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Periodic solutions for a class of Lorenz-Lagrangian systems

by

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ABSTRACT. — The class of Lorenz-Lagrangian systems under consideration are those of the form $-A\ddot{q}=\nabla V(q)$, $q\in\mathbb{R}^n$, where A is a real, symmetric matrix with eigenvalues $\mu_1<0<\mu_2\leqq\ldots\leqq\mu_n$, the corresponding eigenvectors being e_i, $1\leqq i\leqq n$. If M^+ and M^- are disjoint infinite submanifolds of $\mathbb{R}^n$ which are the graphs of bounded real-valued functions on span $\{e_2,\ldots,e_n\}$ with $V=0$ on $M^+\cup M^-$, $V>0$ on the region Ω between M^+ and M^-, and $\langle\nabla V, e_1\rangle\neq 0$ on $\mathbb{R}^n\setminus\Omega$, then we show that there exists a periodic solution of this system, provided that ∇V points towards the e_1 axis outside a large cylinder centred on the e_1 axis.

Key words : Periodic solution, Lorenz-Lagrangian system, indefinite Hamilton system.

RÉSUMÉ. — On étudie une classe de systèmes de Lorentz-Lagrange dans $\mathbb{R}^n$ de la forme $-A\ddot{q}=\nabla V(q)$, où A est une matrice sysmétrique réelle de valeurs propres $\mu_1<0<\mu_2\leqq\ldots\leqq\mu_n$. Les vecteurs propres correspondants sont notés $e_1,\ldots,e_n$. On suppose qu'il existe sur l'hyperplan engendré par $e_2,\ldots,e_n$ deux fonctions bornées, dont les graphes M^+ et M^- sont disjoints, et telles que $V>0$ entre M^+ et M^- et $V=0$ sur M^+ et M^-.

Classification A.M.S. : 34 C 15, 34 C 25, 34 C 28, 34 C 35.

On montre alors que le système possède une solution périodique pourvu que $\langle \nabla V, e_1 \rangle \neq 0$ hors de Ω et que ∇V soit dirigé vers l'axe e_1 loin de celui-ci.

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1. INTRODUCTION

In [1] Hofer and the present author introduced a class of indefinite Hamiltonians of the form $(1/2)\langle Sp, p\rangle + V(q)$, where the symmetric linear operator S has exactly one negative eigenvalue. The corresponding Lagrangian is $(1/2)\langle S^{-1}\dot{q}, \dot{q}\rangle - V(q)$ and Kozlov [2] refers to the corresponding differential equation as a Lorenz-Lagrangian system. The important feature of these equations which is exploited very strongly in [1], [3], [4] is that when $V(q)>0$ and the total energy is zero, then $\langle S^{-1}\dot{q}, \dot{q}\rangle < 0$; in other words the velocity vector is constrained to lie in a cone

$$|\lambda_1^{-1}|\dot{q}_1^2 \geqq \sum_{i=2}^{n} \lambda_i^{-1}\dot{q}_i^2$$

where λ_i denotes an eigenvalue of S and $\lambda_1 < 0$. There we obtained the existence of certain periodic, homoclinic and heteroclinic orbits using a shooting argument and the topological degree of Brouwer.

An essential feature of the analysis was a need to ensure that the shooting-map defined in terms of an exit-time is a continuous function of initial data, and to this end it was more-or-less essential to know that the set where $V>0$ is convex, or at the very least that it was well-behaved with respect to the ordering on $\mathbb{R}^n$ induced by the cone $\langle S^{-1}q, q\rangle < 0$ (see [1], [3], [4] for details). As a consequence of these assumptions the orbits whose existence is proved lie in the set $V>0$.

In this paper we return to these questions with a new idea which dispenses with the need that the shooting map be continuous. There is therefore no longer any need for the set $V>0$ to be convex, and we illustrate the idea with a surprising recurrence theorem in Section 2 and a theorem on the existence of periodic orbits in Section 3. The main idea is contained in Theorem 1 of Section 2.

As was the case with [1] the significance of the results lies in the fact that, provided the kinetic energy term is of the correct form, the existence of periodic orbits is established under hypotheses which only involve the set where $V > 0$. In the present paper we illustrate the method in a context where this set is an unbounded region between two manifolds, and under a hypothesis which is sufficient, yet avoids great technicality. The result can clearly be established under weaker assumptions, and the method of Section 3 can be adapted to give the existence of homoclinic orbits as well.

This general result on the existence of periodic orbits reduces, in the case when $n=1$, to the elementary theory of the periodic motion of a mass on a smooth wire, which oscillates under gravity from a position of rest at $t=0$ to a position of rest at $t=T/2$, where T is the period of the motion, and whose potential energy at any time $t \in (0, T/2)$ is less than its value at $t=0$.

2. THE MAIN RESULT

2.1. Notation and hypotheses

Let S denote a real, symmetric matrix with eigenvalues $\lambda_1 < 0 < \lambda_2 \leqslant \ldots \leqslant \lambda_n$, and let $\{e_1, \ldots, e_n\}$ denote the orthonormal basis comprised of eigenvectors corresponding to $\{\lambda_1, \ldots, \lambda_n\}$. Throughout elements x in $\mathbb{R}^n$ will be identified by $(x_1, \ldots, x_n)$, the tuple of the components of x with respect to this fixed basis. The corresponding *ith* partial derivative of a real-valued function f on $\mathbb{R}^n$ will be denoted by f_i, and its gradient $(f_1, \ldots, f_n)$ will be denoted by (i.e. f'). If F is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then its *ith* component will be denoted by F^i, thus $(f')^i = f_i$ in this notation when f is real-valued on $\mathbb{R}^n$.

Let $Y \subset \mathbb{R}^n$ denote the span of $\{e_2, \ldots, e_n\}$ and let $Q: \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection onto Y parallel to e_1. Note that $S: Y \to Y$ is a positive definite isomorphism which commutes with Q. Let $\langle\ ,\ \rangle$ denote the usual inner product on $\mathbb{R}^n$ with respect to the chosen basis, and let $|x|^2 = \langle x, x \rangle$.

Suppose that M^- and M^+ are smooth connected $(n-1)$ dimensional manifolds such that

(*a*) $M^- \cup M^+ \subset \{x \in \mathbb{R}^n : x_1 \in [a_-, a_+]\}$ *for some* $a_\pm \in \mathbb{R}$;

(*b*) *if* $\Omega^\pm$ *denotes the component of the complement of the manifold* $M^\pm$ *which contains the hyperplane* $x_1 = a_\pm$, *then* $\Omega^\pm$ *is simply connected and*

$$(\Omega^+ \cap M^-) \cup (\Omega^- \cap M^+) = \varnothing.$$

Let $\Omega = \mathbb{R}^n \setminus \overline{(\Omega^+ \cup \Omega^-)}$ *and* $\partial\Omega = M^+ \cup M^-$.

(*c*) *Suppose also that* $V : \mathbb{R}^n \to \mathbb{R}$ *is smooth,* $V > 0$ *in* Ω, $V = 0$ *and* $V' \neq 0$ *on* $M^+ \cup M^-$, *and there exists* $R > 0$ *and a positive definite symmetric operator* $A : Y \to Y$ *such that*

$$\langle QSV'(x), AQx \rangle < 0 \tag{2.1}$$

for all $x \in \mathbb{R}^n$ *with* $\langle AQx, Qx \rangle \geqq \mathbf{R}$.

Remark. — The introduction of the operator A acting on Y in (*c*) is the same as defining a new inner product on Y which behaves rather well with respect to QV′ outside some large cylinder. For example, (2.1) is satisfied if

$$\langle V'(x), Qx \rangle < 0 \quad \text{for all } x \in \mathbb{R}^n \tag{2.2}$$

with $|Qx|$ sufficiently large, taking A to be $S^{-1} : Y \to Y$.

We make the following additional assumption:

(*d*) *there exists a homeomorphism* $h^\pm$: $M_R^\pm = \{x \in M^\pm : |Qx| \leqq R\}$ *onto* B_R, *the closed unit ball of radius* R *in* Y, *which coincides with* Q *on the boundaries of these sets for each* R *sufficiently large.*

Remark. — In what follows it is clear that the behaviour of V outside the bounded region $\{x : a_- \leqq x_1 \leqq x_+,\ |Qx| \leqq R\}$ is irrelevant. We will therefore suppose that for some R sufficiently large V has been modified (if necessary) outside this region to ensure that solutions of (IV) in Section 2 enjoy global existence for all positive time. If (*a*)-(*c*) above hold, then by Lemma 1 this assumption means no loss of generality.

2.2. The main results

The main results, which are proved in a series of lemmas, are contained in the following theorem and its corollaries.

THEOREM 1. — *There exist two maximal closed, connected sets* $C^{\pm}$ *in* $M^{\pm} \times [0, \infty)$ *with the following properties (we state them for* $C^{-} \subset M^{-} \times [0, \infty)$, *and an analogous statement can be made for* C^{+}):

(i) $\{\tau \geqq 0 : (q, \tau) \in C^{-}$ *for some* $q \in M^{-}\} = [0, \infty)$;

(ii) $\{q \in M^{-} : (q, \tau) \in C^{-}$ *for some* $\tau \in [0, \infty)\}$ *is bounded;*

(iii) *there exists* $(q, 0) \in C^{-}$ *with* $V_i(q) = 0$, $i = 2, 3, \ldots, n$;

(iv) *if* $(q, \tau) \in C^{-}$, *then* $\dot{q}_i(\tau) = 0$, $i = 2, 3, \ldots, n$, *where* $q(t)$ *is the solution of the initial value problem*

$$\left.\begin{aligned} -\ddot{q}(t) &= SV'(q(t)), \qquad t > 0, \\ q(0) &= q, \ \dot{q}(0) = 0. \end{aligned}\right\} \tag{IVP}$$

(Here $^{\bullet}$ denotes differentiation with respect to t.)

The first preliminary lemma will be used to prove the main result in this section, and again in Section 3.

LEMMA 1. — *There exists* $\tilde{R} > 0$ *such that if* $q(t)$, $t \geqq 0$, *satisfies* (IVP) *and if* $|Qq(t^*)| \geqq \tilde{R}$ *for some* $t^* \geqq 0$ *then* $Q\dot{q}(t) \neq 0$ *for* $t \geqq t^*$.

Proof. — Since $\|y\|^2 = \langle Ay, y \rangle$ defines a norm on Y which is equivalent to $|.|$, we can choose $\tilde{R} > 0$ such that $|y| \geqq \tilde{R}$ implies that $\|y\| \geqq R$ and $|y| \geqq R$, where R is given by hypothesis (c), for $y \in Y$.

Now if $q(t)$ satisfies (IVP) and $|Qq(t^*)| \geqq \tilde{R}$ for some $t^* \geqq 0$, then $\langle AQq(t^*), Qq(t^*) \rangle \geqq R$ and so there exists $\tilde{t}$ such that

$$\begin{aligned} &\langle AQq(\tilde{t}), Qq(\tilde{t}) \rangle \geqq R \\ \text{and} \qquad & \\ &(d/dt) \langle AQq(t), Qq(t) \rangle |_{t=\tilde{t}} \geqq 0. \end{aligned} \tag{2.3}$$

If $(d/dt) \langle AQq(t), Qq(t) \rangle > 0$ at $t = \tilde{t}$, then $\langle AQq(t), Qq(t) \rangle \geqq R$ on $[\tilde{t}, \hat{t}]$ for some $\hat{t} > \tilde{t}$. If not, then a calculation yields that

$$\begin{aligned} &(d^2/dt^2) \langle AQq(t), Qq(t) \rangle |_{t=\tilde{t}} \\ &\qquad = 2 \{ \langle AQ\dot{q}, Q\dot{q} \rangle - \langle AQSV'(q), Qq \rangle \} |_{t=\tilde{t}} > 0. \end{aligned}$$

Once again $\langle AQq(t), Qq(t) \rangle \geqq R$ on $[\tilde{t}, \hat{t}]$, and in any case $\langle AQq(t), Q\ddot{q}(t) \rangle \geqq 0$ on $[\tilde{t}, \hat{t}]$. Then integration by parts gives

$$\begin{aligned} (d/dt) \langle AQq(t), Qq(t) \rangle &= 2 \langle AQq(t), Q\dot{q}(t) \rangle \\ &= 2 \langle AQq(\tilde{t}), Q\dot{q}(\tilde{t}) \rangle + 2 \int_{\tilde{t}}^{t} \{ \langle AQ\dot{q}(s), Q\dot{q}(s) \rangle \\ &\qquad + \langle AQq(s), Q\ddot{q}(s) \rangle \} \, ds > 0 \quad \text{on } [\hat{t}, \tilde{t}]. \end{aligned}$$

Therefore the supremum of all such $\hat{t}$ is $+\infty$, and $Q\dot{q}(t)\neq 0$ on $[\tilde{t}, \infty)$.
This completes the proof.

Q.E.D.

Let $R>0$ be so large that there exists a homeomorphism $h: M_R^- /\overline{B_R}$, the closure of the unit ball B_R in Y, which coincides with Q on $\partial(M_R^-)$. This is possible by hypotheses (*d*).

Now define a function $F_0: B_R \to Y$ as follows:

$$F_0(y) = QV'(h^{-1}(y)).$$

Let $\deg(\Omega, f, p)$ denote the Brouwer degree of a function f on an open bounded set Ω with respect to a point p.

LEMMA 2. — $\deg(B_R, F_0, 0)$ *is odd for* R *sufficiently large.*

Proof. — On ∂B_R,

$$\begin{aligned}\langle SF_0(y), Ay\rangle &= \langle QSV'(h^{-1}(y)), Ay\rangle \\ &= \langle QSV'(h^{-1}(y)), AQ(h^{-1}(y))\rangle\end{aligned}$$

since $Q=h$ on ∂M_R^-, by (*d*). Therefore by (*c*), $\langle SF_0(y)), A\,y\rangle<0$, $y\in\partial B_R$, and so $\deg(B_R, ASF_0, 0)$ is odd. But $AS: Y\to Y$ is a homeomorphism, and so $\deg(B_R, F_0, 0)$ is odd.

Q.E.D.

Now for each $t>0$ define $F_t: B_R\to Y$ by

$$F_t(y) = -(t^{-1})\,QS^{-1}\dot{q}(t),$$

where $q(t)$ is the solution of (IVP) with $q=h^{-1}(y)$.
The next result is almost immediate.

LEMMA 3. — $\deg(B_R, F_t, 0) = \deg(B_R, F_0, 0)$ *for all* $t>0$.

Proof. — First observe that by the continuity of $h: \bar{B}\to M_R^-$ and the standard theory of continuous dependence for initial value problems $(t, y)\to F_t(y)$ is continuous at each $(t, y)\in(0,\infty)\times\overline{B_R}$. To see that it is also continuous at $(0, y)$, $y\in\overline{B_R}$, suppose that $(t^k, y^k)\to(0, y)$. Then

$$\begin{aligned}F_{t^k}(y^k) &= -(t^k)^{-1}\,QS^{-1}\dot{q}^k(t^k)\\ &= -(t^k)^{-1}\int_0^{t_k} QS^{-1}\ddot{q}^k(t)\,dt\\ &= (t^k)^{-1}\int_0^{t_k} QV'(q^k(t))\,dt \to QV'(h^{-1}(y)) = F_0(y), \quad \text{as } k\to\infty.\end{aligned}$$

Moreover, if R is sufficiently large, then $F_t(y) \neq 0$ for all $(t, y) \in [0, \infty) \times \partial B_R$, by Lemma 1, and the fact that $S: Y \to Y$ is a homeomorphism. The result now follows the homotopy property of Brouwer degree.

Q.E.D.

Proof of Theorem 1. — In the light of these three lemmas the theorem follows as a direct consequence of the result of Rabinowitz, which says that since for R sufficiently large $\deg(B_R, F_0, 0) \neq 0$, the maximal connected subset C of the set $S = \{(y, t) : y \in B_R, t \geqq 0, F_t(y) = 0\}$ which contains the set $\{(y, 0) : y \in B_R, F_0(y) = 0\}$ is either unbounded in $B_R \times [0, \infty)$ or intersects $\partial B_R \times [0, \infty)$. Since the latter possibility is ruled out by Lemma 1, we conclude that the former holds. To obtain the result of the theorem let $C^- = \{(h(y), t) : (y, t) \in C\}$.

Q.E.D.

COROLLARY. — *Either* (i) $\{q(\tau) : (q, \tau) \in C^-, \tau \neq 0\} \subset \Omega$, *or* (ii) *there exists a periodic solution of*

$$-\ddot{q}(t) = SV'(q(t)), \tag{2.4}$$

of period T *where* $(q(0), T) = (q, 2\tau)$ *for some* $(q, \tau) \in C^-$ *and* $q(T/2) = q(\tau) \in M^+ \cup M^-$.

Proof. — We begin by noting that the set $\{q(\tau) : (q, \tau) \in C^-, \tau \geqq 0\}$ is connected, since, by the theory of continuous dependence for initial value problems, it is the continuous image of a connected set. Moreover, if $q(\tau) \in \partial\Omega = M^+ \cup M^-$ for some $(q, \tau) \in C^-$ with $\tau > 0$, then (ii) must hold. To see this it is sufficient to note that, since $(1/2)\langle S^{-1}\dot{q}(t), \dot{q}(t)\rangle + Vq(t))$ is constant on orbits of the differential equation, its value is zero because of the initial data $\dot{q}(0) = 0$, $V(q(0)) = 0$. If $q(\tau) \in \partial\Omega$, then $V(q(\tau)) = 0$ whence $\dot{q}_i(\tau) = 0$, $i = 2, \ldots, n$ and $(1/2)\langle S^{-1}\dot{q}(\tau), \dot{q}(\tau)\rangle = 0$, which implies that $\dot{q}(\tau) = 0$. Now since $V'(q(\tau)) \neq 0$, and $\tilde{q}(t) = q(2\tau - t)$ defines a solution of (2.4) with $\tilde{q}(\tau) = q(\tau)$, $\dot{\tilde{q}}(\tau) = 0$, it is immediate from the uniqueness theorem for initial-value problems that $q(t) = q(2\tau - t)$ for all $t > 0$. Thus a periodic solution satisfying (ii) has been found, where $T = 2\tau$.

So if (ii) is false, it is immediate that

$$\{q(\tau) : (q, \tau) \in C^-, \tau \geqq 0\} \cap \partial\Omega = \{q : (q, 0) \in C^-\}.$$

Now we show that there exists $\delta>0$ sufficiently small that if $(q,\tau)\in C^-$ and $0<\tau<\delta$, then $q(\tau)\in\Omega$. To see this let

$$A=\{q:(q,0)\in C^-\}\subset\{q\in M^-:V_i(q)=0, i=2,3,\ldots,n\}.$$

Then A is compact, and since $V'(q)\neq 0$, $q\in M^-$, it follows that $\alpha>0$ exists such that if $q\in A_\alpha=\{q\in M^-:\operatorname{dist}(q,A)\leqq\alpha\}$, then $\sum_{i=1}^{n}\lambda_i|V_i(q)|^2<-c<0$ for some fixed constant c (the λ_i's are the eigenvalues of S and $\lambda_1<0$). Now if $q\in A_\alpha$, then $V(q)=0$, $(d/dt)V(q)=0$ at $t=0$, and

$$(d^2/dt^2)V(q)=-\langle SV'(q),V'(q)\rangle=-\sum_{i=1}^{n}\lambda_i|V_i(q)|^2>c>0 \quad \text{at } t=0.$$

Hence for each $q\in A_\alpha$,

$$\varepsilon_q=\sup\{t:V(q(s))>0, s\in(0,t)\text{ when } q(0)=q\}>0.$$

Now if ε_q is not bounded below by something strictly positive on A_α, then a compactness argument leads to a contradiction. Hence there exists $\varepsilon>0$ such that $\{q(t):t\in(0,\varepsilon], q(0)=q\in A_\alpha\}\subset\Omega$. Since for τ sufficiently small $\{q:(q,\tau)\in C^-\}\subset A_\alpha$, it is immediate that there exists $\delta>0$ with the required property. Now consider the set $K=\{q(\tau):(q,\tau)\in C^-, \tau\geqq\delta\}$. Then each component of K contains a point $q(\delta)$ of Ω, and since each component is connected and does not intersect $\partial\Omega$ [because (ii) is false] we conclude that each is a subset of Ω. Hence

$$\{q(\tau):\tau>0,(q,\tau)\in C^-\}=\{q(\tau):\tau\in(0,\delta),(q,\tau)\in C^-\}$$
$$\cup\{q(\tau):\tau\geqq\delta,(q,\tau)\in C^-\}\subset\Omega.$$

This completes the proof of the corollary.

Q.E.D.

For emphasis, the conclusion of this corollary is that for any τ there is a solution with $q(0)\in M^-$ and $q(\tau)\in\bar{\Omega}$. Thus under quite general hypotheses we have established a certain type of recurrence theorem. Suppose $(q^k,\tau^k)\in C^-$ and $(q^k,\tau^k,q^k(\tau^k))\to(q,\tau,\hat{q})$; by the corollary such a sequence always exists if there does not exist a periodic orbit of the type described. Then since $(1/2)\langle S^{-1}\dot{q}^k(\tau^k),\ \dot{q}^k(\tau^k)\rangle+V(q^k(\tau^k))=0$, and $\dot{q}_i^k(\tau^k)=0$, $i=2,3,\ldots,n$, then it is immediate that $\dot{q}_1(\tau_k)$ is convergent, to

$\hat{p}=((2|\lambda_1|V(\hat{q})^{(1/2)}, 0, 0, \ldots, 0)$. Hence we have proved the following:

COROLLARY. — *In the phase any solution of the initial value problem with* $q(0)=q$, $\dot{q}(0)=0$ *visits any neighbourhood of* $(q, \dot{q})=(\hat{q}, \hat{p})$ *infinitely often.*

3. PERIODIC ORBITS

To obtain the existence of periodic orbits it is necessary to strengthen the hypotheses somewhat. Suppose that, in addition to $(a)-(d)$ the following hypothesis holds:
(e) *On* $\bar{\Omega}^+$, $V_1<0$ *and on* $\bar{\Omega}^-$, $V_1>0$.

Remark. — As a consequence of $(a)-(e)$ both M^+ and M^- are the graphs of real-valued functions giving x_1 as a function on Y.

THEOREM. — *Under the above hypotheses there exists a non-trivial periodic solution of* $-\ddot{q}=SV'(q)$.

Proof. — It will suffice to suppose that (i) holds and that (ii) is false in the corollary of the previous section. Now in the proof of the corollary we saw (implicitly) that there exists $\delta>0$ such that if $(q, \tau)\in C^-$ and $\tau<\delta$, then $\dot{q}_1(t)\geqq 0$ for all $t\in[0, \tau]$. Let

$$\tilde{C}=\{(q, \tau)\in C^- : \dot{q}_1(t)\geqq 0 \text{ for all } t\in[0, \tau]\}\neq\varnothing.$$

Clearly $\tilde{C}$ is closed by the standard continuous dependence theory for initial-value problems. To see that $\tilde{C}$ is open in C^-, suppose that it is not. Then there exists a sequence $(q^k, \tau^k)\in C^-\setminus\tilde{C}$ such that $(q^k, \tau^k)\to(\tilde{q}, \tilde{\tau})\in\tilde{C}$.

Hence there exists $t^k\in[0, \tau^k]$ such that $t^k\to\tilde{t}$ and $0>\dot{q}_1^k(t^k)\to\dot{\tilde{q}}_1(\tilde{t})\geqq 0$, $\tilde{t}\in[0, \tilde{\tau}]$. Clearly $\tilde{t}>0$ since $\tau^k\geqq\delta>0$. If $\tilde{t}=\tau$, then $\tilde{q}_1(\tilde{t})=\tilde{q}_2(\tilde{t})=\ldots=\tilde{q}_n(\tilde{t})=0$, and so a periodic solution of period $2\tilde{t}$ exists, which is supposed to be false. So $\tilde{t}\in(0, \tilde{\tau})$. Now $\tilde{q}(\tilde{t})\notin\bar{\Omega}$, for if $\tilde{q}(\tilde{t})\in\Omega$, then $V(\tilde{q}(\tilde{t}))>0$ and so

$$|\lambda_1|^{-1}.|\dot{\tilde{q}}_1(\tilde{t})|^2\geqq V(\tilde{q}(\tilde{t}))>0,$$

and if $\tilde{q}(\tilde{t})\in\partial\Omega$, then there exists a period orbit as we have already seen.

So $\dot{\tilde{q}}_1(\tilde{t})=0$ and $\tilde{q}(\tilde{t})\notin\bar{\Omega}$, i.e. $\tilde{q}(\tilde{t})\in\Omega^+\cup\Omega^-$ and $V_1(\tilde{q}(\tilde{t}))\neq 0$. But

$\dot{\tilde{q}}(t)=\int_{\tilde{t}}^{t}|\lambda_1|V_1(\tilde{q}(s))\,ds$ which changes sign at $t=\tilde{t}$. Hence $(\tilde{q},\tilde{\tau})\notin\tilde{C}$, a contradiction. We conclude that $\tilde{C}$ is a non-empty subset of C^- which is both open and closed, and so $\tilde{C}=C^-$. In other words, if $(q,\tau)\in C^-$, then $\dot{q}(t)\geqq 0$, $t\in(0,\tau)$. Moreover $\{q(\tau):(q,\tau)\in C^-,\ \tau>0\}\subset\Omega$, for otherwise there exists $(q,\tau)\in C^-$ with $q(\tau)\in\partial\Omega$, $\tau>0$, which implies the existence of a periodic orbit, and we are assuming that no such exists.

Now let $(q^k,\tau^k)\in C^-$ with $q^k\to q$ and $\tau^k\to\infty$. Then the solution of the initial-value problem (2.1), (2.2) with $q(0)=q$, $\dot{q}(0)=0$ has $\dot{q}_1(t)\geqq 0$ for all $t\in(0,\infty)$. Since $q^k(\tau^k)\in\Omega$ for all k, there exists a sequence $t^k\to\infty$ such that $q(t^k)\in\Omega$ for all k. It follows from Lemma 1 that the set $\{|q^k(t)|:k\in\mathbb{N},\ t\in[0,\tau^k]\}$ is bounded, and so $\{|q(t)|,\ t\geqq 0\}$ is bounded. Hence we may, without loss of generality, suppose that $q(t^k)$ is convergent to q, say. Now because $q(t^k)\in\Omega$, $|\dot{q}_1(t^k)|^2\geqq|2\lambda_1 V(q(t^k))|$, and since $\dot{q}_1(t)\geqq 0$ for all $t\geqq 0$, and $q(t^k)\to q$ it is immediate that $\dot{q}_1(t^k)\to 0$. Hence $q(t^k)\to q$ where $q\in\partial\Omega$.

But since $q\in\partial\Omega$, $V'(q)\neq 0$ and so $\ddot{q}_1(t_k)\leqq\alpha<0$ for all k sufficiently large for some $\alpha<0$. Hence $q(t_k)\to q\in\partial\Omega$ is impossible. This contradiction means that there exists a periodic orbit, and the proof is complete.

Q.E.D.

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