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Remarks on the large time behaviour of a nonlinear diffusion equation

by

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ABSTRACT. — Consider the diffusion equation $u_t - \Delta u = |u|^{p-1} u$ (where $p > 1 + \frac{2}{N}$ *and* $(N-2)\,p < N+2$) on the space $\mathbb{R}^N$. We prove that either $\|u(t)\|_\infty$ blows-up in finite time or $\|u(t)\|_\infty$ goes to zero like $t^{-1/(p-1)}$ as $t \to +\infty$. We give also a new proof to the fact that when $u(t) \geqq 0$ and $1 < p \leqq 1 + \frac{2}{N}$ then $\|u(t)\|_\infty$ blows-up in finite time. Sufficient conditions for global existence or blow-up are given, and the case where instead of $\mathbb{R}^N$ one has a cone like domain is also studied.

RÉSUMÉ. — Nous étudions le comportement en temps des solutions de $u_t - \Delta u = |u|^{p-1} u$. Nous montrons que si $p > 1 + \frac{2}{N}$ et $(N-2)\,p < N+2$, ou bien $\|u(t)\|_\infty$ explose en temps fini, ou bien $\|u(t)\|_\infty$ tend vers zéro comme $t^{-1/(p-1)}$ lorsque $t \to \infty$. On donne également une nouvelle démonstration du fait que si $1 < p \leqq 1 + \frac{2}{N}$ et $u(t) \geqq 0$, alors $\|u(t)\|_\infty$ explose en temps fini. Des conditions suffisantes pour l'existence globale (ou

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l'explosion en temps fini) sont présentées, et le cas où $\mathbb{R}^N$ est remplacé par un cône est également étudié.

Mots clés : Time behaviour, blow-up, global existence, diffusion equation.

1. INTRODUCTION AND MAIN RESULTS

Here we are interested in the time behaviour of solutions to:

$$(1.1)\qquad \begin{cases} u_t - \Delta u = |u|^{p-1} u, & t>0, \quad x \in \mathbb{R}^N \\ u(0,x) := u_0(x), & u_0 \neq 0. \end{cases}$$

By a result of H. Fujita [6] it is known that for $1 < p < 1 + \frac{2}{N}$, if $u_0 \geqq 0$, then the solution blows-up in finite time. The same conclusion holds when $p = 1 + \frac{2}{N}$, and this has been proved by K. Hayakawa [9], K. Kobayashi, T. Sirao and H. Tanaka [10] and alternate proofs have been presented by D. G. Aronson and F. H. Weinberger [1] and also by F. B. Weissler [13].

When $p > 1 + \frac{2}{N}$, positive global solutions to (1.1) exist. For instance, it has been observed by A. Haraux and F. B. Weissler [8], there are self-similar solutions to (1.1) i. e. solutions such as:

$$u(t,x) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right)$$

where f satisfies

$$(1.2)\qquad -\Delta f - \frac{y \cdot \nabla f}{2} = |f|^{p-1} f + \frac{1}{p-1} f \quad \text{on} \quad \mathbb{R}^N$$

and $f > 0$. Then it is clear that for any $\tau_0 > 0$, $u(t+\tau_0)$ is a global solution to (1.1). Actually as it has been observed later on by F. B. Weissler [14] and M. Escobedo and O. Kavian [3], for $p>1$ and $(N-2)p<N+2$, (1.2) has infinitely many solutions which are dominated by a Gaussian, but among these solutions there is a positive one if, and only if, $\frac{1}{p-1}<\frac{N}{2}$. Note that $\frac{N}{2}$ is the least eigenvalue of the self-adjoint operator L (*see* § 2 below) defined by:

$$Lf := -\Delta f - \frac{y.\nabla f}{2} = \frac{-1}{K}\nabla.(K\nabla f)$$

$$K := \exp\left(\frac{|y|^2}{4}\right)$$

$$D(L) \subset L^2(K) := \left\{ f; \int_{\mathbb{R}^N} |f(y)|^2 K(y)\, dy < \infty \right\}.$$

If we consider the evolution equation

$$(1.3)\qquad \begin{aligned} &v_s + Lv = |v|^{p-1} v + \lambda v, \qquad s>0, \quad y \in \mathbb{R}^N \\ &v(0, y) := v_0(y) \end{aligned}$$

then a change of variable transforms (1.1) into (1.3) and solutions to (1.2) appear as stationary points of (1.3). Indeed define for $s \geqq 0$, $y \in \mathbb{R}^N$:

$$(1.4)\qquad v(s, y) := e^{s/(p-1)} u(e^s - 1, e^{s/2} y), \qquad \lambda = \frac{1}{p-1};$$

then if u satisfies (1.1), v satisfies (1.3) with $v_0 = u_0$. Conversely if one knows a solution v of (1.3) $\left(\text{with } \lambda = \frac{1}{p-1}\right)$ then defining for $t \geqq 0$, $x \in \mathbb{R}^N$:

$$(1.5)\qquad u(t, x) := (1+t)^{-1/(p-1)} v\left(\log(1+t), \frac{x}{\sqrt{1+t}}\right),$$

one checks easily that u satisfies (1.1) with $u_0 = v_0$.

The interest in this change of variable lies in the fact that the operator L, defined above, has a compact inverse and therefore equation (1.3) can be studied in the same manner as the heat-equation $u_t - \Delta u = |u|^{p-1} u$ in a bounded region $\Omega \subset \mathbb{R}^N$. This observation has been exploited in M. Escobedo and O. Kavian [4] where it is proved that solutions to $u_t - \Delta u + |u|^{p-1} u = 0$ on $\mathbb{R}^N$ behave like a self-similar solution as $t \to \infty$

(*see* also M. Escobedo, O. Kavian and M. Matano [5] for a complete description of the time behaviour of the positive solutions of this equation).

For example let us prove (rapidly) that, when $1<p<1+\frac{2}{N}$, any positive solution of (1.1) or (1.3) blows-up in finite time. One can check that:

$$\varphi_1 := \exp\left(\frac{-|y|^2}{4}\right) \quad \text{satisfies } L\,\varphi_1 = \frac{N}{2}\varphi_1;$$

then consider $\psi := c(\varepsilon)\,\varphi_1^{1+\varepsilon}$ (for an $\varepsilon>0$ which is going to be fixed) such that $\int \psi\, K\, dy = 1$. Now $L\psi \leqq \frac{N(1+\varepsilon)}{2}\psi$, and multiplying (1.3) by ψK, and integrating by parts we get:

$$\frac{d}{ds}\int_{\mathbb{R}^N} v\,\psi\, K\, dy = \int_{\mathbb{R}^N} v^p.\psi\, K\, dy + \int_{\mathbb{R}^N} v.(\lambda\psi - L\,\psi)\, K\, dy.$$

By Jensen's inequality and the fact that $\lambda\psi - L\,\psi \geqq \left(\lambda - \frac{N(1+\varepsilon)}{2}\right)\psi$ we obtain:

$$\frac{d}{ds}\int v\,\psi\, K\, dy \geqq \left(\int v\,\psi\, K\right)^p + \left(\lambda - \frac{N(1+\varepsilon)}{2}\right)\int v\,\psi\, K.$$

Here we may choose $\varepsilon>0$ small enough in order to have $\lambda - \frac{N(1+\varepsilon)}{2} \geqq 0$ $\left(\text{this is possible because } p<1+\frac{2}{N} \text{ and } \lambda = \frac{1}{p-1} > \frac{N}{2}\right)$. Then the above differential inequality proves that if $\int v_0\,\psi\, K>0$, $v(s)$ cannot exist globally on $s>0$. (In section 3 we shall return to this discussion in more details.)

In this paper we study the large time behaviour of solutions to (1.1) through the evolution equation (1.3). In the sequel we use the following notations:

$$(1.6)\qquad \begin{cases} L^2(K) := \left\{ f;\ \int_{\mathbb{R}^N} |f(y)|^2\, K(y)\, dy < \infty \right\} \\ (f|g) := \int_{\mathbb{R}^N} f(y)\, g(y)\, K(y)\, dy, \end{cases}$$

$$\begin{cases} K(y):=\exp\left(\dfrac{|y|^2}{4}\right) \\ \|f\|:=(f\,|\,f)^{1/2}. \end{cases}$$

(1.7) $$\begin{cases} H^1(K):=\{f\in L^2(K);\nabla f\in L^2(K)\} \\ \text{and} \\ (f\,|\,g)_{H^1}:=(f\,|\,g)+(\nabla f\,|\,\nabla g) \end{cases}$$

[analogously $H^2(K)=\{f\in H^1(K);\ \nabla f\in H^1(K)\}$]

(1.8) $$\begin{cases} Lf:=-\Delta f-\dfrac{y.\nabla f}{2}=\dfrac{-1}{K}\nabla.(K\nabla f) \\ D(L):=H^2(K) \end{cases}$$

For $v\in H^1(K)$ and λ fixed in $\mathbb{R}$ define the energy:

(1.9) $$E_\lambda(v):=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 K(y)\,dy - \frac{\lambda}{2}\int_{\mathbb{R}^N}|v|^2 K(y)\,dy-\frac{1}{p+1}\int_{\mathbb{R}^N}|v|^{p+1}K(y)\,dy$$

$\left(\text{which makes sense when } v\in L^{p+1}(K):=\left\{f\in L^{p+1};\int|f|^{p+1}K\,dy<\infty\right\}\right)$.

The first result is the following:

(1.10) THEOREM. — *Let* $\lambda_1\left(:=\dfrac{N}{2}\right)$ *be the least eigenvalue of* L *on* $H^2(K)$.

Assume that $\lambda<\lambda_1$, $v_0\in L^{p+1}(K)\cap H^1(K)$, *and consider*

$$T_*:=\operatorname{Sup}\{T>0;\ \exists v\in C^1([0,T[,L^2(K))\cap C([0,T[,H^1(K)\cap L^{p+1}(K))\ \textit{Solution of the equation } (1.3)\}.$$

(i) *If* $E_\lambda(v_0)\leqq 0$, *then* T_* *is finite* [*i. e.* $v(s)$ *blows-up in finite time*].

(ii) *For* $p>1$ *and* $(N-2)\,p\leqq N+2$ *define:*

$$a:=a(p):=\inf\left\{\int|\nabla f|^2K\,dy-\lambda\int|f|^2K\,dy;\int|f|^{p+1}K=1,f\in H^1(K)\right\}$$

If

$$0<E_\lambda(v_0)<\frac{p-1}{2(p+1)}a^{(p+1)/(p-1)},$$

and:

$$\int|\nabla v_0|^2 K-\lambda\int|v_0|^2 K<a^{(p+1)/(p-1)},$$

then $T_*=\infty$ [*i. e. the solution* $v(s)$ *is global in time*].

Next, concerning the blow-up we have

(1.11) THEOREM. — *If* $\lambda>\lambda_1\left(:=\frac{N}{2}\right)$ *and* $v_0\neq 0$, $v_0\geqq 0$, *then the maximal solution of* (1.3) *blows-up in finite time.*

[By the "maximal solution" we mean a solution v satisfying (1.3) and such that $v\in C^1([0,T_*[,L^2(K))\cap C([0,T_*[,H^1(K)\cap L^{p+1}(K))$, where T_* is defined in theorem (1.10)].

Going back to the diffusion equation (1.1) we obtain as a corollary of the above theorems:

(1.12) COROLLARY. — *Let* $u_0\in H^1(K)\cap L^{p+1}(K)$, *and consider the maximal solution*

$$u\in C^1([0,T_*[,L^2(\mathbb{R}^N))\cap C([0,T_*[,H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N))$$

of (1.1), *and put* $\lambda:=\frac{1}{p-1}(p>1)$.

(i) *If* $p>1+\frac{2}{N}$ *and* $E_\lambda(u_0)\leqq 0$ *then* $T_*<\infty$ [*i. e.* $u(t)$ *blows-up in finite time*).

(ii) *If* $p>1+\frac{2}{N}$ *and* $(N-2)p\leqq N+2$, *suppose that*:

$$0<E_\lambda(u_0)<\frac{p-1}{2(p+1)}\cdot a^{(p+1)/(p-1)},$$

and

$$\int|\nabla u_0|^2 K-\lambda\int|u_0|^2 K<a^{(p+1)/(p-1)}.$$

Then $T_* = \infty$, *i.e. the solution* $u(t)$ *of* (1.1) *is global.*

(iii) *If* $1 < p \leqq 1 + \frac{2}{N}$ *and* $u_0 \geqq 0$, $u_0 \neq 0$, *then* $u(t)$ *blows-up in finite time.*

[*Here we do not need* $u_0 \in H^1(K) \cap L^{p+1}(K)$.]

We would emphasize that the "critical" exponent $1 + \frac{2}{N}$ is closely related to the least eigenvalue of L on $H^1(K)$. In section 4 we give examples of positive solutions of (1.1) when $\mathbb{R}^N$ is replaced by some cone like domain such as $(\mathbb{R}_+)^N$ (or $\mathbb{R}^{N-1} \times \mathbb{R}_+$) and $1 + \frac{1}{N} < p \leqq 1 + \frac{2}{N}$ $\left(\text{or } 1 + \frac{2}{N+1} < p \leqq 1 + \frac{2}{N}\right)$. The large time behaviour of global solutions of (1.1) is given by:

(1.13) THEOREM. – *Suppose that* $p > 1 + \frac{2}{N}$, $(N-2)p < N+2$, $u_0 \in H^1(K)$, *and consider a global classical solution* $u(t)$ *of the equation* (1.1). *Then* $\sup_{t \geqq 1} t^{1/(p-1)} \| u(t) \|_{C^2(\mathbb{R}^N)} < \infty$. *More precisely define* $v(s, y) := e^{s/(p-1)} u(e^s - 1, e^{s/2} y)$ $\left[\text{\textit{hence } } v \text{\textit{ satisfies} (1.3) \textit{with} } \lambda := \frac{1}{p-1} \text{ \textit{and} } v(0, y) = u_0(y)\right]$. *Then for any* $\sigma > 0$, $\sup_{n \geqq \sigma} \| v(s) \|_{C^2(\mathbb{R}^N)}$ *is finite and the* ω*-limit set of* $v(s)$ *as* $s \to \infty$ *is contained in the set of solution to the equation* (1.2).
[*Here* $\| u \|_{C^2(\mathbb{R}^N)} = \mathrm{Max}(\| u \|_\infty, \| Du \|_\infty, \| D^2 u \|_\infty)$.]

This means that when $t \to \infty$ there is a subsequence $(t_n)_n$ and an $f \in H^1(K)$ satisfying (1.2) such that

$$\lim_{t_n \to \infty} \left\| t_n^{1/(p-1)} u(t_n, .) - f\left(\frac{.}{\sqrt{t_n}}\right) \right\|_\infty = 0,$$

i.e. $u(t_n, .)$ is close to the self-similar solution $t_n^{-1/(p-1)} f\left(\frac{.}{\sqrt{t_n}}\right)$.

The remainder of this paper is organized as follows:

2. Preliminary results and Proofs of theorems (1.10), (1.11).
3. Proof of theorem (1.13).
4. Some observations about the case of cone-like domains.

5. Appendix.

The author wishes to thank Thierry Cazenave for some helpful conversations about this work.

2. PRELIMINARY RESULTS and Proofs of theorems (1.10), (1.11)

We recall here some results about the weighted Sobolev space $H^1(K)$ and the operator L.

(2.1) LEMMA. — (i) *There exists* $C>0$ *such that for any* $u \in H^1(K)$:

$$\int_{\mathbb{R}^N} |u(y)|^2 |y|^2 K(y)\, dy \leqq C \int_{\mathbb{R}^N} |\nabla u|^2 K(y)\, dy.$$

(ii) *The embedding* $H^1(K) \hookrightarrow L^2(K)$ *is compact.*

(iii) $\forall\, u \in H^1(K)$, $\dfrac{N}{2}\displaystyle\int |u|^2 K \leqq \int |\nabla u|^2 K.$

(iv) $u \in H^1(K) \Leftrightarrow K^{1/2} u \in H^1(\mathbb{R}^N)$.

(v) *For any* $f \in L^2(K)$ *there is a unique* $u \in H^2(K)$ *such that* $Lu = f$.

(vi) $\varphi_1 := \exp\left(\dfrac{-|y|^2}{4}\right)$ *is an eigenfunction of* L *corresponding to* $\lambda_1 = \dfrac{N}{2} =$ *the least eigenvalue of* $L\left(L\varphi_i = \dfrac{N}{2}\varphi_1\right)$.

(vii) L *is a positive self-adjoint operator on* $H^2(K) = D(L)$, *with compact inverse.*

(viii) *If* $N=1$, $u \in H^1(K)$, *then* $K^{1/2} u \in L^\infty(\mathbb{R})$.

If $N=2$, $H^1(K) \hookrightarrow L^q(K)$, *for any* $q \geqq 2$ *and* $q < \infty$.

If $N \geqq 3$, $2^* := \dfrac{2N}{N-2}$, $H^1(K) \hookrightarrow L^{2^*}(K)$.

For the proof of this lemma see section 5 below. As a consequence we have:

(2.2) COROLLARY. — *The operator* L *is the generator of an analytic semi-group* $S_*(s)$ *on* $L^2(K)$. *More precisely if*

$$G(t,x) := (4\pi t)^{-N/2} \exp\left(\frac{-|x|^2}{4t}\right)$$

and

$$u(t, x) := (G(t) * v_0)(x),$$

then $v(s, y) = u(e^s - 1, e^{s/2} y) = S_*(s) v_0$, *so that* $\dfrac{dv}{ds} + L v = 0$. *Furthermore we have the following properties:*

(i) $\| v(s) \|_{L^2(K)} \leqq e^{-(N s/2)} \| v_0 \|_{L^2(K)}$;

(ii) $\| v(s) \|_{L^\infty(\mathbb{R}^N)} \leqq (4\pi(e^s - 1))^{-N/2} \| v_0 \|_{L^1(\mathbb{R}^N)}$;

(iii) $\forall q > 1, \| v(s) \|_{L^q(K)} \leqq \| v_0 \|_{L^q(K)}$.

The proof of this corollary will be given in section 5 below. In order to prove theorems (1.10), (1.11) we need some preliminary results.

(2.3) LEMMA. – *Let* $\lambda \in \mathbb{R}$ *be given and assume that for some* $T_* \leqq \infty$, *one has a solution* $v \in C^1([0, T_*[, L^2(K)) \cap C([0, T_*[, D(L))$ *satisfying:*

$$v_s + L v = |v|^{p-1} v + \lambda v, \qquad v(0) = v_0 \in D(L). \tag{2.4}$$

Then

(i) $E_\lambda(v(s)) \leqq E_\lambda(v(0))$, $\forall s \in [0, T_*[$;

(ii) *if there exists* $s_0 \geqq 0$ *such that* $E_\lambda(v(s_0)) \leqq 0$, $v(s_0) \neq 0$, *and* $\lambda \leqq \lambda_1$, *then* $T_* < \infty$.

Proof. – The argument is almost the same as in L. E. Payne and D. H. Sattinger; we repeat it for the reader's convenience. Multiply (2.4) by v_s and v [in the sense of $L^2(K)$] to get:

$$\| v_s \|^2 + \frac{d}{ds} E_\lambda(v(s)) = 0 \tag{2.5}$$

$$\frac{1}{2} \frac{d}{ds} \| v \|^2 + (L v \mid v) = \int | v |^{p+1} K + \lambda \| v \|^2 \tag{2.6}$$

It is clear that (2.5) implies (i). Now if we had, $\| v_s(s_0) \| = 0$, then $v(s_0)$ would satisfy: $L v(s_0) = | v(s_0) |^{p-1} v(s_0) + \lambda v(s_0)$ and $E_\lambda(v(s_0)) \leqq 0$. But multiplying this equation by $v(s_0)$ we get

$$(L v(s_0) \mid v(s_0)) = \int | v(s_0) |^{p+1} K + \lambda \| v(s_0) \|^2$$

$$0 \geqq E_\lambda(v(s_0)) = \frac{p-1}{2(p+1)} [(L v(s_0) \mid v(s_0)) - \lambda \| v(s_0) \|^2] \geqq 0$$

[because $\lambda \leqq \lambda_1$ and $(\mathrm{L}\, v(s_0) \mid v(s_0)) \geqq \lambda_1 \| v(s_0) \|^2$] and this would imply $v(s_0)=0$. Hence we may assume $s_0=0$ and $\mathrm{E}_\lambda(v(0))<0$. Now define $f(t):=\frac{1}{2}\int_0^t \| v(s) \|^2\, ds$. By (2.5), (2.6) we have

$$\int_0^t \| v_s(s) \|^2\, ds = \mathrm{E}_\lambda(v(0)) - \mathrm{E}_\lambda(v(t)) \tag{2.7}$$

$$f''(t) = -(p+1)\,\mathrm{E}_\lambda(v(t)) + \frac{p-1}{2}[(\mathrm{L}\, v \mid v) - \lambda \| v \|^2] \tag{2.8}$$

First observe that as $\mathrm{E}_\lambda(v(t)) \leqq \mathrm{E}_\lambda(v(0)) < 0$:

$$f''(t) \geqq (p-1)(\lambda_1 - \lambda) f'(t) - (p+1)\,\mathrm{E}_\lambda(v(0)).$$

and if we had $\mathrm{T}_* = \infty$, then this inequality would yield $\lim_{t \to \infty} f'(t) = \lim_{t \to \infty} f(t) = +\infty$. Now using (2.7) in (2.8) we get:

$$f''(t) \geqq (p+1) \int_0^t \| v_s(s) \|^2\, ds$$

and multiplying by $f(t)$ we obtain:

$$f(t) f''(t) \geqq \frac{((p+1)}{2} \left(\int_0^t \| v(s) \|^2\, ds \right) \int_0^t \| v_s(s) \|^2\, ds$$

But by Cauchy-Schwarz' inequality:

$$(f'(t) - f'(0))^2 = \left(\int_0^t \int_{\mathbb{R}^\mathrm{N}} v(s,y)\, v_s(s,y)\, \mathrm{K}(y)\, dy\, ds \right)^2$$

$$\leqq \left(\int_0^t \| v(s) \|^2\, ds \right) \left(\int_0^t \| v_s(s) \|^2\, ds \right)$$

Therefore

$$f(t) f''(t) \geqq \frac{(p+1)}{2} (f'(t) - f'(0))^2$$

and as $t \to \infty$ we have for some $\alpha > 0$ and $\forall\, t \geqq t_0$

$$f(t) f''(t) \geqq (1+\alpha)(f'(t))^2$$

Hence $t \mapsto f(t)^{-\alpha}$ is concave on $[t_0, +\infty]$, $f(t)^{-\alpha} > 0$ and $\lim_{t \to \infty} f(t)^{-\alpha} = 0$.
This contradiction proves that $T_* < \infty$. ■

We recall here that due to the compactness of the embedding $H^1(K) \subsetneq L^q(K)$ for $2 \leqq q$ and $(N-2)q < N+2$, one can easily check that $a(p)$ [defined in theorem (1.10) (ii)] is achieved for $p > 1$, $(N-2)p < N+2$. When $(N-2)p = N+2$ and $\operatorname{Max}\left(a, \frac{N}{4}\right) < \lambda < \lambda_1$, $a(p)$ is achieved but hte proof is somewhat more difficult. All these have been proved in [3]. If moreover $\lambda < \lambda_1$ it is easily seen than $a(p) > 0$, for any $p > 1$, $(N-2)\, p \leqq N+2$. The following lemma is contained in D. H. Sattinger [12]:

(2.9) LEMMA. — *Under the hypotheses of theorem* (1.10) (ii), *the solution* $v(s)$ *is global in time.*

Proof. — By lemma (2.3) (i) and the definition of $a(p)$:

$$\frac{1}{2}(Lv \mid v) - \frac{\lambda}{2}\|v\|^2 \leqq E_\lambda(v_0) + \frac{1}{p+1}\int |v|^{p+1} K\, dy$$

$$\frac{1}{2} z(t)^2 \leqq E_\lambda(v_0) + \frac{1}{p+1} a^{-(p+1)/2} z^{p+1}(t)$$

(here for convvenience we put

$$z(t) := [(Lv \mid v) - \lambda \|v\|^2]^{1/2}$$

which makes sense because $\lambda < \lambda_1$).

Hence $\frac{1}{2} z^2 - \frac{a^{-(p+1)/2}}{p+1} \cdot z^{p+1} \leqq E_\lambda(v_0)$; therefore if

$$z(0) < z_* := a^{(p+1)/2\,(p-1)}$$

and

$$E_\lambda(v_0) < \frac{z_*^2}{2} - \frac{a^{-(p+1)/2}}{p+1} \cdot z_*^{p+1},$$

then as long as $v(s)$ exists we know that $z(s) < z_*$. This proves that $\left(1 - \frac{\lambda}{\lambda_1}\right)(Lv(s) \mid v(s)) < z_*^2$ and $v(s)$ exists globally in time. ■

(2.10) *Proof of corollary* (1.12). – It is clear that lemmas (2.3) and (2.9) prove theorem (1.10) which, in turn, implies corollary (1.12) (i), (ii). Now we prove theorem (1.11) [and corollary (1.12) (iii)]. As it was mentioned in the introduction we consider the first eigenfunction of L, namely:

$$\varphi_1(y) := c_0 \exp\left(\frac{-|y|^2}{4}\right)$$

(with c_0 such that $\|\varphi_1\| = 1$)

$$L\varphi_1 = \lambda_1 \varphi_1 = \frac{N}{2}\varphi_1.$$

Then define for an $\varepsilon > 0$ (which will be fixed below):

$$\psi = b_\varepsilon \varphi_1^{1+\varepsilon} \quad \text{with } b_\varepsilon \text{ such that} \int \psi(y) K(y)\, dy = 1$$

One checks that $\psi \in D(L)$ and:

$$L\psi = (1+\varepsilon)\lambda_1 \psi - \varepsilon(1+\varepsilon) b_\varepsilon \varphi_1^{\varepsilon-1} |\nabla \varphi_1|^2$$

hence $L\psi \leqq (1+\varepsilon)\lambda_1 \psi$.

Multiplying in $L^2(K)$ the equation

$$v_s + Lv = v^p + \lambda v$$

by ψ we get [note that as $v_0 \geqq 0$, we have $v(s) \geqq 0$ for all s]:

$$\begin{aligned} \frac{d}{ds}(v(s)\,|\,\psi) &= \int (v(s,y))^p \psi(y) K(y)\, dy \\ &\quad + \lambda (v(s)\,|\,\psi) - (v(s)\,|\,L\psi) \\ &\geqq \left(\int v(s,y)\psi(y) K(y)\, dy\right)^p \\ &\qquad + (\lambda - (1+\varepsilon)\lambda_1)(v(s)\,|\,\psi) \end{aligned}$$

(we use here Jensen's inequality for the probability measure $\psi K\, dy$, and also the choice of ψ). If $\varepsilon > 0$ is fixed in such a way that $\lambda \geqq (1+\varepsilon)\lambda_1$ then the function $g(s) := (v(s)\,|\,\psi)$ satisfies:

$$g' \geqq g^p, \qquad g(0) > 0$$

and this differential inequality proves that $v(s)$ cannot exist on $[0, \infty[$. So theorem (1.11) is proved.

To conclude the proof of corollary (1.12) (iii) we suppose that in (1.3) $\lambda = \dfrac{1}{p-1}$ and $1 < p \leqq 1 + \dfrac{2}{N}$. If $u_0 \geqq 0$, $u_0 \neq 0$ we have [$G(t)$ is defined in lemma (2.2)].

$$u(t, x) \geqq G(t) * u_0 \geqq (4\pi t) - (N/2) \exp\left(\frac{-|x|^2}{2t}\right) \int_{\mathbb{R}^N} v_0(z) \exp\left(\frac{-|z|^2}{2t}\right) dz$$

$\left(\text{we use here } \dfrac{|x-z|^2}{4t} \leqq \dfrac{|x|^2 + |z|^2}{2t}\right)$.

Now assume that the solution $u(t)$ is global. Then there exists $\varepsilon > 0$ $\left[\text{in fact} = (8\pi)^{-N/2} \int v_0(z) \exp\left(\dfrac{-|z|^2}{4}\right) dz\right]$ such that

$$u(2, x) \geqq \varepsilon \exp\left(\frac{-|x|^2}{4}\right) = \frac{\varepsilon}{c_0} \varphi_1(x).$$

Hence if we consider

$$w_s + L w = w^p + \lambda w, \qquad \lambda = \frac{1}{p-1}$$

$$w(0, y) = \frac{\varepsilon}{c_0} \varphi_1(y)$$

we know that [by theorem (1.11)] if $\lambda < \lambda_1$ $\left(\text{i.e. } 1 < p < 1 + \dfrac{2}{N}\right)$, $w(s)$ blows-up in finite time; if $p = 1 + \dfrac{2}{N}$ (i.e. $\lambda = \lambda_1$) we observe that

$$E_\lambda(w(0)) = \frac{-1}{p+1} \left(\frac{\varepsilon}{c_0}\right)^{p+1} \int_{\mathbb{R}^N} |\varphi_1|^{p+1} K(y)\, dy$$

is < 0 and by lemma (2.3) (ii), $w(s)$ blows-up in finite time.

Next consider

$$\tilde{u}(t, x) := (1+t)^{[1/(p+1)]} w\left(\log(1+t), \frac{x}{\sqrt{1+t}}\right);$$

$\tilde{u}(t)$ blows-up in finite time and satisfies:

$$\tilde{u}_t - \Delta\tilde{u} = \tilde{u}^p$$

$$\tilde{u}(0) = \frac{\varepsilon}{c_0}\varphi_1$$

But by the parabolic maximum principle:

$$u(2+t, x) \geqq \tilde{u}''(t, x)$$

and therefore $u(t)$ blows-up in finite time: this contradiction concludes the proof. □

3. PROOF OF THEOREM (1.13)

When Ω is a bounded domain, $p>1$ and $(N-2)\,p<N+2$, Th. Cazenave and P. L. Lions [2] prove that global solutions of the equation $u_t - \Delta u = |u|^{p-1} u + \lambda u$, $u \in H_0^1(\Omega)$ [here $\lambda < \lambda_1 =$ the least eigenvalue of $-\Delta$ on $H^2 \cap H_0^1(\Omega)$] are uniformly bounded in $C^2(\bar{\Omega})$, i.e. $\sup_{t \geqq 1} \| u(t) \|_{C^2(\bar{\Omega})} < \infty$. (*See* also Y. Giga [7] for an alternate proof and a more precise result. Giga's proof uses the boundedness of the domain Ω, while the proof in [2] works also in the case $\Omega =$ unbounded domain and $\lambda < 0$ or in the case $\Omega =$ bounded domain and $\lambda \in \mathbb{R}$ without the above limitation $\lambda < \lambda_1$.)

To prove theorem (1.13) we apply the method used by Th. Cazenave and P. L. Lions [2] (*cf.* lemma 1 and proposition 6 of [2]); however as we work on the equation (1.3)—which is not studied in [2] nor in [7]—, we give below the detailed proof of:

(3.1) Proposition. — *Let*

$$p > 1 + \frac{2}{N}, \qquad (N-2)\,p < N+2$$

so that $\lambda := \frac{1}{p-1} < \lambda_1 = \frac{N}{2}\Big)$ *be given. Consider a solution* $v \in C^1([0, \infty[, L^2(K)) \cap C([0, \infty[, H^1(K))$ *of the equation* (1.3) *i.e.*

$$v_s + \mathrm{L}\, v = |v|^{p-1} v + \lambda v.$$

Then for any $\tau > 0$, $v \in \mathrm{C}^2([\tau, \infty[\times \mathbb{R}^N)$ *and the following hold:*

(i) $$\forall s \geqq 0, \qquad \mathrm{E}_\lambda(v(s)) > 0;$$

(ii) $$\forall s \geqq 0, \quad \|v(s)\|^2 < \frac{2(p+1)\,\mathrm{E}_\lambda(v(0))}{(p-1)(\lambda_1 - \lambda)} \qquad (\|\cdot\| := \|\cdot\|_{\mathrm{L}^2(\mathrm{K})});$$

(iii) $$\int_0^\infty \|v_s(s)\|^2\, ds \leqq \mathrm{E}_\lambda(v(0));$$

(iv) $$\sup_{t \geqq 0} \int_t^{t+1} (\|\nabla v(s)\|^2 + \|v(s)\|_{\mathrm{L}^{p+1}}^{p+1})^2\, ds \leqq \mathrm{Cte};$$

(v) *For any* $\tau > 0$ *there exists* $\mathrm{C} > 0$ *such that* $\sup_{s \geqq \tau} \|v(s)\|_{\mathrm{C}^2(\mathbb{R}^N)} \leqq \mathrm{C}$.

(vi) $\omega(v) := \bigcap_{s \geqq 0} \overline{\{v(\sigma); \sigma \geqq s\}}^{\mathrm{H}^1(\mathrm{K})} \subset \{f \in \mathrm{H}^1(\mathrm{K}); \mathrm{L} f = |f|^{p-1} f + \lambda f\}$.

Proof. — $u(t)$ defined as in (1.5) satisfies $u_t - \Delta u = |u|^{p-1} u$, and by the classical parabolic regularity $u \in \mathrm{C}^2([\tau, \infty[\times \mathbb{R}^N)$. Hence v [given also by (1.4)] is regular i. e. $v \in \mathrm{C}^2([\tau, \infty[\times \mathbb{R}^N)$ for any $\tau > 0$.

(i) By lemma (2.3) if there is an $s_0 \geqq 0$ such that $\mathrm{E}_\lambda(v(s_0)) \leqq 0$ then $v(s)$ cannot exist globally on $[s_0, +\infty[$.

(ii) With the notations used in the proof of lemma (2.3) we have by (2.8), (2.7) and the fact that $(\mathrm{L} v | v) \geqq \lambda_1 \|v\|^2$:

$$f''(t) \geqq (p+1) \int_0^t \|v_s(s)\|^2\, ds + (p-1)(\lambda_1 - \lambda) f'(t) - (p-1)\, \mathrm{E}_\lambda(v(0))$$

Now if there exists $t_0 \geqq 0$ such that:

$$(p-1)(\lambda_1 - \lambda) f'(t_0) \geqq (p+1)\, \mathrm{E}_\lambda(v(0)),$$

then $f''(t_0) > 0$ and $\forall t \geqq t_0$

$$(p-1)(\lambda_1 - \lambda) f'(t) \geqq (p+1)\, \mathrm{E}_\lambda(v(0)).$$

Hence

$$\forall t \geqq t_0, \quad f''(t) \geqq (p+1) \int_0^t \|v_s(s)\|^2\, ds,$$

and by the proof of lemma (2.3) (ii) this leads to a contradiction with the global existence of $v(s)$.

Therefore we have

$$\forall t \geqq 0,$$

$$f'(t)=\frac{1}{2}\|v(t)\|^2_{L^2(K)}<\frac{p+1}{(p-1)(\lambda_1-\lambda)}E_\lambda(v(0)). \blacksquare$$

(iii) By (2.7) we know that:

$$\|v_s(s)\|^2=-\frac{d}{ds}E_\lambda(v(s)),$$

then $\forall T>0$:

$$\int_0^T \|v_s(s)\|^2 ds \leqq E_\lambda(v(0))-E_\lambda(v(T)) \leqq E_\lambda(v(0)) \qquad \text{[we use here (i)]} \quad \blacksquare$$

and (iii) follows.

(iv) By (2.8), using $f''(t) \leqq \|v(t)\| . \|v_s(t)\|$ we have

$$\frac{p-1}{2}[(Lv\,|\,v)-\lambda\|v\|^2] \leqq \|v(t)\| . \|v_s(t)\| + (p+1)E_\lambda(v(0))$$

Therefore by (ii) we have for some $C>0$:

$$(3.2) \qquad \left(\frac{p-1}{2}\left(1-\frac{\lambda}{l_1}\right)\|\nabla v(t)\|^2_{L^2(K)}\right)^2 \leqq C(1+\|v_s(t)\|^2)$$

and

$$\operatorname*{Sup}_{t\geqq 0}\int_t^{t+1}\|\nabla v(s)\|^4_{L^2(K)}ds \leqq \text{Cte}.$$

On the other hand by (i):

$$\|v(s)\|^{p+1}_{L^{p+1}(K)} \leqq \frac{p+1}{2}\|\nabla v(s)\|^2$$

and (iv) follows. $\blacksquare$

(v) Let $A := \{s>0;\ \|v_s(s)\|>1\}$; by (iii), $\lim_{T\to\infty}$ meas $(A \cap [T, +\infty[) = 0$.
First by (3.2) there exist C_0, $C_1>0$ such that (assuming $N\geqq 3$)

$$\forall s\notin A,\quad \|\nabla v(s)\|^2 \leqq C_0 \qquad (3.3)$$
$$\|v(s)\|_{L^{2^*}(K)} \leqq C_1$$

Denoting by $S(t)$ the semi-group of the operator $L-\lambda$ Id [in fact $S(t)=e^{\lambda t}S_*(t)$ with S_* defined in corollary (2.2)], we have by (2.2) (i), (ii):

$$\|S(t)v_0\|_{L^q(K)} \leqq C.t^{-N((1/2)-(1/q))}\|v_0\|_{q/L^{(q-1)}(K)}$$

for $2\leqq q\leqq\infty$.

Moreover for $N\geqq 3$ and $t\leqq 1$:

$$\|S(t)v_0\|_{L^{2^*}(K)} \leqq C\|S(t)v_0\|_{L^{2^*}(K)}$$

hence by interpolation we get:

$$(3.4)\qquad \begin{cases} \text{for } 1<p<2^*, \\ N\geqq 3, \qquad \theta := \dfrac{(p-1)(N-2)}{4} \\ \|S(t)v_0\|_{L^{2^*}(K)} \leqq C.t^{-\theta}\|v_0\|_{L^{2^*/p}(K)} \\ 0<t\leqq 1 \end{cases}$$

(note that $\theta\in]0, 1[$). Now for $0<\sigma\leqq 1$ and $s>0$ we have

$$v(\sigma+s) = S(\sigma)v(s) + \int_0^\sigma S(\sigma-t)|v|^{p-1}v(t+s)\,dt \qquad (3.5)$$

and by (3.4)

$$\|v(\sigma+s)\|_{L^{2^*}(K)} \leqq C\left[\|v(s)\|_{L^{2^*}(K)} + \int_0^\sigma (\sigma-t)^{-\theta}\|v(t+s)\|^p_{L^{2^*}(K)}\,dt\right]$$

So if we put

$$f(\sigma) := \sup_{s\leqq t\leqq s+\sigma} [1+\|u(t)\|_{L^{2^*}(K)}]$$

we have

$$f(\sigma) \leqq f(0) + C(\theta)\, \sigma^{1-\theta} f(\sigma)^p$$

and if there is σ_0 such that

$$f(\sigma_0) = 2\, f(0)$$

then $\sigma_0 \geqq [C(\theta)\, f(\sigma_0)^{(p-1)}]^{-1/1-\theta} = \sigma_*$ and this means that $\forall\, \sigma \leqq \sigma_*$, we have

$$\| v(s+\sigma) \|_{L^{2^*}(K)} \leqq 2\, [1 + \| u(s) \|_{L^{2^*}(K)}]$$

(we use here the fact that f is continuous and non-decreasing).

Now we conclude that there is $s_* > 0$ and $C_2 > 0$ such that [*cf.* (3.3)]

$$\forall\, s \notin A, \quad \forall\, t \in [s, s+s_*],$$
$$\| v(s) \|_{L^{2^*}(K)} \leqq C_2;$$

hence we may fix $T > 0$ large enough to have meas $(A \cap [T, +\infty[) < \frac{1}{2} s_*$ in such a way that:

$$\forall\, s \geqq T, \quad \| v(s) \|_{L^{2^*}(K)} \leqq C_2.$$

Therefore $\sup_{s \geqq \tau} \| v(s) \|_{L^{2^*}(K)} \leqq \text{Cte} = C(\tau)$, for any $\tau > 0$, and $|v(s)|^p$ is uniformly bounded in $L^p(K)$ for $s \geqq \tau$.

Next, to obtain a uniform estimate on $\| v(s) \|_\infty$, we may use the arguments of [2]; nevertheless we prefer the following bootstrap argument. Let $q_0 := 2^*$; by corollary (2.2) we have:

$$\| S(t)\, v_0 \|_{L^q(K)} \leqq C \,.\, t^{-(N/2)((1/r)-(1/q))} \| v_0 \|_{L^r(K)} \quad \text{for} \quad 0 < t \leqq 1, \quad q \geqq r \geqq 1. \tag{3.6}$$

Choosing $r := r_0 := q_0/p$, and $q := q_1 > q_0$, in such a way that $\frac{N}{2}\left(\frac{1}{r} - \frac{1}{q_1}\right) < 1$, we get by (3.5) [note that this choice of q_1 is possible because $(N-2)\,p < N+2$]:

$$\| v(s+\sigma) \|_{L^{q_1}(K)} \leqq C. \Bigg(\sigma^{-(N/2)((1/q_0)-(1/q))} . \| v(s) \|_{L^{q_0}(K)}$$

$$+\int_0^\sigma (\sigma-t)^{-(N/2)((1/r_0)-(1/q_1))}. \| v(s+t)\|_{L^{q_0}(K)}^p \, dt\Big)$$

and using the uniform estimate on $\| v(s)\|_{L^{q_0}(K)}$ for $s \geqq \tau > 0$, we obtain (by putting $\sigma := \tau$ in the above inequality):

$$\forall s \geqq 2\tau, \quad \| v(s)\|_{L^{q_1}(K)} \leqq \text{Cte} = C(\tau).$$

Now, repeating the above argument $\Big[$with $r_{i-1} := q_{i-1}/p$, $q_i > q_{i-1}$ such that $\frac{N}{2}\left(\frac{1}{r_{i-1}} - \frac{1}{q_i}\right) < 1\Big]$, we get —after a finite number of steps—, $r_i > \frac{N}{2}$, and in (3.6) we may choose $q := \infty$, $r := r_i > \frac{N}{2}$; then using this in (3.5) as above, we obtain easily a uniform estimate on $\| v(s)\|_\infty$.

When $N=1$ or $N=2$, one oughts to modify slightly the preceding proof (in a classical fashion) in order to obtain the corresponding uniform estimate.

The uniform estimate in C^2-norm can be derived by using again the equation satisfied by $v(s)$ and the relation (3.5). ■

(iv) This is the classical invariance principle of La Salle (*see* for example C. Dafermos [15]). ■

Proof of theorem (1.13) *concluded.* — With the notations used in the previous sections we have

$$u(t, x) = (1+t)^{-1/(p-1)} v\left(\log(1+t), \frac{x}{\sqrt{1+t}}\right)$$

and by Proposition (3.1) (v), for any $t_0 > 0$, there is $C(t_0)$ such that:

$\forall t \geqq t_0$,

$$\| u(t)\|_{C^2(\mathbb{R}^N)} \leqq (1+t)^{-[1/(p-1)]}$$
$$\times \| v(\log(1+t))\|_{C^2(\mathbb{R}^N)} \leqq C(t_0)(1+t)^{-1/(p-1)} \quad ■$$

(3.6) *Remark.* — When $N=1$ one can prove that the positive solution of :

$$-f'' - \frac{1}{2} y f' = f^p + \lambda f, \qquad \lambda < 1$$

$$f \in H^1(K)$$

is unique (this has been proved by F. B. Weissler; personal communication). Hence if one knows that $\underline{\lim}_{t \to \infty} t^{1/(p-1)} u(t, x) > 0$ for some $x \in \mathbb{R}$, then the ω-limit set of $(v(s))_{s \geqq 0}$ is precisely f (= solution of the above equation). Therefore in this case:

$$\lim_{t \to \infty} t^{1/(p-1)} \| u(t) - w(t) \|_\infty = 0$$

where $w(t, x) := t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right)$ is the positive self-similar solution of (1.1). ■

(3.7) *Remark.* — As an application of theorem (1.13) one may give the following sufficient condition for the blowing-up of positive solutions to (1.1). Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and $\tilde{u}_0 > 0$ such that

$$-\Delta \tilde{u}_0 = \tilde{u}_0^p, \qquad u_0 \in H_0^1(\Omega)$$

then the solution of $u_t - \Delta u = u^p$ with $u(0) = u_0$ ($u_0 := \tilde{u}_0$ on Ω, $u_0 := 0$ on Ω^c) blows-up in finite time. Indeed if $\tilde{u}_t - \Delta \tilde{u} = \tilde{u} = \tilde{u}^p$, $\tilde{u}(t) \in H_0^1(\Omega)$, we have $u(t, x) \geqq \tilde{u}(t, x) . 1_\Omega(x) = u_0 \geqq 0$ [if $u(t)$ were a global solution then $\lim_{t \to \infty} t^{1/(p-1)} \| u(t) \|_\infty = +\infty$ in contradiction with theorem (1.13)].

The same holds if $-\Delta \tilde{u}_0 \leqq \tilde{u}_0^p$, $u_0 \in H_0^1(\Omega) \cap C^2(\bar{\Omega})$: the solution of (1.1) with $u(0) = \tilde{u}_0 . 1_\Omega$ blows up in finite time. ■

4. SOME OBSERVATIONS ABOUT THE CASE OF CONE-LIKE DOMAINS

Consider a domain Ω satisfying:

$$0 \in \bar{\Omega}, \qquad \forall \theta > 0, \qquad \forall x \in \Omega, \qquad \theta x \in \Omega.$$

Then if one considers the evolution equation:

$$(4.1)_\varepsilon \qquad \begin{cases} u_t - \Delta u = \varepsilon |u|^{p-1} u & \text{on } \Omega \\ u(t)_{|\partial\Omega} = 0 \end{cases}$$

where $\varepsilon = \pm 1$, we can study existence or non-existence of self-similar solutions to $(4.1)_\varepsilon$ in the same way as the case of the whole space $\mathbb{R}^N$.

Indeed we can define

$$v(s, y) := e^{s/(p-1)} u(e^2 - 1, e^{s/2} y), \qquad \lambda = \frac{1}{p-1}$$

in a such a way that

$$(4.2)_\varepsilon \qquad \begin{aligned} v_s + \mathrm{L} v &= \varepsilon |v|^{p-1} v + \lambda v \quad \text{on } \Omega \\ v(s)_{|\partial\Omega} &= 0 \end{aligned}$$

Then stationary solutions of $(4.2)_\varepsilon$ yield self-similar solutions of $(4.1)_\varepsilon$. To find such solutions we may consider the problems

$$(4.3) \qquad \begin{aligned} &\mathrm{L} f + |f|^{p-1} f = \lambda f, \\ f \not\equiv 0, \quad &f \in \mathrm{H}_0^1(\mathrm{K}, \Omega) \qquad (\text{if } \varepsilon = -1) \end{aligned}$$

$$(4.4) \qquad \begin{aligned} &\mathrm{L} f = |f|^{p-1} f + \lambda f, \\ f \not\equiv 0, \quad &f \in \mathrm{H}_0^1(\mathrm{K}, \Omega) \qquad (\text{if } \varepsilon = +1) \end{aligned}$$

Now solutions of (4.3) can be found as critical points of

$$\mathrm{F}_1(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{K}(y)\, dy + \frac{1}{p+1}\int_\Omega |v|^{p+1} \mathrm{K}(y)\, dy - \frac{\lambda}{2}\int_\Omega |v|^2 \mathrm{K}(y)\, dy$$

with $v \in \mathrm{H}_0^1(\mathrm{K}, \Omega)$ $(\cap\, \mathrm{L}^{p+1}(\mathrm{K}, \Omega)$ if $(\mathrm{N}-2)p > \mathrm{N}+2)$

$$\mathrm{H}_0^1(\mathrm{K}, \Omega) := \left\{ v;\ \int_\Omega (|v|^2 + |\nabla v|^2) \mathrm{K}(y)\, dy < \infty,\ v_{|\partial\Omega} = 0 \right\}$$

Using the compactness of

$$\mathrm{H}_0^1(\mathrm{K}, \Omega) \hookrightarrow \mathrm{L}^2(\mathrm{K}, \Omega) = \left\{ v;\ \int_\Omega |v|^2 \mathrm{K}(y)\, dy < \infty \right\}$$

[which can be proved easily by lemma (2.1) (i)], one can check that $\inf\{\mathrm{F}_1(v);\ v \in \mathrm{H}_0^1(\mathrm{K}, \Omega) \cap \mathrm{L}^{p+1}(\mathrm{K}, \Omega)\}$ is achieved and that, if $\lambda \leqq \lambda_1 = \lambda_1(\mathrm{L}, \Omega)$ = the least eigenvalue of L on $\mathrm{H}_0^1(\mathrm{K}, \Omega)$, this minimum is equal to zero and F_1 is strictly convex: therefore equation (4.3) does not have any solution. For $\lambda > \lambda_1(\Omega)$ (4.3) has a unique positive solution and if $\lambda > \lambda_k(\Omega)$, (4.3) has at least $2k$ solutions. (For more details see [3] where the case of $\Omega = \mathbb{R}^\mathrm{N}$ is treated but the same arguments hold in the case $\Omega \neq \mathbb{R}^\mathrm{N}$.)

Solutions of (4.4) can be sought as critical points of

$$J(v):=\int_\Omega |\nabla v|^2 K(y)\,dy-\lambda\int_\Omega |v|^2 K(y)\,dy$$

on the set

$$S:=\left\{v\in H_0^1(K,\Omega);\ \int_\Omega |v|^{p+1} K(y)\,dy=1\right\}$$

[we assume here $(N\text{-}2)p<N+2$, $p>1$].

By classical variational methods one can prove that J has a sequence of critical values $C_k\to\infty$ as $k\to\infty$ and to each $C_k>0$ corresponds a solution of (4.4). If $\lambda<\lambda_1(\Omega)$ then (4.4) has a positive solution (which corresponds to Min $\{J(v);\ v\in S\}>0$).

As an example we shall treat only two cases by computing the least eigenvalue of L (with Dirichlet boundary condition) on the two following domains:

$$\Omega_1:=(\mathbb{R}_+)^N:=\{x\in\mathbb{R}^N;\ \forall i_{\leqq N}^{\geqq 1},\ x_i>0\}$$

and

$$\Omega_2:=\mathbb{R}^{N-1}\times\mathbb{R}_+:=\{x\in\mathbb{R}^N;\ x_N>0\}.$$

For Ω_1 we have

$$\lambda_1(\Omega_1)=N,\qquad \varphi_{1,\,\Omega_1}(y):=y_1\ldots y_N\exp\left(\frac{-|y|^2}{4}\right)$$

[This is because one can easily check that

$$L\varphi_{1,\,\Omega_1}=N\varphi_{1,\,\Omega_1}$$
$$\varphi_{1,\,\Omega_1}\in H_0^1(K,\Omega_1),\qquad \varphi_{1,\,\Omega_1}>0\quad\text{on }\Omega_1$$

and by the Krein-Rutman theorem $\varphi_{1,\,\Omega_1}$ corresponds to the least eigenvalue of L on $H_0^1(K,\Omega_1)$].

Therefore if $\lambda<\lambda_1(\Omega_1)=N$, equation (4.4) has a positive solution f. Then the equation $u_t-\Delta u=u^p$ has a positive self-similar solution $w(t,x)$ $\left[\text{namely } w(t,x):=t^{-[1/(p-1)]}f\left(\frac{x}{\sqrt{t}}\right)\right]$ for any $p>1+\frac{1}{N}$,

$(N-2)p<N+2\left[\text{i. e. such that } \lambda:=\frac{1}{p-1}>\lambda_1(\Omega_1)=N\right]$ and this proves that if $1+\frac{1}{N}<p\leqq 1+\frac{2}{N}$ the equation $u_t-\Delta u=u^p$ on $\Omega_1=(\mathbb{R}_+)^N$, $u(t)_{|\partial\Omega_1}=0$ possesses global positive solutions. Using exactly the same arguments as in the proof of theorems (1.10), (1.11) and corollary (1.12) we may prove that the latter equation does not have global positive solutions if $1<p<1+\frac{1}{N}$.

Concerning equation

$$u_t-\Delta u+|u|^{p-1}u=0$$
$$u(t)_{|\partial\Omega_1}=0, \qquad u(0)\in H_0^1(K, \Omega_1)$$

the above observations prove that for $p\geqq 1+\frac{1}{N}$ there is no self-similar solution belonging to $H_0^1(K, \Omega_1)$ and for $1<p<1+\frac{1}{N}$ there are such solutions; using the equation

$$v_s+Lv+|v|^{p-1}v=0$$
$$v(s)\in H_0^1(K, \Omega_1)$$

and the methods used in the previous sections and in [4] one can prove that for this range of p's, as $t\to\infty$, $t^{1/(p-1)}\|u(t)-w(t)\|_\infty\to 0$ for some self-similar solution $w(t, x)=t^{-[1/(p-1)]}f\left(\frac{x}{t}\right)$ $\left(f\right.$ satisfying $\left.Lf+|f|^{p-1}f=\lambda f,\ \lambda=\frac{1}{p-1}\right)$.

For the domain $\Omega_2:=\mathbb{R}^{N-1}\times\mathbb{R}_+$ we can check that

$$\lambda_1(\Omega_2):=\frac{N+1}{2}, \qquad \varphi_{1,\Omega_2}(y):=y_N\exp\left(\frac{-|y|^2}{4}\right)$$

$$L\varphi_{1,\Omega_2}=\frac{N+1}{2}\varphi_{1,\Omega_2}$$

$$\varphi_{1,\Omega_2}\in H_0^1(K, \Omega_2), \qquad \varphi_{1,\Omega_2}>0 \quad \text{on } \Omega_2.$$

and therefore, for instance if $1<p\leqq 1+\frac{2}{N+1}$ there is no global positive solution of $u_t-\Delta u=u^p\, u(t)_{|\,\partial\Omega_2}=0$.

Summing-up these observations and combining the proofs of the preceding sections and those of [3], [4], we are led to the following :

(4.5) THEOREM. – *Let* $\Omega\subset\mathbb{R}^N$ *be a convex cone with* $0\in\partial\Omega$ *and* $\Omega-0$ *convex.*

$$L^2(K,\Omega):=\left\{u;\ \int_\Omega |u|^2 K(y)\,dy<\infty\right\}$$

$$H^1(K,\Omega):=\{u\in L^2(K,\Omega);\ \nabla u\in L^2(K,\Omega)\}$$

and

$$H_0^1(K,\Omega):=\{u\in H^1(K,\Omega);\ u_{|\,\partial\Omega}=0\}$$

For $p>1$, $(N-2)p<N+2$ *consider a maximal solution of*

$$u_t-\Delta u=|u|^{p-1}u,\qquad u(0)\in H_0^1(\Omega),\qquad u(t)\in H_0^1(\Omega).$$

(Dirichlet boundary condition) $\left(\textit{resp. } u(0)\in H^1(K,\Omega),\ \frac{\partial u}{\partial n}(t)=\textit{on } \partial\Omega,\right.$ *Neumann boundary condition*$\Big)$. *Let* λ_1 *denote the least eigenvalue of* $L\left(Lf=-\Delta f-\frac{u.\nabla f}{2}\right)$ *on* $H_0^1(\Omega)$ *(resp. on* $H^1(K,\Omega)$*), then* $\lambda_1\geqq\frac{N}{2}$ *and the following holds:*

(i) *if* $p\leqq 1+\frac{1}{\lambda_1}$ and $u(0)\geqq 0$ *then* $u(t)$ *blows-up in finite time;*

(ii) *if*

$$p\geqq 1+\frac{1}{\lambda_1}$$

and

$$\frac{1}{2}\int_\Omega |\nabla u(0)|^2 K-\frac{1}{p+1}\int_\Omega |u(0)|^{p+1}K(y)\,dy$$
$$-\frac{1}{2(p-1)}\int_\Omega |u(0)|^2 K(y)\,dy=:E(u(0))\leqq 0,$$

then $u(t)$ blows-up in finite time;

(iii) *if* $p>1+\frac{1}{\lambda_1}$ *there exist* C_1, $C_2>0$ *such that if* $0<E(u(0))<C_1$ *and*

$$\int_\Omega |\nabla u(0)|^2 K\,dy-\frac{1}{p-1}\int_\Omega |u(0)|^2 K\,dy<C_2$$

then $u(t)$ is global in time;

(iv) *if* $p>1+\frac{1}{\lambda_1}$ *and $u(t)$ is a global solution, then*

$$\forall t_0>0,\quad \sup_{t\geqq t_0} t^{1/(p-1)}\|u(t)\|_{C^2(\bar{\Omega})}\leqq C(t_0)<\infty.$$

Using self-similar solutions to prove non-uniqueness of solutions to $u_t-\Delta u=|u|^{p-1}u$ one has

(4.6) PROPOSITION. – *For* $(N-2)p<N+2$ *and*

$$p>1+\frac{2}{N}\left[\geqq 1+\frac{1}{\lambda_1},\right.$$

$\left.\lambda_1 \text{ defined in theorem (4.5)}\right]$ *there are infinitely many solutions* $(w_k(t,x))_{k\geqq 1}$ *such that if* $1\leqq q<\frac{N}{2}\cdot\frac{1}{p-1}$ *then* $\lim_{t\downarrow 0}\|w_k(t)\|_{L^q(\Omega)}=0$, *and for* $t>0$,

$$\frac{\partial}{\partial t}w_k-\Delta w_k=|w_k|^{p-1}w_k,$$

$$\|w_k(t)\|_{L^q}>0,\qquad w_k(t)\in H_0^1(K,\Omega)$$

[*resp.* $w_k(t)\in H^1(K,\Omega)$].

Indeed consider

$$-\Delta f-\frac{y.\nabla f}{2}=|f|^{p-1}f+\frac{1}{p-1}f,\qquad f\in H_0^1(K,\Omega)$$

$\left[\text{resp. } H^1(K,\Omega) \text{ and } \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega}=0\right]$ as $\frac{1}{p-1}<\lambda_1$ then this equation has infinitely many solutions (and at least one of them is positive); now $w(t,x):=t^{-1/(p-1)}f\left(\frac{x}{\sqrt{t}}\right)$, satisfies the evolution equation

$$w_t-\Delta w=|w|^{p-1}w,\ w(t)\in H_0^1(K,\Omega)\left[\text{resp. } H^1(\Omega) \text{ and } \frac{dw}{\partial n}\bigg|_{\partial\Omega}=0\right]$$

and:

$$\|w(t)\|_{L^q(\mathbb{R}^N)}=t^{(N/2q)-[1/(p-1)]}\|f\|_{L^q(\mathbb{R}^N)}.$$

[one can prove that if $f\in H_0^1(K,\Omega)$ or $H^1(K,\Omega)$ satisfies $Lf=|f|^{p-1}f+\frac{1}{p-1}f$, then there is $C>0$, $a>0$ such that $|f(y)|\leqq C\exp(-a|y|^2)$].

(4.7) THEOREM. — *With the notations of the theorem* (4.5) *consider a solution of*

$$u_t-\Delta u+|u|^{p-1}u=0,\qquad u(0)\in H_0^1(K,\Omega),\qquad u(t)\in H_0^1(\Omega)$$

$\left(\textit{resp. } u(0)\in H^1(K,\Omega),\ \frac{\partial u}{\partial n}(t)=0 \textit{ on } \partial\Omega\right)$. *Then for any* $t_0>0$

$$\sup_{t\geqq t_0} t^{1/(p-1)}\|u(t)\|_{C^2(\bar{\Omega})}\leqq C(t_0)<\infty.$$

Moreover if $1<p<1+\frac{1}{\lambda_1}$ *and* $u(0)\geqq 0$, *then there is a unique* f *satisfying* $-\Delta f-\frac{y.\nabla f}{2}+f^p=\frac{1}{p-1}f$. *and* $f\in H_0^1(K,\Omega)$ $\left(\textit{resp. } f\in H^1(K,\Omega) \textit{ and } \frac{\partial f}{\partial n}=0 \textit{ on } \partial\Omega\right)$ $0<f(y)<C\exp(-a|y|^2)$ (*for some* $a>0$, $C>0$) *such that*

$$\lim_{t\to\infty}\left\|f^{1/(p-1)}u(t)-f\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_\infty=0.$$

5. APPENDIX

For convenience we present the proof of some of properties described Lemma (2.1) and corollary (2.2).

Proof of lemma (2.1). — In [3] one can find some general results about this kind of inequalities. But we present here an alternative proof based on an observation of P. L. Lions (personal communication). Indeed let $v=K^{1/2}u$; then we have: $\nabla v-\frac{v}{4}v=K^{1/2}\nabla u$, and therefore

$$\int K|\nabla u|^2=\int|\nabla v|^2+\frac{1}{16}\int|y|^2|v|^2-\frac{1}{2}\int yv.\nabla v$$
$$=\int|\nabla v|^2+\frac{1}{16}\int|y|^2|v|^2+\frac{N}{4}\int|v|^2$$

(integrating by parts the third term), and

$$\int K|\nabla u|^2\geqq\frac{1}{16}\int|y|^2K(y)|u|^2;$$

it is clear that: $v\in H^1(\mathbb{R}^N)\Leftrightarrow u\in H^1(K)$ hence (iv).

Then (i) and the classical Rellich's theorem imply (ii) (for more details see [3]). (iii) and (vi) follow by observing that (ii) implies the compactness of $D(L)\hookrightarrow L^2(K)$. The remainder of this lemma —unless (vii) and (v) is in [3].

We prove now (v) and (vii): for $f\in L^2(K)$ given, define [on $H^1(K)$]:

$$F(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2K\,dy-\int_{\mathbb{R}^N}fu.K\,dy$$

by (iii) we see that $\underset{v\in H^1(K)}{\mathrm{Min}}\ F(v)$ is achieved for a unique $u\in H^1(K)$ and that u satisfies:

$$-\Delta u-\frac{v.\nabla u}{2}=f\quad\text{in }\mathscr{D}'(\mathbb{R}^N);$$

as $f\in L^2(K)$ and $u\in H^1(K)\hookrightarrow H^1(\mathbb{R}^N)$, by classical regularity results we conclude that $u\in H^2_{\mathrm{loc}}(\mathbb{R}^N)$. Now define $v:=K^{1/2}u$; one checks easily that v satisfies:

$$-\Delta v+\left(\frac{N}{4}+\frac{|y|^2}{16}\right)v=K^{1/2}f \tag{5.1}$$

$$v\in H^1(\mathbb{R}^N)\quad[\text{by (iv)}],\qquad v\in H^2_{\mathrm{loc}}(\mathbb{R}^N)$$

Let $\varphi_0 \in C^\infty(\mathbb{R}_+, [0, 1])$ be a non increasing function such that

$$\varphi_0(s)=1 \quad \text{if} \quad 0 \leqq s \leqq 1$$
$$\varphi_0(s)=0 \quad \text{if} \quad s \geqq 2,$$

and denote $\varphi_n(y) := \varphi_0\left(\frac{|y|}{n}\right)$ for $n \geqq 1$, $y \in \mathbb{R}^N$. Multiplying (5.1) by $(-\Delta v)\varphi_n$ we get:

$$\int |\Delta v|^2 \varphi_n + \int |\nabla v|^2 \left(\frac{N}{4} + \frac{|y|^2}{16}\right)\varphi_n$$
$$= \int K^{1/2} f.(-\Delta v)\varphi_n - \frac{1}{8}\int y.\nabla v.v\varphi_n - \int v.\nabla v.\left(\frac{N}{4} + \frac{|y|^2}{16}\right)\nabla\varphi_n$$

Hence, using the fact that $K^{1/2} f \in L^2(\mathbb{R}^N)$, $|y|.|v| \in L^2$ we obtain:

$$\int \left(\frac{1}{2}|\Delta v|^2 + \left(\frac{N}{4} + \frac{|y|^2}{16}\right)|\nabla v|^2\right)\varphi_n$$
$$\leqq \frac{1}{2}\int K|f|^2 + C\left(\int |y|^2|v|^2 + \int |\nabla v|^2\right)$$
$$+ \frac{C}{n}\int |v|.|\nabla v| + C\int |yv|.|\nabla v|.|y\nabla\varphi_n|$$

But

$$|y\nabla\varphi_n| \leqq \frac{1}{n}|y|.\left|\varphi_0'\left(\frac{|y|}{n}\right)\right| \leqq \text{Cte}$$
$$|y\nabla\varphi_n| \to 0 \quad \text{a. e.}$$
$$|yv|.|\nabla v|.|y\nabla\varphi_n| \leqq \text{Cte}\,|yv|.|\nabla v| \in L^1(\mathbb{R}^N).$$

As $n \uparrow \infty$, $\varphi_n \uparrow 1$ and we obtain:

$$\int \left(\frac{1}{2}|\Delta v|^2 + \left(\frac{N}{4} + \frac{|y|^2}{16}\right)|\nabla v|^2\right) \leqq \frac{1}{2}\int K|f|^2 + C\int |y|^2|v|^2 + |\nabla v|^2$$

But

$$\nabla v = \frac{y}{4}K^{1/2}u + K^{1/2}\nabla u = \frac{y}{4}v + K^{1/2}\nabla u$$

$$\Delta v = \frac{N}{4} K^{1/2} u + \frac{|y|^2}{16} K^{1/2} u + \frac{v}{2} K^{1/2} \nabla u + K^{1/2} \Delta u$$

By the above inequality we know that $\Delta v \in L^2$ and by (5.1) $|y|^2 v \in L^2$. Therefore $|y| K^{1/2} |\nabla u| \in L^2$ and $K^{1/2} \Delta u \in L^2$ i.e. $u \in H^2(K)$. ■

Proof of corollary (2.2). – The first part is a classical result: $S_*(s) = e^{-sL}$ is well defined on $L^2(K)$ and as the least eigenvalue of L is $\frac{N}{2}$, we have (i).

(ii) follows from the fact that $\| G(t) * v_0 \|_\infty \leqq (4\pi t)^{-N/2} \| v_0 \|_{L^1}$.

(iii) Multiplying $v_s + L v = 0$ by $|v|^{q-2} v K$ we have

$$\frac{1}{q} \frac{d}{ds} \| v(s) \|_{L^q(K)}^q + (q-1) \int |\nabla v|^2 |v|^{q-2} K \, dy = 0$$

and (iii) follows. ■

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