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An asymptotic expansion for the solution of the generalized Riemann problem Part I: General theory

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ABSTRACT. — We consider the generalized Riemann problem for nonlinear hyperbolic systems of conservation laws. We show in this paper that we can find the entropy solution of this problem in the form of an asymptotic expansion in time and we give an explicit method of construction of this asymptotic expansion. Finally, we define from this expansion an approximate solution of the generalized Riemann problem and we give error bounds.

RÉSUMÉ. — On considère le problème de Riemann généralisé pour des systèmes hyperboliques non linéaires de lois de conservation. On trouve la solution entropique de ce problème sous la forme d'un développement asymptotique que l'on construit par une méthode explicite. On en déduit

une solution approchée du problème de Riemann généralisé et l'on estime l'erreur.

I. INTRODUCTION

Let us consider the nonlinear system of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(x,t,u)=g(x,t,u), \qquad x\in\mathbb{R}, \quad t>0. \tag{1.1}$$

In (1.1), $u=u(x,t)\in\mathbb{R}^p$ is a p-vector and f, g are sufficiently smooth functions from $\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^p$ into $\mathbb{R}^p$. We introduce the so-called *generalized Riemann problem* (G.R.P.) for the system (1.1): given two smooth functions $u_L:\mathbb{R}_-\to\mathbb{R}^p$, $u_R:\mathbb{R}_+\to\mathbb{R}^p$, find a function $u:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}^p$ which is an entropy weak solution of (1.1) and satisfies the initial condition

$$u(x,0)=\left\{\begin{array}{ll} u_L(x), & x<0 \\ u_R(x), & x>0. \end{array}\right\} \tag{1.2}$$

For the sake of simplicity, we set:

$$f(u)=f(0,0,u)$$

and

$$u_L^0=u_L(0), \qquad u_R^0=u_R(0).$$

Then, we assume that the eigenvalues $\lambda_i(u)$ of the $p\times p$ Jacobian matrix $A(u)=Df(u)$ are all real and distinct:

$$\lambda_1(u)<\lambda_2(u)<\ldots<\lambda_p(u), \qquad \forall u\in\mathbb{R}^p.$$

We denote by $(r_i(u))_{1\leq i\leq p}$ a basis of corresponding right (column) eigenvectors and by $(l_i(u)^T)_{1\leq i\leq p}$ a basis of left (row) eigenvectors, i. e.,

$$A(u)\,r_i(u)=\lambda_i(u)\,r_i(u), \qquad l_i(u)^T A(u)=\lambda_i(u)\,l_i(u)^T,$$

with the normalization

$$l_i(u)^{\mathrm{T}}.\, r_j(u)=\delta_{ij}, \qquad 1\leqq i, j\leqq p. \tag{1.3}$$

As usual, we suppose that the i-th characteristic field is either genuinely nonlinear, i. e.

$$\mathrm{D}\lambda_i(u).\, r_i(u)=1, \qquad \forall u\in\mathbb{R}^p \tag{1.4}$$

or linearly degenerate, i. e.

$$\mathrm{D}\lambda_i(u).\, r_i(u)=0, \qquad \forall u\in\mathbb{R}^p. \tag{1.5}$$

Then, as it is well known (*cf.* [10] for instance), for $|u_{\mathrm{R}}^0-u_{\mathrm{L}}^0|$ small enough the *classical Riemann problem*

$$\left.\begin{aligned} &\frac{\partial u^0}{\partial t}+\frac{\partial}{\partial x}f(u^0)=0, \qquad x\in\mathbb{R}, \quad t>0\\ &u^0(x, 0)=\begin{cases} u_{\mathrm{L}}^0, & x<0\\ u_{\mathrm{R}}^0, & x>0\end{cases}\end{aligned}\right. \tag{1.6}$$

has an entropy weak solution u^0 which is self-similar, i. e., of the form

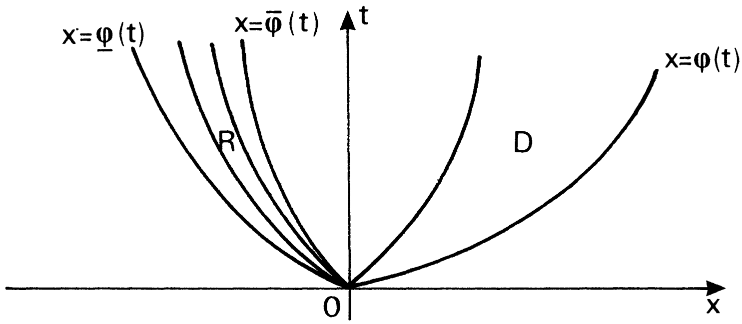
$$u^0(x, t)=v^0\left(\frac{x}{t}\right), \tag{1.7}$$

and consists of at most $(p+1)$ constant states separated by rarefaction waves or shock waves or contact discontinuities. Moreover, a solution of this type is necessarily unique.

Next concerning the G.R.P. (1.1).(1.2), we recall the following result due to Li Ta-tsien and Yu Wen-ci [8]. Again for $|u_{\mathrm{R}}^0-u_{\mathrm{L}}^0|$ small enough, there exists a neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}\times\mathbb{R}_+$ such that the G.R.P. (1.1). (1.2) has a unique entropy weak solution u in $\mathcal{O}$ which has the same structure than u^0. The fonction u consists of $(p+1)$ smoothness open domains D separated either by smooth curves $x=\varphi(t)$ passing through the origin or by rarefaction zones of the form

$$\mathrm{R}=\{(x, t)\in\mathbb{R}\times\mathbb{R}_+;\ \underline{\varphi}(t)<x<\overline{\varphi}(t), 0<t<\delta)$$

where the curves $x=\underline{\varphi}(t)$ and $x=\overline{\varphi}(t)$ are smooth characteristic curves passing through the origin. Moreover, u has a shock or a contact discontinuity across each curve $x=\varphi(t)$ while is continuous across the characteristic curves $x=\underline{\varphi}(t)$ and $x=\bar{\varphi}(t)$. For general results concerning the G.R.P. in d space dimensions, we refer to Harabetian [6].



In fact, the solution u is smooth in the closure $\bar{\mathrm{D}}$ of each smoothness domain D. Hence, using a Taylor expansion of u at the origin, we obtain for $(x, t) \in \mathrm{D}$

$$u(x,t)=v^0\left(\frac{x}{t}\right)+tv^1\left(\frac{x}{t}\right)+\ldots+t^k v^k\left(\frac{x}{t}\right)+\ldots \tag{1.8}$$

where, for each integer $k \geqq 0$, $v^k : \xi \to v^k(\xi)$ is a polynomial of degree k. On the other hand, in a rarefaction zone R, the solution u is singular at the origin. However, setting

$$\underline{\sigma}=\underline{\varphi}'(0), \qquad \bar{\sigma}=\bar{\varphi}'(0)$$

one can check (*cf.* [8], chapter 5 and Lemma 7 below) that the function $\bar{u}$: $(\xi, t) \to \bar{u}(\xi, t)=u(\xi t, t)$ is smooth in a neighborhood of $[\underline{\sigma}, \bar{\sigma}] \times [0, \delta]$. Therefore, we may write for $t>0$ small enough

$$\bar{u}(\xi, t)=v^0(\xi)+tv^1(\xi)+\ldots+t^k v^k(\xi)+\ldots$$

which yields the asymptotic expansion (1.8) of u in the rarefaction zone R.

The purpose of this paper is to derive an *explicit construction of the asymptotic expansion* (1.8) of the entropy solution u of the G.R.P. (1.1). (1.2) in each domain of smoothness of u. We shall also construct the smooth curves $x=\varphi(t)$ [resp. $x=\underline{\varphi}(t)$, $x=\bar{\varphi}(t)$] through expansions of the form

$$\varphi(t)=\sigma^0 t+\sigma^1 t^2+\ldots+\sigma^{k-1} t^k+\ldots$$

$$(\text{resp. } \underline{\varphi}(t)=\underline{\sigma}^0 t+\underline{\sigma}^1 t^2+\ldots+\underline{\sigma}^{k-1} t^k+\ldots,$$

$$\bar{\varphi}(t)=\bar{\sigma}^0 t+\bar{\sigma}^1 t^2+\ldots+\bar{\sigma}^{k-1} t^k+\ldots)$$

Moreover, in many practical problems related to the gas dynamics equations, one can compute explicitely the coefficients σ^1 (resp. $\underline{\sigma}^1, \bar{\sigma}^1$) and the function v^1. This point will be developed in a subsequent paper [4]; see also [3]. For a related approach and its application to the construction of 2nd order difference methods of approximation of the gas dynamics equations based on Van Leer's method, [11], we refer to the work of Ben Artzi and Falcovitz ([1], [2]).

An outline of the paper is as follows. In Section 2, we derive the equations satisfied formally by the functions v^k and the numbers σ^k, $\underline{\sigma}^k$, $\bar{\sigma}^k$. In Section 3, we show how to construct the functions v^k in order to satisfy the equations of Section 2. Section 4 is devoted to the proof of an existence and uniqueness result for the asymptotic expansion (1.8). Finally, we give in Section 5 a L^1-bound for the error $u(.,t)-u^k(.,t)$ where the function $u^k=u^k(x,t)$ is constructed in a suitable way from the truncated expansion

$$\sum_{l=0}^{k} t^l v^l\left(\frac{x}{t}\right).$$

For a different approach of a particular generalized Riemann problem, we refer to Liu [9] and Glimm-Marshall-Plohr [5].

2. PRELIMINARIES

Starting from the asymptotic expansion (1.8), we begin by deriving the *ordinary differential equations* satisfied by the functions v^k. Let us first make the change of independent variables $(x,t) \to (\xi=(x/t,t)$. Setting $\bar{u}(\xi,t)=u(\xi t,t)$ and noticing that

$$\frac{\partial}{\partial x}=\frac{1}{t}\frac{\partial}{\partial \xi}, \qquad \frac{\partial u}{\partial t}=\frac{\partial \bar{u}}{\partial t}-\frac{\xi}{t}\frac{\partial \bar{u}}{\partial \xi},$$

the equation (1.1) gives

$$t\frac{\partial \bar{u}}{\partial t}-\xi\frac{\partial \bar{u}}{\partial \xi}+\frac{\partial}{\partial \xi}f(\xi t, t, \bar{u}(\xi,t))=tg(\xi t, t, \bar{u}(\xi,t)). \tag{2.1}$$

Next, we plug the expansion

$$\bar{u}(\xi, t)=\sum_{k\geqq 0} t^k v^k(\xi) \tag{2.2}$$

in the equation (2.1). On the hand, we have

$$t\frac{\partial \bar{u}}{\partial t}-\xi\frac{\partial \bar{u}}{\partial \xi}=-\xi\frac{dv^0}{d\xi}+\sum_{k\geqq 1} t^k\left(kv^k-\xi\frac{dv^k}{d\xi}\right).$$

On the other hand, we may write

$$f(\xi t, t, \bar{u})=f(0, 0, v^0)+\sum_{k\geqq 1} t^k(\mathrm{D}_u f(0, 0, v^0).v^k+f^k(\xi, v^0, \ldots, v^{k-1}))$$

or

$$f(\xi t, t, \bar{u})=f(v^0)+\sum_{k\geqq 1} t^k(\mathrm{A}(v^0)v^k+f^k(\xi, v^0, \ldots, v^{k-1})) \tag{2.3}$$

and

$$g(\xi t, t, \bar{u})=\sum_{k\geqq 1} t^{k-1} g^k(\xi, v^0, \ldots, v^{k-1}) \tag{2.4}$$

where the functions f^k and g^k depend only on $\xi, v^0, \ldots, v^{k-1}$. Hence, we obtain

$$-\xi\frac{dv^0}{d\xi}+\frac{d}{d\xi}f(v^0)+\sum_{k\geqq 1} t^k\left(kv^k-\xi\frac{dv^k}{d\xi}+\frac{d}{d\xi}(\mathrm{A}(v^0)v^k+f^k)-g^k\right)=0$$

which gives for $k=0$

$$-\xi\frac{dv^0}{d\xi}+\frac{d}{d\xi}f(v^0)=0 \tag{2.5}$$

and for $k\geqq 1$

$$kv^k-\xi\frac{dv^k}{d\xi}+\frac{d}{d\xi}(\mathrm{A}(v^0)v^k+f^k)=g^k. \tag{2.6}$$

Concerning the functions f^k and g^k, we have the following result which will be useful later on.

LEMMA 1. — *Assume that v^l is a polynomial function of degree $\leqq l$ (which values in $\mathbb{R}^p$) for $0\leqq l\leqq k-1$. Then, the function*

$\xi \to f^k(\xi, v^0(\xi), \ldots, v^{k-1}(\xi), \ldots, v^{k-1}(\xi))$ *is a polynomial of degree* $\leqq k-1$.

Proof. – Let us first consider the function f^k. We observe that, in the arguments of the function $f(\xi t, t, v^0 + tv^1 + \ldots + t^{k-1} v^{k-1} + t^k v^k + \ldots)$, the coefficients of t^l are polynomial of degree $\leqq l$ in ξ provided that v^l is a polynomial of degree $\leqq l$, $0 \leqq l \leqq k-1$. In this case, by using a Taylor expansion of the function f at the origin, it is a simple matter to check that, in the corresponding expansion of $f(\xi t, t, v^0 + \ldots + t^k v^k + \ldots)$ in powers of t, the coefficient of t^k is of the form

$$\mathrm{A}(v^0)\, v^k + \text{polynomial of degree} \leqq k \text{ in } \xi.$$

This proves the desired property for f^k. Using exactly the same method of proof gives the corresponding result for the function g^k. ■

In all the sequel, we shall set

$$h^k(\xi) = -\frac{d}{d\xi} f^k(\xi, v^0(\xi), \ldots, v^{k-1}(\xi)) + g^k(\xi, v^0(\xi), \ldots, v^{k-1}(\xi)) \quad (2.7)$$

so that (2.6) becomes

$$k v^k - \xi \frac{dv^k}{d\xi} + \frac{d}{d\xi}(\mathrm{A}(v^0)\, v^k) = h^k. \quad (2.8)$$

Note that the equation (2.8) is valid in each interval of smoothness of the function v^0. Moreover, it follows from Lemma 1 that, in any such interval, the function h^k is a polynomial of degree $\leqq k-1$ if each v^l is a polynomial of degree $\leqq l$, $0 \leqq l \leqq k-1$.

Let us next determine the jump conditions satisfied by the function v^k at the points of discontinuity of the function v^0. Let $x = \varphi(t)$ be a curve passing through the origin which separates two smoothness domains of u. By using a Taylor expansion of the function $t \to \varphi(t)$ at the origin, we can write

$$\varphi(t) = \sigma^0 t + \sigma^1 t^2 + \ldots + \sigma^{k-1} t^k + \ldots \quad (2.9)$$

so that by (1.8)

$$u(\varphi(t), t) = \sum_{k \geqq 0} t^k v^k\left(\frac{\varphi(t)}{t}\right) = \sum_{k \geqq 0} t^k v^k\Big(\sum_{l \geqq 0} t^l \sigma^l\Big).$$

Hence, we obtain

$$u(\varphi(t), t) = v^0(\sigma^0) + \sum_{k \geqq l} t^k \left(v^k(\sigma^0) + \sigma^k \frac{dv^0}{d\xi}(\sigma^0) + z^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1}) \right) \quad (2.10)$$

where $z^k \in \mathbb{R}^p$ depends only on $\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1}$.

Now, we consider the case where u is continuous across the curve $x = \varphi(t)$, i. e.,

$$u(\varphi(t) + 0, t) = u(\varphi(t) - 0, t).$$

If we denote by

$$[w](\sigma^0) = w(\sigma^0 + 0) - w(\sigma^0 - 0)$$

the jump of a function $w = w(\xi)$ across the point σ^0, (2.10) yields for $k = 0$

$$[v^0](\sigma^0) = 0 \quad (2.11)$$

and for $k \geqq 1$

$$\left[v^k + \sigma^k \frac{dv^0}{d\xi} + z^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1}) \right](\sigma^0) = 0. \quad (2.12)$$

Hence the function v^0 is continuous at the point σ^0 while v^k is generally discontinuous at σ^0 for $k \geqq 1$.

Next, we turn to the case where u is discontinuous across the curve $x = \varphi(t)$. We start from the Rankine-Hugoniot jump relations

$$\varphi'(t)[u] = [f(x, t, u)], \qquad x = \varphi(t).$$

On the one hand, using (2.9) and (2.10), we observe that

$$
\begin{aligned}
&f(\varphi(t), t, u(\varphi(t), t)) \\
&\quad = f\Big(t \sum_{k \geqq 0} \sigma^k t^k, t, v^0(\sigma^0) + \sum_{k \geqq 1} t^k \Big(\Big(v^k + \sigma^k \frac{dv^0}{d\xi}\Big)(\sigma^0) \\
&\qquad\qquad + z^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1})\Big)\Big)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&f(\varphi(t), t, u(\varphi(t), t)) \\
&\quad = f(v^0)(\sigma^0) + \sum_{k \geqq 1} t^k \Big\{\Big(A(v^0)\Big(v^k + \sigma^k \frac{dv^0}{d\xi}\Big)\Big)(\sigma^0) \\
&\qquad\qquad + a^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1})\Big\} \qquad (2.13)
\end{aligned}
$$

where $a^k \in \mathbb{R}^p$ depends only on $\sigma^0, \ldots, \sigma^{k-1}$, $v^0, \ldots, v^{k-1}$. On the other hand, using again (2.9) and (2.10), we obtain

$$
\begin{aligned}
&\varphi'(t)\, u(\varphi(t), t) \\
&\quad = \Big(\sum_{k \geqq 0} (k+1)\, \sigma^k t^k\Big)\Big(v^0(\sigma^0) + \sum_{k \geqq 1} t^k \Big(\Big(v^k + \sigma^k \frac{dv^0}{d\xi}\Big)(\sigma^0) \\
&\qquad\qquad + z^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1})\Big)\Big)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\varphi'(t)\, u(\varphi(t), t) \\
&\quad = \sigma^0 v^0(\sigma) + \sum_{k \geqq 1} t^k \Big\{\sigma^0 \Big(v^k + \sigma^k \frac{dv^0}{d\xi}\Big)(\sigma^0) + (k+1)\, \sigma^k v^0(\sigma^0) \\
&\qquad\qquad + b^k(\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1})\Big\} \qquad (2.14)
\end{aligned}
$$

Then, combining (2.13) and (2.14) yields

$$
\begin{aligned}
& f(\varphi(t), t, u(\varphi(t), t)) - \varphi'(t)\, u(\varphi(t), t) \\
& \quad = (f(v^0) - \sigma^0 v^0)(\sigma^0) + \sum_{k \geqq 1} t^k \Big\{ \Big((A(v^0) - \sigma^0)\Big(v^k + \sigma^k \frac{dv^0}{d\xi} \Big) \Big)(\sigma^0) \\
& \qquad - (k+1)\, \sigma^k v^0 (\sigma^0) + w^k (\sigma^0, \ldots, \sigma^{k-1}, v^0, \ldots, v^{k-1}) \Big\} \qquad (2.15)
\end{aligned}
$$

where $w^k \in \mathbb{R}^p$ depends only on $\sigma^0, \ldots, \sigma^{k-1}$, $v^0, \ldots, v^{k-1}$. Therefore the Rankine-Hugoniot jump conditions give for $k=0$

$$\sigma^0 [v^0] = [f(v^0)] \quad \text{at } \sigma^0 \qquad (2.16)$$

and for $k \geqq 1$

$$[(A(v^0) - \sigma^0) v^k] + \sigma^k \Big[(A(v^0) - \sigma^0) \frac{dv^0}{d\xi} - (k+1)\, v^0 \Big] + [w^k] = 0 \qquad (2.17)$$

at σ^0.

Finally, we remark that for $|x/t|$ large enough, say $|x/t| \geqq \xi_0$, $u = u(x, t)$ is a smooth function, so that we may write

$$u(x, t) = u_+^0 + t\Big(\frac{x}{t} \Big(\frac{\partial u}{\partial x} \Big)_+^0 + \Big(\frac{\partial u}{\partial t} \Big)_+^0 \Big) + \ldots, \qquad \frac{x}{t} \geqq \xi_0$$

and

$$u(x, t) = u_-^0 + t\Big(\frac{x}{t} \Big(\frac{\partial u}{\partial x} \Big)_-^0 + \Big(\frac{\partial u}{\partial t} \Big)_-^0 \Big) + \ldots, \qquad \frac{x}{t} \leqq -\xi_0$$

where

$$\psi_+^0 = \lim_{t \to 0,\, x/t \geqq \xi_0} \psi(x, t), \qquad \psi_-^0 = \lim_{t \to 0,\, x/t \leqq -\xi_0} \psi(x, t).$$

Taking into account the initial condition (1.2) gives

$$u_+^0 = u_R^0, \qquad u_-^0 = u_L^0$$

so that by (1.18)

$$v^0(\xi) = \begin{cases} u_R^0 & \text{for } \xi \text{ large enough,} \\ u_L^0 & \text{for } -\xi \text{ large enough.} \end{cases} \qquad (2.18)$$

As a consequence of the above derivation, we find that v^0 satisfies the classical relations (2.5), (2.11), (2.16) and (2.18) which characterize a piecewise smooth self-similar weak solution $u^0(x,t)=v^0(x/t)$ of the Riemann problem (1.6). Thus, we choose u^0 to be the entropy solution of (1.6) introduced in Section 1.

3. CONSTRUCTION OF THE ASYMPTOTIC EXPANSION 1: CHARACTERIZATION OF THE FUNCTION v^k

As we have already recalled it in Section 1, the function u^0 consists of at most $(p+1)$ constant states separated by i-waves, $1 \leqq i \leqq p$. If we denote by $\underline{\sigma}_i^0$ (respectively $\bar{\sigma}_i^0$) the lower bound (resp. the upper bound) of the speeds of the i-wave and by v_i^0, $0 \leqq i \leqq p$, the $(p+1)$ constant states, we have by (2.18)

$$v^0(\xi)=\begin{cases} v_0^0=u_L^0, & \xi<\underline{\sigma}_1^0, \\ v_i^0, & \bar{\sigma}_i^0<\xi<\underline{\sigma}_{i+1}^0, \quad 1 \leqq i \leqq p-1, \\ v_p^0=u_R^0, & \xi>\bar{\sigma}_p^0. \end{cases} \tag{3.1}$$

Moreover, if the i-wave is a rarefaction wave, we find for $\underline{\sigma}_i^0 \leqq \xi \leqq \bar{\sigma}_i^0$

$$\frac{d}{d\xi} v^0(\xi)=r_i(v^0(\xi)), \tag{3.2}$$

$$\lambda_i(v^0(\xi))=\xi. \tag{3.3}$$

On the other hand, if the i-wave is a shock wave or a contact discontinuity, we get

$$\underline{\sigma}_i^0=\bar{\sigma}_i^0=\sigma_i^0. \tag{3.4}$$

For these results, *see* again [10].

Let us determine the structure of the function v^k. We know from the results of Section 2 that the singularities of v^k occur at the points $\underline{\sigma}_i^0$, $\bar{\sigma}_i^0$. $1 \leqq i \leqq p$. Hence, we begin by considering the case of an interval $(\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0)$ where v^0 is constant, with the convention that $\bar{\sigma}_0^0=-\infty$ and $\underline{\sigma}_{p+1}^0=+\infty$.

LEMMA 2. — *Assume that ξ belongs to the interval $(\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0)$, $0 \leqq i \leqq p$. Then, for all $k \geqq 1$, the general solution of the differential equation* (2.8) *is* *given by*

$$v^k(\xi)=(\xi-\mathrm{A}(v_i^0))^k v_i^k+p_i^k(\xi), \tag{3.5}$$

where v_i^k *is an arbitrary vector of* $\mathbb{R}^p$ *and* $\xi \to p_i^k(\xi)$ *is a polynomial function of degree* $\leqq k-1$ *with values in* $\mathbb{R}^p$ *which depends only on* $v^0, \ldots, v^{k-1}$.

Proof. — We proceed by induction. Since v^0 is constant in the interval $(\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0)$, the differential equation (2.8) becomes

$$(\mathrm{A}(v_i^0)-\xi)\frac{dv^k}{d\xi}+kv^k=h^k. \tag{3.6}$$

Assume that v^l is a polynomial of degree $\leqq l$, $0 \leqq l \leqq k-1$, in this interval. Then, using Lemma 1, we obtain that the function h^k defined by (2.7) is a polynomial of degree $\leqq k-1$ in the above interval. Writing

$$h^k(\xi)=\sum_{l=0}^{k-1}\xi^l b_l, \qquad b_l \in \mathbb{R}^p,$$

we look for a particular solution of (3.6) of the form

$$p_i^k(\xi)=\sum_{l=0}^{k-1}\xi^l a_l, \qquad a_l \in \mathbb{R}^p.$$

We have

$$(\mathrm{A}(v_i^0)-\xi)\frac{d}{d\xi}p_i^k+kp_i^k=\sum_{l=0}^{k-2}\xi^l\{(l+1)\mathrm{A}(v_i^0)a_{l+1}+(k-l)a_l\}+\xi^{k-1}a_{k-1}.$$

Hence, p_i^k is a solution of (3.6) if and only if

$$\begin{cases} a_{k-1}=b_{k-1}, & \\ (l+1)\mathrm{A}(v_i^0)a_{l+1}+(k-l)a_l=b_l, & 0 \leqq l \leqq k-2. \end{cases}$$

The above equations determine $a_{k-1}, a_{k-2}, \ldots, a_0$ and therefore the polynomial p_i^k of degree $\leqq k-1$ in a unique way.

It remains to find the general solution of the homogeneous differential equation

$$(\mathrm{A}(v_i^0)-\xi)\frac{dv^k}{d_\xi}+kv^k=0.$$

Clearly, this solution is given by

$$v^k(\xi)=(\xi-A(v_i^0))^k v_i^k$$

where v_i^k is an arbitrary vector of $\mathbb{R}^p$. Thus, the general solution of (3.6) is given by (3.5) and is indeed a polynomial of degree $\leqq k$, which proves the result. ■

Next, we consider the case of an interval $(\underline{\sigma}_i^0, \bar{\sigma}_i^0)$ which corresponds to an i-rarefaction wave for the function u^0. Setting

$$v^k=\sum_{j=1}^{p} \alpha_j^k r_j(v^0), \tag{3.7}$$

we are looking for the functions $\xi \to \alpha_j^k(\xi)$, $1 \leqq j \leqq p$.

LEMMA 3. — *Given the function h^k and the scalars $\beta_j \in \mathbb{R}$, $1 \leqq j \leqq p$, $j \neq 1$, there exists a unique function v^k of the form* (3.7) *solution of the differential equation* (2.8) *in the interval* $(\underline{\sigma}_i^0, \bar{\sigma}_i^0)$ *which satisfies the initial conditions*

$$\alpha_j^k(\underline{\sigma}_i+0)=\beta_j, \qquad 1 \leqq j \leqq p, \quad j \neq i. \tag{3.8}$$

Proof. — Writting

$$h^k=\sum_{j=1}^{p} \gamma_j^k r_j(v^0)$$

and using (3.7), the equation (2.8) gives

$$\frac{d}{d\xi}\left(\sum_{j=1}^{p} \lambda_j(v^0)\,\alpha_j^k r_j(v^0)\right)-\xi\frac{d}{d\xi}\left(\sum_{j=1}^{p} \alpha_j^k r_j(v^0)\right)+k\sum_{j=1}^{p} \alpha_j^k r_j(v^0)=\sum_{j=1}^{p} \gamma_j^k r_j(v^0).$$

On the one hand, we have by (3.2)

$$\frac{d}{d\xi}\left(\sum_{j=1}^{p} \alpha_j r_j(v^0)\right)=\sum_{j=1}^{p}\left\{\frac{d}{d\xi}\alpha_j^k r_j(v^0)+\alpha_j^k \,\mathrm{D}r_j(v^0).r_j(v^0)\right\}$$

On the other hand, we can write using again (3.2)

$$\frac{d}{d\xi}\left(\sum_{j=1}^{p} \lambda_j(v^0)\,\alpha_j^k r_j(v^0)\right)$$

$$=\sum_{j=1}^{p}\left\{\mathrm{D}\lambda_j(v^0).r_i(v^0)\,\alpha_j^k+\lambda_j(v^0)\frac{d}{d\xi}\alpha_j^k\right\} r_j(v^0)$$

$$+\sum_{j=1}^{p} \lambda_j(v^0)\,\alpha_j^k\, \mathrm{D}r_j(v^0)\,.\,r_i(v^0).$$

Hence, setting

$$\mathrm{D}r_j(v^0)\,.\,r_i(v^0)=\sum_{l=1}^{p} \omega_{ijl}\, r_l(v^0),$$

the differential equation (2.8) becomes

$$\sum_{j=1}^{p}\left\{(\lambda_j(v^0)-\xi)\frac{d}{d\xi}\alpha_j^k+(k+\mathrm{D}\lambda_j(v^0)\,.\,r_i(v^0))\,\alpha_j^k\right\} r_j(v^0)$$

$$+\sum_{j,\,l=1}^{p}(\lambda_j(v^0)-\xi)\,\omega_{ijl}\,\alpha_j^k\, r_l(v^0)=\sum_{j=1}^{p}\gamma_j^k\, r_j(v^0).$$

Using (3.3), this gives for $1 \leqq j \leqq p$

$$(\lambda_j(v^0)-\xi)\frac{d}{d\xi}\alpha_j^k+(k+\mathrm{D}\lambda_j(v^0)\,.\,r_i(v^0))\,\alpha_j^k$$

$$+\sum_{\substack{l=1\\ l\neq 1}}^{p}(\lambda_l(v^0)-\xi)\,\omega_{ilj}\,\alpha_l^k=\gamma_j^k. \qquad (3.9)$$

Consider first the equations (3.9) for $j \neq i$. We obtain a system of $(p-1)$ linear differential equations in the unknown fonctions α_j^k, $j \neq i$. Since $\lambda_j(v^0(\xi))-\xi \neq 0$ in the interval $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$ for $j \neq i$, this differential system is nondegenerate so that the Cauchy problem corresponding to the initial conditions (3.8) is well posed in $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$. Hence the functions $\xi \to \alpha_j^k(\xi)$, $j \neq i$, are uniquely determined in this interval.

Consider next the equation (3.9) for $j=i$. Using (1.4) (the i-th characteristic field is genuinely nonlinear) and (3.3) again, we find

$$(k+1)\,\alpha_i^k+\sum_{\substack{l=1\\ l\neq 1}}^{p}(\lambda_l(v^0)-\xi)\,\omega_{ili}\,\alpha_l^k=\gamma_i^k, \qquad (3.10)$$

which gives the function $\xi \to \alpha_i^k(\xi)$ in $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$. ∎

In order to obtain the function v^k, we need to determine the vectors $v_i^k \in \mathbb{R}^p$, $0 \leqq i \leqq p$, which appear in Lemma 2. A first step consists in relating two consecutive vectors v_{i-l}^k and v_i^k for $1 \leqq i \leqq p$. Consider first the case where the i-wave of u^0 is a discontinuity wave. We denote by $x=\varphi_i(t)$ the smooth curve such that

$$\varphi_i(0)=0, \qquad \varphi_i'(0)=\sigma_i^0$$

across which the function u is discontinuous. We write

$$\varphi_i(t)=\sigma_i^0 t+\sigma_i^1 t^2+\ldots+\sigma_i^k t^{k+1}+\ldots \tag{3.11}$$

LEMMA 4. — *Assume that u^0 contains an i-shock wave or an i-contact discontinuity. Then, for all $k\geqq 1$, there exists a vector $q_i^k\in\mathbb{R}^p$ which depends only on $v^0, \ldots, v^{k-1}, \sigma_i^0, \ldots, \sigma_i^{k-1}$ such that*

$$(\mathrm{A}(v_i^0)-\sigma_i^0)^{k+1} v_i^k = (\mathrm{A}(v_{i-l}^0)-\sigma_i^0)^{k+1} v_{i-1}^k+(-1)^k(k+1)\sigma_i^k(v_i^0-v_{i-1}^0)+q_i^k \tag{3.12}$$

Proof. — The jump condition (2.17) becomes here

$$[(\mathrm{A}(v^0)-\sigma_i^k)v^k]+\sigma_i^k\left[(\mathrm{A}(v^0)-\sigma_i^0)\frac{dv^0}{d\xi}-(k+1)v^0\right]+[w_i^k]=0 \quad \text{at } \sigma_i^0,$$

where w_i^k depends only on $\sigma_i^0, \ldots, \sigma_i^{k-1}, v^0, \ldots, v^{k-1}$. Since

$$v^0(\sigma_i^0-0)=v_{i-1}^0, \qquad v^0(\sigma_i^0+0)=v_i^0, \qquad \frac{dv^0}{d\xi}(\sigma_i^0+0)=0,$$

this gives

$$(\mathrm{A}(v_i^0)-\sigma_i^0)v^k(\sigma_i^0+0)-(\mathrm{A}(v_{i-1}^0)-\sigma_i^0)v^k(\sigma_i^0-0) -(k+1)\sigma_i^k(v_i^0-v_{i-1}^0)+w_i^k(\sigma_i^0+0)-w_i^k(\sigma_i^0-0)=0.$$

Next, using (3.5), we have

$$v^k(\sigma_i^0-0)=(-1)^k(\mathrm{A}(v_{i-1}^0)-\sigma_i^0)^k v_{i-1}^k+p_{i-1}^k(\sigma_i^0)$$

and

$$v^k(\sigma_i+0)=(-1)^k(\mathrm{A}(v_i^0)-\sigma_i^0)^k v_i^k+p_i^k(\sigma_i^0)$$

so that (3.12) holds with

$$q_i^k=(-1)^{k+1}((\mathrm{A}(v_i^0)-\sigma_i^0)-p_i^k(\sigma_{i-1}^0)-(\mathrm{A}(v_{i-1}^0)-\sigma_i^0)p_{i-1}^k(\sigma_i^0)) -(w_i^k(\sigma_i^0+0)-w_i^k(\sigma_i^0-0)). \quad \blacksquare$$

Let us turn to the case where the i-wave of u^0 is a rarefaction wave. We denote here by $x=\underline{\varphi}_i(t)$ and $x=\bar{\varphi}_i(t)$ the smooth characreristic curves which bound the corresponding rarefaction zone of u and we set:

$$\underline{\varphi}_i(t)=\underline{\sigma}_i^0\, t+\underline{\sigma}_i^1\, t^2+\ldots+\underline{\sigma}_i^k\, t^{k+1}+\ldots, \tag{3.13}$$

$$\bar{\varphi}_i(t)=\bar{\sigma}_i^0\, t+\bar{\sigma}_i^1\, t^2+\ldots+\bar{\sigma}_i^k\, t^{k+1}+\ldots \tag{3.14}$$

LEMMA 5. — *Assume that u^0 contains an i-rarefaction wave. Then, for all $k \geqq 1$, there exist two vectors $\underline{q}_i^k=\underline{q}_i^k(\underline{\sigma}_i^0, \ldots, \underline{\sigma}_i^{k-1}, v^0, \ldots, v^{k-1})$ and $\bar{q}_i^k=\bar{q}_i^k(\bar{\sigma}_i^0, \ldots, \bar{\sigma}_i^{k-1}, v^0, \ldots, v^{k-1})$ such that*

$$v^k(\underline{\sigma}_i^0+0)+\underline{\sigma}_i^k\, r_i(v_{i-1}^0)=(\underline{\sigma}_i^0-\mathrm{A}(v_{i-1}^0))^k\, v_{i-1}^k+\underline{q}_i^k, \tag{3.15}$$

$$(\bar{\sigma}_i^0-\mathrm{A}(v_i^0))^k\, v_i^k-\bar{\sigma}_i^k\, r_i(v_i^0)=v^k(\bar{\sigma}_i^0-0)+\bar{q}_i^k \tag{3.16}$$

Proof. — Let us derive (3.15). The jump condition (2.12) gives

$$[v^k]+\underline{\sigma}_i^k\left[\frac{dv^0}{d\xi}\right]+[\underline{z}_i^k]=0 \quad \text{at } \underline{\sigma}_i^0,$$

where $\underline{z}^k$ depends only on $\underline{\sigma}_i^0, \ldots, \underline{\sigma}_i^{k-1}, v^0, \ldots, v^{k-1}$. Since

$$v^0(\underline{\sigma}_i^0-0)=v^0(\underline{\sigma}_i^0+0)=v_{i-1}^0, \qquad \frac{dv^0}{d\xi}(\underline{\sigma}_i^0-0)=0$$

and by (3.2)

$$\frac{dv^0}{d\xi}(\underline{\sigma}_i^0+0)=r_i(v^0(\underline{\sigma}_i^0+0))=r_i(v_{i-1}^0),$$

we obtain

$$v^k(\underline{\sigma}_i^0+0)+\underline{\sigma}_i^k\, r_i(v_{i-1}^0)=v^k(\underline{\sigma}_i^0-0)-[\underline{z}_i^k](\underline{\sigma}_i^0).$$

But we have by (3.5)

$$v^k(\underline{\sigma}_i^0-0)=(\underline{\sigma}_i^0-\mathrm{A}(v_{i-1}^0))^k\, v_{i-1}^k+p_{i-1}^k(\underline{\sigma}_i^0)$$

so that (3.15) holds with

$$\underline{q}_i^k=p_{i-1}^k(\underline{\sigma}_i^0)-[\underline{z}_i^k](\underline{\sigma}_i^0).$$

The relation (3.16) is derived in a completely similar way. ■

Finally, we determine the vectors v_0^k and v_p^k. Assuming that the functions u_L and u_R are smooth, we may write in a neighborhood of the origin

$$u_L(x) = u_L^0 + x u_L^1 + \ldots + x^k u_L^k + \ldots, \tag{3.17}$$

$$u_R(x) = u_R^0 + x u_R^1 + \ldots + x^k u_R^k + \ldots \tag{3.18}$$

LEMMA 6. — *We have for all* $k \geqq 1$

$$v_0^k = u_L^k, \qquad v_p^k = u_R^k. \tag{3.19}$$

Proof. — Using Lemma 2, we get for $\xi < \underline{\sigma}_1^0$

$$v^k(\xi) = (\xi - A(v_0^0))^k v_0^k + p_0^k(\xi).$$

Since p_0^k is a polynomial of degree $\leqq k-1$, we find

$$\lim_{(x/t) < \underline{\sigma}_1^0,\, t \to 0} t^k v^k\left(\frac{x}{t}\right) = x^k v_0^k.$$

Hence, it follows from (1.8) that

$$u(x, 0) = \lim_{(x/t) < \underline{\sigma}_1^0,\, t \to 0} u(x, t) = v_0^0 + x v_0^1 + \ldots + x^k v_0^k + \ldots$$

$$x < 0$$

in a neighborhood of the origin. Comparing with (3.17), this yields $v_0^k = u_L^k$ for all $k \geqq 1$.

The 2nd conclusion (3.19) $v_p^k = u_R^k$ is derived analogously by considering the case $\xi > \bar{\sigma}_p^0$. ■

4. CONSTRUCTION OF THE ASYMPTOTIC EXPANSION II : THE EXISTENCE AND UNIQUENESS RESULT

Assume now that, for $0 \leqq l \leqq k-1$, we have constructed the functions v^l and computed the numbers σ_i^l or the pairs $(\underline{\sigma}_i^l, \bar{\sigma}_i^l)$, $1 \leqq i \leqq p$. We want to determine the function v^k together with the numbers σ_i^k or the pairs $(\underline{\sigma}_i^k, \bar{\sigma}_i^k)$, $1 \leqq i \leqq p$, in order to satisfy the conditions of the previous section. We shall show that this is indeed possible when the states u_L^0 and u_R^0 are sufficiently close. In this case, we now (*cf.* again [10]) that there exists a vector $\varepsilon=(\varepsilon_1, \ldots, \varepsilon_p)^T \in \mathbb{R}^p$, $|\varepsilon|$ small enough, such that

$$v_i^0 = v_{i-1}^0 + \varepsilon_i r_i(v_{i-1}^0) + O(\varepsilon_i^2). \tag{4.1}$$

Moreover, if the i-wave of u^0 is a shock wave, we have $\varepsilon_i<0$ and

$$\sigma_i^0 = \lambda_i(v_{i-1}^0) + \frac{\varepsilon_i}{2} + O(\varepsilon_i^2). \tag{4.2}$$

On the other hand, if the i-wave is a contact discontinuity, the sign of ε_i is arbitrary and

$$\sigma_i^0 = \lambda_i(v_{i-1}^0) = \lambda_i(v_i^0). \tag{4.3}$$

Finally, if the i-wave is a rarefaction wave, we have $\varepsilon_i>0$ and

$$\underline{\sigma}_i^0 = \lambda_i(v_{i-1}^0), \qquad \bar{\sigma}_i^0 = \lambda_i(v_i^0). \tag{4.4}$$

Then, we can state the main result of this paper which gives a practical procedure of the asymptotic expansion (1.8).

THEOREM 1. — *Let $k \geqq 1$ be an integer and suppose that the functions v^l and the numbers σ_i^l or the pairs $(\underline{\sigma}_i^l, \bar{\sigma}_i^l)$, $1 \leqq i \leqq p$, have been already determined for $l=1, \ldots, k-1$. Then, if $|u_R^0 - u_L^0|$ is small enough, there exists a unique function v^k and a unique set of numbers σ_i^k or pairs $(\underline{\sigma}_i^k, \bar{\sigma}_i^k)$, $1 \leqq i \leqq p$, solution of the equations* (2.8), (3.12), (3.15), (3.16) *and* (3.19).

Proof. — Using Lemma 2, we have only to determine the vectors v_i^k and the scalars σ_i^k or $(\underline{\sigma}_i^k, \bar{\sigma}_i^k)$, $1 \leqq i \leqq p$. In the sequel, we shall look for the vectors v_i^k in the form

$$v_i^k = \sum_{j=1}^{p} \alpha_{ij}^k r_j(v_i^0). \tag{4.5}$$

(1) We begin by considering the case where the i-wave of u^0 is either a shock wave or a contact discontinuity. Let us then check that the equation (3.12) is equivalent to a system of $(p-1)$ linear equations of the form

$$\alpha_{ij}^k - \alpha_{i-1,j}^k = \varepsilon_i \sum_{m=1}^{p} \{ a_{ijm} \alpha_{im}^k + b_{ijm} \alpha_{i-1,m}^k \} + c_{ij}, \qquad j \neq i \tag{4.6}$$

where the coefficients a_{ijm}, b_{ijm} and c_{ij} depend on ε_i (and k) but remain bounded as ε_i tends to zero.

Setting

$$\eta_i^k = (-1)^k (k+1)\, \sigma_i^k, \tag{4.7}$$

and using (4.5), the equation (3.12) becomes

$$\sum_{m=1}^{p} (\lambda_m (v_i^0) - \sigma_i^0)^{k+1}\, \alpha_{im}^k\, r_m (v_i^0)$$
$$= \sum_{m=1}^{p} (\lambda_m (v_{i-1}^0) - \sigma_i^0)^{k+1}\, \alpha_{i-1,\,m}^k\, r_m (v_{i-1}^0)$$
$$+ \eta_i^k (v_i^0 - v_{i-1}^0) + q_i^k. \tag{4.8}$$

By multiplying (4.8) on the left by $l_j (v_{i-1}^0)^{\mathrm{T}}$ and taking into account the normalization (1.3), we obtain for $j=1, \ldots, p$

$$\sum_{m=1}^{p} (\lambda_m (v_i^0) - \sigma_i^0)^{k+1}\, \alpha_{im}^k\, l_j (v_{i-1}^0)^{\mathrm{T}} . r_m (v_i^0)$$
$$= (\lambda_j (v_{i-1}^0) - \sigma_i^0)^{k+1}\, \alpha_{i-1,\,j}^k + \eta_i^k\, l_j (v_{i-1}^0 - v_{i-1}^0) + l_j (v_{i-1}^0)^{\mathrm{T}} . q_i^k. \tag{4.9}$$

Now, it follows from (4.1) that

$$r_m (v_i^0) = r_m (v_{i-1}^0) + O (\varepsilon_i)$$

and therefore

$$l_j (v_{i-1}^0)^{\mathrm{T}} . r_m (v_i^0) = \delta_{jm} + O (\varepsilon_j). \tag{4.10}$$

Similarly, we find

$$l_j (v_{i-1}^0)^{\mathrm{T}} . (v_i^0 - v_{i-1}^0) = \varepsilon_i\, \delta_{ij} + O (\varepsilon_i^2). \tag{4.11}$$

On the other hand, we have in the case of a shock wave

$$\lambda_m (v_i^0) - \sigma_i^0 = \lambda_m (v_{i-1}^0) - \sigma_i^0 + O (\varepsilon_i)$$

and

$$\lambda_m (v_i^0) - \sigma_i^0 = \lambda_m (v_i^0) - \lambda_i (v_i^0) + O (\varepsilon_i)$$

which gives

$$\frac{1}{(\lambda_m (v_i^0) - \sigma_i^0} = O (1), \qquad m \neq i \tag{4.12}$$

and

$$\frac{\lambda_m(v_i^0-1)-\sigma_i^0}{\lambda_m(v_i^0)-\sigma_i^0}=1+O(\varepsilon_i), \qquad m\neq i. \tag{4.13}$$

Note that (4.12) and (4.13) hold trivially in the case of a contact discontinuity.

It follows from (4.11) that for $|\varepsilon_i|$ small enough

$$l_i(v_{i-1}^0)^T.(v_i^0-v_{i-1}^0)\neq 0.$$

Hence (4.9) gives for $j=i$

$$\eta_i^k=\frac{1}{l_i(v_{i-1}^0)^T.(v_i^0-v_{i-1}^0)}\times\left\{\sum_{m=1}^{p}(\lambda_m(v_i^0)-\sigma_i^0)^{k+1}\,l_i(v_{i-1}^0)^T r_m(v_i^0)\,\alpha_{im}^k-(\lambda_i(v_{i-1}^0)-\sigma_i^0)^{k+1}\,\alpha_{i-1,i}^k-l_i(v_{i-1}^0).q_i^k\right\} \tag{4.14}$$

Replacing η_i^k by its value (4.14) in (4.9), we obtain for $j\neq i$

$$\sum_{m=1}^{p}(\lambda_m(v_i^0)-\sigma_i^0)^{k+1}\,l_j(v_{i-1}^0)^T.r_m(v_i^0)\,\alpha_{im}^k-(\lambda_j(v_{i-1}^0)-\sigma_i^0)^{k+1}\,\alpha_{i-1,j}^k=\frac{l_j(v_{i-1}^0)^T.(v_i^0-v_{i-1}^0)}{l_i(v_{i-1}^0)^T.(v_i^0-v_{i-1}^0)}\times\left\{\sum_{m=1}^{p}(\lambda_m(v_i^0)-\sigma_i^0)^{k+1}\,l_i(v_{i-1}^0)^T.r_m(v_i^0)\,\alpha_{im}^k-(\lambda_i(v_{i-1}^0)-\sigma_i^0)^{k+1}\,\alpha_{i-1,i}^k-l_i(v_{i-1}^0)^T.q_i^k\right\}+l_j(v_{i-1}^0)^T q_i^k.$$

Now, dividing the above equation by $(\lambda_j(v_{i-1}^0)-\sigma_i^0)^{k+1}$ and using (4.10)-(4.13), we obtain (4.6) with the indicated properties of the coefficients a_{ijm}, b_{ijm} and c_{im} so that the equations (4.6) and (4.14) are indeed equivalent to (3.12).

(2) Next consider the case where the i-wave of u^0 is a rarefaction wave. Let us check that a system of the form (4.6) still holds. Using (3.7) and

(4.5), the jump relation (3.15) can be written

$$\sum_{j=1}^{p} \alpha_j^k(\underline{\sigma}_i^0+0)\, r_j(v_{i-1}^0)+\underline{\sigma}_i^k r_i(v_{i-1}^0)=\sum_{j=1}^{p}(\underline{\sigma}_i^0-\lambda_j(\mathrm{v}_{i-i}^0))^k \alpha_{i-1,j}^k r_j(v_{i-1}^0)+\underline{q}_i^k$$

which gives by (1.3)

$$\alpha_i^k(\underline{\sigma}_i^0+0)+\underline{\sigma}_i^k=l_i(v_{i-1}^0)^{\mathrm{T}}\cdot \underline{q}_i^k \tag{4.15}$$

and

$$\alpha_j^k(\underline{\sigma}_i+0)=(\underline{\sigma}_i^0-\lambda_j(v_{i-1}^0))^k \alpha_{i-1,j}^k+l_j(v_{i-1}^0)^{\mathrm{T}}\cdot \underline{q}_i^k, \qquad j\neq i \tag{4.16}$$

on the other hand, the jump relation (3.16) becomes

$$\sum_{j=1}^{p}(\bar{\sigma}_i^0-\lambda_j(v_i^0))^k \alpha_{ij}^k r_j(v_i^0)-\bar{\sigma}_i^k r_i(v_i^0)=\sum_{j=1}^{p}\alpha_i^k(\bar{\sigma}_i^0-0)\, r_j(v_i^0)+\bar{q}_{io}^k.$$

Therefore, we obtain using again (1.3)

$$-\bar{\sigma}_i^k=\alpha_i^k(\bar{\sigma}_i^0-0)+l_i(v_i^0)^{\mathrm{T}}\cdot \bar{q}_i^k \tag{4.17}$$

and

$$(\bar{\sigma}_i^0-\lambda_j(v_i^0))^k \alpha_{ij}^k=\alpha_j^k(\bar{\sigma}_i^0-0)+l_j(v_i^0)^{\mathrm{T}}\cdot \bar{q}_i^k, \qquad j\neq i. \tag{4.18}$$

Hence, combining (4.16) and (4.18), we obtain for $j\neq i$

$$\begin{aligned}(\bar{\sigma}_i^0-\lambda_j(v_i^0))^k \alpha_{ij}^k-(\underline{\sigma}_i^0-\lambda_j(v_{i-1}^0))_{i-1,y}^k\\ =\alpha_j^k(\bar{\sigma}_i^0-0)-\alpha_j^k(\underline{\sigma}_i^0+0)+l_j(v_i^0)^{\mathrm{T}}\cdot \bar{q}_i^k+l_j(v_{i-1}^0)^{\mathrm{T}}\cdot \bar{q}_i^k.\end{aligned} \tag{4.19}$$

It remains to relate $\alpha_j^k(\underline{\sigma}_i^0+0)$ and $\alpha_j^k(\bar{\sigma}_i^0-0)$, $j\neq i$. This is achieved by solving the differential equation (2.8) in the interval $(\underline{\sigma}_i^0, \bar{\sigma}_i^0)$. In fact, we have already seen in the proof of Lemma 3 that the equations (3.9), $j\neq i$, represent a linear differential system of the form

$$(\Lambda(v_i^0)-\xi)\frac{d\alpha}{d\xi}+\mathrm{A}(\xi)\,\alpha=f(\xi)$$

where

$$\alpha^{\mathrm{T}}=(\alpha_1^k, \ldots, \alpha_{i-1}^k, \alpha_{i+1}^k, \ldots, \alpha_p^k),$$
$$\Lambda(v_i^0)=\operatorname{diag}(\lambda_1(v_i^0), \ldots, \lambda_{i+1}(v_i^0), \ldots, \lambda_p(v_i^0)).$$

Observe that, for $\xi \in [\underline{\sigma}_i^0, \bar{\sigma}_i^0]$, the matrix $(\Lambda(v_i^0)-\xi)$ is invertible and the $(p-1)\times(p-1)$ matrices $(\Lambda(v_i^0)-\xi)^{-1}$ and $A(\xi)$ together with the $(p-1)$-vector $f(\xi)$ are bounded independently of ε_i. Moreover, we have by (4.1) and (4.4)

$$\bar{\sigma}_i^0-\underline{\sigma}_i^0=\lambda_i(v_i^0)-\lambda_i(v_{i-1}^0)=O(\varepsilon_i).$$

Hence, we obtain

$$\alpha_j^k(\bar{\sigma}_i^0-0)-\alpha_j^k(\underline{\sigma}_j^0+0)=\varepsilon_i\left\{\sum_{\substack{m=1\\ m\neq 1}}^{p} d_{ijm}\,\alpha_m^k(\underline{\sigma}_i^0+0)+e_{im}\right\} \tag{4.20}$$

where the coefficients d_{ijm} and e_{im} remain bounded as ε_i tends to zero.

Finally, we notice that by (4.1) and (4.4) again

$$\frac{1}{\bar{\sigma}_i^0-\lambda_j(v_i^0)}=\frac{1}{\lambda_i(v_i^0)-\lambda_j(v_i^0)}=O(1),\qquad j\neq i$$

and

$$\frac{\underline{\sigma}_i^0-\lambda_j(v_{i-1}^0)}{\bar{\sigma}_i^0-\lambda_j(v_i^0)}=\frac{\lambda_i(v_{i-1}^0)-\lambda_i(v_{i-1}^0)}{\lambda_i(v_i^0)-\lambda_j(v_i^0)}=1+O(\varepsilon_i),\qquad j\neq i.$$

Thus, dividing (4.19) by $(\bar{\sigma}_i^0-\lambda_j(v_i^0))^k$ and using (4.20) together with (4.16) yield (4.6), where again the coefficients a_{ijm}, b_{ijm} and c_{im} remain bounded as ε_i tends to zero. Hence, in the case of an i-rarefaction wave, solving the equations (2.8), (3.15) and (3.16) amounts to solve a system of the form (4.6).

(3) It follows from the first two parts of the proof that the vectors v_i^k, $0\leqq i\leqq p$, may be characterized as the solutions of the equations (4.6), $1\leqq i,j\leqq p$, $j\neq 1$, and (3.19). Therefore, we obtain a linear system of $p(p-1)+2p=p(p+1)$ equations in the $p(p+1)$ unknowns α_{ij}^k, $0\leqq i\leqq p$, $1\leqq j\leqq p$. For proving that this linear system has a unique solution, it suffices to check that the corresponding homogeneous system

$$\begin{aligned}&\alpha_{y}^k-\alpha_{i-1,j}^k=\varepsilon\sum_{m=1}^{p}\{a_{ijm}\,\alpha_{im}^k+b_{ijm}\,\alpha_{i-1,m}^k)\\&1\leqq i,j\leqq p,\qquad j\neq i,\\&\alpha_{oj}^k=0,\qquad 1\leqq j\leqq p,\\&\alpha_{pj}^k=0,\qquad 1\leqq j\leqq p\end{aligned} \tag{4.21}$$

admits only the trivial solution. Now, it is a simple matter to show that this is indeed the case when $\varepsilon_i=0$, $1\leqq i\leqq p$. Hence, the matrix associated with the left-hand side of (4.21) is invertible so that (4.21) admits only the trivial solution at least for $|\varepsilon|$ small enough. Therefore, the vector v_i^k, $0\leqq i\leqq p$, are uniquely determined for $|\varepsilon|$ small enough.

Finally, if the i-wave of u^0 is a shock wave or a contact discontinuity, (4.7) and (4.14) give σ_i^k. On the other hand, if the i-wave is a rarefaction wave, Lemma 3 enables us to determine v^k in the interval $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$. Moreover, (4.15) and (4.17) give $\underline{\sigma}_i^k$ and $\bar{\sigma}_i^k$ respectively. This ends the proof of the theorem. ■

5. AN APPROXIMATION RESULT

Up to now, we have derived an asymptotic expansion (1.8) of the solution u of the G.R.P. (1.1).(1.2) which is valid in the domains of smoothness of u. In this section, we want to use the truncated expansion

$$\sum_{l=0}^{k} t^l v^l\left(\frac{x}{t}\right)$$

in order to construct a function $u^k=u^k(x,t)$ which approximates u in a neighborhood of the origin.

First, we set

$$\varphi_i^k(t)=\sum_{l=0}^{k} \sigma_i^l t^{l+1} \tag{5.1}$$

in the case of an i-shock wave or an i-contact discontinuity and

$$\underline{\varphi}_i^k(t)=\sum_{l=0}^{k} \underline{\sigma}_i^l t^{l+1}, \qquad \bar{\varphi}_i^k(t)=\sum_{l=0}^{k} \bar{\sigma}_i^l t^{l+1} \tag{5.2}$$

in the case of an i-rarefaction wave. In fact, in all the sequel, it will be convenient to set

$$\underline{\varphi}_i(t)=\bar{\varphi}_i(t)=\varphi_i(t), \qquad \underline{\varphi}_i^k(t)=\bar{\varphi}_i^k(t)=\varphi_i^k(t)$$

for an i-shock wave or an i-contact discontinuity. Observe that $\underline{\varphi}_i^k(t)$ [resp. $\bar{\varphi}_i^k(t)$] is a Taylor expansion of $\underline{\varphi}_i(t)$ [resp. $\bar{\varphi}_i(t)$] at the origin so that

$$\underline{\varphi}_i(t)-\underline{\varphi}_i^k(t)=O(t^{k+2}), \qquad \bar{\varphi}_i(t)-\bar{\varphi}_i^k(t)=O(t^{k+2}). \tag{5.3}$$

Next, for $0\leqq i\leqq p$, we introduce the domain

$$D_i=\{(x,t);\ \bar{\varphi}_i(t)<x<\underline{\varphi}_{i+1}(t),\ 0<t\leqq\delta)$$

with the convention that

$$\bar{\varphi}^0(t)=-\infty, \qquad \underline{\varphi}_{p+1}(t)=+\infty.$$

On the one hand, the function u is smooth in the closure $\bar{D}_i$ of D_i. On the other hand, using Lemma 2, we know that the restriction $v^{l,i}$ of the function v^l to the interval $(\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0)$ is a polynomial of degree $\leqq l$ which can be trivially extended on a larger interval. Hence the function

$$\sum_{l=0}^{k} t^l v^{l,i}\left(\frac{x}{t}\right)$$

is a polynomial of degree $\leqq k$ in the two variables (x,t) which coincides with a truncated Taylor expansion at the origin of the restriction of u to the domain $\bar{D}_i$. Hence, we have

$$\left|u(x,t)-\sum_{l=0}^{k} t^l v^{l,i}\left(\frac{x}{t}\right)\right|=O(|x|^{k+1}+t^{k+1}) \quad \text{in } D_i, \tag{5.4}$$

Then, we define the domain

$$D_i^k=\{(x,t);\ \bar{\varphi}_i^k(t)<x<\underline{\varphi}_{i+1}^k(t),\ 0<t\leqq\delta\} \tag{5.5}$$

and we set

$$u^k(x,t)=\sum_{l=0}^{k} t^l v^{l,i}\left(\frac{x}{t}\right) \quad \text{in } D_i^k. \tag{5.6}$$

Finally, we consider a rarefaction zone

$$R_i=\{(x,t);\ \underline{\varphi}_i(t)\leqq x\leqq\bar{\varphi}_i(t),\ 0<t<\delta\}.$$

In such a zone R_i, the solution u is singular at the origin but we have

LEMMA 7. — *The function* $\bar{u}$: $(\xi, t) \rightarrow \bar{u}(\xi, t) \in \mathbb{R}^p$ *defined by*

$$\bar{u}(\xi, t) = u(\xi t, t) \tag{5.7}$$

is smooth in the closed domain $\bar{\mathrm{R}}_i$.

Proof. — Let us introduce the one-parameter family of i-characteristic curves of $\bar{\mathrm{R}}_i$, $t \rightarrow \mathrm{X}(t; \sigma)$, defined for $\underline{\sigma}_i^0 \leqq \sigma \leqq \bar{\sigma}_i^0$ by

$$\frac{\partial \mathrm{X}}{\partial t} = \lambda_i(\mathrm{X}, t, u(\mathrm{X}, t)),$$

$$\mathrm{X}(0) = 0,$$

$$\frac{\partial \mathrm{X}}{\partial t}(0) = \sigma.$$

We know (*cf.* [8]) that the functions $(\sigma, t) \rightarrow \mathrm{X}(t; \sigma)$ and $(\sigma, t) \rightarrow v(\sigma, t) = u(\mathrm{X}(t; \sigma), t)$ are smooth in $[\underline{\sigma}_i^0, \bar{\sigma}_i^0] \times [0, \delta]$. Next, we observe that the equation

$$\xi = \frac{\mathrm{X}(t, \sigma)}{t}$$

defines a smooth function $(\xi, t) \rightarrow \sigma(\xi, t)$ for ξ in a neighborhood of $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$ and $t \in [0, \delta]$, δ small enough. In fact, we have

$$\mathrm{X}(t; \sigma) = \sigma t + O(t^2)$$

so that

$$\frac{\partial \xi}{\partial \sigma} = 1 + O(t)$$

is positive in a neighborhood of $[\underline{\sigma}_i^0, \bar{\sigma}_i^0] \times \{0\}$, which proves our assertion.

Hence the function

$$\bar{u}(\xi, t) = v(\sigma(\xi, t), t)$$

is smooth. The desired conclusion follows from

$$\bar{u}(\xi, t) = u(\mathrm{X}(t; \sigma(\xi, t)), t) = u(t\xi, t). \quad \blacksquare$$

Note that the restriction $\hat{v}^{k,i}$ of v^k to the interval $(\underline{\sigma}_i^0, \bar{\sigma}_i^0)$ can be extended in a smooth function defined in a neighborhood of $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$ and that

$$\sum_{l=0}^{k} t^l \hat{v}^{l,i}(\xi)$$

is a truncated Taylor expansion of the function $t \to \bar{u}(\xi, t)$ at the origin, ξ being considered as a parameter. Hence, we obtain for all ξ in a neighborhood of $[\underline{\sigma}_i^0, \bar{\sigma}_i^0]$

$$\left| \bar{u}(\xi, t) - \sum_{l=0}^{k} t^l \hat{v}^{l,i}(\xi) \right| = O(t^{k+i}). \tag{5.7}$$

Then, we introduce the approximate rarefaction zone

$$R_i^k = \{ (x, t); \underline{\varphi}_i^k(t) \leqq x \leqq \bar{\varphi}_i^k(t), 0 < t \leqq \delta \} \tag{5.8}$$

and we set

$$u^k(x, t) = \sum_{l=0}^{k} t^l \hat{v}^{l,i}\left(\frac{x}{t}\right) \quad \text{in } R_i^k \tag{5.9}$$

Using (5.6) and (5.9), we have thus defined a piecewise smooth approximation u_k of u for $t \leqq \delta$. Let us now state

THEOREM 2. — *We have for x, $t > 0$ small enough*

$$\int_{-x}^{+x} | u(y, t) - u^k(y, t) | \, dy = O(x^{k+2} + t^{k+2}) \tag{5.10}$$

and

$$| f(0, t, u(0, t)) - f(0, t, u^k(0, t)) | = O(t^{k+1}). \tag{5.11}$$

Proof. — Assume that

$$-x < \min(\underline{\varphi}_1(t), \underline{\varphi}_1^k(t)) \leqq \ldots \leqq \max(\bar{\varphi}_p(t), \bar{\varphi}_p^k(t)) < x.$$

Then setting

$$a_i = \min(\varphi_i, \varphi_i^k), \qquad b_i = \max(\varphi_i, \varphi_i^k), \qquad \underline{a}_i = \ldots,$$

we can write

$$\int_{-x}^{x}=\int_{-x}^{\underline{a}_1(t)}+\sum_{i=1}^{p}\int_{\underline{a}_i(t)}^{b_i(t)}+\sum_{i=1}^{p-1}\int_{\overline{b}_i(t)}^{\underline{a}_{i+1}(t)}+\int_{\overline{b}_p(t)}^{x}.$$

First, using (5.4), (5.6), we obtain if $e^k=u-u^k$

$$\int_{-x}^{\underline{a}_i(t)}|e^k(y,t)|\,dy+\int_{\overline{b}_p(t)}^{x}|e^k(y,t)|\,dy=O\,(x^{k+2}+t^{k+2}).$$

and

$$\int_{\overline{b}_i(t)}^{\underline{a}_{i+1}(t)}|e^k(y,t)|\,dy=O\,(t^{k+2}),\qquad 1\leqq i\leqq p-1.$$

Next, we write for $1\leqq i\leqq p$

$$\int_{\underline{a}_i(t)}^{b_i(t)}=\int_{\underline{a}_i(t)}^{\underline{b}_i(t)}+\int_{\underline{b}_i(t)}^{\overline{a}_i(t)}+\int_{\overline{a}_i(t)}^{\overline{b}_i(t)}$$

On the one hand, we have by (5.3)

$$\int_{a_i(t)}^{\underline{b}_i(t)}|e^k(y,t)|\,dy+\int_{\overline{a}_i(t)}^{\overline{b}_i(t)}|e^k(y,t)|\,dy=O\,(t^{k+2}).$$

On the other hand, it follows from (5.7) and (5.9) that

$$\int_{\underline{b}_i(t)}^{\overline{a}_i(t)}|e^k(y,t)|\,dy=O\,(t^{k+2}),$$

which yields the estimate (5.10).

We turn to the proof of the estimate (5.11). When the numbers $\sigma_i^0, \underline{\sigma}_i^0, \overline{\sigma}_i^0$ are $\neq 0$ for all $1\leqq i\leqq p$, the whole segment $I_\delta=\{(0,t);\ 0<t\leqq\delta\}$, δ small enough, is contained in one of the sets $D_i\cap D_i^k$ or $R_i\cap R_i^k$ so that (5.11) is an obvious consequence of (5.4) or (5.7). It remains only to consider the case where there exists a curve $x=\varphi(t)$ such that $\varphi(0)=\varphi'(0)=0$ which separates two dimains of smoothness of u. Assume first that there exists an integer m with $2\leqq m\leqq k+1$ such that $\varphi^{(m)}(0)\neq 0$. Then, we have for t small enough

$$\operatorname{sgn}\varphi^k(t)=\operatorname{sgn}\varphi(t)=\operatorname{sgn}\varphi^{(m)}(0)$$

and again I_δ is contained in some set $D_i \cap D_i^k$ or $R_i \cap R_i^k$ which implies (5.11).

Assume next that $\varphi^{(m)}(0)=0$, $0 \leqq m \leqq k+1$, and therefore

$$\varphi^k(t) \equiv 0, \qquad \varphi(t) = O(t^{k+2}).$$

Assume in addition that the exact solution u is discontinuous across the curve $x=\varphi(t)$ and consider, for specificity, the case where $\varphi(t)$ is positive, $0<t\leqq\delta$. We want to check that

$$f(0, t, u(0, t)) - f(0, t, u^k(0\pm, t)) = O(t^{k+1}). \tag{5.12}$$

We have by (5.4) and (5.7)

$$u(x, t) = u^k(x, t) + O(|x|^{k+1} + t^{k+1}) \qquad \text{for} \quad x > \varphi(t) \text{ and } x < 0. \tag{5.13}$$

On the one hand, using (5.13) gives

$$u(0-, t) = u^k(0-, t) + O(t^{k+1})$$

so that

$$u(\varphi(t) - 0, t) = u(0-, t) + O(t^{k+2}) = u^k(0-, t) + O(t^{k+1}).$$

On the other hand, using again (5.13), we find

$$u(\varphi(t)+0, t) = u^k(\varphi(t)+0, t) + O(t^{k+1}) = u^k(0+, t) + O(t^{k+1}).$$

Now, since $\varphi'(t) = O(t^{k+1})$, we have by the Rankine-Hugoniot jump conditions

$$[f(\varphi(t), t, u(\varphi(t), t))] = \varphi'(t)[u(\varphi(t), t)] = O(t^{k+1}).$$

As

$$f(\varphi(t), t, u(\varphi(t)+0, t)) = f(0, t, u^k(0+, t)) + O(t^{k+1})$$

and

$$f(\varphi(t), t, u(\varphi(t)-0, t)) = f(0, t, u^k(0-, t)) + O(t^{k+1})$$
$$= f(0, t, u(0-, t)) + O(t^{k+2}),$$

the assertion (5.12) follows.

In the case where $\varphi(t)\equiv 0$, we obtain

$$f(0, t, u(0\pm, t))-f(0, t, u^k(0\pm, t))=O(t^{k+1}).$$

Finally, we assume that the exact solution u is continuous across the curve $x=\varphi(t)$. Argueing as above, we obtain

$$u(0, t)-u^k(0, t)=O(t^{k+1})$$

which implies (5.11). ■

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