ANNALES DE L'I. H. P., SECTION C

MARIANO GIAQUINTA
GIUSEPPE MODICA

Partial regularity of minimizers of quasiconvex integrals

Annales de l'I. H. P., section C, tome 3, nº 3 (1986), p. 185-208
<http://www.numdam.org/item?id=AIHPC_1986__3_3_185_0>

Ann. Inst. Henri Poincaré,
Vol. 3, n° 3, 1986, p. 185-208.

Echanges Annales
Analyse non linéaire

Partial regularity of minimizers of quasiconvex integrals

by

Mariano GIAQUINTA and Giuseppe MODICA
Istituto di Matematica Applicata « G. Sansone »
Università degli Studi di Firenze Via di S. Marta, 3,
I-50139, Firenze, Italia

ABSTRACT. — We consider variational integrals

$$\int_\Omega F(x, u, Du)dx$$

with integrands $F(x, u, p)$ growing polynomially and of class C^2 in p and Hölder-continuous in (x, u). Under the main assumption that $F(x, u, p)$ is strictly quasiconvex we prove that each minimizer is of Class $C^{1,\mu}$ in an open set $\Omega_0 \subset \Omega$ with meas $(\Omega - \Omega_0) = 0$.

RÉSUMÉ. — On considère des fonctionnelles du Calcul des Variations

$$\int_\Omega F(x, u, Du)dx$$

et on suppose que $F(x, u, p)$ ait une croissance polynomiale en p et soit de classe C^2 en p et Hölderienne en (x, u). Sous l'hypothèse que $F(x, u, p)$ soit strictement quasiconvexe nous démontrons que les minima ont les dérivées premières Hölderiennes dans un ouvert $\Omega_0 \subset \Omega$ de mesure totale égale à Ω.

Mots-clés : Calculus of variations, quasiconvex integrands, Caccioppoli inequality, Hölder regularity.

1. INTRODUCTION

In this paper we study the regularity of derivatives of the minimizers of variational integrals

$$(1.1) \qquad \mathcal{F}[u;\Omega] = \int_\Omega F(x, u, Du)dx$$

with integrands $F(x, u, p)$ *uniformily strictly quasiconvex.*

Here Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, $u:\Omega \to \mathbb{R}^N$, $N \geq 1$, $Du = \{D_\alpha u^i\}$, $1 \leq \alpha \leq n$, $1 \leq i \leq N$, stands for the gradient matrix of u and F: $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a function satisfying

$$(1.2) \qquad \lambda |p|^m - a \leq F(x, u, p) \leq \Lambda |p|^m + a \qquad \lambda > 0$$

where m is a real number larger than or equal to 2.

A *minimizer* of the functional $\mathcal{F}$ is a function $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ such that for every $\phi \in W^{1,m}(\Omega, \mathbb{R}^N)$ with supp $u \subset \Omega$

$$(1.3) \qquad \mathcal{F}[u;\text{supp}\,\phi] \leq \mathcal{F}[u+\phi, \text{supp}\,\phi]$$

The regularity of minimizers of differentiable functionals and of the weak solutions of related nonlinear elliptic systems has been intensively studied in the last twenty years. It would be very difficult to list the various contributions and we refer to M. Giaquinta [*8*] for that.

Except for the classical two dimensional result by C. B. Morrey, the regularity of minimizers of non-differentiable functionals has been studied only recently, see M. Giaquinta and E. Giusti [*9*] [*10*] [*11*], M. Giaquinta, P. A. Ivert [*13*], see also [*8*].

In both cases the main assumption was the *strong ellipticity* :

$$(1.4) \quad F_{p^i_\alpha p^j_\beta}(x, u, p)\xi^\alpha_i \xi^\beta_j \geq \nu(1 + |p|^2)^{\frac{m-2}{2}} |\xi|^2 \quad \forall \xi \in \mathbb{R}^{nN}; \qquad \nu > 0$$

This is a natural strengthening of the *convexity* condition of $F(x, u, p)$ with respect to p. A typical integrand F which satisfies (1.4) is

$$(1.5) \qquad F(p) := (1 + |p|^2)^{\frac{m}{2}} \qquad m \geq 2$$

As it is well known, the convexity of $F(x, u, p)$ with respect to p is a sufficient condition for the sequential weak lower semicontinuity of $\mathcal{F}$ in $W^{1,m}(\Omega, \mathbb{R}^N)$ and therefore, together with the coercivity condition (1.2), for the existence of a minimizer (subject to given boundary conditions) for $\mathcal{F}$. But in general it is a necessary condition only in the scalar case, $N = 1$.

In 1952 C. B. Morrey [*17*] showed that a necessary and sufficient condition for the weak sequential lower semicontinuity of $\mathcal{F}$ is that F be *quasi-*

convex. This means that *for almost every* $x_0 \in \Omega$, *for all* $u_0 \in \mathbb{R}^N$, $p_0 \in \mathbb{R}^{nN}$ *and for all* $\phi \in C_0^1(\Omega, \mathbb{R}^N)$ *we have*

$$(1.6) \qquad \frac{1}{|\Omega|}\int_\Omega F(x_0, u_0, p_0 + D\phi(x)dx \geq F(x_0, u_0, p_0)$$

i. e. *the frozen functional*

$$\mathscr{F}^0[u;\Omega] = \int_\Omega F(x_0, u_0, Du)dx$$

has the (affine) linear functions as minimizers (see also C. B. Morrey [*18*] (4.4), E. Acerbi, N. Fusco [*1*], J. Ball [*2*]).

Quasiconvexity is strictly weaker than convexity if $N > 1$ while it reduces to convexity if $N = 1$. Note that it is a global condition; but if F is of class C^2 in p, it implies the pointwise *Legendre-Hadamard condition:*

$$(1.7) \qquad F_{p_\alpha^i p_\beta^j}(x, u, p)\xi^\alpha\xi^\beta\pi_i\pi_j \geq 0 \qquad \forall\xi \in \mathbb{R}^n, \qquad \forall\pi \in \mathbb{R}^N$$

It is an open problem whether the converse is true in general.

As in the convex case, in order to study the regularity of minimizers, it is natural, and in a certain sense necessary, to strengthen condition (1.6).

DEFINITION 1.1. — *We say that* $F(x, u, p)$ *is uniformly strictly quasiconvex if for almost every* $x_0 \in \Omega$, *for all* $u_0 \in \mathbb{R}^N$. $p_0 \in \mathbb{R}^{nN}$ *and for all* $\phi \in C_0^1(\Omega, \mathbb{R}^N)$ *we have*

$$(1.8 \quad \int_\Omega [F(x_0, u_0, p_0 + D\phi) - F(x_0, u_0, p_0)]dx \geq \nu \int_\Omega (1 + |p_0|^2 + |D\phi|^2)^{\frac{m-2}{2}} |D\phi|^2 dx.$$

We moreover suppose $m \geqslant 2$.

Recently L. C. Evans [*6*], adapting the so-called indirect approach in [*8*], showed partial regularity of minimizers of the functional (1.1) in the case that the integrand F was uniformly strictly quasiconvex with $m \geqslant 2$ and moreover it did not depend on x and u.

In this paper we extend Evans' result proving the following theorem.

THEOREM 1.1. — *Suppose that* $F(x, u, p)$ *satisfies* (1.2). *Suppose moreover that*

i) $F(x, u, p)$ *is uniformly strictly quasiconvex*

ii) for every $(x, u) \in \Omega \times \mathbb{R}^N$, $F(x, u, p)$ *is twice continuously differentiable in p and we have*

$$|F_{pp}(x, u, p)| \leq L(1 + |p|^2)^{\frac{m-2}{2}}$$

iii) for every $p \in \mathbb{R}^{nN}$, *the function* $(1 + |p|^2)^{-\frac{m}{2}} F(x, u, p)$ *is Hölder continuous in* $\Omega \times \mathbb{R}^N$ *uniformly in p.*
Let $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *be a minimizer of the functional* (1.1). *Then there exists an open set* $\Omega_0 \subset \Omega$ *such that u has Hölder continuous first derivatives in* Ω_0. *Moreover we have* meas $(\Omega - \Omega_0) = 0$.

We conclude this introduction with a few comments on the method of proof, the so-called direct approach in [*8*], which strongly relies on Caccioppoli's type inequalities.

From the previous work, see [*8*], and especially from [*9*] [*10*] [*11*] the crucial role of the so-called Caccioppoli's inequality clearly appears, in dealing with the regularity of minimizers and solutions of nonlinear elliptic systems. This inequality, in the simplest case, amounts to the following: *Let u be a minimizer. Then for all balls* $B_R(x_0) \subset \Omega$ *we have*

$$(1.9) \qquad \int_{B_{R/2}(x_0)} |Du|^2 dx \leq \frac{c}{R^2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx$$

Here $u_{x_0,R}$ denotes the average of u on $B_R(x_0)$ i. e.

$$u_{x_0,R} = \fint_{B_R(x_0)} u dx = \frac{1}{B_R(x_0)} \int_{B_R(x_0)} u dx$$

On the basis of a result on reverse Hölder inequalities with increasing supports, see F. W. Gehring [*7*], M. Giaquinta, G. Modica [*14*] and [*8*], chap. V (1.9) implies that Du *lies in some* $L^p_{loc}(\Omega)$ *with* $p > 2$, *and moreover we have*

$$\left(\fint_{B_{R/2}} |Du|^p dx \right)^{1/p} \leq c \left(\fint_{B_R} |Du|^2 dx \right)^{1/2}.$$

This is actually the main point.

In the scalar case, a modified version of (1.9) implies even Hölder-continuity, see [*4*] [*8*] [*11*], and a Harnack inequality [*5*] for the minimizers.

The work of L. C. Evans [*6*], see also [*15*], and this paper shows that a second Caccioppoli's inequality is crucial for the regularity: *for any* $p_0 \in \mathbb{R}^{nN}$ *and any* $u_0 \in \mathbb{R}^N$ *we have*

$$(1.10) \quad \int_{B_{R/2}(x_0)} |Du - p_0|^2 dx \leq \frac{c}{R^2} \int_{B_R(x_0)} |u - u_0 - p_0(x - x_0)|^2 dx$$

We shall prove in section 4, that it implies the following reverse Hölder inequality: *for some* $p > 2$ *and for every* B_R

$$\left(\fint_{B_{R/2}(x_0)} |Du - (Du)_{x_0,R/2}|^p dx \right)^{1/p} \leq c \left(\fint_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 dx \right)^{1/2}$$

The fact that the uniform strict quasiconvexity permits a proof of inequality (1.10) was pointed out by L. C. Evans [*6*], see section 4 in our situation.

We would like to remark that inequality (1.10) relies heavily on the minimizing property of u and on the quasiconvexity assumption. As a matter of fact, even the first Caccioppoli's inequality (1.9) may not be true for solutions of quasilinear systems

$$D_\beta(A_{ij}^{\alpha\beta}(x, u)D_\alpha u^i) = 0 \qquad i = 1, \ldots, N$$

with coefficients satisfying the strengthened Legendre-Hadamard condition

$$A_{ij}^{\alpha\beta}(x, u)\xi^\alpha\xi^\beta\pi_i\pi_j \geq |\xi|^2 |\pi|^2 \qquad \forall\xi \in \mathbb{R}^n, \qquad \forall\pi \in \mathbb{R}^N$$

see M. Giaquinta, J. Souček [*16*].

2. TECHNICAL PRELIMINARIES

In this section we collect as lemmata a few simple remarks, mainly of algebraic nature, that we shall use in the sequel.

LEMMA 2.1. — *For $\delta \geq 0$, and for all $a, b \in \mathbb{R}^k$ we have*

$$(2.1) \qquad 2^{-2(1+\delta)} \leq \frac{\int_0^1 (1 + |ta + (1-t)b|^2)^{\delta/2} dt}{(1 + |a|^2 + |b|^2)^{\delta/2}} \leq 2^{\delta/2}$$

and

$$(2.2) \qquad 4^{-(1+2\delta)} \leq \frac{\int_0^1 (1 + |ta + (1-t)b|^2)^{\delta/2} dt}{(1 + |a|^2 + |b - a|^2)^{\delta/2}} \leq 4^{\delta}$$

Proof. — Let us prove inequality (2.1); (2.2) follows at once, as

$$\frac{1}{2}(1 + |a|^2 + |b|^2) \leqslant 1 + |a|^2 + |b - a|^2 \leq 2(1 + |a|^2 + |b|^2)$$

The inequality on the right follows immediately since for any $t \in [0, 1]$

$$1 + |ta + (1-t)b|^2 \leq 2(1 + |a|^2 + |b|^2)$$

In order to show the inequality on the left it suffices to notice that we may assume that $|a| \geq |b|$, so for $t \in \left(\frac{3}{4}, 1\right)$ we have

$$|ta + (1-t)b| \geq t|a| - (1-t)|b| \geq \frac{1}{4}(|a| + |b|) \quad \text{Q.E.D.}$$

For $\delta > 0$, and any $p \in \mathbb{R}^k$ define the vector valued function

$$V_\delta(p) = (1 + |p|^2)^{\delta/2} p \tag{2.3}$$

LEMMA 2.2. — *We have*

$$5^{-\left(1+\frac{\delta}{2}\right)} \leqslant \frac{|V_\delta(p) - V_\delta(q)|}{(1 + |p|^2 + |q|^2)^{\delta/2} |p - q|} \leq (1 + \delta) 2^{\delta/2} \tag{2.4}$$

Proof. — The inequality on the right follows at once, using lemma 2.1, since

$$\begin{aligned} |V_\delta(p) - V_\delta(q)| &\leq \int_0^1 \left| \frac{d}{dt} V_\delta(tp + (1 - t)q) \right| dt \\ &\leq (1 + \delta) \int_0^1 (1 + |tp + (1 - t)b|^2)^{\delta/2} dt \, |p - q| \end{aligned}$$

In order to prove the inequality on the left we suppose without loss in generality that $|p| \geq |q|$, and we distinguish the case $|p| \geq 2|q|$ in which we have

$$|p - q| \leq \frac{3}{2} |p|$$

$$(1 + |p|^2 + |q|^2)^{\delta/2} \leq 2^{\delta/2}(1 + |p|^2)^{\delta/2}$$

and the case $|q| \leq |p| \leq 2|q|$.

In the first case, since $|V(p)|$ is an increasing function of $|p|$, we have

$$|V_\delta(p) - V_\delta(q)| \geq |V(p)| - |V(q)| \geq |V(p)| - \left| V\left(\frac{1}{2} p\right) \right| \geq \frac{1}{2} |V(p)|$$

hence the inequality follows immediately using (2.5). In the second case we note that for every $\tau \geq 1$ we have $|\tau p - q| \geq |p - q|$ hence, setting

$$W(q) = (1 + |q|^2)^{\delta/2}$$

we get

$$|V_\delta(p) - V_\delta(q)| = W(q) \left| \frac{W(p)}{W(q)} p - q \right| \geq W(q) |p - q|$$

The result then follows, because

$$(1 + |p|^2 + |q|^2)^{\delta/2} \leq 5^{\delta/2} W(q) \qquad \text{Q.E.D.}$$

The next lemma is an easy consequence of lemma 1.1 of [*9*].

LEMMA 2.3. — *Let $f(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t \leq s \leq T_1$ we have*

$$f(t) \leq A(s - t)^{-\alpha} + B(s - t)^{-\beta} + C + \theta f(s)$$

where A, B, α, β, θ are nonnegative constants, and $\theta < 1$. Then there exists *a constant* $c_0 = c_0(\theta, \alpha, \beta)$ *such that for every* $\rho, R, T_0 \leq \rho < R \leq T_1$, *we have*

$$f(\rho) \leq c_0 [A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C].$$

The next lemma will be used quite often in the sequel. Consider the vector valued function

$$V(p) = (1 + |p|^2)^{\frac{m-2}{4}} p$$

defined for $p \in \mathbb{R}^{nN}$, and $m \geq 2$; and denote by

$$\phi_{x_0,R} = \fint_{B_R(x_0)} \phi dx = \frac{1}{|B_R|} \int_{B_R(x_0)} \phi dx$$

the mean value over the ball $B_R(x_0)$ in $\mathbb{R}^n$ of the vector valued function $\phi : B_R(x_0) \to \mathbb{R}^{nN}$.

We have

LEMMA 2.4. — *For any* $p \geq 1$ *there exists a constant c such that for any* $\lambda \in \mathbb{R}^{nN}$

$$\fint_{B_R(x_0)} |V(\phi) - V(\phi_{x_0,R})|^p dx \leq c(p) \fint_{B_R(x_0)} |V(\phi) - V(\lambda)|^p dx$$

In particular

$$\fint_{B_R(x_0)} |V(\phi) - V(\phi_{x_0,R})|^p dx \leq c(p) \fint_{B_R(x_0)} |V(\phi) - V(\phi)_{x_0,R}|^p dx$$

Proof. — From lemma 2.1 we have

$$\fint_{B_r(x_0)} |V(\phi) - V(\phi_{x_0,R})|^p dx$$

$$\leq c_1 (1 + |\phi_{x_0,R}|^2)^{\frac{m-2}{4} p} \fint_{B_R(x_0)} |\phi - \phi_{x_0,R}|^p dx + c_1 \fint_{B_R(x_0)} |\phi - \phi_{x_0,R}|^{\frac{m}{2} p} dx$$

$$\leq c_2 (1 + |\lambda|^2 + |\phi_{x_0,R} - \lambda|^2)^{\frac{m-2}{4} p} \fint_{B_R(x_0)} |\phi - \phi_{x_0 R}|^p dx + c_1 \fint_{B_R(x_0)} |\phi - \phi_{x_0 R}|^{\frac{m}{2} p} dx$$

$$\leq c_3 (1 + |\lambda|^2)^{\frac{m-2}{4} p} \fint_{B_R(x_0)} |\phi - \phi_{x_0,R}|^p dx + c_1 \fint_{B_R(x_0)} |\phi - \phi_{x_0,R}|^{\frac{m}{2} p} dx$$

$$+ c_3 \left(\fint_{B_R(x_0)} |\phi - \lambda| dx \right)^{\frac{m-2}{4} p} \fint_{B_R(x_0)} |\phi - \lambda|^p dx$$

$$\leq c_4 (1 + |\lambda|^2)^{\frac{m-2}{5} p} \fint_{B_R(x_0)} |\phi - \lambda|^p dx + c_4 \fint_{B_R(x_0)} |\phi - \lambda|^{\frac{m}{2} p} dx$$

$$\leq c_5 \fint_{B_R(x_0)} |V(\phi) - V(\lambda)|^p dx$$

the second statement follows at once choosing λ in such a way that

$$V(\lambda) = V(\phi)_{x_0,R}. \qquad \text{Q.E.D.}$$

3. REVERSE HÖLDER INEQUALITIES

We shall denote by $Q_R(x_0)$ the cube in $\mathbb{R}^n$ centered at x_0 with sides of length 2R parallel to the axes, i. e.

$$Q_R(x_0) = \{ x \in \mathbb{R}^n : | x_i - x_{0i} | < R \qquad i = 1, 2, \ldots, n \}$$

while by $B_R(x_0)$ we shall denote, as usual, the ball of radius R centered at x_0, i. e.

$$B_R(x_0) = \{ x\mathbb{R}^n : | x - x_0 | < R \}$$

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $g \in L^q(\Omega)$. We say that g satisfies a *reverse Hölder inequality with increasing supports in* Ω if for some $r < q$ we have:

$$\left(\fint_{Q_{R/2}(x_0)} | g |^q dx \right)^{1/q} \leq b \left(\fint_{Q_R(x_0)} | g |^r dx \right)^{1/r} \tag{3.1}$$

for all $Q_R(x_0) \subset \Omega$.

We recall the usual notations

$$g_E = \fint_E g dx = \frac{1}{|E|} \int_E g dx$$

Reverse Hölder inequalities with increasing supports play an important role in the theory of the regularity of solutions to nonlinear elliptic differential equations, see [*8*], chapters V, VI. In [*14*] we proved (see also [*8*], chap. V, [*7*] [*8*]) that whenever (3.1) is satisfied for all $Q_R(x_0) \subset \Omega$, then g has higher integrability and satisfies a reverse Hölder inequality with increasing supports and exponents $q + \varepsilon$, q. More precisely we proved the following theorem which we now state in a slightly more general form, and which will be used in the sequel.

THEOREM 3.1. — *Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose we have*

$$\left(\fint_{Q_{R/2}} | g |^q dx \right)^{1/q} \leq b \left(\fint_{Q_R} | g |^r dx \right)^{1/r} + \left(\fint_{Q_R} | f |^q dx \right)^{1/q} \tag{3.2}$$

for all $Q_R \subset \Omega$, where $g \in L^q(\Omega)$, $f \in L^s(\Omega)$ and $0 < r < q < s < +\infty$. Then *there exists a positive* $\varepsilon = \varepsilon(n, q, r, s, b)$ *such that* $g \in L^{q+\varepsilon}_{loc}(\Omega)$. *Moreover for any* $\Omega' \subset\subset \Omega$ *we have*

$$(3.3) \quad \left(\fint_{\Omega'} |g|^{q+\varepsilon} dx \right)^{\frac{1}{q+\varepsilon}} \leq c \left[\left(\fint_{\Omega} |g|^q dx \right)^{\frac{1}{q}} + \left(\fint_{\Omega} |f|^{q+\varepsilon} dx \right)^{\frac{1}{q+\varepsilon}} \right]$$

where c is a constant depending on n, q, r, s, b and on $|\Omega|/\text{dist}(\Omega', \partial\Omega)^n$ and $|\Omega|/|\Omega'|$.

REMARK 3.1. — On the left-hand side of the inequality (3.2) we may have any cube $Q_{\tau R}$, $0 < \tau < 1$, instead of $Q_{R/2}$. Then the conclusions of theorem 3.1 remain true of course with the constants c and ε in (3.3) depending on τ.

REMARK 3.2. — Obviously in (3.2) we can have balls B_R instead of cubes Q_R, and the same conclusions holds.

4. CACCIOPPOLI'S INEQUALITIES AND HIGHER INTEGRABILITY

Let $F(x, u, p) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ be a Carathéodory function (i. e. measurable in x and continuous in u, p) satisfying condition (1.2), which we rewrite, for simplicity, as

(H.1) $F(x, u, p)$ *satisfies the inequalities*

$$|p|^m \leq F(x, u, p) \leq \Lambda(1 + |p|^2)^{\frac{m}{2}}$$

where $\lambda > 0$, *and* $m \geq 2$.

In M. Giaquinta and E. Giusti [*10*], see also [*11*] for a more general statement, the following theorem was proved

THEOREM 4.1. — *Let* $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *be a minimizer of the functional*

$$(4.1) \qquad \mathcal{F}[u; \Omega] = \int_{\Omega} F(x, u, Du) dx$$

where the integrand F *satisfies* (H.1).
Then there exists an $\varepsilon > 0$ *such that* $Du \in L^{m+\varepsilon}_{loc}(\Omega, \mathbb{R}^{nN})$.
Moreover, for every $x_0 \in \Omega$ *and* R, *with* $0 < R < \text{dist}(x_0, \partial\Omega)$ *we have*

$$(4.2) \quad \left(\fint_{B_{R/2}(x_0)} (1 + |Du|^2)^{\frac{m+\varepsilon}{2}} dx \right)^{\frac{1}{m+\varepsilon}} \leq c \left(\fint_{B_R(x_0)} (1 + |Du|^2)^{\frac{m}{2}} dx \right)^{\frac{1}{m}}$$

with c independent of R *and* u.

Theorem 4.1 is a consequence, see [9], of theorem 3.1 and of the following inequality proved in [9].

Caccioppoli's first inequality: Under the assumption of theorem 4.1, for every $x_0 \in \Omega$, ρ, R, *with* $0 < \rho < R <$ dist $(x_0, \partial\Omega)$, *we have*

$$\int_{B_\rho(x_0)} |Du|^m dx \leq \frac{c}{(R-\rho)^m} \int_{B_R(x_0)} |u - u_{x_0,R}|^m dx + c\,|(B_R(x_0)| \tag{4.3}$$

In the next proposition we shall prove, using an idea of L. C. Evans [6], an inequality for the mean oscillation of Du of the same nature of (4.3). In order to do that, we need some additional hypotheses on F. We collect here these hypotheses and some simple consequences as (H.2) ... (H.6); as in (H.1) m is larger than or equal to 2.

(H.2) F(x, u, p) *is of class* C^2 *with respect to* p *and*

$$|F_{pp}(x, u, p)| \leq c_1(1 + |p|^2)^{\frac{m-2}{2}}$$

Taking into account lemma 2.1, (H.2) implies immediately the following statement that we number as (H.3)

(H.3) *The derivatives of* F(x, u, p) *with respect to* p *satisfy*

$$|F_p(x, u, p) - F_p(x, u, q)| \leq c_2(1 + |p|^2 + |q|^2)^{\frac{m-2}{2}} |p - q|$$

In the next section we need a stronger version of (H.2).

(H.4) F(x, u, p) *is twice continuously differentiable in* p, *uniformly with respect to* x, u*; more precisely there exists a continuous, non negative, bounded function* $\omega(t, s)$ *increasing in* t *for fixed* s *and in* s *for fixed* t, *concave in* s, *with* $\omega(t, 0) = 0$, *and such that for every* $(x, u) \in \Omega \times \mathbb{R}^N$ *and* $p, q \in \mathbb{R}^{nN}$ *we have*

$$|F_{pp}(x, u, p) - F_{pp}(x, u, q)| \leq c_3(1 + |p|^2 + |q|^2)^{\frac{m-2}{2}} \omega(|p|, |p - q|)$$

(H.5) $(1 + |p|^2)^{-\frac{m}{2}}F(x, u, p)$ *is Hölder-continuous in* (x, u) *uniformly with respect to* p, i. e.

$$|F(x, u, p) - F(y, v, p)| \leq c_4(1 + |p|^2)^{\frac{m}{2}} \eta(|u|, |x - y| + |u - v|)$$

where

$$\eta(t, s) = \mathscr{K}(t) \min(s^\delta, L)$$

for some δ, $0 < \delta < 1$, *and* $L > 0$, *and* $\mathscr{K}(t)$ *is an increasing function.* Notice that $\eta(t, s)$ is concave in s for fixed t. Finally

(H.6) F(x, u, p) *is uniformly strictly quasiconvex in the sense of definition 1.1, or equivalently (compare lemma 2.2) there exist a positive* *constant* ν *such that for almost every* $x_0 \in \Omega$, *for all* $u_0 \in \mathbb{R}^N$, $p_0 \in \mathbb{R}^{nN}$ *and for all* $\Phi \in C_0^1(\Omega, \mathbb{R}^N)$ *we have*

$$\nu \int_\Omega [V(p_0 + D\Phi) - V(p_0)]^2 dx \leq \int_\Omega [F(x_0, u_0, p_0 + D\Phi) - F x_0, u_0, p_0)] dx$$

where, for all $p \in \mathbb{R}^{nN}$, $V(p)$ *is the vector valued function defined by*

$$V(p) = (1 + |p|^2)^{\frac{m-2}{4}} p$$

Let $u \in {}^{1,m}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a minimizer for the functional $\mathscr{F}[u;\Omega]$ in (4.1). For x_0, $y \in \Omega$, u_0, $v_0 \in \mathbb{R}^N$, $p_0 \in \mathbb{R}^{nN}$, we define

$$P(x) = \{ P_i(x) \} \; i = 1, \ldots, N; \qquad P_i(x) = u_0^i + p_{0\alpha}^i (x_\alpha - x_{0\alpha})$$

and we simply write

$$P(x) = u_0 + p_0(x - x_0) \tag{4.4}$$

Moreover we set

$$\alpha(x) = \mathscr{K}(|v_0|) \min \{ [|x - y| + |u(x) - v_0| + |u(x) - P(x)|]^\delta, L \} \tag{4.5}$$

$$G^2(x) = (1 + |p_0|^2 + |Du|^2)^{m/2} \min(1, \alpha(x)) \tag{4.6}$$

We have

PROPOSITION 4.1. — *(Caccioppoli's second inequality). Let* $u \in W^{1,m}_{\text{loc}}(\Omega, \mathbb{R}^n)$ *be a minimizer of the functional* $\mathscr{F}[u;\Omega]$ *in* (4.1). *Suppose that the integrand* F *satisfies* (H.1) (H.2) (H.3) (H.5) *and* (H.6). *Then for every* x_0, $y \in \Omega$, u_0, $v_0 \in \mathbb{R}^N$, $p_0 \in \mathbb{R}^{nN}$ *and every* ρ, R *with* $0 < \rho < R < \text{dist}(x_0, \partial\Omega)$ *we have*

$$\int_{B_\rho(x_0)} [(1 + |p_0|^2)^{\frac{m-2}{2}} |Du - p_0|^2 + |Du - p_0|^m] dx \tag{4.7}$$

$$\leq c \left\{ \frac{1}{(R-\rho)^2} \int_{B_R(x_0)} (1 + |p_0|^2)^{\frac{m-2}{2}} |u - P|^2 dx + \frac{1}{(R-\rho)^m} \int_{B_R(x_0)} |u - P|^m dx + \int_{B_R(x_0)} G^2 dx \right\}.$$

If moreover the integrand F *does not depend explicitly on* x *and* u, *then we may take* $G = 0$ *in* (4.7).

Proof. — Let $0 \leq \rho \leq s < t \leq R$ and choose $\xi \in C_0^\infty(B_t(x_0))$ satisfying $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $B_s(x_0)$, $|D\xi| \leq c/(t - s)$. Define

$$\Phi = \xi(u - P) \qquad \Psi = (1 - \xi) \qquad (u - P)$$

so that

$$\Phi + \Psi = u - P \qquad D\Phi + D\Psi = Du - p_0$$

As $\xi = 0$ on ∂B_t, the quasiconvexity assumption (H.6) implies for every $y \in \Omega$ and $v_0 \in \mathbb{R}^N$

$$(4.8) \quad \nu \int_{B_t} [(1 + |p_0|^2)^{\frac{m-2}{2}} |D\Phi|^2 + |D\Phi|^m] dx \leq \int_{B_t} [F(y, v_0, p_0 + D\Phi) - F(y, v_0, p_0)] dx$$

We now rewrite the right-hand side of (4.8) as

$$(4.9) \quad \int_{B_t} [F(y, v_0, p_0 + D\Phi) - F(y, v_0, p_0)] dx$$
$$= \int_{B_t} [F(x_0, u_0, Du - D\Psi) - F(x_0, u_0, Du)] dx$$
$$+ \int_{B_t} [F(y, v_0, Du) - F(x, u, Du)] dx$$
$$+ \int_{B_t} [F(x, u, Du) - F(x, u - \Phi, Du - D\Phi)] dx$$
$$+ \int_{B_t} [F(x, u - \Phi, p_0 + D\Psi) - F(y, v_0, p_0 + D\Psi)] dx$$
$$+ \int_{B_t} [F(y, v_0, p_0 + D\Psi) - F(y, v_0, p_0)] dx = (I) + (II) + (III) + (IV) + (V)$$

and we notice that (III) ≤ 0 because u is a minimizer, and that (II) and (IV) are zero if the integrand F does not depend explicitly on x and u.
We have, using (H.3) and lemma 2.1,

$$(I) + (V) = \int_{B_t} dx \int_0^1 [F_{p^i_\alpha}(y, v_0, p_0 + \tau D\Psi) - F_{p^i_\alpha}(y, v_0, Du - \tau D\Psi)] D_\alpha \Psi^i d\tau$$
$$\leq c_5 \int_{B_t} (1 + |p_0|^2 + |Du|^2 + |D\Psi|^2)^{\frac{m-2}{2}} (|Du - p_0| + |D\Psi|) |D\Psi| dx$$

Then we observe that $|D\Psi|$ has support in $B_t - B_s$ and that

$$|D\Psi| \leq c_6 |Du - p_0| + \frac{c_6}{t - s} |u - P|$$

so using the elementary inequality $a^{m-2}b^2 \leq a^m + b^m$, we conclude

$$(4.10) \quad (I) + (V) \leq c_7 \left\{ \frac{1}{(t-s)^2} \int_{B_t} (1 + |p_0|^2)^{\frac{m-2}{2}} |u - P|^2 dx + \frac{1}{(t-s)^m} \int_{B_t} |u - P|^m dx \right\} + c_8 \int_{B_t - B_s} [(1 + |p_0|^2)^{\frac{m-2}{2}} |Du - p_0|^2 + |Du - p_0|^m] dx$$

On the other hand, (H.5) and (H.1) imply

$$(4.11)\quad ((\mathrm{II})+(\mathrm{IV})\leq c_9\left\{\int_{\mathrm{B}_t}\min(\alpha(x),1)(1+|\mathrm{D}u|^2)^{\frac{m}{2}}dx\right.$$

$$\left.+\int_{\mathrm{B}_t}\min(\alpha(x),1)(1+|p_0+\mathrm{D}\Psi|^2)^{\frac{m}{2}}dx\right\}\leq c_{10}\left\{\int_{\mathrm{B}_t}\mathrm{G}^2dx+\int_{\mathrm{B}_t}|\mathrm{D}\Psi|^m dx\right\}$$

$$\leq c_{11}\left\{\int_{\mathrm{B}_t}\mathrm{G}^2dx+\frac{1}{(t-s)^m}\int_{\mathrm{B}_t}|u-\mathrm{P}|^m dx+\int_{\mathrm{B}_t-\mathrm{B}_s}|\mathrm{D}u-p_0|^m dx\right\}$$

Therefore, from (4.8) ... (4.11) we conclude

$$\int_{\mathrm{B}_s}[(1+|p_0|^2)^{\frac{m-2}{2}}|\mathrm{D}u-p_0|^2+|\mathrm{D}u-p_0|^m]dx$$

$$\leq c_{12}\left\{\frac{1}{(t-s)^2}\int_{\mathrm{B}_t}(1+|p_0|^2)^{\frac{m-2}{2}}|u-\mathrm{P}|^2dx+\frac{1}{(t-s)^m}\int_{\mathrm{B}_t}|u-\mathrm{P}|^m dx\right.$$

$$\left.+\int_{\mathrm{B}_t}\mathrm{G}^2dx\right\}+c_{12}\int_{\mathrm{B}_t-\mathrm{B}_2}[(1+|p_0|^2)^{\frac{m-2}{2}}|\mathrm{D}u-p_0|^2+|\mathrm{D}u-p_0|^m]dx\,.$$

Now we fill the hole, i. e. we add c_{12} times the left-hand side of (4.12) to both sides of (4.12) and we get

$$f(s)\leq\frac{\mathrm{A}}{(t-s)^2}+\frac{\mathrm{B}}{(t-s)^m}+\mathrm{C}+\theta f(t)$$

with

$$f(t)=\int_{\mathrm{B}_t}[(1+|p_0|^2)^{\frac{m-2}{2}}|\mathrm{D}u-p_0|^2+|\mathrm{D}u-p_0|^m]dx$$

$$\mathrm{A}=c_{12}\int_{\mathrm{B_R}}(1+|p_0|^2)^{\frac{m-2}{2}}|u-\mathrm{P}|^2dx$$

$$\mathrm{B}=c_{12}\int_{\mathrm{B_R}}|u-\mathrm{P}|^m dx$$

$$\mathrm{C}=c_{12}\int_{\mathrm{B_R}}\mathrm{G}^2$$

$$\theta=\frac{c_{12}}{1+c_{12}}<1$$

Hence the result follows at once from lemma 2.3. Q. E. D.

Now we show how Caccioppoli's second inequality permits us to prove a useful reverse Hölder inequality for $\mathrm{V}(\mathrm{D}u)-\mathrm{V}(\mathrm{D}u)_{x_0,\mathrm{R}}$ V being the function defined in (H.6).

We choose in (4.7), $\rho = R/2$, $u_0 = u_{x_0,R}$, and we recall the well-known Sobolev-Poincaré inequality; *for any* $p \geq \frac{n}{n-1}$ *define*

$$p_* = \frac{np}{n+p}$$

then there is a constant $c = c(n, p)$ *such that*

$$\int_{B_R(x_0)} |u - u_{x_0,R} - p_0 .(x - x_0)|^p dx \leq c\left(\int_{B_R(x_0)} |Du - p_0|^{p_*} dx\right)^{p/p_*}.$$

We estimate the first two terms on the right-hand side of (4.7) first by

$$\frac{c_{13}}{R^2}(1 + |p_0|^2)^{\frac{m-2}{2}}\left(\int_{B_R(x_0)} |Du - p_0|^{2_*} dx\right)^{2/2_*} + \frac{c_{13}}{R^m}\left(\int_{B_R} |Du - p_0|^{m_*} dx\right)^{\frac{m}{m_*}}$$

and then, noticing that $m_* \leq 2_* m$, with a simple use of Hölder inequality by

$$\frac{c_{14}}{R^2}\left\{\int_{B_R} [(1 + |p_0|^2)^{\frac{m-2}{2}} |Du - p_0|^2 + |Du - p_0|^m]^{2_*} dx\right\}^{2/2_*}.$$

Taking into account lemma 2.2 and 2.4, Caccioppoli's second inequality implies

$$(4.13) \quad \left(\fint_{B_{R/2}(x_0)} |V(Du) - V(p_0)|^2 dx\right)^{1/2}$$
$$\leq c\left\{\left(\fint_{B_R(x_0)} |V(Du) - V(p_0)|^{2_*} dx\right)^{1/2_*} + \left(\fint_{B_R(x_0)} G^2 dx\right)^{1/2}\right\}.$$

In case F does not depend explicitly on x and u, and therefore $G = 0$, inequality (4.13) together with theorem 3.2 (applied to $|V(Du) - V(p_0)|$) implies that

$$\left(\fint_{B_{R/2}} |V(Du) - V(p_0)|^p dx\right)^{1/p} \leqslant c\left(\fint_{B_R} |V(Du) - V(p_0)|^2 dx\right)^{1/2}$$

for some $p > 2$. Therefore, choosing for each $B_R(x_0)$ p_0 in such a way that $V(p_0) = V(Du)_{x_0,R}$ we get

THEOREM 4.2. — *Let* $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *be a minimizer of* $\mathcal{F}[u : \Omega]$ *in* (4.1) *suppose that the integrand* F *does not depend explicitly on x and u and satisfies* (H.2) (H.3) (H.6). *Then there exists a* $p > 2$ *such that for all* $B_R(x_0) \subset \Omega$ *we have*

$$(4.14) \quad \left(\fint_{B_{R/2}(x_0)} |V(Du) - V(Du)_{x_0,R/2}|^p dx\right)^{1/p}$$
$$\leq c\left(\fint_{B_R(x_0)} |V(Du) - V(Du)_{x_0,R}|^2 dx\right)^{1/2}.$$

In general we have

THEOREM 4.3. — *Let $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ be a minimizer of $\mathcal{F}[u;\Omega]$ in* (4.1), *suppose that the integrand* F *satisfies* (H.1) (H.2) (H.5) *and* (H.6). *Then there exists a $p > 2$, a $\gamma > 0$ and a constant c such that for all* $B_R(x_0) \subset \Omega$ *we have*

$$(4.15)\quad \left(\fint_{B_{R/2}(x_0)} |V(Du) - V(Du)_{x_0,R/2}|^p dx\right)^{2/p}$$

$$\leq \fint_{B_R(x_0)} |V(Du) - V(Du)_{x_0,R}|^2 dx + R^{2\gamma} h(|u_{x_0,R}| + |(Du)_{x_0,R}| + \psi(x_0,R)^{1/m})$$

where $h(t)$ is an increasing function and $\psi(x_0, R)$ is defined by

$$\psi(x_0, R) = \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^m dx.$$

Proof. — Choosing $u_0 = u_{x_0,\frac{3}{2}R}$, from Caccioppoli's inequality (4.7); using as before Sobolev-Poincaré inequality, we get

$$(4.16)\quad \left(\fint_{B_{R/2}} |V(Du) - V(p_0)|^2 dx\right)^{1/2} \leq c\left(\fint_{B_{\frac{2}{3}R}} |V(Du) - V(p_0)|^{2_*} dx\right)^{1/2_*} + c\left(\fint_{B_{\frac{2}{3}R}} G^2 dx\right)^{1/2}.$$

We note, theorem 4.1, that $(1 + |Du|^2)^{m/2} \in L^{1+\varepsilon}_{loc}(\Omega)$, so that $G \in L^s_{loc}(\Omega)$ $2 < s \leq 2\dfrac{m+\varepsilon}{m}$. Set

$$\bar{\alpha}(x) = K(|v_0|) \min\{[|x - y| + |u(x) - v_0|]^\delta, L\}$$

We have

$$(4.17)\quad \fint_{B_{\frac{2}{3}R}} G^2 dx \leq \fint_{B_{\frac{2}{3}R}} \bar{\alpha}(x)(1 + |p_0|^2 + |Du|^2)^{\frac{m}{2}} dx$$

$$+ \fint_{B_{\frac{2}{3}R}} K(|v_0|) \min\{L, (|u - u_0| + |p_0||x - x_0|)^\delta\}(1 + |p_0|^2 + |Du|^2)^{\frac{m}{2}} dx.$$

We estimate the second integral on the right-hand side using theorem 4.1 and Poincaré inequality as follows:

$$(4.18)\quad \fint_{B_{\frac{2}{3}R}} K(|v_0|)\min\{L,(|u-u_0|+|p_0||x-x_0|)^\delta\}(1+|p_0|^2+|Du|^2)^{\frac{m}{2}}dx$$

$$\leq k(|v_0|)\left(\fint_{B_{\frac{2}{3}R}}(1+|p_0|^2+|Du|^2)^{\frac{m}{2}\frac{s}{2}}dx\right)^{2/s}$$

$$\left[\fint_{B_{\frac{2}{3}R}}\min\{L,(|u-u_0|+|p_0||x-x_0|)^\delta\}^{\frac{1}{1-\frac{2}{s}}}dx\right]^{1-\frac{2}{s}}$$

$$\leq ck(|v_0|)\left(\fint_{B_R}(1+|p_0|^2+|Du|^2)^{\frac{m}{2}}dx\left(R^m\fint_{B_R}|Du|^m+|p_0|^m\right)dx\right)^{1-\frac{2}{s}}$$

$$\leq ck(|v_0|)R^{\frac{m}{2}\left(1-\frac{2}{s}\right)}\left(\fint_{B_R}(1+|p_0|^2+|Du|^2)^{\frac{m}{2}}dx\right)^{2-\frac{2}{s}}$$

$$\leq ck(|v_0|)R^{2\gamma}\fint_{B_R}(1+|p_0|^2+|Du|^2)^{m\left(1-\frac{1}{s}\right)}dx \quad \gamma=\frac{m}{2}\left(1-\frac{1}{s}\right).$$

Note that for s close to 2, $m\left(1-\frac{1}{s}\right)$ is close to $\frac{m}{2}$.
Set now for R_0 fixed

$g^2(x)=|V(Du)-V(p_0)|^2$

$f^2(x)=\bar{\alpha}(x)(1+|p_0|^2+|Du(x)|^2)^{\frac{m}{2}}+k(|v_0|)R_0^{2\gamma}(1+|p_0|^2+|Du|^2)^{m\left(1-\frac{1}{s}\right)}$

From (4.16) (4.17) (4.18) we deduce

$$f^2\in L^t_{loc}(\Omega)\quad \text{for some}\quad t>1$$

and for all $R\leq R_0$, $v_0\in\mathbb{R}^N$, $y\in\Omega$, $p_0\in\mathbb{R}^{nN}$ we have

$$\left(\fint_{B_{R/2}}g^2dx\right)^{1/2}\leq c\left(\fint_{B_R}g^{2_*}dx\right)^{1/2_*}+c\left(\fint_{B_R}f^2dx\right)^{1/2}.$$

Applying theorem 4.1 we conclude that there is an exponent $p>2$ such that

$$(4.19)\qquad \left(\fint_{B_{R_0/2}}g^pdx\right)^{1/p}\leq c\left(\fint_{B_{R_0}}g^2dx\right)^{1/2}+\left(\fint_{B_{R_0}}f^pdx\right)^{1/p}$$

Now for each $B_{R_0}(x_0)$ we choose $y=x_0$, $v_0=u_{x_0,R_0}$, p_0 in such a way that $V(p_0)=V(Du)_{x_0,R_0}$ and we estimate the last integral in (4.19) as in (4.18). Then the result follows at once. Q. E. D.

5. PARTIAL REGULARITY

In this section we shall prove theorem 1.1 which was stated in the introduction. Through the whole section $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ will denote a minimizer of the functional

$$\mathscr{F}[u;\Omega] = \int_\Omega F(x, u, Du)dx \tag{5.1}$$

with integrand F satisfying the main assumptions (H.1) ... (H.6) of section 4. V(Du), simply written V, when no confusion may arise, will denote the vector valued function in (H.6) defined as

$$V(Du) = (1 + |Du|^2)^{\frac{m-2}{2}} Du$$

Let us first consider the case that the integrand F does not depend explicitly on x and u. So let $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ be a minimizer of

$$\int_\Omega F(Du)dx \tag{5.2}$$

with F satisfying (H.2) (H.4) (H.6). We have, see also [*6*].

THEOREM 5.1. — *There exists an open set* $\Omega_0 \subset \Omega$ *such that*

$$V(Du) \in C^{0,\sigma}(\Omega_0, \mathbb{R}^{nN}) \qquad \textit{for every} \quad \sigma \in (0.1).$$

We have $\Omega - \Omega_0 = \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{ x_0 \in \Omega : \sup_{R>0} (|u_{x_0,R}| + |(Du)_{x_0,R}|) = +\infty \}$$

$$\Sigma_2 = \left\{ x_0 \in \Omega : \liminf_{\rho \to 0^+} \fint_{B\rho(x_0)} |V - V_{x_0,\rho}|^2 dx > 0 \right\}$$

In particular meas $(\Omega - \Omega_0) = 0$.

Moreover, for every fixed $\sigma \in (0, 1)$ *and* M_c *there exist positive constants* $\varepsilon_0(M_0)$, $R_0(M_0)$ *such that, if for some* $x_0 \in \Omega$, $R \leq R_0$ *we have*

$$|u_{x_0,R}| + |(Du)_{x_0,R}| < M_0, \fint_{B_R(x_0)} |V - V_{x_0,R}|^2 dx < \varepsilon_0 \tag{5.3}$$

then for all ρ, R, $0 < \rho < R \leq R_0$, *we have*

$$\fint_{B_\rho(x_0)} |V - V_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^{2\sigma} \fint_{B_R(x_0)} |V - V_{x_0,R}|^2 dx \tag{5.4}$$

REMARK 5.1. — The first part of theorem 5.1 follows actually from the second part. In fact, (5.3) clearly holds almost everywhere in Ω, and since

$$|u_{x,R}| + |(Du)_{x,R}| \qquad \text{and} \qquad \int_{B_R(x_0)} |V - V_{x,R}|^2 dy$$

are continuous functions of x, the inequalities (5.3) are satisfied in a neighbourhood of x_0 whenever they hold for x_0. Therefore (5.4) holds, with x_0 replaced by x for x in a neighbourhood of x_0. The first part of the theorem then follows immediately taking into account Campanato's characterization of Hölder-continuous functions, see e. g. [*8*], p. 70.

REMARK 5.2. — Theorem 5.1 implies that $u \in C^{1,\sigma}(\Omega_0, \mathbb{R}^N)$ for every $\sigma \in (0, 1)$.

In the general case we have, compare also remark 5.3.

THEOREM 5.2. — *Let* $u \in W^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *be a minimizer of the functional in* (5.1) *and* (H.1) ... (H.6) *hold. Then there exists an open set* $\Omega_0 \subset \Omega$ *and a* $\sigma \in (0.1)$ *such that* $V(Du) \in C^{0,\sigma}(\Omega_0, \mathbb{R}^N)$. *We have* $\Omega - \Omega_0 = \Sigma_1 \cup \Sigma_2$, *hence* meas $(\Omega - \Omega_0) = 0$. *Moreover there exist positive constants* ε_0, M_0, R_0 *such that, if for some* $x_0 \in \Omega$, $R \leq R_0$ (5.3) *hold, then* (5.4) *holds.*

Both theorems follow in a standard way, see e. g. [*8*], p. 197-199, from the following proposition.

PROPOSITION 5.1. — *Set*

$$\Phi(x_0, R) = \int_{B_R(x_0)} |V - V_{x_0,R}|^2 dx$$

Then for every $x_0 \in \Omega$, $\varepsilon > 0$, *and for every* ρ, R, $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$ *we have*

$$\Phi(x_0, \rho) \leq c \left\{ \left(\frac{\rho}{R}\right)^2 + \varepsilon + \left(\frac{R}{\rho}\right)^n c(x)\chi(x_0, R) \right\} \Phi(x_0, R) + R^{2\gamma} H(x_0, R)$$

where $\gamma > 0$, *and*

$$(5.5) \quad \chi(x_0, R) = \gamma(|u_{x_0,R}| + |(Du)_{x_0,R}| + \Phi(x_0, R)^{\frac{1}{2}}, \Phi(x_0, R))$$

$$H(x_0, R) = \begin{cases} 0 & \textit{if } F \textit{ is independent of } x \textit{ and } u \\ \tilde{h}(|u_{x_0,R}| + |(Du)_{x_0,R}| + \Phi(x_0, R)^{\frac{1}{2}} & \textit{otherwise} \end{cases}$$

$\gamma(t, s)$ *being an increasing function in* t *going to zero as* s *goes to zero uniformly for* t *in a bounded set, and* $\tilde{h}(t)$ *an increasing function of* t.

Proof. — Fix a point $x_0 \in \Omega$ and a radius $R < \operatorname{dist}(x_0, \partial\Omega)$. Set

$$u_0 = u_{x_0,R/2}$$
$$p_0 = (Du)_{x_0,R/2}$$

and denote by $F^0(p)$ the frozen integrand

$$F^0(p) = F(x_0, u_0, p)$$

Define the integrand $G: \mathbb{R}^{nN} \to \mathbb{R}$ by

$$G(p) = F^0(p_0) + F^0_{p^i_\alpha}(p_0)(p^i - p^i_{0\alpha}) + \frac{1}{2} F^0_{p^i_\alpha p^j_\beta}(p_0)(p^i_\alpha - p^i_{0\alpha})(p^j_\beta - p^j_{0\beta})$$

and notice that (H.4) implies

(5.6) $$|F^0(p) - G(p)| \le c\,[(1 + |p_0|^2)^{\frac{m-2}{2}} |p - p_0|^2 + |p - p_0|^m]\,\omega(|p_0|, |p - p_0|)$$

Since (H.3) (H.6) imply (see [*29*], theor. 4.4.1) that

$$\begin{aligned} \nu(1 + |p_0|^2)^{\frac{m-2}{2}} |\xi|^2 |\pi|^2 &\le F^0_{p^i_\alpha p^j_\beta}(p_0)\xi^\alpha \xi^\beta \pi_i \pi_j \\ &\le c(1 + |p_0|^2)^{\frac{m-2}{2}} |\xi|^2 |\pi|^2 \qquad \forall \xi \in \mathbb{R}^n \qquad \forall \pi \in \mathbb{R}^N \end{aligned}$$

the elementary Hilbert space theory together with Gårding inequality ensure the existence and uniqueness of a solution $v \in W^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ of

$$\begin{cases} \displaystyle\int_{B_{R/2}(x_0)} G(Dv)dx \to \min \\ v - u \in W^{1,2}_0(B_{R/2}(x_0), \mathbb{R}^N). \end{cases}$$

Note that v is the solution of the Dirichlet boundary value problem

$$\begin{cases} F^0_{p^i_\alpha p^j_\beta}(p_0) D_\beta D_\alpha v^i = 0 \quad j = 1, \dots, N \quad \text{in} \quad B_{R/2}(x_0) \\ v - u = 0 \qquad \text{on} \qquad \partial B_{R/2}(x_0) \end{cases}$$

for an elliptic system with constant coefficients. Therefore we have, see e. g. [*8*], chap. III, for any $A \in \mathbb{R}^{nN}$ and for any $\rho \le \frac{R}{2}$

(5.7) $$\int_{B_{R/2}(x_0)} |Dv - A|^2 dx \le c \int_{B_{R/2}(x_0)} |Du - A|^2 dx$$

(5.8) $$⨍_{B_\rho(x_0)} |Dv|^2 dx \le c ⨍_{B_{R/2}(x_0)} |Dv|^2 dx$$

(5.9) $$⨍_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho}|^2 dx \le c\left(\frac{\rho}{R}\right)^2 ⨍_{B_{R/2}(x_0)} |Dv - (Dv)_{x_0,\frac{R}{2}}|^2 dx$$

Moreover, from the L^p-theory for elliptic systems we deduce that if $u \in W^{1,p}(B_{R/2}(x_0), \mathbb{R}^N)$, $p \ge 2$, then $v \in W^{1,p}(B_{R/2}(x_0), \mathbb{R}^N)$ and

(5.10) $$\int_{B_{R/2}(x_0)} |Dv - A|^p dx \le c \int_{B_{R/2}(x_0)} |Du - A|^p dx$$

(5.11) $$⨍_{B_\rho(x_0)} |Dv - (Dv)_{x_0,\rho}|^p dx \le c\left(\frac{\rho}{R}\right)^p ⨍_{B_{R/2}(x_0)} |Dv - (Dv)_{x_0,\frac{R}{2}}|^p dx$$

Notice that the constant c appearing in (5.7) ... (5.11) does not depend on p_0.

From lemma 2.2, (5.9) (5.11) we deduce

$$\fint_{B_\rho(x_0)} |V(Dv) - V(Dv)_{x_0,\rho}|^2 dx$$
$$\leq c(1+|(Dv)_{x_0,\rho}|^2)^{\frac{m-2}{2}} \fint_{B_\rho(x_0)} |Dv-(Dv)_{x_0,\rho}|^2 dx + c\fint_{B_\rho(x_0)} |Dv-(Dv)_{x_0,\rho}|^m dx$$
$$\leq c\left(\frac{\rho}{R}\right)^2 (1+|(Dv)_{x_0,\rho}|^2)^{\frac{m-2}{2}} \int_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^2 dx$$
$$+c\left(\frac{\rho}{R}\right)^m \int_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^m dx.$$

On the other hand from (5.8) we get

$$|(Dv)_{x_0,\rho}| \leq \fint_{B_\rho(x_0)} |Dv|^2 dx \leq c\fint_{B_{R/2}(x_0)} |Dv|^2 dx$$
$$\leq c|(Dv)_{x_0,\frac{R}{2}}| + c\int_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^2 dx.$$

Hence we have

$$\int_{B_\rho(x_0)} |V(Dv) - V(Dv)_{x_0,\rho}|^2 dx$$
$$\leq c\left(\frac{\rho}{R}\right)^2 (1+|(Dv)_{x_0,\frac{R}{2}}|^2)^{\frac{m-2}{2}} \fint_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^2 dx$$
$$+c\left(\frac{\rho}{R}\right)^2 \left(\fint_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^2 dx\right)^{m/2} + c\left(\frac{\rho}{R}\right)^m \fint_{B_{R/2}(x_0)} |Dv-(Dv)_{x_0,\frac{R}{2}}|^m dx$$

Taking into account lemmata 2.2 and 2.4, we then conclude

$$(5.12)\quad \fint_{B_\rho(x_0)} |V(Dv)-V(Dv)_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^2 \fint_{B_{R/2}(x_0)} |V(Dv)-V(Dv)_{x_0,\frac{R}{2}}|^2 dx$$

and therefore

$$(5.13)\quad \int_{B_\rho(x_0)} |V(Du) - V(Du)_{x_0,\rho}|^2 dx$$
$$\leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_{R/2}(x_0)} |V(Du) - V(Du)_{x_0,\frac{R}{2}}|^2 dx + c\int_{B_{R/2}(x_0)} |V(Du)-V(Dv)|^2 dx.$$

Now we estimate the last term in (5.13). Using lemma 2.2, it is not difficult to see that $\forall \varepsilon > 0$

$$(5.14)\quad \int_{B_{R/2}(x_0)} |V(Du)-V(Dv)|^2 dx$$
$$\leq c(\varepsilon)\int_{B_{R/2}(x_0)} (1+|p_0|^2)^{\frac{m-2}{2}} |Dw|^2+|Dw|^m dx+\varepsilon\int_{B_{R/2}(x_0)} |Du-p_0|^m$$

where $w = u - v$.

From the quasiconvexity assumption (H.6) we have

$$(5.15)\quad \nu\int_{B_{R/2}(x_0)} (1 + |p_0|^2)^{\frac{m-2}{2}} |Dw|^2 + |Dw|^m dx$$
$$\leq \int_{B_{R/2}(x_0)} F^0(p_0+Dw)-F^0(p_0)dx=\int_{B_{R/2}(x_0)} [F^0(p_0+Dw)-G(p_0+Dw)]dx$$
$$+\frac{1}{2}\int_{B_{R/2}(x_0)} F^0_{p^i_\alpha p^j_\beta}(p_0)D_\alpha w^i D_\beta w^i dx=(I)+\frac{1}{2}\int_{B_{R/2}(x_0)} F_{p^i_\alpha p^j_\beta}(p_0)D_\alpha w^i D_\beta w^j dx.$$

On the other hand we have

$$\int_{B_{R/2}(x_0)} F_{p^i_\alpha p^j_\beta}(p_0)D_\alpha w^i D_\beta w^j dx=\int_{B_{R/2}(x_0)} [G(Du)-G(Dv)]dx$$
$$=\int_{B_{R/2}(x_0)} [G(Du) - F^0(Du)]dx + \int_{B_{R/2}(x_0)} [F^0(Du)-F(x, u, Du)]dx$$
$$+\int_{B_{R/2}(x_0)} [F(x, u, Du)-F(x, v, Dv)]dx+\int_{B_{R/2}(x_0)} [F(x, v, Dv)-F^0(Dv)]dx$$
$$+\int_{B_{R/2}(x_0)} [F^0(Dv) - G(Dv)]dx = (II) + (III) + (IV) + (V) + (VI).$$

Therefore from (5.14) (5.15) (5.16) we conclude

$$(5.17)\quad \int_{B_{R/2}(x_0)} |V(Du)-V(Dv)|^2 dx \leq c((I)+(II)+(III)+(IV)+(V)+(VI)).$$

Notice that (IV) ≤ 0, since u is a minimizer, while (III) and (V) are zero if F does not depend on x and u.

In order to estimate the terms on the right-hand side of (5.17), it is convenient to distinguish two situations.

a) $F = F(p)$; in this case (III). (V) are zero. So we need to estimate (I)

(II) (VI). Using (5.6), the boundedness of ω, lemmata 2.2 and 2.4, the reverse Hölder inequality (4.14) we obtain

$$
\begin{aligned}
(\mathrm{II}) &\leq c \int_{B_{R/2}(x_0)} [(1+|p_0|^2)^{\frac{m-2}{2}} |Du-p_0|^2 + |Du-p_0|^m]\omega(|p_0|, |Du-p_0|)dx \\
&\leq c \int_{B_{R/2}(x_0)} |V(Du)-V((Du)_{x_0,\frac{R}{2}})|^2 \omega(|(Du)_{x_0,\frac{R}{2}}|, |Du-(Du)_{x_0,\frac{R}{2}}|)dx \\
&\leq c R^n \left(\fint_{B_{R/2}(x_0)} |V(Du)-V((Du)_{x_0,\frac{R}{2}})|^p dx \right)^{2/p} \\
&\qquad \left(\fint_{B_{R/2}(x_0)} \omega(|(Du)_{x_0,\frac{R}{2}}|, |Du-(Du)_{x_0,\frac{R}{2}}|)dx \right)^{1-\frac{2}{p}} \\
&\leq c \int_{B_R(x_0)} |V-V_{x_0,\frac{R}{2}}|^2 dx \left(\fint_{B_{R/2}(x_0)} \omega(|(Du)_{x_0,\frac{R}{2}}|, |Du-(Du)_{x_0,\frac{R}{2}}|)dx \right)^{1-\frac{2}{p}}
\end{aligned}
$$

and by Jensen's inequality, since $\omega(t, s)$ is concave in s

$$
(5.18) \quad (\mathrm{II}) \leq c\omega\left(|(Du)_{x_0,R}|, \left(\fint_{B_R(x_0)} |Du-(Du)_{x_0,R}|^m dx \right)^{\frac{1}{m}} \right)^{1-\frac{2}{p}} \int_{B_R(x_0)} |V - V_{x_0,R}|^2 dx .
$$

Exactly in the same way, taking also into account (5.7) (5.10) we obtain

$$
(\mathrm{I}), (\mathrm{VI}) \leq c\omega\left(|(Du)_{x_0,R}|, c\left(\fint_{B_R(x_0)} |Du-(Du)_{x_0,R}|^m dx \right)^{\frac{1}{m}} \right)^{1-\frac{2}{p}} \int_{B_R(x_0)} |V - V_{x_0,R}|^2 dx .
$$

This, together with (5.13) (5.17) (5.18), gives (5.5) with $H=0$ and $\rho < \frac{R}{2}$. Since (5.5) is trivial for $\frac{R}{2} \leq \rho \leq R$, the proof of proposition 5.1 is concluded in this case.

b) The general case $F(x, u, Du)-(\mathrm{I})$ (II) (III) (VI) can be estimated as before, using (4.15) instead of (4.14). In this way on the right-hand side of (5.18) the following extra terms appears

$$
R^{n+2\gamma} h(|u_0| + |(Du)_{x_0,R}| + \psi(x_0, R)^{\frac{1}{m}}) \omega(|(Du)_{x_0,R}|, \psi(x_0, R)^{\frac{1}{m}})^{1-\frac{2}{p}}
$$

which gives a term of type $R^{2\sigma} H(x_0, R)$ in (5.5).

So, in order to conclude the proof, it suffices to estimate (III) and (V). Assumption (H.5) implies, using also theorem 4.1

$$(\mathrm{III}) \leq c \int_{\mathrm{B}_{\mathrm{R}/2}(x_0)} (1+|\mathrm{D}u|^2)^{\frac{m}{2}} \eta(|u_{x_0,\frac{\mathrm{R}}{2}}|, |x-x_0|+|u-u_{x_0,\frac{\mathrm{R}}{2}}|)dx$$

$$\leq c\mathrm{R}^n \left(\fint_{\mathrm{B}_{\mathrm{R}/2}(x_0)} (1+|\mathrm{D}u|^2)^{\frac{m+\varepsilon}{2}} dx \right)^{\frac{m}{m+\varepsilon}} \left(\fint_{\mathrm{B}_{\mathrm{R}/2}(x_0)} \eta(|u_{x_0,\frac{\mathrm{R}}{2}}|, |x-x_0| + |u-u_{x_0,\frac{\mathrm{R}}{2}}|)dx \right)^{\frac{\varepsilon}{m+\varepsilon}}$$

$$\leq c \int_{\mathrm{B}_{\mathrm{R}}(x_0)} (1+|\mathrm{D}u|^2)^{\frac{m}{2}} dx \mathscr{K}(|u_{x_0,\mathrm{R}}|) \left[\mathrm{R} + \fint_{\mathrm{B}_{\mathrm{R}/2}(x_0)} |u-u_{x_0,\frac{\mathrm{R}}{2}}| dx \right]^{\frac{\delta\varepsilon}{m+\varepsilon}}.$$

Since

$$\fint_{\mathrm{B}_{\mathrm{R}/2}(x_0)} |u-u_{x_0,\frac{\mathrm{R}}{2}}| dx \leq c\mathrm{R} \fint_{\mathrm{B}_{\mathrm{R}/2}} |\mathrm{D}u| dx,$$

we then conclude

$$(\mathrm{III}) \leq c\mathrm{R}^{n+2\gamma} \mathscr{K}(|u_{x_0,\mathrm{R}}|) \fint_{\mathrm{B}_{\mathrm{R}}(x_0)} (1+|\mathrm{D}u|^2)^{\frac{m}{2}} dx \left(1 + \fint_{\mathrm{B}_{\mathrm{R}}(x_0)} |\mathrm{D}u| dx \right)^{\delta\frac{\varepsilon}{m+\varepsilon}}$$
$$\leq \mathrm{R}^{m+2\gamma} \tilde{\mathrm{H}}(x_0, \mathrm{R})$$

for $\gamma > 0$, when $\tilde{\mathrm{H}}$ is of the same type as H described in the statement of the theorem.

This concludes the proof of the proposition, since, taking into account (5.10), (V) can be estimated in the same way. Q. E. D.

REMARK 5.3. — Actually if $\mathrm{F}(x, u, p)$ is Hölder-continuous in x with exponent δ and in u with exponent γ then $\mathrm{V}(\mathrm{D}u) \in \mathrm{C}^{0,\sigma}(\Omega)$ with

$$2\sigma = \min\left(\tau, \frac{m\gamma}{m-\gamma}\right)$$

compare with [*12*].

REFERENCES

[*1*] E. ACERBI, N. FUSCO, Semicontinuity problems in the calculus of variations. *Arch. Rat. Mech. Anal.*, t. **86**, 1984, p. 125-145.

[*2*] J. M. BALL, Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, t. **63**, 1977, p. 337-403.

[*3*] R. R. COIFFMAN, C. FEFFERMAN, Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, t. **51**, 1974, p. 241-250.

[*4*] E. DE GIORGI, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino, cl. Sci. Fis. Mat. Nat.*, (3), t. **3**, 1957, p. 25-43.

[*5*] E. DI BENEDETTO, T. S. TRUDINGER, Harnack inequalities for quasi-minima of variational integrals. *Ann. Inst. H. Poincaré. Analyse non linéaire.*

[6] C. L. EVANS, Quasiconvexity and partial regularity in the calculus of variations. Preprint.

[7] F. W. GEHRING, The L^p integrability of the partial derivatives of a quasi conformal mapping. *Acta Math.*, t. **130**, 1973, p. 265-277.

[8] M. GIAQUINTA, *Multiple integrals in the Calculus of Variations and Nonlinear Elliptic Systems*. Princeton Univ. Press, Princeton, 1983.

[9] M. GIAQUINTA, E. GIUSTI, On the regularity of minima of variational integrals. *Acta Math.*, t. **148**, 1982, p. 31-46.

[10] M. GIAQUINTA, E. GIUSTI, Differentiability of minima of non-differentiable functionals. *Inventiones Math.*, t. **72**, 1983, p. 285-298.

[11] M. GIAQUINTA, E. GIUSTI, Quasi-minima. *Ann. Inst. H. Poincaré, Analyse non linéaire*, t. **1**, 1984, p. 79-107.

[12] M. GIAQUINTA, E. GIUSTI, Sharp estimates for the derivatives of local minima of variational integrals. *Boll. U. M. I.* (6), t. **3-A**, 1984, p. 239-248.

[13] M. GIAQUINTA, P. A. IVERT, *Partial regularity for minima of variational integrals*. Preprint FIM, ETH Zürich, 1984.

[14] M. GIAQUINTA, G. MODICA, Regularity results for some classes of higher order nonlinear elliptic systems. *J. reine u. angew. Math.*, t. **311-312**, 1979, p. 145-169.

[15] M. GIAQUINTA, G. MODICA, Almost-everywhere regularity results for solutions of nonlinear elliptic systems. *Manuscripta math.*, t. **28**, 1979, p. 109-158.

[16] M. GIAQUINTA, J. SOUCEK, Caccioppoli's inequality and Legendre-Hadamard condition. *Math. Ann.*, t. **270**, 1985, p. 105-107.

[17] C. B. MORREY, jr., Quasi convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, t. **2**, 1952, p. 25-53.

[16] C. B. MORREY, jr., *Multiple integrals in the Calculus of Variations*. Springer Verlag, Heidelberg, 1966.

(Manuscrit reçu le 7 mars 1985)