ satisfy the bounds (4.50). The desired results (4.48) and (4.49) follow finally from (4.53) and lemma 2.1.

To check the second assertion of the lemma we start again from (2.9) and observe that if $|y| \leqq R$ then, by definition of the characteristics, $|X(t; y, s)| \leqq R(t)$ for $s \leqq t$. Therefore $\langle \xi(.,t), \varphi \rangle = 0$ as soon as φ has its support out of a ball with center 0 and radius $R(t)$. ■

Remark. – Lemma 4.6 can obviously be generalized to any value of m, leading to more involved expressions for the constants $C_{k,m}$. Briefly speaking its meaning is that, when one solves the system 2.2, high order derivatives of v only interact with smooth components of the solution. This idea can be used to give an alternative analysis to the one given in section 3 (and also in the foregoing section 5).

In order to derive estimates for $\omega - \omega^h$ we shall proceed in a sligthly different way than in section 3. We define $\tilde{\lambda}^h$ and $\tilde{\mu}^h$ to be the respective solutions of

$$\begin{gathered} \frac{\partial \tilde{\mu}^h}{\partial t} + \nabla . (u^h \tilde{\mu}^h) = 0 \\ \tilde{u}^h(.,0) = \omega_0 - \omega_0^h \end{gathered} \tag{4.54}$$

$$\begin{gathered} \frac{\partial \tilde{\lambda}^h}{\partial t} + \nabla . (u^h \tilde{\lambda}^h) = -\nabla . ((u - u^h)\,\omega) \\ \tilde{\lambda}^h(.,0) = 0. \end{gathered} \tag{4.55}$$

Then we introduce

$$\tau_M^h = \sup [t \leqq \tau;\ \| u^h(.,t) \|_{1,\infty} + \varepsilon \| u^h(.,t) \|_{2,\infty} \leqq M]$$

where the constant M will be defined below.

We first state the

LEMMA 4.7. – *Assume that* $\omega_0 \in W^{3,1}(\mathbb{R}^2) \cap W^{3,\infty}(\mathbb{R}^2)$ *and has compact support. Then the following assertions hold:*

(i) *There exist two constants* R_0 *and* ε_0 *depending only on* τ *such that* $\tilde{\lambda}^h(.,t)$ and $\tilde{\mu}^h(.,t)$ *have their support contained in the square* $[-R+\varepsilon, R+\varepsilon]^2$ *for* $R > R_0$, $\varepsilon \leqq \varepsilon_0$ *and* $t \leqq \tau_M^h$.

(ii) *Let p be in* $]2, +\infty[$. *We may write*

$$\tilde{\mu}^h = \tilde{\mu}_1^h + \tilde{\mu}_2^h, \qquad \tilde{\mu}_1^h \in W^{-1,p}(\mathbb{R}^2), \quad \tilde{\mu}_2^h \in W^{-2,p}(\mathbb{R}^2) \tag{4.56}$$

with

$$\left.\begin{aligned} \| \tilde{\mu}_1^h(.,t) \|_{-1,p} &\leqq C h^2/\varepsilon \\ \| \tilde{\mu}_2^h(.,t) \|_{-2,p} &\leqq C h^2 \end{aligned}\right\} t \leqq \tau_M^h. \tag{4.57}$$

(iii) *We have* $\tilde{\lambda}^h \in W^{-1,p}(\mathbb{R}^2)$ *and*

$$\|\tilde{\lambda}^h(.,t)\|_{-1,p} \leqq C \int_0^t \|(u-u^h)(.,s)\|_{0,p}\, ds,$$

$$t \leqq \tau_M^h, \qquad p \in]1, +\infty[. \quad (4.58)$$

Proof. — Let A denote a constant such that

$$\text{supp}\, \omega(.,t) \subset B(0, A), \qquad 0 \leqq t \leqq \tau.$$

In particular we have also

$$\text{supp}\, \tilde{\mu}^h(.,0) \subset B(0, A), \qquad 0 \leqq t \leqq \tau.$$

$$\text{supp}\, \nabla.((u-u^h)\,\omega)(.,t) \subset B(0, A), \qquad 0 \leqq t \leqq \tau.$$

Whence by lemma 4.6 we get

$$\text{supp}\, \tilde{\mu}^h(.,t) \subset B(0, A+t\,M)$$

$$\text{supp}\, \tilde{\lambda}^h(.,t) \subset B(0, A+t\,M).$$

To check the second assertion we notice as in the proof of lemma 3.4 that

$$\|\tilde{\mu}^h(.,0)\|_{-2,p} \leqq C(\|\omega_0\|_{2,1} + \|\omega_0\|_{2,\infty})\, h^2.$$

Since $\tilde{\mu}^h$ is solution of (4.54) we get by virtue of lemma 4.6

$$\tilde{\mu}^h = \tilde{\mu}_1^h + \mu_2^h, \qquad \tilde{\mu}_1^h \in W^{-1,p}(\mathbb{R}^2), \quad \tilde{\mu}_2^h \in W^{-2,p}(\mathbb{R}^2) \quad (4.59)$$

with

$$\left.\begin{array}{l} \|\tilde{\mu}_1^h(.,t)\|_{-1,p} \leqq C_{1,2}\,\|\tilde{\mu}_2^h(.,0)\|_{-2,p} \\ \|\tilde{\mu}_2^h(.,t)\|_{-2,p} \leqq C_{2,2}\,\|\tilde{\mu}_2^h(.,0)\|_{-2,p}. \end{array}\right\} \quad (4.60)$$

But for $t \leqq \tau_M^h$ we have

$$\left.\begin{array}{l} C_{1,2} \leqq C \\ C_{2,2} \leqq C\,\varepsilon^{-1} \end{array}\right\} \quad (4.61)$$

where C is a constant depending only on τ; (4.57) results now easily from (4.60) and (4.61).

For the last assertion of the lemma, we first observe that

$$\|\nabla.((u-u^h)\,\omega)\|_{-1,p} \leqq \|\omega\|_{0,\infty}\,\|u-u^h\|_{0,p}.$$

Then, applying lemma 4.6 leads to the following consequence of (4.55)

$$\|\tilde{\lambda}^h(.,t)\|_{-1,p} \leqq C_{1,1} \int_0^t \|(u-u^h)(.,s)\|_{0,p} \|\omega(.,s)\|_{0,\infty} ds,$$
$$p \in]1, +\infty[.$$

which yields (4.58). ■

Now we are in position to prove the theorem.

Proof of Theorem 4.5. — We write:

$$\begin{aligned} u-u^h &= u - \mathscr{L}_R^\varepsilon(\omega) + \mathscr{L}_R^\varepsilon(\omega-\omega^h) \\ &= u - \mathscr{L}_R^\varepsilon(\omega) + \mathscr{L}_R^\varepsilon(\tilde{\lambda}^h) + \mathscr{L}_R^\varepsilon(\tilde{\mu}_1^h) + \mathscr{L}_R^\varepsilon(\tilde{\mu}_2^h). \end{aligned} \quad (4.62)$$

Since by lemma 4.6 the distributions $\tilde{\lambda}^h$, $\tilde{\mu}_1^h$ and $\tilde{\mu}_2^h$ have their compact support contained in $[-R+\varepsilon, R-\varepsilon]^2$, for R large enough and ε small enough we may use lemma 4.3 and 4.4; we obtain first by the estimate (4.19) of lemma 4.3:

$$\left.\begin{aligned} \|\mathscr{L}_R^\varepsilon(\tilde{\lambda}^h) + \mathscr{L}_R^\varepsilon(\tilde{\mu}_2^h)\|_{0,p} &\leqq C(\|\tilde{\lambda}^h\|_{-1,p} + \|\tilde{\mu}_1^h\|_{-1,p}) \\ \|\mathscr{L}_R^\varepsilon(\tilde{\mu}_2^h)\|_{0,p} &\leqq C\varepsilon^{-1}\|\mathscr{L}_R^\varepsilon(\tilde{\mu}_2^h)\|_{-2,p} \end{aligned}\right\} p \in]1, +\infty[, \quad (4.63)$$

whereas, by lemma 4.4

$$\|u - \mathscr{L}_R^\varepsilon(\omega)\|_{0,p} \leqq C(\varepsilon^2 + R^{2/p-1}), \qquad p \in]2, +\infty[. \quad (4.64)$$

Combining (4.62) to (4.64) implies

$$\begin{aligned} \|(u-u^h)(.,t)\|_{0,p} \leqq C[\varepsilon^2 + R^{2/p-1} + \|\tilde{\lambda}^h(.,t)\|_{-1,p} \\ + \|\tilde{\mu}_1^h(.,t)\|_{-1,p} + \|\tilde{\mu}_2^h(.,t)\|_{-2,p}], \qquad t \leqq \tau_M^h, \quad p \in]2, +\infty[. \end{aligned}$$

Therefore we obtain by the estimates proved in lemma 4.7

$$\|(u-u^h)(.,t)\|_{0,p} \leqq C\left[\varepsilon^2 + h^2/\varepsilon + R^{2/p-1} + \int_0^t \|(u-u^h)(.,s)\|_{0,p} ds\right],$$
$$t \leqq \tau_M^h,$$

which implies, by Gronwall's lemma

$$\left.\begin{aligned} \|(u-u^h)(.,t)\|_{0,p} \leqq C[\varepsilon^2 + h^2/\varepsilon + R^{2/p-1}], \\ t \leqq \tau_M^h, \qquad p \in]2, +\infty[. \end{aligned}\right\} \quad (4.65)$$

It remains now to prove that $\tau_M^h=\tau$. First we get by combining (4.57), (4.58) and (4.65)

$$\begin{gathered}\|\tilde{\lambda}^h(.,t)\|_{-1,p}\leqq C(\varepsilon^2+R^{-s})\\ \|\tilde{\mu}_1^h(.,t)\|_{-1,p}\leqq C h^2/\varepsilon\\ \|\tilde{\mu}_2^h(.,t)\|_{-2,p}\leqq C h^2\\ s=1-2/p,\qquad p\in]2,+\infty[,\quad t\leqq\tau_M^h.\end{gathered}\tag{4.66}$$

Due to (4.20), (4.66) leads to

$$\|\mathscr{L}_R^\varepsilon(\tilde{\lambda}^h)+\mathscr{L}_R^\varepsilon(\tilde{\mu}^h)\|_{1,\infty}\leqq C(\varepsilon^s+R^{-s}\varepsilon^{s-2}+h^2\varepsilon^{s-3})\tag{4.67}$$

and, by (4.28)

$$\|u-\mathscr{L}_R^\varepsilon(\omega)\|_{1,\infty}\leqq C(\varepsilon+\varepsilon^{-1}R^{-s}).\tag{4.68}$$

Inserting (4.46) in (4.67) and (4.68) and using (4.62) give

$$\|(u-u^h)(.,t)\|_{1,\infty}\leqq C(\varepsilon^{2s+as-2}+\varepsilon^{s-1+as}+\varepsilon^{as}+\varepsilon^s).$$

Taking p sufficiently large and thus s close to 1 in the above estimate yields the following estimate, valid as soon as ε is small enough

$$\|(u-u^h)(.,t)\|_{1,\infty}\leqq C\varepsilon^{a/2}.\tag{4.69}$$

The same arguments apply to give the following bound

$$\|(u-u^h)(.,t)\|_{2,\infty}\leqq C\varepsilon^{a/2-1}.\tag{4.70}$$

Finally if we specify now our choice of the constant M by setting

$$M=2(\sup_{t\leqq\tau}\|u(.,t)\|_{1,\infty}+\sup_{t\leqq\tau}(\|u(.,t)\|_{2,\infty})$$

we obtain as a consequence of (4.69) and (4.70)

$$\|u^h(.,\tau_M^h)\|_{1,\infty}+\varepsilon\|u^h(.,\tau_M^h)\|_{2,\infty}\leqq 3M/4.$$

This proves that $\tau_M^h=\tau$ and ends the proof of the theorem.

IV.4. Back to the finite difference approach

In this section we briefly discuss a more conventional finite difference-particle scheme which was already analyzed in [6] and we show the links between the analysis of the convergence of the two methods.

We assume here that the triangulations $\mathscr{T}_\varepsilon$ are defined on uniform grids (x_{ij}) of $\mathbb{R}^2$, with $x_{ij}=(i\varepsilon, j\varepsilon)$, $i, j\in$. We denote by w_ε the piecewise linear function such that the family $\{w_\varepsilon(.-x_{ij})\}_{i,j}$ is the natural basis of Ξ_ε.

Given a bounded measure ω, we set

$$\omega_{ij}=\varepsilon^{-2}\,\omega * (\chi_\varepsilon * w_\varepsilon)(x_{ij}). \tag{4.71}$$

If ω is a Dirac mass located on a particle, the above formula defines an assignment procedure from the particle towards the mesh.

Denoting by Δ^ε the usual five points approximation of Δ on the mesh x_{ij}, we consider the solution of the following system

$$-(\Delta^\varepsilon \psi^\varepsilon)_{i,j}=\omega_{ij} \qquad \text{if} \quad x_{ij}\in\Omega_R$$

$$\varepsilon^{-2}[-1/2\,\psi^\varepsilon_{i,j-1}-1/2\,\psi^\varepsilon_{i,j+1}-\psi^\varepsilon_{i-1,j}+2\,\psi^\varepsilon_{i,j}]$$
$$=\omega_{i,j}+(8\,R\,\varepsilon)\int_{\Omega_R}\omega\,ds \qquad \text{if} \quad i\varepsilon=R, \quad |j\varepsilon|<R \tag{4.72}$$

$$\varepsilon^{-2}[-1/2\,\psi^\varepsilon_{i,j-1}-1/2\,\psi^\varepsilon_{i-1,j}+\psi^\varepsilon_{i,j}]$$
$$=\omega_{i,j}+(8\,R\,\varepsilon)\int_{\Omega_R}\omega\,dx \qquad \text{if} \quad i\varepsilon=R, \quad j\varepsilon=R.$$

$$\dots$$

(the dots mean that we do not write the 12 equations needed to describe all the corners and all the sides of Γ_R). The above system turns out to be a natural finite difference discretization of the Neuman problem:

$$-\Delta\psi=\omega \quad \text{on}\,\Gamma_R$$
$$\frac{\partial\psi}{\partial n}=\frac{1}{\operatorname{meas}\Gamma_R}\int_{\Omega_R}\omega\,dx.$$

Next we set

$$v^\varepsilon_{ij}=\begin{cases}(2\varepsilon)^{-1}(\psi^\varepsilon_{i,j+1}-\psi^\varepsilon_{i,j-1}),\ -(2\varepsilon)^{-1}(\psi^\varepsilon_{i+1,j}-\psi^\varepsilon_{i-1,j}) & \text{if}\, x\in\Omega_R\\ 0 \quad \text{otherwise,}\end{cases} \tag{4.73}$$

and

$$v_{\mathrm{R}}^{\varepsilon}(x)=\sum_{i,j} v_{ij}^{\varepsilon}\, w_{\varepsilon}(x-x_{ij}). \tag{4.74}$$

The equations (4.71) to (4.74) define a mapping $\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon}$ such that

$$v_{\mathrm{R}}^{\varepsilon}=\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon}(\omega).$$

Now let (v^h, ω^h) be the solution of

$$\begin{aligned} &\frac{\partial \omega^h}{\partial t}+\nabla.(v^h \omega^h)=0 \\ &\omega^h(.,0)=\sum_j \alpha_j(x-x_j) \\ &v^h=\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon}(\omega^h). \end{aligned} \tag{4.75}$$

Observe that, unlike u^h defined in section 4.1, v^h is divergence free. We have

THEOREM 4.8. — *Under the assumptions of theorem* 4.5, *we get*

$$\|(u-v^h)(.,t)\|_{0,p} \leqq C\{\varepsilon^2+h^2/\varepsilon+R^{2/p-1}\}, \qquad p\in]2,+\infty[.$$

Proof. — We start from (4.54) and (4.55) with v^h instead of u^h. We introduce also the modified value of τ_{M}^h. Then we obviously obtain the analogue of lemma 4.7 with u^h replaced by v^h.

Therefore the proof reduces to the one given for theorem 4.5, provided we are able to estimate $u-v^h$ in terms of $\omega-\omega^h$.

For that, we proceed as in (4.62) and we write

$$u-v^h=u-\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}(\omega)+\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}(\omega-\omega^h)+(\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}-\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon})(\omega^h). \tag{4.76}$$

Now we assert the following result established in [6]: there exists a constant C^h depending only on the size of the support of ω^h and on the mass of ω^h such that, for R sufficiently large

$$\left.\begin{aligned} &\|(\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}-\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon})(\omega^h)\|_{0,\infty,\mathrm{R}/2} \leqq C^h/\mathrm{R} \\ &|(\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}-\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon})(\omega^h)|_{1,\infty,\mathrm{R}/2} \leqq C^h/\mathrm{R}^2. \end{aligned}\right\} \tag{4.77}$$

In addition, we claim that the following estimate results also from the proof given in [6]

$$|(\mathscr{L}_{\mathrm{R}/2}^{\varepsilon}-\tilde{\mathscr{L}}_{\mathrm{R}}^{\varepsilon})(\omega^h)|_{2,\infty,\mathrm{R}/2} \leqq C^h/\mathrm{R}^3. \tag{4.78}$$

Next we observe that the total mass of ω^h is obviously bounded by $\sum |\alpha_j|$, whereas, for $t \leqq \tau_M^h$, the support of ω^h is contained in a ball $B(0, A)$, where A depends only on τ. Therefore the constant C^h involved in the right hand sides of the estimates (4.77) and (4.78) is bounded independently of h.

Now the proof can go along the same lines as for theorem 4.5 and we obtain the same error estimates. ■

V. VORTEX METHODS IN THREE DIMENSIONS

In three dimensions the vorticity-velocity formulation of the Euler equations has the form

$$\frac{\partial \omega}{\partial t} + \nabla . (u \otimes \omega) - (\omega . \nabla) u = 0 \tag{5.1}$$

$$\omega(., 0) = \omega_0 \tag{5.2}$$

$$u = \mathbf{K} * \omega \tag{5.3}$$

where in the above equations: $u = (u_1, u_2, u_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ are vector-valued functions defined in $\mathbb{R}^3$; $\nabla . (u \otimes \omega)$ is the vector with components $\sum_j \partial (u_j \omega_i)/\partial x_j$; $\mathbf{K}$ is a kernel which takes its values in the space of linear mappings in $\mathbb{R}^3$ and can be written in the following way:

$$\mathbf{K}(x) . y = \frac{-x}{4\pi |x|^3} \times y.$$

The design of grid-free three dimensional Vortex methods is based on the following lemma

LEMMA 5.1. – *Assume that* $u \in \mathbf{C}^0(0, \tau; \mathbf{C}^1(\mathbb{R}^3))$ *and* $\omega_0 = \alpha_0 \delta(x - x_0)$, $\alpha_0 \in \mathbb{R}^3$. *Then the unique measure solution of* (5.1), (5.2) *is the vector-valued measure*

$$\omega(x, t) = \alpha(t) \delta(x - X(t))$$

where X *and* α *are respectively the solutions of the following differential systems*

$$\left.\begin{aligned}\frac{\partial X}{\partial s} &= u(X, s)\\ X(0) &= x_0\end{aligned}\right. \tag{5.4}$$

$$\left.\begin{aligned}\frac{\partial \alpha}{\partial s} &= \nabla u(X, s) . \alpha(s)\\ \alpha(0) &= \alpha_0.\end{aligned}\right. \tag{5.5}$$

In (5.5) $\nabla u . \alpha$ means the product of the matrix $(\partial u_i/\partial x_j)_{i,j}$ by the vector α. The proof of this result follows from simple calculations in distributions spaces (*see* [16]).

The design of the method is then as follows: we set

$$\omega_0^h(x) = \sum_j \alpha_j \delta(x - x_j)$$

where

$$\begin{cases} x_j = (j_1 h, j_2 h, j_3 h), \qquad j = (j_1, j_2, j_3) \in \mathbb{Z}^3, \quad h > 0\\ \alpha_j = h^3 \omega_0(x_j). \end{cases}$$

Next, u^h being an approximate velocity field, we consider the measure solution of

$$\left.\begin{aligned}\frac{\partial \omega^h}{\partial t} + \nabla . (u^h \otimes \omega^h) - (\omega^h . \nabla) u^h &= 0\\ \omega^h(., 0) &= \omega_0^h\end{aligned}\right\} \tag{5.6}$$

which by virtue of lemma 5.1 can be written

$$\omega^h(x, t) = \sum_j \alpha_j^h(t) \delta(x - X_j^h(t))$$

where $(X_j^h)_j$ and $(\alpha_j^h)_j$ are obtained through the ordinary differential systems

$$\begin{aligned} \frac{dX_j^h}{ds} &= u^h(X_j^h, s) \\ X_j^h(0) &= x_j \end{aligned} \qquad (5.7)$$

$$\begin{aligned} \frac{d\alpha_j^h}{ds} &= \nabla u^h(X_j^h, s) \,.\, \alpha_j^h \\ \alpha_j^h(0) &= \alpha_j. \end{aligned} \qquad (5.8)$$

Then we define a regularization of the kernel $\mathbf{K}$ as in the two dimensional case by setting, if ζ is a function in $\mathbf{L}^\infty(\mathbb{R}^3) \cap \mathbf{L}^1(\mathbb{R}^3)$

$$\zeta_\varepsilon(x) = \varepsilon^{-3}\zeta(x/\varepsilon), \qquad \varepsilon > 0,$$

and

$$\mathbf{K}_\varepsilon = \mathbf{K} * \zeta_\varepsilon.$$

Finally we write

$$u^h = \mathbf{K}_\varepsilon * \omega^h \qquad (5.9)$$

and the approximation is defined by (5.6) [or equivalently (5.7) and (5.8)], and (5.9).

An interesting question would be to wonder wether these equations lead to a well-posed problem. Such an analysis is straightforward in two dimensions because the system under consideration is lipschitz (or quasi lipschitz if the cut off is only bounded). In the present situation we only want to underline that combining (5.8) and (5.9) leads to a nonlinear differential system in the α_j^h's. Therefore it rapidly becomes clear that existence and uniqueness of the approximate solution only hold for small time, just like in the continuous case. This time could happen to depend on ε and actually it will be a consequence of the foregoing analysis that it can be bounded below independently of ε.

Before stating our convergences result let us also comment somewhat the physical relevance of the proposed numerical method. In the continuous problem the vorticity ω is obviously divergence free, a property which is unfortunately not shared by ω_0^h. In particular (5.9) does not imply that $\omega^h = \operatorname{curl} u^h$. This could introduce undesirable features on the approximate solutions like, for instance, distorsions of the vortex lines. However, starting from (5.6), it is not difficult to check that div ω^h is a weak solution of the following hyperbolic equation

$$\frac{\partial}{\partial t}(\operatorname{div}\omega^h)+\nabla(u^h.\operatorname{div}\omega^h)=0$$

which means that the method is (in a weak sense) conservative with respect to $\operatorname{div}\omega^h$. Therefore provided the initialization is correct, it is unlikely that large distorsions of the vortex lines could develop.

Our main result is the

THEOREM 5.2. — *Assume that* ω_0 *is smooth enough. Assume that the following conditions hold:*

$$\zeta\in W^{m,\infty}(\mathbb{R}^3)\cap W^{m,1}(\mathbb{R}^3),\qquad \forall m>0, \tag{5.10}$$

$$\int\zeta(x)\,dx=1$$

$$\int x^\alpha\zeta(x)\,ds=0,\qquad 1\leqq|\alpha|\leqq d-1; \tag{5.11}$$

$$\int|x|^d|\zeta(x)|\,dx<+\infty$$

and that there exist two strictly positive constants C *and* s *such that*

$$h\leqq C\varepsilon^{1+s}. \tag{5.12}$$

Then there exists a time τ *and constants* C *depending only on* ω_0 *such that for* h *and* ε *small enough*

$$\left.\begin{array}{l}\|(u-u^h)(.,t)\|_{0,p}\leqq C\varepsilon^d,\\ p\in]3/2,+\infty],\qquad t\in[0,\tau].\end{array}\right\} \tag{5.13}$$

In fact, it can be proved (but the argument below must be slightly modified) that (5.13) holds as long as there exists a smooth solution to the Euler equations. This proves in particular that, at least for h and ε small enough satisfying (5.12), existence of solutions for the system (5.7), (5.8) holds for any time interval in which the Euler equations have a smooth solution.

First of all we need a result analogue to lemma 2.2 dealing with convection equations with terms of order 0. Consider the following problem

$$\frac{\partial \xi}{\partial t}+\nabla.(v \otimes \xi)+(\xi.\nabla)\, a=\theta \qquad (5.14)$$
$$\xi(.,0)=\xi_0$$

where ξ_0, $\xi(.,t)$ (resp. a and v) are vector-valued measures (resp. continuous functions). We get

LEMMA 5.3. — *Let $m \geqq 1$, $p \in]1, +\infty[$, and $\tau>0$. Let v and a be vector valued functions respectively in $\mathbf{L}^\infty(0,\tau;(\mathbf{W}^{m,\infty}(\mathbb{R}^n))^n)$ and in $\mathbf{L}^\infty(0,\tau;(\mathbf{W}^{m+1,\infty}(\mathbb{R}^n))^n)$; given ξ_0 in $(\mathbf{W}^{-m,p}(\mathbb{R}^n))^n$ and θ in $\mathbf{L}^1(0,\tau;(\mathbf{W}^{-m,p}(\mathbb{R}^n))^n)$, the problem (2.2) has a unique solution ξ in $\mathbf{L}^\infty(0,\tau;(\mathbf{W}^{-m,p}(\mathbb{R}^n))^n)$ and there exists a constant C only depending on τ and v such that*

$$\|\xi(.,t)\|_{-m,p} \leqq C\left\{\|\xi_0\|_{-m,p}+\int_0^t \|\theta(.,s)\|_{-m,p}\,ds\right\}, \qquad (5.15)$$
$$t \in [0,\tau].$$

Moreover, assume that ξ_0 and $\theta(.,t)$ belong to $\vartheta_{m,p}$ with $p>3/2$ and that $a \in \mathbf{L}^\infty(0,\tau;(\mathbf{W}^{m+1,3}(\mathbb{R}^n))^n)$; let q be such that $1/q=1/p+1/3$. Then we can write

$$\xi(.,t)=\bar{\xi}(.,t)+\xi'(.,t)$$

where $\bar{\xi} \in \mathbf{L}^q(\mathbb{R}^n)$ and $\xi' \in \vartheta_{m,p}$ satisfy

$$\|\bar{\xi}(.,t)\|_{0,q}+[\xi'(.,t)]_{-m,p} \leqq C\left\{[\xi_0]_{-m,p}+\int_0^t [\theta(.,s)]_{-m,p}\,ds\right\}, \qquad (5.16)$$
$$t \in [0,\tau].$$

Proof. — We only give the proof of (5.16). We proceed as in lemma 2.2; to begin with, we define the operators **L** and **L*** as follows

$$\mathbf{L}\,\varphi=\frac{\partial\varphi}{\partial t}+\nabla.(v\,\varphi)+(\varphi.\nabla)\,a$$

$$\mathbf{L}^*\,\varphi=-\left(\frac{\partial\varphi}{\partial t}+(v.\nabla)\,\varphi\right)+(\varphi\times\nabla)\,a)$$

where $(\varphi\times\nabla)\,a)$ stands for the vector $(\sum_j \varphi_j\,\partial a_j/\partial x_i)_i$.

Let us denote by $[\nabla a]^*$ the matrix $(\partial a_j/\partial x_i)_{i,\,j}$ and by Exp the exponential of matrices. We observe that it is possible to write

$$-\mathbf{L}^*\,\varphi(\mathrm{X}(t;y,s),t)=\left[\operatorname{Exp}\left(\int_s^t[\nabla a]^*(\mathrm{X}(\sigma;y,s),\sigma)\right)d\sigma\right]$$

$$\frac{d}{dt}\left\{\left[\operatorname{Exp}\left(-\int_s^t[\nabla a]^*(\mathrm{X}(\sigma;y,s),\sigma)\right)d\sigma\right][\varphi(\mathrm{X}(t;y,s),t)]\right\}$$

Next, starting from the following decomposition of ξ_0 and θ in $\vartheta_{m,\,p}$

$$\theta=\sum_{1\leq|\alpha|\leq m}\partial^\alpha\theta_\alpha,\qquad \xi_0=\sum_{1\leq|\alpha|\leq m}\partial^\alpha\xi_{0,\,\alpha},$$

it is straightforward to check that the distribution $\xi(.\,,t)$ defined by

$$\langle\xi(.\,,t),\varphi\rangle=\sum_\alpha(-1)^{|\alpha|}\left\{\int\xi_{0,\,\alpha}(y)\right.$$

$$\partial_y^\alpha\left\{\left[\operatorname{Exp}\left(-\int_0^t[\nabla a]^*(\mathrm{X}(\sigma;y,0),\sigma)\right)d\sigma\right][\varphi(\mathrm{X}(t;y,0))]\right\}dy$$

$$+\int_0^t ds\int\theta_\alpha(y,s)$$

$$\left.\partial_y^\alpha\left\{\left[\operatorname{Exp}\left(-\int_0^t[\nabla a]^*(\mathrm{X}(\sigma;y,s),\sigma)\right)d\sigma\right][\varphi(\mathrm{X}(t;y,s))]\right\}dy\right\}$$

is indeed the weak solution [still in the sense of (2.6), (2.7)] of (5.26).

Now we use the smoothness of X and Holder's inequality to estimate the integrals in the above right hand side; we obtain for each value of α

$$\left|\int \xi_{0,\alpha}(y)\,\partial_y^\alpha\left\{\left[\operatorname{Exp}\left(-\int_0^t [\nabla a]^*(X(\sigma; y, 0), \sigma)\right)d\sigma\right][\varphi(X(t; y, 0))]\right\}dy\right|$$
$$\leqq C(\alpha, a)\,\|\xi_{0,\alpha}\|_{0,p}\sum_{k\leqq m}\|\nabla a\|_{m,p_k^1}\|\varphi\|_{k,p_k^2}$$

$$\left|\int_0^t ds\int \theta_\alpha(y, s)\,\partial_y^\alpha\left\{\left[\operatorname{Exp}\left(-\int_0^t [\nabla a]^*(X(\sigma; y, s), \sigma)\right)d\sigma\right]\right.\right.$$
$$\left.\left.\times[\varphi(X(t; y, s))]\right\}dy\right|$$
$$\leqq C(\alpha, a)\int_0^t \|\theta_\alpha(.\,, s)\|_{0,p}\sum_{k\leqq m}\|\nabla a\|_{m,p_k^1}\|\varphi\|_{k,p_k^2}\,ds$$

where in the above formulas: the p_k^i satisfy $1/p_k^1+1/p_k^2=1-1/p$, $\|\nabla a\|$ stands for Max $(\|\nabla a(.\,,\sigma)\|,\ \sigma\leqq t)$, $C(\alpha, a)$ is a constant which can be written in terms of $\|\nabla a\|_{m,\infty}$.

We choose $p_0^1=3$, $p_0^2=q^*$ and $p_k^1=\infty$, $p_k^2=p^*$ if $k>0$ (p^* and q^* denote the conjugate exponents of p and q) and we obtain finally

$$|\langle\xi(.\,,t),\varphi\rangle|\leqq C\left\{[\xi_0]_{-m,p}+\int_0^t[\theta(.\,,s)]_{-m,p}\,ds\right\}$$
$$\Big\{\|\varphi\|_{0,q^*}+\sum_{1\leqq k\leqq m}|\varphi|_{k,p^*}\Big\}.$$

The decomposition $\xi(.\,,t)=\bar{\xi}(.\,,t)+\xi'(.\,,t)$ and the estimate (5.16) result now easily from lemma 2.1 with the particular choice $p_0=q$, $p_k=p$ if $k\neq 0$. ■

The proof keeps going like in two dimensions. We state now a lemma which is the 3 D analogue of lemma 3.2.

LEMMA 5.4. — (i) *Let* $\mathbf{T}$ *be in* $\mathbf{L}^q(\mathbb{R}^3)$ *with* $q<3$; *then*

$$\|\mathbf{K}*\mathbf{T}\|_{0,p}\leqq C\|\mathbf{T}\|_{0,q},\qquad 1/p=1/q-1/3. \tag{5.17}$$

(ii) *Let* ζ *ve in* $\mathbf{L}^\infty(\mathbb{R}^3)\cap\mathbf{L}^1(\mathbb{R}^3)$ *and* $\mathbf{T}$ *be in* $\mathbf{L}^p(\mathbb{R}^3)$ *with* $q<3$ *and let* q *such that* $1/q\leqq 1/p-1/3$; *then*

$$\|\mathbf{K}_\varepsilon*\mathbf{T}\|_{0,p}\leqq C\varepsilon^{-\alpha}\|\mathbf{T}\|_{0,q},\qquad \alpha=3(1/p-1/q)-1. \tag{5.18}$$

(iii) *Let* ζ *be in* $W^{m,\infty}(\mathbb{R}^3)\cap W^{m,1}(\mathbb{R}^3)$ *and* $\mathbf{T}$ *be in* $\vartheta_{m,p}$, $m\geqq 1$, $1<p<+\infty$; *then*

$$\|\mathbf{K}_\varepsilon * \mathbf{T}\|_{0,p}\leqq C/\varepsilon^{m-1}[\mathbf{T}]_{-m,p}. \tag{5.19}$$

(iv) *Under the assumption* (5.11) *we get*

$$\begin{aligned}&\|(\mathbf{K}-\mathbf{K}_\varepsilon) * \mathbf{T}\|_{0,p}\leqq C\varepsilon^d(\|\mathbf{T}\|_{d,1}+\|\mathbf{T}\|_{d,\infty}),\\ &\quad \text{if}\quad \mathbf{T}\in W^{d,\infty}(\mathbb{R}^3)\cap W^{d,1}(\mathbb{R}^3);\\ &\|(\mathbf{K}-\mathbf{K}_\varepsilon) * \mathbf{T}\|_{0,p}\leqq C\varepsilon^d\|\mathbf{T}\|_{d-1,p},\\ &\quad \text{if}\quad \mathbf{T}\in W^{d-1,p}(\mathbb{R}^3),\quad 1<p<\infty.\end{aligned} \tag{5.20}$$

We shall not prove these results; the only difference with the 2 D situation appears in the value of q in (5.17) and α in (5.18). These modifications result from the modifications in the Sobolev imbedding in 3 D.

Next we introduce the solution $(\omega_\varepsilon, u_\varepsilon)$ of

$$\begin{aligned}&\frac{\partial\omega_\varepsilon}{\partial t}+\nabla.(u_\varepsilon\otimes\omega_\varepsilon)-(\omega_\varepsilon.\nabla)u_\varepsilon=0\\ &\omega_\varepsilon(.,0)=\omega_0\\ &u_\varepsilon=\mathbf{K}_\varepsilon * \omega_\varepsilon\end{aligned} \tag{5.21}$$

and we have

LEMMA 5.5. — *Assume that* (5.11) *holds and that* $\omega_0\in W^{m,\infty}(\mathbb{R}^3)\cap W^{m,1}(\mathbb{R}^3)$ *with* $m\geqq d$. *Then there exist a time* τ *and constants* C *depending only on* ω_0 *such that for* $0\leqq t\leqq\tau$

$$\left.\begin{aligned}\|\omega(.,t)\|_{m,1}+\|\omega(.,t)\|_{m,\infty}\leqq C\\ \|\omega_\varepsilon(.,t)\|_{m,1}+\|\omega_\varepsilon(.,t)\|_{m,\infty}\leqq C\end{aligned}\right\} \tag{5.22}$$

$$\|(u-u_\varepsilon)(.,t)\|_{0,p}\leqq C\varepsilon^d,\qquad 3/2<p<+\infty. \tag{5.23}$$

Proof. — Like in two dimensions we omit the proof of (5.22) because it is obviously related to the existence of smooth solutions for the original Euler equations [corresponding to the case $\varepsilon=0$ in (5.21)] for small time (*see* [12] for a proof of this result). Thus let us focus on (5.23).

Starting from (5.1)-(5.3) and (5.21) we obtain

$$\frac{\partial}{\partial t}(\omega-\omega_\varepsilon)+\nabla.(u_\varepsilon\otimes(\omega-\omega_\varepsilon))-(\omega_\varepsilon.\nabla)(u-u_\varepsilon)$$
$$=\nabla.((u_\varepsilon-u)\otimes\omega)-((\omega_\varepsilon-\omega).\nabla)u$$
$$(\omega-\omega_\varepsilon)(.,0)=0$$

Let us then consider some real number $q\in[1,+\infty]$. Argueing like in two dimensions we get

$$\|(\omega-\omega_\varepsilon)(.,t)\|_{0,q}\leqq\int_0^t\|\nabla.((u-u_\varepsilon)\,\omega_\varepsilon)(.,s)\|_{0,q}\,ds$$
$$+\int_0^t\|(\omega_\varepsilon.\nabla)(u-u_\varepsilon)](.,s)\|_{0,q}\,ds$$
$$+\int_0^t\|((\omega-\omega_\varepsilon).\nabla)u_\varepsilon](.,s)\|_{0,q}\,ds. \quad (5.24)$$

Next we write

$$u-u_\varepsilon=\mathbf{K}_\varepsilon*(\omega-\omega_\varepsilon)+(\mathbf{K}-\mathbf{K}_\varepsilon)*\omega.$$

By (5.19) and (5.20) we have then, for $1<q<+\infty$

$$|u-u_\varepsilon|_{1,q}\leqq C(\|\omega-\omega_\varepsilon\|_{0,q}+\varepsilon^d\|\omega\|_{d,q})$$

and therefore for $t\leqq\tau$, using (5.22) we get:

$$\|(\omega_\varepsilon.\nabla)(u-u_\varepsilon)(.,s)\|_{0,q}\leqq C(\varepsilon^d+\|(\omega-\omega_\varepsilon)(.,s)\|_{0,q}). \quad (5.25)$$

Now assume that $q<3$ and let p be such that $1/p+1/3=1/q$; we can write, using (5.17) and (5.20)

$$\|u-u_\varepsilon\|_{0,p}\leqq C(\varepsilon^d\|\omega\|_{d-1,q}+\|\omega-\omega_\varepsilon\|_{0,q}). \quad (5.26)$$

By Holder's inequality and (5.22) this implies, for $t\leqq\tau$:

$$\|((u-u_\varepsilon).\nabla)\,\omega_\varepsilon\|_{0,q}\leqq\|u-u_\varepsilon\|_{0,p}\|\nabla\omega_\varepsilon\|_{0,3} \quad (5.27)$$
$$\leqq C(\varepsilon^d+\|\omega-\omega_\varepsilon\|_{0,q}).$$

Obviously we have also

$$\|((\omega-\omega_\varepsilon).\nabla)u\|_{0,q}\leqq C\|\omega-\omega_\varepsilon\|_{0,q}. \quad (5.28)$$

Thus, combining (5.22), (5.23), (5.25) and (5.26) yields for $q<3$ and $t \leqq \tau$

$$\|(\omega-\omega_\varepsilon)(.,t)\|_{0,q} \leqq C\left(\varepsilon^d+\int_0^t \|(\omega-\omega_\varepsilon)(.,s)\|_{0,q}\,ds\right)$$

which by Gronwall's lemma implies

$$\|(\omega-\omega_\varepsilon)(.,t)\|_{0,q} \leqq C\,\varepsilon^d, \qquad q \in]1,3[, \quad t \leqq \tau.$$

To obtain the desired estimates on the velocity it remains only to combine (5.26) and the above estimate for the appropriate value of q. ■

Like in the two dimensional case we write now

$$\omega_\varepsilon-\omega^h=\lambda^h+\mu^h$$

where λ^h and μ^h are respectively solutions of

$$\begin{aligned} &\frac{\partial \mu^h}{\partial t}+\nabla.(u_\varepsilon \otimes \mu^h)-(\mu^h.\nabla)\,u_\varepsilon=0 \\ &\mu^h(.,0)=\omega_0-\omega_0^h \end{aligned} \tag{5.29}$$

$$\begin{aligned} &\frac{\partial \lambda^h}{\partial t}+\nabla.(u^h \otimes \lambda^h)-(\lambda^h.\nabla)\,u^h \\ &\quad =\nabla.((u_\varepsilon-u^h)\otimes(\mu^h-\omega_\varepsilon))-((\mu^h-\omega_\varepsilon).\nabla)(u_\varepsilon-u^h) \\ &\lambda^h(.,0)=0. \end{aligned} \tag{5.30}$$

In all the sequel the real number τ will take the above specified value. Denote now by X_ε the solutions of (2.3) with u_ε instead of v; we set $X_{j,\varepsilon}(t)=X_\varepsilon(t; x_j, 0)$; we get

LEMMA 5.6. — (i) *Let* $m \geqq 3$. *Assume* $\omega_0 \in W^{m+1,\infty}(\mathbb{R}^3) \cap W^{m+1,1}(\mathbb{R}^3)$; *then we get:*

$$\|\mu^h(.,t)\|_{-m,p} \leqq C\,h^m, \qquad t \in [0,\tau], \quad p \in]1,+\infty[. \tag{5.31}$$

Moreover we can write for any $p \in]3/2,+\infty[$ *and* q *such that* $1/q=1/p+1/3$

$$\mu^h=\bar{\mu}^h+\mu'^h \tag{5.32}$$

where $\bar{\mu}^h(.,t) \in L^q(\mathbb{R}^3)$, $\mu'^h(.,t) \in \vartheta_{m,p}$ *satisfy*

$$\|\bar{\mu}^h(.,t)\|_{0,q} \leqq C\,h^m, \qquad [\mu'^h(.,t)]_{-m,p} \leqq C\,h^m. \tag{5.33}$$

(ii) *The following identity holds:*

$$(\omega_\varepsilon - \mu^h)(x, t) = h^3 \sum_j \omega_\varepsilon(X_{j,\varepsilon}(t), t)\, \delta(x - X_{j,\varepsilon}(t)). \tag{5.34}$$

Proof. — Let us consider $p, q \in]1, +\infty[$ and $m \geqq 3$. We observe that, as in the proof of lemma 3.4, lemma 2.3 implies that, using the same arguments as in two dimensions, we can write

$$\mu^h(.,0) = \sum_{0 \leqq |\alpha| \leqq m} \xi_{0,\alpha}$$

where

$$\| \xi_{0,\theta} \|_{0,q} \leqq C\, h^m; \qquad [\xi_{0,\alpha}]_{-|\alpha|,p} \leqq C\, h^m.$$

Then, since (5.30) is a linear problem, we write μ^h as $\sum \xi_\alpha$ where ξ_α is the weak solution in $W^{-|\alpha|,p}(\mathbb{R}^3)$ of the system

$$\frac{\partial \xi_\alpha}{\partial t} + \nabla . (u_\varepsilon \otimes \xi_\alpha) - (\xi_\alpha . \nabla)\, u_\varepsilon = 0.$$

$$\xi_\alpha(.,0) = \xi_{0,\alpha}.$$

Since $\omega_0 \in W^{m+1,\infty}(\mathbb{R}^3) \cap W^{m+1,1}(\mathbb{R}^3)$ we have by (5.23)

$$\| u_\varepsilon(.,t) \|_{m+1,\infty} \leqq C, \qquad t \leqq \tau.$$

Therefore lemma 5.3 applies and gives

$$\| \xi_\alpha \|_{-|\alpha|,p} \leqq C\, h^m.$$

Furthermore, by (5.16) we can write $\xi_\alpha = \bar{\xi}_\alpha + \xi'_\alpha$ with

$$\| \bar{\xi}_\alpha \|_{0,q} + [\xi'_\alpha]_{-|\alpha|,p} \leqq C\, h^m.$$

Writing $\bar{\mu}^h = \sum \bar{\xi}_\alpha$, $\mu'^h = \sum \xi'_\alpha$, this last estimate yields (5.33).

To check (5.34) we first observe that $\omega_\varepsilon - \mu^h$ is solution of

$$\begin{aligned} &\frac{\partial}{\partial t}(\omega_\varepsilon - \mu^h) + \nabla (u_\varepsilon \otimes (\omega_\varepsilon - \mu^h)) - (\omega_\varepsilon - \mu^h) . \nabla u_\varepsilon = 0 \\ &(\omega_\varepsilon - \mu^h)(x, 0) = h^3 \sum_j \omega_0(x_j)\, \delta(x - x_j). \end{aligned} \tag{5.35}$$

Next since $(u_\varepsilon, \omega_\varepsilon)$ is solution of (5.18) we have clearly

$$\frac{d}{dt}(\omega_\varepsilon(X_{j,\varepsilon}(t), t) = \left(\frac{\partial \omega_\varepsilon}{\partial t} + (u_\varepsilon . \nabla)\,\omega_\varepsilon\right)(X_{j,\varepsilon}(t), t)$$
$$= ([\nabla u_\varepsilon]\,\omega_\varepsilon)\,(X_{j,\varepsilon}(t), t).$$

Therefore by lemma 5.1 the distribution

$$h^3 \sum_j \omega_\varepsilon(X_{j,\varepsilon}(t), t)\,\delta(x - X_{j,\varepsilon}(t))$$

is also a measure solution of (5.29) and thus coincides with $\omega_\varepsilon - \mu^h$. ■

As in the two dimensional case the main part of the proof is now devoted to estimating λ^h. First we write

$$\rho^h = \nabla . ((u_\varepsilon - u^h) \otimes (\omega_\varepsilon - \mu^h))$$
$$\nu^h = ((\omega_\varepsilon - \mu^h) . \nabla)\,(u_\varepsilon - u^h).$$

For brevity we introduce the following notation: if a, b, c are three vector fields we shall denote by $[a, b, c]$ the scalar function

$$\sum_{i,j=1}^{3} a_i\, b_j\, \partial c_i / \partial x_j.$$

Next we set $\rho^h = \rho_1^h + \rho_2^h$ where, for any smooth vector valued function φ with compact support

$$\langle \rho_1^h(.\,, t), \varphi \rangle = \int [\omega_\varepsilon, (u_\varepsilon - u^h), \varphi]\,(X_\varepsilon(t; y, 0), t)\, dy \qquad (5.36)$$

$$\langle \rho_2^h(.\,, t), \varphi \rangle = \sum_j \int_{B_j} [\omega_0, (u_\varepsilon - u^h), \varphi]\,(X_\varepsilon(t; y, 0), t)\, dy$$
$$- \sum_j [\alpha_j, (u_\varepsilon - u^h), \varphi]\,(X_{j,\varepsilon}(t), t). \qquad (5.37)$$

We write also $\nu^h = \nu_1^h + \nu_2^h$ where

$$\langle \nu_1^h(.\,, t), \varphi \rangle = \int [\varphi, \omega_\varepsilon, (u_\varepsilon - u^h)]\,(X_\varepsilon(t, y, 0), t)\, dy \qquad (5.38)$$

$$\langle v_2^h(.,t), \varphi \rangle = \sum_j \int_{B_j} [\varphi, \omega_\varepsilon, (u_\varepsilon - u^h)] (X_\varepsilon(t, y, 0), t)\, dy$$
$$- \sum_j [\varphi, \alpha_j, (u_\varepsilon - u^h)] (X_{j,\varepsilon}(t), t). \quad (5.39)$$

As expected, estimating v^h, which indeed takes into account the effect of the stretching term on the error, is almost straigthforward. In particular the predominent part v_1^h is directly related to the L^p norm of $u - u^h$:

LEMMA 5.7. — *Let* $p \in]3/2, +\infty[$, q *such that* $1/q = 1/p + 1/3$ *and* $r \geqq 3$. *Assume that* $\omega_0 \in W^{r,\infty}(\mathbb{R}^3) \cap W^{r,1}(\mathbb{R}^3)$. *Then the following assertions hold for* $t \in [0, \tau]$:

(i) *We have:* $v_1^h \in \vartheta_{1,p} \cap L^p(\mathbb{R}^3)$, *and:*

$$\| v_1^h(.,t) \|_{-1,p} \leqq C \| (u_\varepsilon - u^h)(.,t) \|_{0,p} \quad (5.40)$$

$$\| v_1^h(.,t) \|_{0,p} \leqq C \| (u_\varepsilon - u^h)(.,t) \|_{1,p} \quad (5.41)$$

(ii) *we can write*

$$v_2^h = \sum_{0 \leqq k \leqq r} v_{2,k}^h,$$

with

$$v_{2,k}^h(.,t) \in W^{-k,p}(\mathbb{R}^2) \quad \text{for } 1 \leqq k \leqq r, \qquad v_{2,0}^h(.,t) \in L^q(\mathbb{R}^3)$$

and

$$\| v_{2,k}^h(.,t) \|_{-k,p} \leqq C h^r \| (u_\varepsilon - u^h)(.,t) \|_{r+1-k,p},$$
$$0 \leqq k \leqq r. \quad (5.42)$$
$$\| v_{2,0}^h(.,t) \|_{0,q} \leqq C h^r \| (u_\varepsilon - u^h)(.,t) \|_{r+1,p}.$$

Proof. — Using the change of variables $z = X_\varepsilon(t; y, 0)$ and integrating by parts gives

$$\langle v_1^h(.,t), \varphi \rangle = \int \left(\sum_{i,j} (u_\varepsilon - u^h)_i \frac{\partial}{\partial x_j} (\omega_{\varepsilon,j}\, \varphi_i) \right) (z, t)\, dz$$

and thus, denoting by p^* the conjugate exponent of p

$$| \langle v_1^h(.,t), \varphi \rangle | \leqq \| \omega_\varepsilon(.,t) \|_{0,\infty} \| (u_\varepsilon - u^h)(.,t) \|_{0,p} \| \nabla \varphi \|_{0,p^*}$$

which gives (5.40) and implies that $v_1^h \in \vartheta_{1,p}$; (5.41) is derived in a similar way (without integrating by parts).

Next using the quadrature formula of order $r \geqq 3$ in $\mathbb{R}^3$ given in lemma 2.3 yields immediatley:

$$|\langle v_2^h(.,t), \varphi \rangle| \leqq C h^r \| (\omega_\varepsilon . \nabla (u_\varepsilon - u^h) . \varphi)(X_\varepsilon(t; ., 0), t) \|_{r,1}$$

and, since by lemma 5.5 ω_ε and X_ε are smooth, it follows from the Holder's inequality that, if q^* denotes the conjugate exponent of q:

$$|\langle v_2^h(.,t), \varphi \rangle| \leqq C h^t \sum_{1 \leqq k \leqq r} \| \omega_\varepsilon(.,t) \|_{r,\infty}$$

$$\times \| (u_\varepsilon - u^h)(.,t) \|_{r+1-k,p} | \varphi |_{k,p^*}$$

$$+ \| \omega_\varepsilon(.,t) \|_{r,3} \| (u_\varepsilon - u^h)(.,t) \|_{r+1,p} | \varphi |_{k,q^*}.$$

The estimate given in the assertion (ii) follows then from lemma 2.1. ■

Argueing as in the two dimensional case we can also prove

LEMMA 5.8. — (i) *We have* $\rho_1^h(.,t) \in \vartheta_{1,p}$ and

$$\| \rho_1^h(.,t) \|_{-1,p} \leqq C \| (u_\varepsilon - u^h)(.,t) \|_{0,p}. \tag{5.43}$$

(ii) *The following decomposition holds*

$$\rho_2^h = \sum_{1 \leqq k \leqq r} \rho_{2,k}^h, \rho_{2,k}^h(.,t) \in \vartheta_{k,p}$$

with the bounds

$$\left. \begin{array}{c} \| \rho_{2,k}^h(.,t) \|_{-k,p} \leqq C h^r \| (u_\varepsilon - u^h)(.,t) \|_{r+1-k,p}, \\ 1 \leqq k \leqq r+1. \end{array} \right\} \tag{5.44}$$

We can now put an end to the

Proof of the theorem. — Let us first fix the value of the integer r introduced in lemma 5.6. We choose r such that

$$h^r \leqq C \varepsilon^{r+1} \tag{5.45}$$

for ε small enough, which is indeed possible by (5.12). Moreover ω_0 will be assumed to be in $W^{r+1,\infty}(\mathbb{R}^3) \cap W^{r+1,1}(\mathbb{R}^3)$. In all the sequel p will be in $]3/2, +\infty[$ and q will be such that $1/q = 1/p + 1/3$.

Let us consider the solution λ_1^h of (5.30) with right hand side $\rho_1^h+\nu_1^h+\rho_{2,1}^h+\nu_{2,1}^h$, that is

$$\frac{\partial\lambda_1^h}{\partial t}+\nabla.(u^h\otimes\lambda_1^h)-(\lambda_1^h.\nabla)\,u^h=\rho_1^h+\nu_1^h+\rho_{2,1}^h+\nu_{2,1}^h$$
$$\lambda_1^h(.,0)=0. \tag{5.46}$$

Since, by lemma 5.6 and 5.7, $\rho_1^h+\nu_1^h+\rho_{2,1}^h+\nu_{2,1}^h$ is in $\mathbf{W}^{-1,p}(\mathbb{R}^3)$, lemma 5.3 applies and gives

$$\lambda_1^h=\bar{\lambda}_1^h+\lambda_1^{h\prime},$$

where

$$\bar{\lambda}_1^h(.,t)\in\mathbf{L}^q(\mathbb{R}^3),\qquad \lambda_1^{h\prime}(.,t)\in\vartheta_{1,p}.$$

Proceeding similarily, we introduce for $k\in[2,r]$ λ_k^h, the solution of (5.30) with right hand side $\rho_{2,k}^h+\nu_{2,k}^h$. In addition, for $k=r+1, \lambda_{r+1}^h$ will be the solution of (5.30) with right hand side $\rho_{2,r+1}^h$. Then we write

$$\lambda_k^h=\bar{\lambda}_k^h+\lambda_k^{h\prime},$$

where

$$\bar{\lambda}_k^h(.,t)\in\mathbf{L}^q(\mathbb{R}^3),\qquad \lambda_k^{h\prime}(.,t)\in\vartheta_{k,p}.$$

For $k=0$, λ_0^h will denote the solution of (5.30) with right hand side $\nu_{2,0}^h$ and therefore $\lambda_0^h(.,t)\in\mathbf{L}^q(\mathbb{R}^3)$.

Finally we introduce the following notation:

$$\bar{\lambda}^h=\sum_{k=1}^{r+1}\bar{\lambda}_k^h+\lambda_0^h.$$

It results from the above definitions that we get

$$\lambda^h=\sum_{k=1}^{r+1}\lambda_k^{h\prime}+\bar{\lambda}^h, \tag{5.47}$$

where

$$\bar{\lambda}^h(.,t)\in\mathbf{L}^q(\mathbb{R}^3),\qquad \lambda_k^{h\prime}(.,t)\in\vartheta_{k,p},\quad k\in[1,r+1].$$

Now let us define

$$\tau_M^h = \sup\{t \in [0, \tau]; \|u^h(., t)\|_{r+1, \infty} \leqq M\}$$

where M is a constant to be defined below.

Starting from the estimates proved in lemmata 5.6 and 5.7 on the one hand, and lemma 5.3 on the other hand, it is easily seen that there exist positive constants only depending on ω_0, τ and M such that the following estimates hold

$$\|\bar{\lambda}^h(., t)\|_{0, q} \leqq C \int_0^t \{\|(u_\varepsilon - u^h)(., s)\|_{0, p} + h^r \|(u_\varepsilon - u^h)(., s)\|_{r+1, p}\} ds$$

$$[\lambda_1^{h\prime}(., t)]_{0, q} \leqq C \int_0^t \{\|(u_\varepsilon - u^h)(., s)\|_{0, p} + h^r \|(u_\varepsilon - u^h)(., s)\|_{r, p}\} ds \quad (5.48)$$

$$[\lambda_k^{h\prime}(., t)]_{-k, p} \leqq C \int_0^t h^r \|(u_\varepsilon - u^h)(., s)\|_{r-k+1, p} ds, \quad k \in [2, r+1].$$

Let

$$y_p(t) = \int_0^t (\|\bar{\lambda}^h(., s)\|_{0, q} + \sum_{1 \leqq k \leqq r+1} \varepsilon^{1-k} [\lambda_k^{h\prime}(., s)]_{-k, p}) ds.$$

Our purpose is now to bound $\|u_\varepsilon - u^h\|_{k, p}$ in terms of y_p. For this we write

$$u_\varepsilon - u^h = \mathbf{K}_\varepsilon * (\omega_\varepsilon - \omega^h) = \mathbf{K}_\varepsilon * (\lambda^h + \mu^h).$$

On the one hand, combining (5.32), (5.33) and (5.17), (5.19) yields

$$\|\mathbf{K}_\varepsilon * \mu^h\|_{k, p} \leqq C h^m / \varepsilon^{m-1+k}. \quad (5.49)$$

On the other hand, we get, still by (5.17) and (5.19)

$$\left.\begin{aligned} \|\mathbf{K}_\varepsilon * \lambda_i^{h\prime}\|_{k, p} &\leqq C \varepsilon^{-k-i+1} [\lambda_i^{h\prime}]_{-i, p} \quad \text{for} \quad i \neq 0 \\ \|\mathbf{K}_\varepsilon * \bar{\lambda}^h\|_{k, p} &\leqq C \varepsilon^{-k} \|\bar{\lambda}^h\|_{0, q}. \end{aligned}\right\} \quad (5.50)$$

Therefore combining (5.40), (5.41), (5.42) leads after straigthforward calculations very similar to those made in the two-dimensional case to the following differential inequality

$$\frac{\partial}{\partial t} y_p(t) \leqq C \left\{ y_p(t) \left(1 + \frac{h^r}{\varepsilon^{r+1}} \right) + \frac{h^m}{\varepsilon^{m-1}} \right\}, \qquad t \in [0, \tau_M^h].$$

But by (5.45) this yields

$$\frac{\partial}{\partial t} y_p(t) \leqq C \left\{ y_p(t) + \frac{h^m}{\varepsilon^{m-1}} \right\}, \qquad t \in [0, \tau_M^h],$$

which by Gronwall's theorem finally gives

$$y_p(t) \leqq C\, h^m / \varepsilon^{m-1}, \qquad t \in [0, \tau_M^h].$$

Taking then

$$M = 2 \max_{0 < t < \tau;\ \varepsilon > 0} \| u_\varepsilon(\cdot, t) \|_{r+1, \infty}$$

and using (5.49) and (5.50) with $k = r + 1$, combined with (5.22) allows us to check as in the two dimensional case that $\tau_M^h = \tau$.

We have thus proved that

$$\| u - u_\varepsilon(\cdot, t) \|_{0, p} \leqq C\, h^m / \varepsilon^{m-1}, \qquad t \in [0, \tau].$$

Using in addition (5.23) completes now easily the proof of the theorem.

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