ANNALES DE L'I. H. P., SECTION C

PATRICK BONCKAERT

Smooth invariant curves of singularities of vector fields on $\mathbb{R}^3$

Annales de l'I. H. P., section C, tome 3, nº 2 (1986), p. 111-183

<http://www.numdam.org/item?id=AIHPC_1986__3_2_111_0>

Ann. Inst. Henri Poincaré,
Vol. 3, n° 2, 1986, p. 111-183.

Echanges Annales
Analyse non linéaire

Smooth invariant curves of singularities of vector fields on $\mathbb{R}^3$

by

Patrick BONCKAERT
Limburgs Universitair Centrum, Universitaire Campus,
B-3610 Diepenbeek (Belgium)

ABSTRACT. — We describe singularities of C^∞ vector fields, mainly in $\mathbb{R}^3$, in the neighbourhood of a given « generalized » direction (this is: the image of a C^∞ germ $\gamma:(\mathbb{R}^+,0) \to (\mathbb{R}^3,0)$ with $\gamma'(0) \neq 0$). It is a local study. One of the major results is: if X is a germ in $0 \in \mathbb{R}^3$ of a C^∞ vector field and if X is not infinitely flat along a direction D then there exists a cone of finite contact around D in which four specific situations can occur. In three of these situations D is formally invariant under X (with formally we mean: up to the level of formal Taylor series) and there exists a C^∞ one-dimensional invariant manifold having infinite contact with D. In particular, we obtain that the existence of a formally invariant direction D always implies the existence of a « real life » invariant direction having infinite contact with D, provided that X is not infinitely flat along D. Using the blowing up method for singularities of vector fields we reduce a singularity always to either a « flow box » or to a singularity with nonzero 1-jet and with a formally invariant direction. Finally we give topological models for the obtained situations.

I would like to thank Freddy Dumortier for suggesting me the problem and for his valuable help.

RÉSUMÉ. — Nous décrivons les singularités de champs de vecteurs C^∞ dans $\mathbb{R}^3$ au voisinage d'une « direction » donnée, c'est-à-dire de l'image

Liste de mots-clés : Vector Field. Singularity. Invariant Manifold.

Classification AMS : 34C05 Ordinary Differential Equations. Qualitative Theory. Location of integral curves, singular points, limit cycles.

d'un germe $\gamma:(\mathbb{R}^+,0) \to (\mathbb{R}^3,0)$, avec $\gamma'(0) \neq 0$. C'est une étude locale. Nous montrons en particulier que si X est le germe à l'origine d'un champ C^∞ et si X n'est pas plat dans une direction D, alors il existe un cône de contact d'ordre fini autour de D où quatre situations distinctes peuvent se produire. Dans trois d'entre elles, D est formellement invariant par X et il existe une courbe invariante C^∞ présentant un contact d'ordre infini avec D. En utilisant la méthode de blow-up pour les singularités de champs de vecteurs, nous réduisons toutes les singularités soit à un « flow box », soit à une singularité au 1-jet non trivial possédant une direction formellement invariante. Enfin, nous fournissons des modèles topologiques pour les diverses situations obtenues.

Je remercie Freddy Dumortier pour m'avoir suggéré ce problème et pour son aide.

I. INTRODUCTION, PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

§ 1. Elementary definitions and useful theorems.

(1.1) Definition. — Let X be a vector field on $\mathbb{R}^n$. We say that X *has a singularity in* $p \in \mathbb{R}^n$ if $X(p) = 0$.

If we only investigate local properties of a singularity in p, it is no restriction to put p in 0, the origin.

(1.2) Definition. — Two vector fields X and Y on $\mathbb{R}^n$ are called *germ-equivalent* in 0 if there is a neighbourhood U of 0 on which they coincide, that is: $X|_U = Y|_U$.

(1.3) Definition. — The set of all vector fields on $\mathbb{R}^n$ which are germ-equivalent with X (in 0) is called the *germ* of X (in 0). In the same way we can define germs in 0 of functions, diffeomorphism,...

The germ in 0 of a set $A \subset \mathbb{R}^n$ is the germ in 0 of its characteristic function χ_A. A germ will often be confused with a representative of it if this is without danger.

(1.4) Notation. — G^n denotes the set of all germs in 0 of C^∞ vector fields X on $\mathbb{R}^n$ with $X(0) = 0$.

(1.5) Definition. — Let X, $Y \in G^n$, $k \in \mathbb{N}$. X and Y are called *k-jet equivalent* if their derivatives in 0 up to, and including, order k are equal: $D^iX(0) = D^iY(0)$, $\forall i$, $0 \leq i \leq k$.

(1.6) DEFINITION. — The set of all $Y \in G^n$ which are k-jet equivalent with $X \in G^n$ is called the *k-jet of* X and denoted $j_k X(0)$. It is important to observe that the k-th order Taylor approximation of X in 0 belongs to $j_k X(0)$; we often will make no distinction between these two objects. In fact there is a $1 - 1$ correspondence between k-jets and vector fields X on $\mathbb{R}^n$ with $X(0) = 0$ and with polynomial component functions of degree $\leq k$.

(1.7) NOTATION $J_k^n = \{ j_k X(0) \mid X \in G^n \}$. — We can take the inverse limit of the sets J_k^n for the mappings $\pi_{lk} : J_l^n \to J_k^n : j_l X(0) \to j_k X(0)$ $(l \geq k)$.

(1.8) DEFINITION. — This inverse limit is denoted J_∞^n; its elements are called ∞*-jets*; the ∞-jet corresponding to $X \in G^n$ is denoted $j_\infty X(0)$. Elements of J_∞^n can be regarded as n-tupels of formal power series in n variables; also $j_\infty X(0)$ represents the Taylor series of X at 0.

In the same way we define jets of functions, diffeomorphisms...

(1.9) DEFINITION. — $X \in G^n$ is said to have *flatness* k if $j_k X(0) = 0$ and $j_{k+1} X(0) \neq 0$.

(1.10) DEFINITION. — Let X, $Y \in G^n$. X and Y are said to be C^r *conjugate* ($r \in \mathbb{N} \cup \{ \infty, \omega \}$, $r \geq 1$) if for some (and hence for all) representatives $\tilde{X}$ and $\tilde{Y}$ of X resp. Y there are open neighbourhoods U and V of 0 in $\mathbb{R}^n$ and a C^r diffeomorphism $\phi : U \to V$ such that $\forall x \in U$: $D\phi(x).\tilde{X}(x) = \tilde{Y}(\phi(x))$.

We also write: $\phi_* X = Y$ in this case.

(1.11) DEFINITION. — Let X, $Y \in G^n$. X and Y are said to be C^r *equivalent* ($r \in \mathbb{N} \cup \{ \infty, \omega \}$) if for some (and hence for all) representatives $\tilde{X}$ and $\tilde{Y}$ of X resp. Y there are open neighbourhoods U and V of 0 in $\mathbb{R}^n$ and a C^r diffeomorphism $h: U \to V$ which maps integral curves of $\tilde{X}$ to integral curves of $\tilde{Y}$ preserving the « sense » but not necessarily the parametrization; more precisely: if $p \in U$ and $\phi_{\tilde{X}}(p, [0, t]) \subset U$, $t > 0$, then there is some $t' > 0$ such that $\phi_{\tilde{Y}}(h(p), [0, t']) = h(\phi_{\tilde{X}}(p, [0, t]))$. ($\phi_{\tilde{X}}$ and $\phi_{\tilde{Y}}$ denote the flow of $\tilde{X}$ resp. $\tilde{Y}$). Time preserving C^r-equivalence is called C^r conjugacy and coincides with definition 1.10 for $r \geq 1$.

With C^0 diffeomorphism we mean a homeomorphism.

(1.12) THEOREM (Borel) [Di, Na]. — For all $T \in J_\infty^n$ there exists an $X \in G^n$ such that $T = j_\infty X(0)$. (Or: $j_\infty : G^n \to J_\infty^n$ is surjective.). □

(1.13) DEFINITION. — A *generalized direction in 0*, or shortly a *direction*, is a C^∞-diffeomorphic image of the germ in $0 \in \mathbb{R}^n$ of a halfline with endpoint 0. The diffeomorphism is assumed to preserve 0.

(1.14) DEFINITION. — A germ in 0 of a set $K \subset \mathbb{R}^{m+1}$ is called a C^l (solid) *cone of contact* k, $k \in \mathbb{N}$, $l > k$, if there exists a germ of a C^l function h: $([0, \infty[, 0) \to (\mathbb{R}, 0)$ with $j_k h(0) = 0$, $j_{k+1} h(0) \neq 0$ and a germ of a C^∞ diffeomorphism ϕ: $(\mathbb{R}^{m+1}, 0) \to (\mathbb{R}^{m+1}, 0)$ such that

$$K = \phi(\{ (x, z) \in \mathbb{R}^m \times [0, \infty[\mid |x| \leq h(z) \}).$$

K is called a cone of contact *k around the direction* $D = \phi(\{0\}^m \times [0, \infty[)$. The intersection of a C^r diffeomorphic image of a hyperplane, passing through D, with K is called a *C^r m-dimensional subcone of* K $(0 \leq r \leq l)$.

(1.15) PROPERTY. — If D is a direction in $\mathbb{R}^n$, then there exists a germ of a C^∞ map $\gamma: ([0, \infty[, 0) \to (\mathbb{R}^n, 0)$ with $\gamma'(0) \neq 0$ such that the image of γ is D. Conversely, if $\gamma: ([0, \infty[, 0) \to (\mathbb{R}^n, 0)$ is C^∞ and if $\gamma'(0) \neq 0$, then the image of γ is a direction.

Proof. — Easy. □

(1.16) DEFINITION. — For $X \in G^n$ and D a direction we say that X *is non-flat along* D if for some (and hence for all) C^∞ germ $\gamma: ([0, \infty[0) \to (\mathbb{R}^n, 0)$ with $\gamma'(0) \neq 0$ and image D we have $j_\infty(X \circ \gamma)(0) \neq 0$. In the other case we say that X *is flat along* D.

(1.17) DEFINITION. — We say that $X \in G^n$ satisfies a *Łojasiewicz inequality* if there exist $k \in \mathbb{N}$ and C, $\delta \in]0, \infty[$ such that

$$\forall x \in \mathbb{R}^n \quad \text{with} \quad |x| < \delta : |X(x)| \geq C |x|^k.$$

(1.18) PROPERTY. — If $X \in G^n$ satisfies a Łojasiewicz inequality then X is non-flat along every direction.

(1.19) CONVENTION. — We write (x, z) for an element of $\mathbb{R}^m \times \mathbb{R}$; for a vector field X on $\mathbb{R}^m \times \mathbb{R}$ we write $X = (X_x, X_z)$.

(1.20) DEFINITION. — We say that $X \in G^{m+1}$ leaves the z-axis $\{0\}^m \times \mathbb{R}$ *formally invariant* if for all $k \in \mathbb{N}$: $\dfrac{\partial^k X_x}{\partial z^k}(0) = 0$. This is the same as saying that $j_\infty X_x(0)$ does not contain pure z-terms.

A direction $D = \phi(\{0\}^m \times [0, \infty[)$ is formally invariant under $X \in G^{m+1}$ if $\phi_*^{-1} X$ leaves the z-axis formally invariant.

(1.21) THEOREM (Normal Form Theorem, Poincaré and Dulac). — Let $X \in G^n$ and let X_1 be the linear vector field on $\mathbb{R}^n$ such that $j_1 X_1(0) = j_1 X(0)$.

Let, for $h \in \mathbb{N}$, H^h denote the vector space of those vector fields on $\mathbb{R}^n$ whose coefficient functions are homogeneous polynomials of degree h.

Denote $[X_1, -]_h : H^h \to H^h : Y \to [X_1, Y]$ and let $B^h = \operatorname{Im}([X_1, -]_h)$.

Choose for each $h \geq 2$ an arbitrary supplementary space G^h for B^h in H^h (that is: $H^h = B^h \oplus G^h$).

Then there exists a germ of a C^∞ diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\phi_* X$ has an ∞ jet of the form

$$j_\infty(\phi_* X)(0) = X_1 + g_2 + g_3 + \dots$$

where $g_h \in G^h$, $h = 2, 3, \dots$.

Application (to be used later on).

Let $X \in G^3$ with $j_1 X(0) = ax\dfrac{\partial}{\partial x}$, $a \neq 0$. If the z-axis $\{0\}^2 \times \mathbb{R}$ is formally invariant under X, then there exists a C^∞ diffeomorphism $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\phi_* X = X'$ has an ∞ jet of the form

$$j_\infty X'(0) = ax\frac{\partial}{\partial x} + \sum_{j+k=1}^{\prime} \alpha_{jk} x y^j z^k \frac{\partial}{\partial x} + \sum_{\substack{j+k=2\\ j\geq 1}}^{\infty} \beta_{jk} y^j z^k \frac{\partial}{\partial y} + \sum_{j+k=2}^{\infty} \gamma_{jk} y^j z^k \frac{\partial}{\partial z}$$

Proof [Ta]. — Suppose, by induction, that for $i \in \mathbb{N}$, $i \geq 1$, one can write $X = X_1 + g_2 + \dots + g_{i-1} + R_{i-1}$ with the $g_h \in G^h$ and with $j_{i-1} R_{i-1}(0) = 0$. We can write $R_{i-1} = g_i + b_i$ with $g_i \in G^i$, $b_i \in B^i$, $j_i R_i(0) = 0$. Take an $Y \in H^i$ with $[X_1, Y] = b_i$.

Taking $\phi_i = \phi_Y(., -1)$ (the time -1 mapping of Y) one can check that $(\phi_i)_*(X) = X_1 + g_2 + \dots + g_i + R_i$.

Since $j_{i-1}\phi_i(0) = j_{i-1}\,\mathrm{Id}\,(0)$ we can apply the theorem of Borel (for diffeomorphisms) to obtain the desired diffeomorphism ϕ.

Proof of the application. — For $X_1 = ax\dfrac{\partial}{\partial x}$ it follows from a straightforward calculation that for all $p, q, r \in \mathbb{N}$:

$$\left[X_1, x^p y^q z^r \frac{\partial}{\partial x}\right] = a(p-1) x^p y^q z^r \frac{\partial}{\partial x}$$

$$\left[X_1, x^p y^q z^r \frac{\partial}{\partial y}\right] = apx^p y^q z^r \frac{\partial}{\partial y}$$

$$\left[X_1, x^p y^q z^r \frac{\partial}{\partial z}\right] = apx^p y^q z^r \frac{\partial}{\partial z}$$

We see that for each $h \geq 2$ the linear map $[X_1, -]_h$ has a diagonal matrix with respect to the basis

$$\left\{ x^p y^q z^r \frac{\partial}{\partial x}, x^p y^q z^r \frac{\partial}{\partial y}, x^p y^q z^r \frac{\partial}{\partial z} \mid p, q, r \in \mathbb{N} \quad \text{and} \quad p + q + r = h \right\}$$

Hence $\operatorname{Im} [X_1, -]_h \oplus \operatorname{Ker} [X_1, -]_h = H^h$.

A basis for $\operatorname{Ker} [X_1, -]_h$ is obviously

$$\left\{ xy^q z^r \frac{\partial}{\partial x} \mid 1 + q + r = h \right\} \cup \left\{ y^q z^r \frac{\partial}{\partial y} \mid q + r = h \right\} \cup \left\{ y^q z^r \frac{\partial}{\partial z} \mid q + r = h \right\}.$$

So, applying the normal form theorem, we have for $\phi_* X = X'$ an ∞ jet of the form

$$j_\infty X'(0) = ax\frac{\partial}{\partial x} + \sum_{j+k=1}^{\infty} \alpha_{jk} x y^j z^k \frac{\partial}{\partial x} + \sum_{j+k=2}^{\infty} \beta_{jk} y^j z^k \frac{\partial}{\partial y} + \sum_{j+k=2}^{\infty} \gamma_{jk} y^j z^k \frac{\partial}{\partial z}.$$

It suffices to prove that ϕ can be chosen in such a way that it formally preserves the z-axis. Because then X' also leaves the z-axis formally invariant, and hence the $\frac{\partial}{\partial y}$ component of $j_\infty X'(0)$ cannot contain pure z-terms. Take any $Y' \in H^i$ with $[X_1, Y'] = b_i$. If we separate the pure z^i terms of the $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ components of Y', we can write it in the form:

$$Y' = Y + Az^i \frac{\partial}{\partial x} + Bz^i \frac{\partial}{\partial y}$$

where $Y \in H^i$ has no pure z^i terms in its $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ components. We have

$$b_i = [X_1, Y] + \left(- Az^i \frac{\partial}{\partial x} \right).$$

But as b_i and $[X_1, Y]$ don't contain pure z^i terms in their $\frac{\partial}{\partial x}$ component, necessarily $A = 0$. So $b_i = [X_1, Y]$.

As $\phi_i = \phi_{Y(.,-1)}$ (the time -1 mapping of Y) we obtain the result. □

In chapter IV we will make extensively use of the following result.

(1.22) THEOREM (Brouwer, Leray-Schauder-Tychonoff fixed point theorem) [Sm]. — Let E be a locally convex topological vector space, K a nonvoid compact convex subset of E, f any continuous map of K into K. Then f admits at least one fixed point in K, that is: a point $x \in K$ with $f(x) = x$.

§ 2. The main result.

One can pose the following question: suppose that a (generalized) direction D is formally invariant under a germ $X \in G^3$ (by formally we mean: up to the level of ∞ jets, or equivalently: up to the level of formal Taylor series); does there exist a C^∞ invariant one-dimensional manifold having ∞ contact with D? The answer is yes, provided X is non-flat along D. This is one of the major ingredients of the following theorem.

Let us emphasize that the results of this theorem are stated in terms of germs in $0 \in \mathbb{R}^3$.

(2.1) THEOREM. — Let $X \in G^3$ and let D be a direction. If X is non-flat along D, then there exists a C^∞ cone K of finite contact around D such that one of the following situations occurs:

I. all the orbits of X hitting K enter K and leave K, except (of course) $\{(0, 0, 0)\}$;

II. D is formally invariant under X; there exists a direction D′ in K, having ∞ contact with D, which is invariant under X; if we arrange (by changing the sign of X if necessary) that the orbit of X in D′ tends to 0 then either

A. the only orbit of X in K tending to 0 is contained in D′; all the other orbits of X starting in K leave K;

B. all the orbits of X in K tend to 0; if we add 0 to such an orbit, we obtain a direction which has ∞ contact with D;

C. there exists a unique C^0 2-dimensional subcone S of K such that

i) S is invariant under X;

ii) all the orbits starting in S tend to 0; if we add 0 to such an orbit, we obtain a direction which has ∞ contact with D;

iii) all the orbits of X starting in K\S leave K.

Always assuming that along the invariant directions the orbits tend to 0 for $t \to \infty$, we find in cases II.A and II.C a unique model up to C^0 equivalence and in case II.B even up to C^0 conjugacy.

The *proof* of this theorem will be given in chapters IV and V.

(2.2) PICTURES of the situations occuring in the theorem.

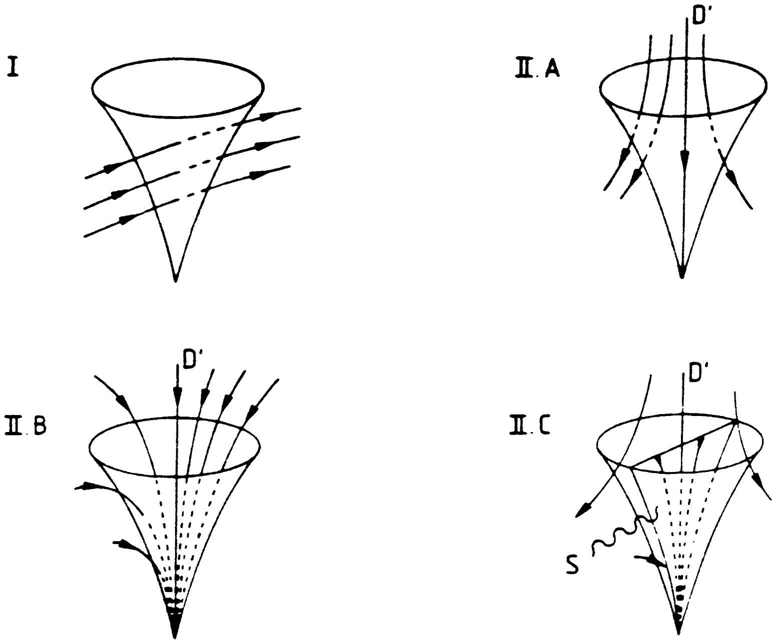

FIG. 1.

(2.3) REMARK. — Perhaps in some cases the claimed results in (2.1) are not evident at first sight.

Take for example the linear vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z}$ and $D = z$-axis. All the orbits tend to 0 in negative time and have parabolic contact with the z-axis. Nevertheless one can find a cone of finite contact (two in this case) such that alle the orbits leave it, except of course the one contained in the z-axis. This example indicates in which way the main theorem must be considered: once an orbit leaves the cone, we have no further information about its future, which may be for example to tend to zero or even to re-enter the cone.

(2.4) REMARK. — One can formulate and prove an analogon of theorem (2.1) in dimension 2, this time of course without situation II.C. On the other hand it is not yet clear to me how to generalize the theorem to arbitrary dimensions; more specifically one could ask: does a formally invariant direction always hide a « real life » invariant direction? The same question can be posed for diffeomorphisms.

II. PREPARATORY CALCULATIONS CONCERNING BLOWING UP OF VECTOR FIELDS

Although the blowing up method for singularities of vector fields is a known technique [Ta, Du1, Du2, D.R.R., Go] we recall it here because it is of fundamental importance in the sequel. Especially for blowing up in a direction some specific calculations will be elaborated in full detail.

Roughly spoken, blowing up a vector field is: write it down in spherical coordinates and divide the result as much as possible by a power of r (= Euclidean norm of x) such that one can take the limit for $r \to 0$.

§ 1. One spherical blowing up.

(1.1) NOTATIONS. — We denote, for $m \in \mathbb{N}$:

$$S^m = \left\{ (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \,\middle|\, \sum_{i=1}^{m+1} x_i^2 = 1 \right\}$$

and $\Phi : S^m \times \mathbb{R} \to \mathbb{R}^{m+1} : (x, r) \to rx$.

(1.2) PROPERTIES. — *i)* $\Phi|_{S_m \times]0,\infty[}$ is a C^∞ diffeomorphism onto $\mathbb{R}^{m+1} \setminus \{0\}$

ii) $$S^m \times \{0\} = \Phi^{-1}(\{0\}). \quad \square$$

So in fact with Φ we introduced on $\mathbb{R}^{m+1} \setminus \{0\}$ a sort of spherical coordinates.

(1.3) PROPOSITION [Ta]. — Let X be a C^p vector field on $\mathbb{R}^{m+1} (p \in \mathbb{N} \cup \{\infty\})$ with $X(0) = 0$.

Then there is a C^{p-1} field $\tilde{X}$ on $S^m \times \mathbb{R}$ such that in each $q \in S^m \times \mathbb{R}$: $\Phi_*(\tilde{X}(q)) = X(\Phi(q))$ (or: $\Phi_* \tilde{X} = X$). $\square$

If X has flatness k, that is, $j_k X(0) = 0$ and $j_{k+1} X(0) \neq 0$, then for the following vector field

$$\bar{X} = \frac{1}{r^k} \tilde{X}$$

we can take the limit for $r \to 0$ and we obtain a vector field of class C^{p-k-1} on $S^m \times \mathbb{R}$ which we still denote $\bar{X}$.

(1.4) DEFINITION. — $\bar{X}$ is called the vector field obtained by *blowing up* X (*once*) *and dividing bij* r^k.

$\overline{X}$ restricted to $S^m \times \{0\}$ is a vector field tangent to $S^m \times \{0\}$. The study of this vector field sometimes gives information on the asymptotic behavior of the orbits when they tend to 0 [Go].

§ 2. One directional blowing up.

In high dimensions the explicit calculations for spherical blowing up quickly become complicated. If we restrict our attention to one open half-sphere of S^m we use the better computable directional blowing-up. We can assume that the half-sphere has $(0, 0, \ldots, 0, 1)$ as « north-pole ».

Let us replace $\mathbb{R}^m \times \mathbb{R}$ by $E \times \mathbb{R}$ where E is an arbitrary normed vector-space. We do this because later on we shall use this form and after all the construction remains the same.

(2.1) CONSTRUCTION.

A point of $E \times \mathbb{R}$ will be denoted (x, z). With « the z-axis » we mean $\{0\} \times \mathbb{R}$. Let, for $n \in \mathbb{N}$, Ψ^n be the following map: $\Psi^n : E \times \mathbb{R} \to E \times \mathbb{R}$: $(x, z) \to (xz^n, z)$. Similarly to proposition (1.3) one proves

(2.1.1) PROPOSITION. — Let X be a C^p vector field on $E \times \mathbb{R}$ $(p \in \mathbb{N} \cup \{\infty\})$ with $X(0) = 0$. Then there is a C^{p-1} vector field $\tilde{X}^1$ on $E \times \mathbb{R}$ such that in each $q \in E \times \mathbb{R}$: $\Psi^1_*(\tilde{X}^1(q)) = X(\Psi^1(q))$ (or: $\Psi^1_*\tilde{X}^1 = X$). Again if $j_k X(0) = 0$ and $j_{k+1}X(0) \neq 0$, we devide $\tilde{X}^1$ by z^k; we denote the resulting vector field $\overline{X}^1 = \frac{1}{z^k}\tilde{X}^1$ and again we can take the limit for $z \to 0$ to obtain a C^{p-k-1} vectorfield on $E \times \mathbb{R}$ still denoted $\overline{X}^1$.

(2.1.2) DEFINITION. — $\overline{X}^1$ is called the vectorfield obtained by *blowing up* X (*once*) *in the z-direction and dividing by* z^k. Like in § 1, $E \times \{0\}$ is invariant under $\overline{X}^1$, so we can consider the restriction of $\overline{X}^1$ to $E \times \{0\}$.

(2.2) RELATION BETWEEN DIRECTIONAL AND SPHERICAL BLOWING UP IN THE $\mathbb{R}^{m+1}$ CASE.

Let S^m_+ denote the « upper hemisphere », that is:

$$S^m_+ = \{(x_1, \ldots, x_m, z) \in S^m \mid z > 0\}.$$

Consider the bijection $F : S^m_+ \times \mathbb{R} \to \mathbb{R}^{m+1}$:

$$(x_1, \ldots, x_m, z, r) \to \left(\frac{x_1}{z}, \ldots, \frac{x_m}{z}, zr\right).$$

One immediately checks that $\Phi = \Psi^1 \circ F$.

Hence $\tilde{X}|_{S^m_+ \times \mathbb{R}}$ and $\tilde{X}^1$ are C^ω conjugated: $F_*\tilde{X} = \tilde{X}^1$.

If we denote $F(x_1, \ldots, x_m, z, r) = \left(\frac{x_1}{z}, \ldots, \frac{x_m}{z}, rz\right) = (X_1, \ldots, X_m, Z)$ then, using $\sum_{i=1}^m x_i^2 + z^2 = 1$, we get $z = \frac{1}{(1+X_1^2+\ldots+X_m^2)^{1/2}}$ and $r = \frac{Z}{z}$.

Hence $F_*\overline{X} = \frac{1}{(1 + X_1^2 + \ldots + X_m^2)^{\frac{k}{2}}}\overline{X}^1$.

As $\frac{1}{(1 + X_1^2 + \ldots + X_m^2)^{\frac{k}{2}}}$ is an everywhere positive function, the orbits of $\overline{X}^1$ and $F_*\overline{X}$ coincide. They also have the same orientation.

(2.3) CALCULATIONS CONCERNING ONE DIRECTIONAL BLOWING UP.

(2.3.1) GENERAL FORMULAS. — If X is a vector field on $E \times \mathbb{R}$ we write $X = (X_x, X_z)$.

It is an easy calculation to see that $\Psi^1_*\tilde{X}^1 = X$ if only if

$$\tilde{X}^1 = \left(\frac{1}{z}(X_x \circ \Psi^1) - \frac{x}{z}(X_z \circ \Psi^1), X_z \circ \Psi^1\right)$$

and thus $\overline{X}^1 = \left(\frac{1}{z^k}\left[\frac{1}{z}(X_x \circ \Psi^1) - \frac{x}{z}(X_z \circ \Psi^1)\right], \frac{1}{z^k}(X_z \circ \Psi^1)\right)$.

From now on we assume X to be C^∞.

(2.3.2) THE ∞ JET OF $\overline{X}^1$ IN 0 (CASE $\mathbb{R}^{m+1}$). — Let us indicate here some (usual) multiindex abbreviations.

For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we write $|\alpha| = \sum_{i=1}^m \alpha_i$; if $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ then we denote $x^\alpha = \prod_{i=1}^m x_i^{\alpha_i}$; also $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$ and $\alpha! = \prod_{i=1}^m (\alpha_i!)$.

We write the ∞ jet of X in 0 down as

$$j_\infty X(0) = \sum_{i=1}^m \sum_{n=k+1}^m \sum_{|\alpha|=0}^n a^n_{i,\alpha} x^\alpha z^{n-|\alpha|} \frac{\partial}{\partial x_i} + \sum_{n=k+1}^\infty \sum_{|\alpha|=0}^n c^n_\alpha x^\alpha z^{n-|\alpha|} \frac{\partial}{\partial z}$$

if X has flatness k in 0.

For the ∞ jet of $\overline{X}^1$ in 0 we write

$$j_\infty \overline{X}^1(0) = \sum_{i=1}^{m} \sum_{n=0}^{\infty} \sum_{|\alpha|=0}^{n} \overline{a}_{i,\alpha}^n x^\alpha z^{n-|\alpha|} \frac{\partial}{\partial x_i} + \sum_{n=0}^{\infty} \sum_{|\alpha|=0}^{n} \overline{c}_\alpha^n x^\alpha z^{n-|\alpha|} \frac{\partial}{\partial z}.$$

Using (2.3.1) and calculating straightforward we find:

i) $\overline{a}_{i,\alpha}^n = a_{i,\alpha}^{n+k+1-|\alpha|} - c_{(\alpha_1,\dots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\dots,\alpha_m)}^{n+k+1-|\alpha|}$ for $1 \leq i \leq m$, $1 \leq \alpha_i$, $1 \leq |\alpha| \leq n$

ii) $\overline{a}_{i,(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_m)}^n = a_{i,\alpha}^{n+k+1-|\alpha|}$ for $1 \leq i \leq m$, $0 \leq |\alpha| \leq n$

iii) $\overline{c}_\alpha^n = c_\alpha^{n+k-|\alpha|}$ for $0 \leq |\alpha| \leq n$;

in *i*), *ii*), *iii*) we may take the right hand side zero whenever it is undefined.

(2.3.3) REMARK. — In particular in (*iii*) we see that for each $n \in \mathbb{N}$ and for each α with $|\alpha| = n$: $\overline{c}_\alpha^n = 0$. This expresses the fact that the $z = 0$ hyperplane is invariant under $\overline{X}^1$, or equivalently that the $\dfrac{\partial}{\partial z}$ component of $\overline{X}^1$ does not contain pure x^α terms.

§ 3. Successive directional blowing ups.

(3.1) *Construction.*

Let $\overline{X}^1$ be obtained by blowing up X in the z-direction and dividing by z^k.

Suppose that $\overline{X}^1$ has again a singularity say in $(a, 0) \in E \times \{0\}$ and that $j_p\overline{X}^1(a, 0) = 0$, $j_{p+1}\overline{X}^1(a, 0) \neq 0$.

Then we can blow up $\overline{X}^1$ in $(a, 0)$ in the z-direction in the following natural sense. Denote $T: E \times \mathbb{R} \to E \times \mathbb{R}: (x, z) \to (x - a, z)$; since $T_*\overline{X}^1$ has a singularity of flatness p in 0, we can consider the vector field $\overline{T_*\overline{X}^1}^1$ obtained by blowing up $T_*\overline{X}^1$ in the z-direction and dividing by z^p.

(3.1.1) DEFINITION. — $\overline{T_*\overline{X}^1}^1$ is called the vector field obtained by *blowing up* X *twice* in the z-direction first in (0, 0) then in $(a, 0)$ and dividing first by z^k then by z^p; we denote it $\overline{X}^2$.

Of course $\overline{X}^2$ depends on the choice of the singularity of $\overline{X}^1$. In this way, we can of course go on with the construction and define successively $\overline{X}^n$ by blowing up $\overline{X}^{n-1}$ in some singularity. Obviously $\overline{X}^n$ depends on the choice of the singularities in which we blow up. Let us formalize the idea giving a prescribed sequence of singularities in which must be blown up.

(3.1.2) DEFINITION. — Let X be a C^∞ vector field on $E \times \mathbb{R}$ with $X(0) = 0$. A *directed sequence* of successive blowing ups of X is a sequence of triples $(\overline{X}^n, (x_n, 0), k_n)_{0 \leq n < N}$ $(N \in \mathbb{N} \cup \{\infty\})$ such that

i) $\overline{X}^0 = X$ and $(x_0, 0) = (0, 0)$;

ii) $\forall n, 0 \leq n < N : \overline{X}^n$ has a singularity of flatness k_n in $(x_n, 0)$ and $\overline{X}^{n+1}$ is obtained by blowing up $\overline{X}^n$ in $(x_n, 0)$ and dividing by z^{k_n}.

(3.1.3) DEFINITION. — A directed sequence of successive blowing ups of X where $\forall n, 0 \leq n < N : (x_n, 0) = (0, 0)$ will for brevity be called a *sequence of blowing ups in* 0 of X (in the z-direction). Such a sequence is henceforth denoted by $(\overline{X}^n, 0, k_n)_{0 \leq n < N}$.

(3.1.4) REMARK. — Further on we will show that there exists a $1-1$ correspondence between directed sequences of successive blowing ups $(\overline{X}^n, (x_n, 0), k_n)_{0 \leq n < N}$ and $N - 1$ jets of directions ($N = \infty$ included).

(3.2) *Calculations concerning successive directional blowing-ups.*

We will give explicit formulas only for a sequence of blowing ups in 0. As we have already announced this is, at least for our purposes, no restriction. In this case we can define $\overline{X}^n$ in one step:

(3.2.1) LEMMA. — Let $\overline{X}^n$ be obtained by a sequence of blowing ups in 0 of X as defined in (3.1).

Then $\overline{X}^n = \dfrac{1}{z^{k_0 + \ldots + k_{n-1}}} \tilde{X}^n$ where $\tilde{X}^n$ has the property $\Psi^n_* (\tilde{X}^n(p)) = X(\Psi^n(p))$, $\forall p \in E \times \mathbb{R}$.

Proof. — Straightforward. □

(3.2.2) REMARK. — We see that properties of $\overline{X}^n$ in a cylinder-shaped neighbourhood of $\{ 0_E \} \times [0, \infty[$ are transformed, by Ψ^n, to simular proporties of X in a cone of contact $n-1$ around $\{ 0_E \} \times [0, \infty[$. It is in this sense that we will obtain the results in the main theorem (I.2.1).

(3.2.3) FORMULAS. — Just like in (2.3.1) for $X = (X_x, X_z)$ we have $\Psi^n_* \tilde{X}^n = X$ if and only if $\tilde{X}^n = \left(\dfrac{1}{z^n} (X_x \circ \Psi^n) - \dfrac{nx}{z} (X_z \circ \Psi^n), X_z \circ \Psi^n \right)$ and thus

$$\overline{X}^n = \frac{1}{z^{k_0 + k_1 + \ldots + k_{n-1}}} \left(\frac{1}{z^n} (X_x \circ \Psi^n) - \frac{nx}{z} (X_z \circ \Psi^n), X_z \circ \Psi^n \right).$$

Further on we will need the following result.

(3.2.4) LEMMA. — Let X be a C^∞ vector field on $E \times \mathbb{R}$. If X is non-flat along $\{ 0_E \} \times [0, \infty[$ and if $\{ 0_E \} \times \mathbb{R}$ is formally invariant, then there exists a $Q \in \mathbb{N}$ and a C^∞ function $\gamma : E \times \mathbb{R} \to \mathbb{R}$ with $\gamma(0, 0) \neq 0$ such that the $\mathbb{R}$-component of $\tilde{X}^{Q+1}$ is of the form $z^Q \gamma(x, z)$. ($\tilde{X}^{Q+1}$ is the vector field obtained by blowing up X $Q + 1$ times in 0 in the z-direction without dividing by a power of z.)

Proof. — Put $X=(X_x, X_z)$. As, by our assumptions, the map $z \to X_z(0, z)$ has a nonzero ∞ jet, there exists a $Q \in \mathbb{N}$ and C^∞ maps $A_0, A_1, \ldots, A_Q$: $E \to \mathbb{R}$ with $A_0(0) = A_1(0) = \ldots = A_{Q-1}(0) = 0$ and $A_Q(0) \neq 0$ such that we can write

$$X_z(x, z) = A_0(x) + zA_1(x) + \ldots + z^{Q-1}A_{Q-1}(x) + z^Q A_Q(x) + 0(z^{Q+1}).$$

The $\mathbb{R}$-component of $\tilde{X}^{Q+1}$ is $X_z \circ \Psi^{Q+1}$.

More explicitely:

$$\begin{aligned}(X_z \circ \Psi^{Q+1})(x, z) &= A_0(z^{Q+1}x) + zA_1(z^{Q+1}x) + \ldots \\ &\quad + z^{Q-1}A_{Q-1}(z^{Q+1}x) + z^Q A_Q(z^{Q+1}x) \\ &\quad + 0(z^{Q+1}) \\ &= z^Q(A_Q(z^{Q+1}x) + 0(z)).\end{aligned}$$

Put $\gamma(x, z) = A_Q(z^{Q+1}x) + 0(z)$. □

III. REDUCTION OF A SINGULARITY TO A SINGULARITY WITH NONZERO 1-JET BY SUCCESSIVE DIRECTIONAL BLOWING UPS

When blowing up a singularity of flatness k one might hope that the flatness of the resulting vector field in a singularity is not strictly bigger than k. Up to one type of singularities, this will in fact be the case.

If we consider a sequence of blowing ups $(\overline{X}^n, 0, k_n)_{0 \le n < N}$, we will see, also for that exceptional type of singularities, that after at most one step the flatness k_n becomes decreasing (perhaps constant). In fact, a vector field which is the « blown up » of another one cannot be of that exceptional type (mentioned above).

Next we will prove that if the sequence k_n does not decrease to zero, then the vector field is flat along the z-axis.

Let us start by showing that, up to a C^∞ change of coordinates, it is no restriction to assume that a directed sequence consists of blowing ups along the z-axis.

§ 1. Definitions and properties of directed sequences.

(1.1) PROPOSITION. — Let $N \in \mathbb{N} \cup \{\infty\}$ and suppose that $(\overline{X}^n, (x_n, 0), k_n)_{0 \le n < N}$ is a directed sequence of successive blowing ups of $X \in G^{m+1}$. Then there exists a C^∞ change of coordinates $\phi : (\mathbb{R}^{m+1}, 0) \to (\mathbb{R}^{m+1}, 0)$ such that in the new coordinates this sequence is $(\overline{\phi_*(X)}^n, (0, 0), k_n)_{0 \le n < N}$.

Proof. — We define inductively some transformations $(\alpha_n)_{1 \le n < N}$, $(\phi_n)_{1 \le n < N}$ and $(T_n)_{1 \le n < N}$ on $\mathbb{R}^{m+1}$ as follows.

Put

$$\alpha_1 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x, z) \to (x - zx_1, z)$$
$$\phi_1 = \alpha_1$$
$$T_1 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x, z) \to (x - x_1, z).$$

Suppose that α_i, ϕ_i, T_i are defined for $1 \leq i \leq n$ with $n + 1 < N$. We denote $(a_{n+1}, 0) \in \mathbb{R}^n \times \mathbb{R}$ the singularity of $\overline{(\phi_n)_*(X)}^{n+1}$ corresponding to $(x_{n+1}, 0)$ in the new coordinates induced by ϕ_n. Put

$$\alpha_{n+1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x, z) \to (x - z^{n+1}a_{n+1}, z)$$
$$\phi_{n+1} = \alpha_{n+1} \circ \phi_n$$
$$T_{n+1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x, z) \to (x - a_{n+1}, z)$$

If $\Psi^n : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x, z) \to (z^n x, z)$ denotes the « blowing up map », observe that $\Psi^n \circ T_n = \alpha_n \circ \Psi^n$, $1 \leq n < N$.

If $Y \in G^{m+1}$ can be blown up $n + 1$ times in $(0, 0)$ and if $\overline{Y}^{n+1}$ must be blown up in a singularity $(a_{n+1}, 0)$ then:

$$\begin{aligned} \Psi_*^{n+1}(\tilde{Y}^{n+1}) = Y &\Rightarrow (\alpha_{n+1})_* \Psi_*^{n+1} \tilde{Y}^{n+1} = (\alpha_{n+1})_* Y \\ &\Rightarrow \Psi_*^{n+1}((T_{n+1})_* \tilde{Y}^{n+1}) = (\alpha_{n+1})_* Y \\ &\Rightarrow (T_{n+1})_* \tilde{Y}^{n+1} = \widetilde{(\alpha_{n+1})_* Y}^{n+1} \\ &\Rightarrow (T_{n+1})_* \overline{Y}^{n+1} = \overline{(\alpha_{n+1})_* Y}^{n+1} \end{aligned}$$

(the z component is not altered by T_{n+1} nor by α_{n+1}) so $\overline{(\alpha_{n+1})_* Y}^{n+1}$ must be blown up in $(0, 0)$.

Applying the foregoing observation to $Y = (\phi_n)_* X$, we obtain that $\overline{(\alpha_{n+1})_*(\phi_n)_* X}^{n+1}$ (which is $\overline{(\phi_{n+1})_* X}^{n+1}$) must be blown up in $(0, 0)$. For $n \geq 1$, $j_n \alpha_{n+1}(0) = j_n \mathrm{Id}\,(0)$ so

$$j_n \phi_{n+1}(0) = j_n \phi_n(0). \tag{1}$$

For finite N, ϕ_{N-1} is the desired ϕ.

For $N = \infty$ we can, by (1), consider the inverse limit of the ϕ_n, and theorem (I.1.12) of Borel together with the inverse function theorem [Di] gives the desired ϕ. □

(1.2) COROLLARY. — For each directed sequence $(\overline{X}^n, (x_n, 0), k_n)_{0 \leq n < N}$ there exists a C^∞ change of coordinates $(x, z) = \phi(x', z')$ and a direction $D = \phi^{-1}$ (z-axis) such that if we blow up $\phi_*^{-1} X$ successively in 0 then points corresponding to $(x, z) = (0, 0)$ like in the proof of (1.1), are precisely $(x', z') = (x_n, 0)$. Conversely, given a direction $D = \phi^{-1}$ (z-axis), there exists a sequence x_n as above. Moreover, D is unique up to $N-1$-jet equivalence. In other words: there is a $1-1$ correspondence between directed sequences and $N-1$ jets of directions. So we can speak of a sequence of blowing ups along a direction.

If we blow up X successively along the z-axis it is of course possible that after some finite time there is no singularity any more in $(0, 0)$. Let us formalize this as follows.

(1.3) DEFINITION. — $(\overline{X}^n, 0, k_n)_{0 \leq n < N}$ is called a *maximal directed sequence* of blowing ups along the z-axis if either

i) $N = \infty$

ii) $N \in \mathbb{N}$ and $\overline{X}^{N-1}(0) \neq 0$.

Property. — Such a sequence always exists.

(1.4) PROPOSITION. — If $(\overline{X}^n, 0, k_n)_{0 \leq n < N}$ is a maximal directed sequence with $N \in \mathbb{N}$, then there exists a cone K of contact $N-2$ around the z-axis such that all orbits inside K enter K and leave K after a finite time.

Proof. — As $\overline{X}^{N-1}(0) \neq 0$, we can construct a cylinder shaped neighbourhood of 0 in $\mathbb{R}^m \times [0, \infty[$ of the form $C = \{ (x, z) \in \mathbb{R}^m \times \mathbb{R} \mid \| x \| \leq R, z \in [0, \delta[\}$ such that all orbits of $\overline{X}^{N-1}$ inside C enter C and leave C after a finite time. Then

$$\begin{aligned} \Psi^{N-1}(C) &= \{ (z^{N-1}x, z) \mid \| x \| \leq R, z \in [0, \delta[\} \\ &= \{ (x', z) \mid \| x' \| \leq z^{N-1}R, z \in [0, \delta[\} . \end{aligned}$$

Let K be the germ in 0 of this set. □

So from now on we will consider infinite sequences of blowing ups. Let us indicate what it means for X that a maximal directed sequence of blowing ups of it along the z-axis is infinite.

(1.5) PROPOSITION. — Let $(\overline{X}^n, 0, k_n)_{0 \leq n < N}$ be a maximal directed sequence of blowing ups of X along the z-axis. The following statements are equivalent:

i) $N = \infty$

ii) the z-axis is formally invariant under X (see definition (I.1.20)).

Proof. — *i*) $\Rightarrow$ *ii*). Let us write down the ∞ jets as follows

$$j_\infty X(0) = \sum_{i=1}^{m} \sum_{K=k_0+1}^{\infty} \sum_{|\alpha|=0}^{K} a_{i,\alpha}^K x^\alpha z^{K-|\alpha|} \frac{\partial}{\partial x_i} + \sum_{K=k_0+1}^{\infty} \sum_{|\alpha|=0}^{K} c_\alpha^K x^\alpha z^{K-|\alpha|} \frac{\partial}{\partial z}$$

$$j_\infty \overline{X}^n(0) = \sum_{i=1}^{m} \sum_{K=k_n+1}^{\infty} \sum_{|\alpha|=0}^{K} {}^n\overline{a}_{i,\alpha}^K x^\alpha z^{K-|\alpha|} \frac{\partial}{\partial x_i} + \sum_{K=k_n+1}^{\infty} \sum_{|\alpha|=0}^{K} {}^n\overline{c}_\alpha^K x^\alpha z^{K-|\alpha|} \frac{\partial}{\partial z}.$$

We must prove that $a_{i,0}^K = 0$ for all $K \geq k_0 + 1$ and $i \in \{1, \ldots, m\}$. With the formula in (II.2.3.2) *ii*) we find that for all $K \geq k_0 + 1$ and

$i \in \{1, \ldots, m\}$: ${}^1\bar{a}_{i,0}^{K-k_0-1} = a_{i,0}^K$ and further by induction on n we get, using the same formula, for all $n \in \mathbb{N}$: ${}^n\bar{a}_{i,0}^{K-k_0-\ldots-k_{n-1}-n} = a_{i,0}^K$.

Suppose, by contradiction, that $a_{i,0}^K \neq 0$. Then ${}^1\bar{a}_{i,0}^{K-k_0-1} \neq 0$ so $k_1 + 1 \leq K - k_0 - 1$ (remember the choise of k_1 in definition (II.3.1.2)) or $0 \leq K - k_0 - k_1 - 2$; further by induction we find ${}^n\bar{a}_{i,0}^{K-k_0-k_1-\ldots-k_{n-1}-n} \neq 0$ with always, as it should be, $0 \leq K - k_0 - \ldots - k_{n-1} - n$.

Certainly sooner or later $0 = K - k_0 - \ldots - k_{n-1} - n$ for some n. But then $\bar{X}^n(0) = \sum_{i=1}^{m} {}^n\bar{a}_{i,0}^0 \frac{\partial}{\partial x} \neq 0$, contradicting $N = \infty$.

ii) $\Rightarrow$ *i)*. Let us use the same notations. We have $a_{i,0}^K = 0$, for all $K \geq k_0 + 1$ and $i \in \{1, \ldots, m\}$. The formula (II.2.3.2) gives immediately ${}^n\bar{a}_{i,0}^0 = 0$ for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$ whence $\bar{X}^n(0) = 0$, $\forall n$ and the result. $\square$

§ 2. **Reduction to a singularity with nonzero 1-jet.**

The main purpose of this section is to prove the following:

(2.1) THEOREM. — If $(\bar{X}^n, 0, k_n)_{0 \leq n < \infty}$ is a maximal directed sequence of blowing ups of $X \in G^{m+1}$ along the z-axis $\{0\}^m \times [0, \infty[$ and if X is non-flat along the z-axis then there exists a $N \in \mathbb{N}$ such that $\bar{X}^{N+n}$ has flatness zero $\forall n \in \mathbb{N}$.

The *proof* of this theorem will be a consequence of the following propositions (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8). $\square$

(2.2) PROPOSITION. — Let $X \in G^{m+1}$ have flatness k. If $\bar{X}^1$ has flatness $\geq k + 1$ then $j_{k+1}(X)(0) = \sum_{|\alpha|=k+1} c_\alpha^{k+1} x^\alpha \frac{\partial}{\partial z}$.

Proof. — As $j_k X(0) = 0$ we only have to look at the $a_{i,\alpha}^{k+1}$ and the c_α^{k+1}, $0 \leq |\alpha| \leq k + 1$, $1 \leq i \leq m$. Out of (II.2.3.2) *iii)* we obtain for each n with $1 \leq n \leq k$ and α with $|\alpha| = n-1$: $0 = \bar{c}_\alpha^n = c_\alpha^{n+k-n+1} = c_\alpha^{k+1}$ so $c_\alpha^{k+1} = 0$, for each α with $0 \leq |\alpha| \leq k$. (2)

From (II.2.3.2) *i)* we get for each n with $1 \leq n \leq k + 1$ and α with $|\alpha| = n$ and $1 \leq \alpha_i$:

$$\begin{aligned} 0 = \bar{a}_{i,\alpha}^n &= a_{i,\alpha}^{n+k+1-n} - c_{(\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\ldots,\alpha_m)}^{n+k+1-n} \\ &= a_{i,\alpha}^{k+1} - c_{(\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\ldots,\alpha_m)}^{k+1} . \end{aligned}$$

But by (2) above those c's are zero. Consequently $a_{i,\alpha}^{k+1} = 0$ for each α with $0 \leq |\alpha| \leq k + 1$ and $1 \leq \alpha_i$.

In the same way, using (II.2.3.2) *ii*), we obtain that all the coefficients in $j_{k+1}X(0)$ vanish except possibly c_α^{k+1} with $|\alpha| = k+1$. □

(2.3) PROPOSITION. — If $X \in G^{m+1}$ and if $\overline{X}^1$ has flatness k, then $\overline{X}^2$ has flatness $\leq k$.

Proof. — Suppose that $j_{k+1}\overline{X}^2(0) = 0$. Then proposition (2.2) would imply that

$$j_{k+1}\overline{X}^1(0) = \sum_{|\alpha|=k+1} \overline{c}_\alpha^{k+1} x^\alpha \frac{\partial}{\partial z}.$$

By our assumptions one of the $\overline{c}_\alpha^{k+1}, |\alpha| = k+1$, must be different from zero. But this is impossible because of the remark (II.2.3.3) (invariance of the $z=0$ hyperplane). □

(2.4) PROPOSITION. — If $X \in G^{m+1}$ has flatness k and $j_{k+1}\overline{X}^1(0) = 0$ then $\overline{X}^1$ has flatness $k+1$.

Proof. — We must show that $j_{k+2}\overline{X}^1(0) \neq 0$. From proposition (2.2) and from $j_{k+1}\overline{X}^1(0) = 0$ we get

$$j_{k+1}X(0) = \sum_{|\alpha|=k+1} c_\alpha^{k+1} x^\alpha \frac{\partial}{\partial z}.$$

Because X has flatness k, there exists an α with $|\alpha| = k+1$ and $c_\alpha^{k+1} \neq 0$. Using (II.2.3.2) *iii*) we observe that

$$\overline{c}_\alpha^{k+2} = c_\alpha^{k+2+k-k-1} = c_\alpha^{k+1}.$$

So $j_{k+2}\overline{X}^1(0) \neq 0$. □

(2.5) COROLLARY. — If $X \in G^{m+1}$, if $(\overline{X}^n, 0, k_n)_{0 \leq n < N}$ is a directed sequence of blowing ups in 0 of X in the z-direction ($N \in \mathbb{N} \cup \{\infty\}$) and if X has flatness k then either.

i) $\overline{X}^1$ has flatness smaller than or equal to k and for all n with $0 \leq n < N-1$ the flatness of $\overline{X}^{n+1}$ is smaller than or equal to the flatness of $\overline{X}^n$;

ii) $\overline{X}^1$ has flatness $k+1$ and for all n with $1 \leq n < N-1$ the flatness of $\overline{X}^{n+1}$ is smaller than or equal to the flatness of $\overline{X}^n$.

(2.6) PROPOSITION. — Let $X \in G^{m+1}$. Suppose that there exists an integer $k \geq 1$ and an infinite directed sequence $(\overline{X}^n, 0, k)_{n \in \mathbb{N}}$ of blowing ups in 0 of X in the z direction such that $\forall n \in \mathbb{N} : \overline{X}^n$ has flatness k and at each step we divide by z^k.

Denote $i : \mathbb{R} \to \mathbb{R}^{m+1} : z \to (0, \ldots, 0, z)$.

Then $j_\infty(j_{k-1}(X) \circ i)(0) = 0$.

Proof. — Let us write again X for its formal part. Then $j_{k-1}(X)(0,\ldots,0,z)$ is formally determined by

$$\left(\frac{\partial^M X}{\partial x^\beta \partial z^{M-|\beta|}}(0,\ldots,0,z)\right)_{\substack{0\le M\le k-1\\ 0\le|\beta|\le M}}.$$

We use the notations as in (II.2.3.2) and obtain

$$\frac{\partial^M X}{\partial x^\beta \partial z^{M-|\beta|}}(x_1,\ldots,x_m,z) = \sum_{n=k+1}^{\infty}\sum_{|\alpha|=0}^{n}\left(a^n_{1,\alpha}\frac{\partial}{\partial x_1}+\ldots\right.$$

$$\left.+\,a^n_{m,\alpha}\frac{\partial}{\partial x_m}+c^n_\alpha\frac{\partial}{\partial x}\right)\frac{\alpha!}{(\alpha-\beta)!}x^{\alpha-\beta}\frac{(n-|\alpha|)!}{(n-|\alpha|-M+|\beta|)!}z^{n-|\alpha|-M+|\beta|}$$

So

$$\frac{\partial^M X}{\partial x^\beta \partial z^{M-|\beta|}}(0,\ldots,0,z) = \sum_{n=k+1}^{\infty}\left(a^n_{1,\beta}\frac{\partial}{\partial x_1}+\ldots\right.$$

$$\left.+\,a^n_{m,\beta}\frac{\partial}{\partial x_m}+c^n_\beta\frac{\partial}{\partial z}\right)\frac{\beta!(n-|\beta|)!}{(n-M)!}z^{n-M}.$$

Hence the terms of $j_\infty X(0)$ playing a role in $j_\infty(j_{k-1}(X)\circ i)(0)$ are

$$\sum_{N=k+1}^{\infty}\sum_{|\beta|=0}^{k-1}\left(a^N_{1,\beta}\frac{\partial}{\partial x_1}+\ldots+a^N_{m,\beta}\frac{\partial}{\partial x_m}+c^N_\beta\frac{\partial}{\partial z}\right)x^\beta z^{N-|\beta|}.$$

We look at what these terms give when blowing up as in the proposition.

We use $\Psi^n:\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}:(x_1,\ldots,x_m,z)\to(x_1z^n,\ldots,x_mz^n,z)$ and the formula in (II.3.2) which becomes here

$$\bar{X}^n=\frac{1}{z^{kn}}\left[\sum_{i=1}^{m}\left(\frac{1}{z^n}(X_i\circ\Psi^n)-\frac{nx_i}{z}(X_z\circ\Psi^n)\right)\frac{\partial}{\partial x_i}+(X_z\circ\Psi^n)\frac{\partial}{\partial z}\right].$$

For the $\dfrac{\partial}{\partial x_i}$ component we obtain

$$\frac{1}{z^{kn}}\frac{1}{z^n}\sum_{N=k+1}^{\infty}\sum_{|\beta|=0}^{k-1}a^N_{i,\beta}x^\beta z^{n|\beta|}z^{N-|\beta|}-\frac{1}{z^{kn}}\frac{nx_i}{z}\sum_{N=k+1}^{\infty}\sum_{|\beta|=0}^{k-1}c^N_\beta x^\beta z^{n|\beta|}z^{N-|\beta|}$$

$$=\sum_{N=k+1}^{\infty}\sum_{|\beta|=0}^{k-1}a^N_{i,\beta}x^\beta z^{N-|\beta|-n(k+1-|\beta|)}$$

$$-\,n\sum_{N=k+1}^{\infty}\sum_{|\beta|=0}^{k-1}c^N_\beta x^\beta z^{N-|\beta|-1-n(k-|\beta|)} \qquad (3)$$

and for the $\frac{\partial}{\partial z}$ component we have

$$\sum_{N=k+1}^{\infty} \sum_{|\beta|=0}^{k-1} c_\beta^N x^\beta z^{N-|\beta|-n(k-|\beta|)} \tag{4}$$

First look at (4). If this construction is possible for each $n \in \mathbb{N}$ then $c_\beta^N = 0$ for all $N \geq k+1$, $0 \leq |\beta| \leq k-1$.

Thus the second $\Sigma\Sigma$ in (3) vanishes. For the same reasons as just mentioned we must have $a_{i,\beta}^N = 0$ for all $N \geq k+1$, $0 \leq i \leq m$, $0 \leq |\beta| \leq k-1$.

This means that $j_\infty(j_{k-1}(X) \circ i)(0) = 0$. $\square$

(2.7) PROPOSITION. — Let $X \in G^{m+1}$ be non-flat along the z-axis. Let $\overline{X}^n$ be obtained from X by n successive blowing ups in 0 in the z direction. Then $\overline{X}^n$ is non-flat along the z-axis.

Proof. — Let $\Psi^n : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} : (x_1, \ldots, x_m, z) \to (x_1 z^n, \ldots, x_m z^n, z)$ be the blowing up mapping leading to $\overline{X}^n$.

Then for some analytic function G_n we have $\tilde{X}^n = G_n \overline{X}^n$ and $X \circ \Psi^n = \Psi_*^n \circ \tilde{X}^n$. Denote i: $[0, \infty[\to \mathbb{R}^{m+1} : z \to (0, \ldots, 0, z)$.

Suppose that $j_\infty(\overline{X}^n \circ i)(0) = 0$.

Then $j_\infty(\tilde{X}^n \circ i)(0) = 0$ and hence $j_\infty(X \circ \Psi^n \circ i)(0) = 0$.

But $\Psi^n \circ i = i$. This would imply that $j_\infty(X \circ i)(0) = 0$, contradicting our assumptions. $\square$

(2.8) PROPOSITION. — Suppose that $X \in G^{m+1}$ is non-flat along the z-axis. Then it impossible that there exists an infinite directed sequence $(\overline{X}^n, 0, k_n)_{n \in \mathbb{N}}$ of blowing ups in 0 of X in the z-direction with $\forall n \geq 1 : k_n \geq 1$.

Proof. — Suppose that there exists such a sequence. Then there exist integers $k \geq 1$ and $N \geq 1$ such that $\forall n \geq 0 : \overline{X}^{N+n}$ has flatness k.

Hence proposition (2.6) gives $j_\infty(j_{k-1}(\overline{X}^N) \circ i)(0) = 0$ contradicting proposition (2.7). $\square$

This completes the proof of theorem (2.1).

IV. TREATMENT OF ALL THE FLATNESS ZERO CASES AND PROOF OF THE MAIN THEOREM, WITH EXCEPTION OF THE C^0 RESULTS

We study germs in 0 of vectorfields which

i) leave the $\mathbb{R}^m \times \{0\}$ hyperplane invariant

ii) leave the z-axis $\{0\}^m \times \mathbb{R}$ formally invariant

iii) have a nonzero 1-jet.

(Sometimes we will only consider the case $m = 2$.)

The 1-jet of such a vector field has a matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & c_0^1 \end{bmatrix}$$

with $A \in \mathbb{R}^{m \times m}$ and where c_0^1 has the same meaning as in (II.2.3.2).

Such vector fields appear as a result of the reduction by successive blowing up in part III. Even if the 1-jet is nonzero, we will sometimes blow up the vector field some more times in order to obtain situations like in the main theorem (I.2.1); also in some cases the proof of the existence of the C^∞ 1-dimensional invariant manifold is made easier by making extra blowing ups.

We must show that all possibilities for c_0^1 and A (both not simultaneously zero) lead to a situation as in the main theorem (I.2.1).

We first distinguish in § 1 the case $c_0^1 \neq 0$. This turns out to be fairly easy. If $c_0^1 = 0$, (§ 2) we restrict ourselves to the case $m = 2$. The possibilities are:

i) A is hyperbolic: see (2.1)

ii) A has eigenvalues $i\lambda$, $-i\lambda$, $\lambda \in \mathbb{R}\backslash\{0\}$ (« the rotation case »): see (2.2)

iii) A has eigenvalues a, 0, $a \in \mathbb{R}\backslash\{0\}$: see (2.3)

iv) both eigenvalues of A are zero: see (2.4).

§ 1. The eigenvalues in the z-direction is nonzero.

(1.1) PROPOSITION. — Suppose that $X \in G^{m+1}$ satisfies:

i) the z-axis $\{0\}^m \times \mathbb{R}$ is formally invariant under X

ii) $c_0^1 \neq 0$ (see the notations of (II.2.3.2) or above).

Then

a) there exists a C^∞ germ $h: ([0, \infty[, 0) \to (\mathbb{R}^m, 0)$ whose graph (germ) $\{(h(z), z) \in \mathbb{R}^m \times \mathbb{R} \mid z \in [0, \infty[\}$ is invariant under X and with $j_\infty h(0) = 0$;

b) supposing $c_0^1 < 0$—change the sign of X if necessary—there exists a cone K of finite contact around the z-direction $\{0\}^m \times [0, \infty[$ such that the only orbit of X in K tending to 0 is contained in $\{(h(z), z) \in \mathbb{R}^m \times \mathbb{R} \mid z \in [0, \infty[\}$; all the other orbits starting in K leave K after a finite amount of time.

Proof. — If we use the formulas in (II.3.2.3) (with of course $k_0 = k_1 = \ldots = k_{n-1} = 0$) we find that for all $n \geq 1$ the 1-jet of $\overline{X}^n$ has a matrix of the form

$$\begin{bmatrix} A - nc_0^1 \,\mathrm{Id}_{m \times m} & 0 \\ 0 & c_0^1 \end{bmatrix}$$

Notice that λ is an eigenvalue of A if and only if λ-nc_0^1 is an eigenvalue of $A - nc_0^1 \,\mathrm{Id}_{m \times m}$.

Hence for large n, $\overline{X}^n$ is a hyperbolic saddle for which we can apply the (un-) stable manifold theorem (see for example [H.P.S., Ke]) to obtain the results. □

(1.2) REMARK. — The methods we develop further on for the more delicate situations would also work here; the calculations would even be much simpler here.

(1.3) REMARK. — In (1.1) we may replace $\mathbb{R}^m$ by any Banach space.

§ 2. The eigenvalue in the z-direction is zero.

From now on we work in $\mathbb{R}^3$, thus: $m = 2$.

(2.1) THE HYPERBOLIC CASE.

(2.1.1) PROPOSITION. — Suppose $X \in G^3$ satisfies

i) the z-axis is formally invariant under X

ii) $j_1X(0) = ax\frac{\partial}{\partial x} - \text{by }\frac{\partial}{\partial y}$, $a, b > 0$

iii) X is non-flat along the z-axis.

Then there exists a cone K of finite contact around the z-axis, a unique $C^0$2-dimensional subcone S of K and a C^∞ germ $h:([0, \infty[, 0) \to (\mathbb{R}^2, 0)$ such that $j_\infty h(0) = 0$ and:

a) S is invariant under X

b) the graph of h, $\{ (h(z), z) \mid z \in [0, \infty[\}$, is invariant under X and lies in S

c) in K we have situation II.C of the main theorem (I.2.1).

Proof. — By lemma (II.3.2.4) we may assume that the $\frac{\partial}{\partial z}$ component of X is of the form $z^Q\gamma(x, y, z)$ with $\gamma(0, 0, 0) \neq 0$ and $Q \geq 2$.

We may, and do, assume that $\gamma(0, 0, 0) > 0$, because if not, replace X by $-$ X which is of the same type.

If we use the center manifold theory such as in [H.P.S., Ke] we obtain the existence of a unique center-unstable-manifold.

Although it is general not necessarily C^∞, here we can use the uniqueness of the center-unstable manifold as well as the fact that X is non flat along the z-axis to obtain that the center-unstable manifold is C^∞ in this case, as follows.

For each $m \in \mathbb{N}$ there exists a C^m center-unstable manifold W_m defined on some neighbourhood U_m of 0; we can take care that $U_{m+1} \subset U_m$ for all $m \in \mathbb{N}$. By uniqueness of the center-unstable manifold we have

$W_m \cap U_{m+1} = W_{m+1}$. If we take U_1 small enough then the following holds for fixed m.

Since $\gamma(0,0,0) > 0$ and since $a > 0$ there is a $T > 0$ such that

$$\phi_X(-T, W_1 \cap U_1) \subset W_1 \cap U_m = W_m \cap U_m$$

(ϕ_X denotes the flow of X). So $U_1 \cap W_1$ is a C^m manifold.

But as m is arbitrary, we obtain that $U_1 \cap W_1$ is C^∞.

The behavior of X restricted to this center-unstable manifold can be found in full detail in [D.R.R.]. □

(2.1.2) REMARK. — The methods to obtain invariant manifolds, developped further on for the more delicate cases, could also be applied here.

(2.1.3) PROPOSITION. — Let E be a Banach space, X a C^∞ vector field defined on a neighbourhood U of $0 \in E \times \mathbb{R}$ with $X(0) = 0$. Suppose that

i) the z-axis is formally invariant under X;

ii) $E \times \{0\}$ is invariant for X and $D(X|_{E\times\{0\}})(0)$ is a hyperbolic contraction or expansion, that is: the spectrum of this linear operator is contained in a subset of $\mathbb{C}$ of the form $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq -\lambda\}$ or $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda\}$ for some $\lambda \in \mathbb{R}$, $\lambda > 0$;

iii) X is non flat along the z-axis.

iv) X is bounded on U together with all its derivatives.

Then there exists a cone K of finite contact around the z-axis and a C^∞ germ h: $([0, \infty[, 0) \to (E, 0)$ with

a) the graph of h, $\{(h(z), z) \mid z \in [0, \infty[\}$, is invariant under X

b) in K we have situation II.A or II.B of the main theorem (I.2.1).

Proof. — Let us write $X = (X_x, X_z)$.

By lemma (II.3.2.4) we may assume that X_z is of the form:

$$X_z(x, z) = z^Q \gamma(x, z)$$

with $\gamma(0,0) \neq 0$ and $Q \geq 2$.

There exist $A \in L_c(E, E)$ and $B: (E \times \mathbb{R}, 0) \to (L_c(E, E), 0)$ and R_∞: $(E \times \mathbb{R}, 0) \to (E, 0)$ such that

$$X_x(x, z) = (A + B(x, z)).x + R_\infty(x, z)$$

and such that (up to changing the sign of X).

i) the spectrum of A lies in $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq \lambda < 0\}$

ii) B and R_∞ are C^∞ germs; especially $B(0,0) = 0$

iii) $j_\infty R_\infty(0, 0) = 0$.

This because the z-axis is formally invariant. We also may assume that R_∞ is ∞ flat along $E \times \{0\}$ because if not, blow up once.

The existence of the invariant graph follows from the center manifold theory. Its smoothness follows from the fact that $\gamma(0,0) \neq 0$. If $\gamma(0,0) > 0$ then we have situation II.A and if $\gamma(0,0) < 0$ then we have situation II.B; this can be proved by standard techniques like in [D.R.R.] or like further on in the more delicate cases. □

(2.2) THE ROTATION CASE.

We study germs in 0 of vector fields on $\mathbb{R}^3$ which leave the z-axis $\mathbb{R}^2 \times \{0\}$ formally invariant and for which the 1-jet has eigenvalues $i\lambda$, $-i\lambda$, 0 ($\lambda \in \mathbb{R}\backslash\{0\}$). Such a vector field is completely non-hyperbolic, so here the « classical » theorems about invariant manifolds as in [Ke, H.P.S.] are not applicable. First of all, such a vector field has, up to a linear change of coordinates preserving $\mathbb{R}^2 \times \{0\}$ and $\{0\}^2 \times \mathbb{R}$, a 1-jet of the form

$$j_1X(0) = \lambda\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \qquad \lambda \neq 0.$$

In [B.D.] we obtained the following result:

(2.2.1) PROPOSITION. — Suppose $X \in G^3$ satisfies

i) the z-axis $\{0\}^2 \times \mathbb{R}$ is formally invariant under X

ii) $j_1X(0) = \lambda\left(y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}\right)$, $\lambda \in \mathbb{R}\backslash\{0\}$

iii) X is non flat along the z-axis

then

a) there exists a C^∞ germ $h:([0,\infty[,0) \to (\mathbb{R}^2,0)$ whose graph (germ) $\{(h(z),z) \mid \in [0,\infty[\}$ is invariant under X and with $j_\infty h(0) = 0$

b) there exists a cone K of finite contact around $\{0\}^2 \times [0,\infty[$ such that in K we have situation II.A or II.B of the main theorem (I.2.1).

(2.3) ONLY ONE NONZERO EIGENVALUE.

Here we will meet all situations II.A, II.B, II.C of the main theorem. In situation II.C we will have to construct invariant surfaces. Sometimes we could make use of the classical center manifold theory, but for other cases, where the requested invariant manifold is not a stable, center-stable, center, center-unstable or unstable manifold, we will have to apply our own methods. But if we set up the whole machinery anyway it can effortlessly be applied to the « classical » cases.

The methods to obtain smooth invariant manifolds developped in this section, can also be used to obtain the results in (2.1) and (2.2).

Up to a linear change of coordinates preserving $\mathbb{R}^2 \times \{0\}$ and $\{0\}^2 \times \mathbb{R}$, we may assume that the 1-jet is of the form

$$j_1X(0) = ax\frac{\partial}{\partial x}, \qquad a \neq 0$$

(2.3.1) PROPOSITION. — Suppose $X \in G^3$ satisfies

i) the z-axis is formally invariant under X

ii) $j_1X(0) = ax\frac{\partial}{\partial x}$, $a \neq 0$

iii) X is non-flat along the z-axis

then there exists a cone K of finite contact around the z-axis and a C^∞ germ $h : ([0, \infty[, 0) \to (\mathbb{R}^2, 0)$ with $j_\infty h(0) = 0$ such that

a) the graph (germ) $\{ (h(z), z) \mid z \in [0, \infty[\}$ is invariant under X

b) in K we have situation II.A or II.B or II.C of the main theorem (I.2.1).

Proof. — We may assume that $j_\infty X(0)$ has the form as in (I.1.21).

Because X is non-flat along the z-axis, we can apply lemma (II.3.2.4) and hence we assume that there exists a $Q \in \mathbb{N}$ such that the $\frac{\partial}{\partial z}$ component of X is of the form

$$z^Q\gamma(x, y, z)\frac{\partial}{\partial z}$$

with $\gamma(0, 0, 0) \neq 0$.

So X can be written as follows:

$$X(x, y, z) = (a + f_1(y, z))x\frac{\partial}{\partial x} + f_2(z)y\frac{\partial}{\partial y}$$
$$+ f_3(y, z)y^2\frac{\partial}{\partial y} + z^Q\gamma(x, y, z)\frac{\partial}{\partial z} + S_\infty(x, y, z)$$

where

i) $f_1(0, 0) = f_2(0) = 0$

ii) $\gamma(0, 0, 0) \neq 0$

iii) S_∞ is a vector field which is ∞ flat along the $z = 0$ plane and with zero $\frac{\partial}{\partial z}$-component (we absorbed it in γ; we also assumed at least one blowing up)

iv) $Q \geq 2$.

We only pay attention to the upper half space $\mathbb{R}^2 \times [0, \infty[$. If we blow up X n times, without dividing by a power of z of course since the 1-jet is nonzero, then we get

$$\begin{aligned}\tilde{X}^n(x, y, z) = &\ [a + f_1(yz^n, z) - nz^{Q-1}\gamma(xz^n, yz^n, z)]x \frac{\partial}{\partial x} \\ &+ [f_2(z) - nz^{Q-1}\gamma(xz^n, yz^n, z)]y \frac{\partial}{\partial y} \\ &+ f_3(yz^n, z)y^2 z^n \frac{\partial}{\partial y} \\ &+ z^Q\gamma(xz^n, yz^n, z) \frac{\partial}{\partial z} + \tilde{S}_\infty(x, y, z)\end{aligned}$$

for some $\tilde{S}_\infty$ which is ∞ flat along the $z = 0$ plane.

Because $\gamma(0, 0, 0) \neq 0$ we can assume that $\tilde{X}^n$, denoted again X, has the following expression (new $f_1, \gamma, \ldots$) provided that n is big enough:

$$\begin{aligned}X(x, y, z) = &\ (a + f_1(x, y, z))x \frac{\partial}{\partial x} + z^P f(x, y, z)y \frac{\partial}{\partial y} \\ &+ z^Q\gamma(x, y, z) \frac{\partial}{\partial z} + S_\infty(x, y, z)\end{aligned}$$

where

i) $f_1(0, 0, 0) = 0$
ii) $f(0, 0, 0) = b \neq 0$
iii) $P \in \{ 1, \ldots, Q - 1 \}$
iv) S_∞ is ∞ flat along the $z = 0$ plane and has zero $\frac{\partial}{\partial z}$ component.
v) $\gamma(0, 0, 0) = c \neq 0$.

Furthermore, in order to make the calculations a bit simpler, we divide X by $1 + \frac{1}{a} f_1$. The resulting vector field, again denoted X, is of the form (new $f, \ldots$):

$$\begin{aligned}X(x, y, z) = &\ ax \frac{\partial}{\partial x} + z^P f(x, y, z)y \frac{\partial}{\partial y} + z^Q\gamma(x, y, z) \frac{\partial}{\partial z} \\ &+ S_\infty(x, y, z)\end{aligned}$$

where f, γ, S_∞ satisfy the same properties (*ii*) to (*v*) as above. The foregoing manipulation is not essential.

If we look back to the expression for $\tilde{X}^n$ hereabove, we see that in case $P = Q - 1$ we may assume that $b.c < 0$ provided n is chosen big enough. So up to a change of the sign of X the four following situations can occur:

I. $P < Q - 1, a < 0, b < 0, c < 0$
II. $P \in \{ 1, \ldots, Q - 1 \}, a < 0, b < 0, c > 0$
III. $P < Q - 1, a < 0, b > 0, c > 0$
IV. $P \in \{ 1, \ldots, Q - 1 \}, a > 0, b < 0, c > 0.$

Here we already announce that I will lead to situation II.B of the main theorem, II to situation II.A, III to situation II.C and IV to II.C. We treat each case separately. We will refer to some lemmas which will be proved after this proposition.

Case I : $P < Q - 1$, $a < 0$, $b < 0$, $c < 0$.

In order to detect invariant graphs for X we consider the graph transformation defined by the time one mapping of X (precise definitions will follow). Roughly spoken this is: take a graph of a map $(x, y) = h(z)$ and transform it by the time one mapping of X.

First we modify X, without modifying its germ in $(0, 0, 0)$, with a C^∞ « bump » function $\tau : [0, \infty[\to [0, 1]$, where $\tau(u) = 1$ on $\left[0, \frac{1}{2}\right]$ and $\tau(u) = 0$ on $[1, \infty[$ as follows.

For all $\varepsilon > 0$ we define

$$X_\varepsilon(x, y, z) = X(x, y, z) - S_\infty(x, y, z) + \tau\left(\frac{z}{\varepsilon}\right)S_\infty(x, y, z).$$

Then X_ε has the same germ in 0 as X can be defined on a neighbourhood of the form $\overline{B}(0, \mu) \times [0, \infty[$, with $\overline{B}(0, \mu) = \{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq \mu \}$. Moreover the set $\{(0, 0, z) \mid z \geq \varepsilon\}$ is invariant under $-X_\varepsilon$. We can modify X to X_ε in such a way that the (new) f, γ satisfy satisfy inequalities like

$$\tilde{a}_1 \leq f(x, y, z) \leq \tilde{a}_2 < 0$$
$$\tilde{b}_1 \leq \gamma(x, y, z) \leq \tilde{b}_2 < 0$$

on some neighbourhood $\overline{B}(0, \mu) \times [0, \infty[$, where $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, $\tilde{b}_2$ can be chosen independent of ε (for this reason we don't attach an index ε to f or γ).

Let us show that the « flatness condition » of the term $\tau\left(\frac{z}{\varepsilon}\right)S_\infty(x, y, z)$ is independent of ε, that is: for all $r, s \in \mathbb{N}$ there exist $\delta > 0$ and $G_{r,s} > 0$ such that for all $\varepsilon > 0$ and for all $(x, y, z) \in \overline{B}(0, \mu) \times [0, \delta]$:

$$\left| \frac{\partial^s}{\partial z^s}\left(\tau\left(\frac{z}{\varepsilon}\right).S_\infty(x, y, z)\right)\right| \leq G_{r,s} z^r.$$

Let in fact $r, s \in \mathbb{N}$. We have:

$$\frac{\partial^s}{\partial z^s}\left(\tau\left(\frac{z}{\varepsilon}\right).S_\infty(x, y, z)\right) = \sum_{j=0}^{s} \binom{s}{j} \frac{\partial^j}{\partial z^j}\left(\tau\left(\frac{z}{\varepsilon}\right)\right) \frac{\partial^{s-j} S_\infty}{\partial z^{s-j}}(x, y, z)$$
$$= \sum_{j=0}^{s} \binom{s}{j} \frac{1}{\varepsilon^j} \tau^{(j)}\left(\frac{z}{\varepsilon}\right) \frac{\partial^{s-j} S_\infty}{\partial z^{s-j}}(x, y, z).$$

There exists a $\delta > 0$ such that for all $p \in \{ 0, \ldots, r+s \}$ and all $i \in \{ 0, \ldots, s \}$ there exists a constant $L_{i,p} > 0$ such that on $\overline{B}(0, \mu) \times [0, \delta]$:

$$\left| \frac{\partial^i S_\infty}{\partial z^i}(x, y, z) \right| \leq L_{i,p} z^p .$$

Furthermore for all $j \in \mathbb{N}$ there exists a $K_j > 0$ such that for all $u \in [0, \infty[$: $| \tau^{(j)}(u) | \leq K_j$. Hence for all $z \leq \varepsilon$:

$$\left| \frac{\partial^s}{\partial z^s}\left(\tau\left(\frac{z}{\varepsilon}\right) S_\infty(x, y, z) \right) \right| \leq \sum_{j=0}^{s} \binom{s}{j} \frac{1}{\varepsilon j} K_j L_{s-j,r+j} z^{r+j}$$

$$\leq \left(\sum_{j=0}^{s} \binom{s}{j} K_j L_{s-j,r+j} \right) z^r$$

and for all $z \geq \varepsilon$: $\tau\left(\dfrac{z}{\varepsilon}\right) = 0$ so the problem is trivial there.

The time one mapping of X_ε can be written down as:

$$F_\varepsilon(x, y, z) = (e^a x, (1 + z^P \alpha(x, y, z))y, z + z^Q g(x, y, z)) + R_{\infty,\varepsilon}(x, y, z)$$

where

i) $R_{\infty,\varepsilon}$ is ∞-flat along $\mathbb{R}^2 \times \{ 0 \}$ and the flatness condition is independent of ε

ii) $R_{\infty,\varepsilon}$ has zero z-component (we absorbed its z-component in g);

iii) there exist constants a_1, a_2, b_1, b_2 and a neighbourhood

$$V = \overline{B}(0, \mu) \times [0, \infty[,$$

independent of ε, such that on V:

$$a_1 \leq \alpha(x, y, z) \leq a_2 < 0$$
$$b_1 \leq g(x, y, z) \leq b_2 < 0 .$$

We write $F_\varepsilon = (F_{x,\varepsilon}, F_{y,\varepsilon}, F_{z,\varepsilon})$. We will show in lemma (2.3.5) that there exist $\delta_1 > 0$ and $\delta \in]0, \delta_1[$ such that if $h: [0, \delta] \to \mathbb{R}^2$ satisfies $h(0) = 0$ and for all $z \in [0, \delta]: \| h'(z) \| \leq 1$ then $Z: [0, \delta] \to \mathbb{R}: z \to F_{z,\varepsilon}(h(z), z)$ is a diffeomorphism onto (at least) $[0, \delta_1]$.

Let $z(Z)$ denote the inverse. We define, for $0 < \varepsilon < \delta_1$, the following function spaces:

$$F^m_{0,\varepsilon} = \{ h \mid h: [0, \delta] \to \mathbb{R}^2 \text{ is } C^m,\ h([0, \delta]) \subset \overline{B}(0, \mu),\ \| h'(z) \| \leq 1,\ \ h(0)=0 \text{ and for all } z \in [\varepsilon, \delta]: h(z) = 0 \}$$

$$B^m_\varepsilon = \{ h \mid h: [0, \delta] \to \mathbb{R}^2 \text{ is } C^m,\ h(0) = 0 \text{ and for all } z \in [\varepsilon, \delta]: h(z)=0 \}$$

$$F^m_\varepsilon = \{ h \mid h \in F^m_{0,\varepsilon},\ \ \| h^{(i)}(z) \| \leq z^{(P+1)(m-i)} \text{ for all } i \in \{ 0, \ldots, m \}\} .$$

Define $H = \Gamma_\varepsilon h$ as $H(Z) = (F_{x,\varepsilon}(h(z(Z)), z(Z)), F_{y,\varepsilon}(h(z(Z)), z(Z)))$ if $Z \in [0, \delta_1]$

and $H(Z) = 0$ if $Z \in]\delta_1, \delta]$. Γ_ε is a well defined map from $F^m_{0,\varepsilon}$ into B^m_ε provided ε is small and Γ_ε is continuous for the C^m topology.

From lemma (2.3.6) we will obtain that for all $m \geq 1$ there exists an $\varepsilon > 0$ such that $\Gamma_\varepsilon(F^m_\varepsilon) \subset F^m_\varepsilon$.

Let $\overline{F}^m_\varepsilon$ be the closure of F^m_ε in B^{m-1}_ε for the C^{m-1} topology. Each $D^i h$ for $h \in F^m_\varepsilon$ and $0 \leq i \leq m-1$ is a Lipschitz function with a Lipschitz constant independent of h; this is still true for $h \in \overline{F}^m_\varepsilon$.

The space $\overline{F}^m_\varepsilon$ is hence *compact* for the C^{m-1} topology, as a consequence of a theorem of Ascoli-Arzela [Di].

$\overline{F}^m_\varepsilon$ is also *convex*.

As Γ is continuous on $F^{m-1}_{0,\varepsilon} \supset \overline{F}^m_\varepsilon$ for the C^{m-1} topology, we still have $\Gamma(\overline{F}^m_\varepsilon) \subset \overline{F}^m_\varepsilon$.

Hence it follows from the Leray-Schauder-Tychonov fixed point theorem (I.1.22) that Γ has at least one fixed point in $\overline{F}^m_\varepsilon$.

We hence obtain for each $m \in \mathbb{N}$ an $\varepsilon_m > 0$ and a C^m map $h_m : [0, \delta] \to \mathbb{R}^2$ with $j_m h_m(0) = 0$ whose graph is invariant under F_{ε_m}.

Because $h_m(z) = 0$ on $[\varepsilon_m, \delta]$ and because $\lim_{N \to \infty} F^N_{\varepsilon_m}(0, 0, z) = (0, 0, 0)$ uniformly in z (see the estimates on g) we obtain that h_m is C^∞ outside 0.

In lemma (2.3.7) we will prove that if $h : [0, \delta] \to \overline{B}(0, \mu)$ is C^1 on $[0, \delta]$ and C^∞ on $]0, \delta]$ and if the graph of h is invariant under F_{ε_m}, then h is C^∞ on $[0, \delta]$ and $j_\infty h(0) = 0$.

Hence h_m will be C^∞. So we can consider for example F_{ε_1} and h_1. As a matter of fact we want h_1 to be invariant under X_{ε_1}. To see this, observe that for all $t \in [0, 1]$: $\phi_{X_{\varepsilon_1}}((0, \delta), t)$ lies on the z-axis ($\phi_{X_{\varepsilon_1}}$ denotes the flow of X_{ε_1}) since $\varepsilon_1 \leq \delta_1$; for all $t' > 0$ we write $t' = n + t$ where $t \in [0, 1]$ and $n \in \mathbb{N}$; we have

$$\phi_{X_{\varepsilon_1}}((0, \delta), t') = \phi_{X_{\varepsilon_1}}(\phi_{X_{\varepsilon_1}}((0, \delta), t)n)$$

hence the graph of h_1 is invariant under $\phi_{X_{\varepsilon_1}}(., t')$ for all $t' > 0$.

Finally in lemma (2.3.8) we show that in some neighbourhood of $(0, 0, 0)$ all orbits tend to $(0, 0, 0)$ and that those outside $\mathbb{R}^2 \times \{0\}$ (in the blown up situation) are graphs of maps h satisfying the hypothesis of lemma (2.3.7).

Hence we are in situation II.B of the main theorem (I.2.1).

Case II : $P \in \{1, \ldots, Q-1\}$, $a < 0$, $b < 0$, $c > 0$.

Here the graph transformation can be defined as follows. The time one mapping of X is again of the form

$$F(x, y, z) = (e^a x, (1 + z^P \alpha(x, y, z))y, z + z^Q g(x, y, z)) + R_\infty(x, y, z)$$

where R_∞ is ∞ flat along the $z = 0$ plane and has a zero z-component.

This time there exist constants a_1, a_2, b_1, b_2 and a neighbourhood V of (0, 0, 0) such that on V we have

$$a_1 \leq \alpha(x, y, z) \leq a_2 < 0$$
$$0 < b_1 \leq g(x, y, z) \leq b_2 .$$

Here we take the function spaces

$$B_\varepsilon^m = \{ h \mid h: [0, \varepsilon] \to \mathbb{R}^2 \text{ is } C^m \text{ and } h(0) = 0 \}$$
$$F_\varepsilon^m = \{ h \mid h: [0, \varepsilon] \to \mathbb{R}^2 \text{ is } C^m \text{ and } \| h^{(i)}(z) \| \leq z^{(P+1)(m-i)} \text{ for } 0 \leq i \leq m \} .$$

Denote $F = (F_x, F_y, F_z)$.

If $\varepsilon > 0$ is sufficiently small and if $h \in F_\varepsilon^m$ then the map

$$Z : [0, \varepsilon] \to \mathbb{R}$$
$$z \to F_z(h(z), z)$$

is a diffeomorphism onto (at least) $[0, \varepsilon]$. To prove this, remark first that for small z:

$$Z'(z) \geq 1 + Qz^{Q-1}g(h(z), z) + 0(z^Q)$$
$$\geq 1$$

because we can estimate the first order derivatives of g and h by constants independent of $h \in F_\varepsilon^m$. Second observe that for small ε:

$$Z(\varepsilon) \geq \varepsilon + \varepsilon^Q b_1$$
$$\geq \varepsilon .$$

Denote z the inverse of Z; put $H = \Gamma h$ where $H(Z) = (F_x(h(z), z), F_y(h(z), z))$.

Again:

i) $\Gamma : F_\varepsilon^m \to B_\varepsilon^m$ is well defined if ε is small and Γ is continuous for the C^m topology

ii) for all $m \geq 1$ there exists an $\varepsilon > 0$ such that $\Gamma(F_\varepsilon^m) \subset F_\varepsilon^m$; proof see lemma (2.3.9) hereafter

iii) Γ has a fixed point in $\overline{F_\varepsilon^m}$ (the closure of F_ε^m for the C^{m-1} topology).

In order to prove that h_Q is C^∞ we will show in lemma (2.3.10) that there exist $\varepsilon, \mu > 0$ such that if $(x_i, y_i, z_i)_{i \in \mathbb{N}}$ is a sequence in $\overline{B}(0, \mu) \times [0, \varepsilon]$ with $\lim_{i \to \infty} (x_i, y_i, z_i) = (0, 0, 0)$ and $F(x_i, y_i, z_i) = (x_{i-1}, y_{i-1}, z_{i-1})$ for $i \geq 1$ then this sequence must lie on the graph on h_Q.

This implies that for all $m \geq Q$ there exists an $\varepsilon'_m > 0$ with $\varepsilon'_m \leq \min \{ \varepsilon_m, \varepsilon_Q \}$ such that h_m and h_Q coincide on $[0, \varepsilon'_m]$. As a consequence of the movement of F in the z-direction (see the estimates on g above) we see that h_Q is C^m on $[0, \varepsilon_Q]$ and that $j_m h_Q(0) = 0$. Hence h_Q is C^∞ and $j_\infty h(0) = 0$.

Because of the unicity of h_Q, in the sense of lemma (2.3.10), we see as follows that the graph of h_Q is also invariant under X. Let $q \in \mathbb{N} \setminus \{ 0 \}$. Then, by similar arguments as above, we find an $\tilde{\varepsilon} \in]0, \varepsilon_Q[$ and a C^∞ map $\tilde{h}$:

$[0, \tilde{\varepsilon}] \to \mathbb{R}^2$ with $j_\infty \tilde{h}(0) = 0$ whose graph is invariant under $\phi_X\left(\frac{1}{q}, .\right)$ $\left(\text{the time } \frac{1}{q} \text{ mapping of } X\right)$. Take $z \in [0, \varepsilon]$. Then the sequence

$$\phi_X\left(-\frac{i}{q}, (h(z), z)\right)_{i \in \mathbb{N}}$$

must. by lemma (2.3.10), lie on the graph of h_Q.

So $\tilde{h}\,|_{[0,\tilde{\varepsilon}]} = h_Q\,|_{[0,\tilde{\varepsilon}]}$. Hence the graph of h_Q is invariant under $\phi_X\left(\frac{1}{q}, .\right)$, for each q. Thus under X.

Finally from lemma (2.3.11) it will follow that we are in situation II.A of the main theorem.

Case III : $P < Q - 1$, $a < 0$, $b > 0$, $c > 0$.

Let us first search for an invariant surface tangent to the yz-plane. This will be a so-called center manifold. If we would apply the « classical » center manifold theorem, we would obtain for each $r \in \mathbb{N}$ the existence of a C^r center manifold: see for example [Car, Gu, H.P.S., Ke, M.M.]. But we will show that in this particular case the center manifold is C^∞, at least in a blown up situation. Further we will show that the orbits starting in some neighbourhood of 0, but starting outside the invariant surface, will leave the neighbourhood. The behavior of X restricted to this invariant 2-dimensional manifold tangent to the yz-plane is well known and is as follows: there exists an invariant direction D′ having ∞ contact with the z-axis; all the orbits starting in the invariant surface tend to 0 in negative time and have ∞ contact with D′.

So we will be in situation II.C of the main theorem.

But let us now come to the proof of all this. First we prove the existence of the invariant C^∞ surface.

For reasons which will become clear later in the lemmas we first perform a rescaling of the z-axis as follows. We want in fact that $P \geqslant 2$.

Let $R : \mathbb{R}^2 \times]0, \infty[\to \mathbb{R}^2 \times]0, \infty[: (x, y, u) \to (x, y, u^2)$.

We have that $R_* X' = X$ if and only if for

$$X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y} + X_3 \frac{\partial}{\partial z} \quad \text{and } X' = X'_1 \frac{\partial}{\partial x} + X'_2 \frac{\partial}{\partial y} + X'_3 \frac{\partial}{\partial u} :$$

$$X'_1(x, y, u) = X_1(x, y, u^2)$$
$$X'_2(x, y, u) = X_2(x, y, u^2)$$
$$X'_3(x, y, u) = \frac{1}{2u} X_3(x, y, u^2)\,.$$

Here in particular:

$$X'_3(x, y, u) = \frac{1}{2} u^{2Q-1} \gamma(x, y, u^2).$$

Let us write again z instead of u, X instead of X′.

We obtain again a vector field of the form (new P, Q, f, . . .).

$$X(x, y, z) = ax \frac{\partial}{\partial x} + z^P f(x, y, z) y \frac{\partial}{\partial y} + z^Q \gamma(x, y, z) \frac{\partial}{\partial z} + S_\infty(x, y, z)$$

with this time $P \geq 2$ and still $P < Q - 1$.

Let us already remark here that such a rescaling will have no influence on our result, since it is our aim to construct an invariant graph of a C^∞ function $x = h(y, z)$ with for all y: $j_x h(y, 0) = 0$: it is easy to show that then $x = h(y, \sqrt{u})$ is also a C^∞ function of (x, u) ∞-flat along the y-axis. Also all the other results we shall obtain are preserved by this rescaling.

The time one mapping of X can be written as

$$F(x, y, z) = (e^a x, (1 + z^P \alpha(x, y, z)) y, z + z^Q g(x, y, z)) + R_\infty(x, y, z)$$

where $\alpha(0, 0, 0) > 0$, $g(0, 0, 0) > 0$, R_∞ is ∞ flat along the $z = 0$ plane and R_∞ has zero z-component.

There exist constants a_1, a_2, b_1, b_2 and a neighbourhood V of $(0, 0, 0)$ such that on V:

$$0 < a_1 \leq \alpha(x, y, z) \leq a_2$$
$$0 < b_1 \leq g(x, y, z) \leq b_2 .$$

We look for a C^∞ invariant graph of the form

$$\{ (h(y, z), y, z) \mid (y, z) \in [-\mu, \mu] \times [0, \varepsilon] \}$$

with in particular $h(y, 0) = 0$ for all $y \in [-\mu, \mu]$.

The $\mu > 0$ will be rechosen some times for our purposes. If μ, $\delta > 0$ are small then $V_{\delta,\mu} := [-\mu, \mu] \times [-\mu, \mu] \times [0, \delta] \subset V$ and $F(V_{\delta,\mu}) \subset V$.

Let us define some function spaces for $\varepsilon \leq \delta$:

$$F^1_{0,\varepsilon} = \{ h \mid h: [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^1,\ h([-\mu, \mu] \times [0, \varepsilon]) \subset [-\mu, \mu], \text{ for all } y \in [-\mu, \mu]: h(y, 0) = 0 \text{ and } \| Dh(y, z) \| \leq 1 \}$$

$$B^m_\varepsilon = \{ h \mid h: [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^m \text{ and for all } y \in [-\mu, \mu]: h(y, 0) = 0 \}$$

$$F^m_{0,\varepsilon} = F^1_{0,\varepsilon} \cap B^m_\varepsilon$$

$$F^m_\varepsilon = \{ h \mid h: [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^m \text{ and } \| D^i h(y, z) \| \leq z^{m-i} \text{ for } 0 \leq i \leq m \}.$$

Let us write $F = (F_x, F_y, F_z)$ and $F_1 = F_x$, $F_2 = (F_y, F_z)$. We would like that the surface determined by $(y, z) \to F(h(y, z), y, z)$ is again the graph of some function. For that purpose we shall prove in lemma (2.3.12) that for small μ, $\varepsilon > 0$ and $h \in F^1_{0,\varepsilon}$ the map (Y, Z): $[-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R}^2$: $(y, z) \to F_2(h(y, z), y, z)$ is a diffeomorphism onto (at least) $[-\mu, \mu] \times [0, \varepsilon]$. In that case we can take the inverse of (Y, Z), and denote it (y, z). So (y, z) is defined on (at least) $[-\mu, \mu] \times [0, \varepsilon]$.

We put $H = \Gamma h$ where

$$H(Y, Z) = F_1(h(y, z), y, z).$$

H is defined on (at least) $[-\mu, \mu] \times [0, \varepsilon]$. So, for ε small, Γ is a map defined on $\Gamma^m_{0,\varepsilon}$ and takes values in B^m_ε.

Moreover Γ is continuous for the C^m topology.

As $F^m_\varepsilon \subset F^m_{0,\varepsilon}$ for small ε, Γ is defined and continuous on F^m_ε. In lemma (2.3.13) we shall prove that there exists a (fixed) $\mu > 0$ such that for each $m \in \mathbb{N}$, $m \geq 1$, there exists an $\varepsilon > 0$ such that $\Gamma(F^m_\varepsilon) \subset F^m_\varepsilon$.

From this fact, from the theorem of Ascoli-Arzela and from the Leray-Schauder-Tychonoff fixed point theorem (I.1.22) we derive, in an analoguous way as in case I, that for all $m \in \mathbb{N}$ there exists an $\varepsilon_m > 0$ and a C^m map h_m: $[-\mu, \mu] \times [0, \varepsilon_m] \to \mathbb{R}$ with $j_m h_m(y, 0) = 0$ for all $y \in [-\mu, \mu]$ whose graph is invariant under F. In lemma (2.3.14) we shall show that there exists an $\varepsilon > 0$ such that if $(x_i, y_i, z_i)_{i \in \mathbb{N}}$ is a sequence in $[-\mu, \mu] \times [-\mu, \mu] \times [0, \varepsilon]$ with $\lim_{i \to \infty} (x_i, z_i) = (0, 0)$ and $F(x_i, y_i, z_i) = (x_{i-1}, y_{i-1}, z_{i-1})(i \geq 1)$ then this sequence must lie on the graph of h_Q. Again this uniqueness of the invariant graph and the movement in the z-direction (see the estimates on g) imply that h_Q is in fact C^∞ and that its graph is invariant under the vector field X. Also $j_\infty h_Q(y, 0) = 0$ for all $y \in [-\mu, \mu]$.

Next we consider the restriction of X to this invariant C^∞ 2-dimensional manifold obtained above (always restricted to the upper half space).

This 2-dimensional situation is treated in full detail in [D.R.R.], and is hence omitted.

Finally, from the expression of X we can now easily show that we are in situation II.C of the main theorem (I.2.1).

Case IV : $P \in \{1, \ldots, Q-1\}$, $a > 0$, $b < 0$, $c > 0$.

Here we also look first for an invariant surface, this time tangent to the xz-plane and passing through the x-axis.

In contrast with case III this surface is *not* a center manifold. But nevertheless we will prove that we are in situation II.C of the main theorem, that is: the orbits starting outside the invariant surface and starting in some small neighbourhood of 0 leave this neighbourhood, on the other hand the orbits in the invariant surface just mentioned tend to 0 in negative time and have ∞ contact with the z-axis.

One can observe in the sequel that in this case it is crucial that the xz-plane is formally invariant, thanks to the normal form.

If this were not the case, our method wouldn't work.

The time one mapping F of X has the same form as in case III with this time $\alpha(0,0,0) < 0$ and $g(0,0,0) > 0$.

Hence there exists constants a_1, a_2, b_1, b_2 and a neighbourhood V such that on V:

$$a_1 \leq \alpha(x, y, z) \leq a_2 < 0$$
$$0 < b_1 \leq g(x, y, z) \leq b_2 .$$

If $\mu, \delta > 0$ are small then $V_{\mu,\delta} := [-\mu, \mu] \times [-\mu, \mu] \times [0, \delta] \subset V$ and $F(V_{\delta,\mu}) \subset V$.

We define for $\varepsilon \leq \delta$:

$$F^1_{0,\varepsilon} = \{ h \mid h\colon [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^1, \ h([-\mu, \mu] \times [0, \varepsilon]) \subset [-\mu, \mu] \text{ and for all } x \in [-\mu, \mu]: h(x, 0) = 0 \text{ and } \| Dh(x, z) \| \leq 1 \}.$$

$$B^m_\varepsilon = \{ h \mid h\colon [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^m \text{ and for all } x \in [-\mu, \mu]: h(x,0) = 0 \}$$
$$F^m_{0,\varepsilon} = F^1_{0,\varepsilon} \cap B^m_\varepsilon$$
$$F^m_\varepsilon = \{ h \mid h\colon [-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R} \text{ is } C^m \text{ and } \| D^i h(x, z) \| \leq z^{(P+1)(m-i)} \text{ for } 0 \leq i \leq m \}.$$

Again:

i) for small ε we can define the graph transformation Γ on $F^1_{0,\varepsilon}$: see lemma (2.3.15) here after.

ii) $\Gamma : F^m_{0,\varepsilon} \to B^m_\varepsilon$ is continuous for the C^m topology;

iii) there exists a (fixed) $\mu > 0$ such that for each $m \in \mathbb{N}$, $m \geq 1$, there exists an $\varepsilon > 0$ such that $\Gamma(F^m_\varepsilon) \subset F^m_\varepsilon$: see lemma (2.3.16);

iv) applying the theorems of Ascoli-Arzela and of Leray-Schauder-Tychonoff yields for all $m \in \mathbb{N}$ the existence of an ε_m and a C^m map h_m: $[-\mu, \mu] \times [0, \varepsilon_m] \to \mathbb{R}$ with $j_m h_m(x, 0) = 0$ for all $x \in [-\mu, \mu]$ whose graph is invariant under F;

v) the uniqueness in the sense of lemma (2.3.17), and the movement of F in the z-direction (see the estimates on g) imply that h_Q is in fact C^∞ and invariant under the vector field X. Also $j_\infty h_Q(x, 0) = 0$, for all $x \in [-\mu, \mu]$;

vi) the results of [D.R.R.] concerning the behavior of X restricted to the invariant surface together with lemma (2.3.18) hereafter imply that we are in situation II.C of the main theorem. □

Lemmas used in the proof of proposition (2.3.1), case I.

(2.3.5) LEMMA. — There exist $\delta > 0$ and $\delta_1 \in]0, \delta[$ such that if h: $[0, \delta] \to \overline{B}(0, \mu)$ satisfies $h(0) = 0$ and $\| h'(z) \| \leq 1$ on $[0, \delta]$ then for all $\varepsilon > 0$ the map

$$Z\colon [0, \delta] \to \mathbb{R} : z \to F_{z,\varepsilon}(h(z), z)$$

is a diffeomorphism onto at least $[0, \delta_1]$

Proof. — Choose $\delta < (-b_1)^{\frac{-1}{Q-1}}$ then

$$\begin{aligned} F_{z,\varepsilon}(h(\delta), \delta) &= \delta + \delta^Q g(h(\delta), \delta) \\ &\geq \delta + \delta^Q b_1 > 0 \end{aligned}$$

Put $\delta_1 = \delta + \delta^Q b_1$. Further $Z'(z) \geq 1 + Qz^{Q-1}b_1 + 0(z^Q) \geq \frac{1}{2}$ for small δ (independent of h). □

(2.3.6) LEMMA. — For each $m \in \mathbb{N}$, $m \geq 1$, there exists an $\varepsilon \in]0, \delta_1[$ such that $\Gamma_\varepsilon(F_\varepsilon^m) \subset F_\varepsilon^m$.

Proof. — Let $h \in F_\varepsilon^m$ and put $H = \Gamma_\varepsilon h$. For $z = z(Z)$ and $(x, y) = h(z)$ we have

$$\begin{aligned} \| H(Z) \| &= \| (e^a x, (1 + z^P \alpha(x, y, z))y) \| + 0(z^\infty) \\ &\leq (1 + z^P a_2) \| (x, y) \| + 0(z^\infty) \quad \text{for small } z \\ &\leq (1 + z^P a_2) Z^{(P+1)m}(1 - (P+1)mb_1 z^{Q-1} + 0(z^Q)) \\ &\leq Z^{(P+1)m} \quad \text{for small } z \end{aligned}$$

since $a_2 < 0$ and $P < Q - 1$ and $a < 0$.

Suppose, by induction on i, that $\| H^{(j)}(Z) \| \leq Z^{(P+1)(m-j)}$ for all $j \in \{0, 1, \dots, i-1\}$. Let us abbreviate $\bar{h}(z) := (h(z), z)$ and $F_1 := (F_{x,\varepsilon}, F_{y,\varepsilon})$ and $F_2 := F_{z,\varepsilon}$.

We differentiate the equality $H \circ F_2 \circ \bar{h} = F_1 \circ \bar{h}$ i times and obtain for the left hand side, using the higher order chain rule [A.R., Ya]:

$$D^i(H \circ (F_2 \circ \bar{h}))(z) = \sum_{k=1}^{i} \sum_{(*)} D^k H(F_2(\bar{h}(z))) . (D^{j_1}(F_2 \circ \bar{h})(z), \dots, D^{j_k}(F_2 \circ \bar{h})(z))$$

where (*) is some « universal » summation over the $j_1, \dots, j_k$ with the properties $j_1 + \dots + j_k = i$ and $j_p \geq 1$ for all $p \in \{1, \dots, k\}$. Let us isolate the term with $k = i$ in the summation:

$$\begin{aligned} D^i(H \circ (F_2 \circ \bar{h}))(z) &= D^i H(F_2(\bar{h}(z))) . (D(F_2 \circ \bar{h})(z), \dots, D(F_2 \circ h)(z)) \\ &+ \sum_{k=1}^{i-1} \sum_{(*)} D^k H(F_2(\bar{h}(z))) . (D^{j_1}(F_2 \circ \bar{h})(z), \dots, D^{j_k}(F_2 \circ \bar{h})(z)) . \end{aligned}$$

There exist C^∞ functions A_k, $k \in \{0, 1, \dots, i-1\}$, such that we can write

$$\begin{aligned} D^i(H \circ (F_2 \circ \bar{h}))(z) &= D^i H(F_2(\bar{h}(z))) . (D(F_2 \circ \bar{h})(z), \dots, D(F_2 \circ \bar{h})(z)) \\ &+ \sum_{k=1}^{i-1} D^k H(F_2(\bar{h}(z))) . A_k(h(z), Dh(z), \dots, D^i h(z), z) . \end{aligned}$$

For the right hand side we get

$$\begin{aligned} D^i(F_1 \circ \bar{h})(z) &= \sum_{k=1}^{i} \sum_{(*)} D^k F_1(\bar{h}(z)) . (D^{j_1}\bar{h}(z), \ldots, D^{j_k}\bar{h}(z)) \\ &= DF_1(\bar{h}(z)) . D^i\bar{h}(z) \\ &+ \sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(z)) . (D^{j_1}\bar{h}(z), \ldots, D^{j_k}\bar{h}(z)) . \end{aligned}$$

Intermezzo:

SUBLEMMA. — If E is a normed space, if $L : E \to E$ is an invertible continuous linear map and if $B : E^i \to E$ is a continuous i-linear map then

$$\| B \| \leq \| B \circ (L, \ldots, L) \| . \| L^{-1} \|^i$$

where $B \circ (L, \ldots, L)$ is the i-linear map defined by

$$(B \circ (L, \ldots, L))(w_1, \ldots, w_i) = B(Lw_1, \ldots, Lw_i) .$$

Proof. — Obvious./.

If we apply this in the equality

$$\begin{aligned} & D^i H(F_2(\bar{h}(z))) . (D(F_2 \circ \bar{h})(z), \ldots, D(F_2 \circ \bar{h})(z)) \\ &\quad + \sum_{k=1}^{i-1} D^k H(F_2(\bar{h}(z))) . A_k(h(z), \ldots, z) = DF_1(\bar{h}(z)) . D^i\bar{h}(z) \\ &\quad + \sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(z)) . (D^{j_1}\bar{h}(z), \ldots, D^{j_k}\bar{h}(z)) \end{aligned}$$

to $L = D(F_2 \circ \bar{h})(z)$ and $B = D^i H(F_2(\bar{h}(z)))$ we get (replacing $F_2(\bar{h}(z))$ by Z):

$$\begin{aligned} \| D^i H(Z) \| \leq \Big[& \| DF_1(\bar{h}(z)) . D^i\bar{h}(z) \| \\ & + \Big\| \sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(z)) . (D^{j_1}\bar{h}(z), \ldots, D^{j_k}\bar{h}(z)) \Big\| \\ & + \Big\| \sum_{k=1}^{i-1} D^k H(F_2(\bar{h}(z))) . A_k(h(z), \ldots, D^i h(z), z) \Big\| \Big] \\ & \| (D(F_2 \circ \bar{h})(z))^{-1} \|^i . \end{aligned}$$

Let us make an estimate for each term or factor separately

a) $\underline{\| (D(F_2 \circ \bar{h})(z))^{-1} \|}$

As we already saw in lemma (2.3.5):

$$\| D(F_2 \circ \bar{h})(z) \|^{-i} \leq 1 - iQz^{Q-1}b_1 + 0(z^Q).$$

Remark that this 0 symbol, as well as the ones following, are independent of $h \in F_\varepsilon^m$.

b) $\underline{\left\| \sum_{k=1}^{i-1} D^k H(F_2(\bar{h}(z))) \,.\, A_k(h(z), \ldots, D^i h(z), z) \right\|}$

$= 0(Z^{(P+1)(m-i+1)})$ because of the induction hypothesis; this 0-symbol is independent of h since we can bound the $\| A_k(h(z), \ldots, D^i h(z), z) \|$ by a constant independent of $h \in F_\varepsilon^m$.

c) $\underline{\sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(z)) \,.\, (D^{j_1}\bar{h}(z), \ldots, D^{j_k}\bar{h}(z))}$

In each term of this summation we have $j_1 + \ldots + j_k = i$ and $k \geq 2$. So there must be a $p \in \{1, 2, \ldots, k\}$ with $j_p \leq i - 1$. Hence, except for the case that $j_1 = j_2 = \ldots = j_k = 1$ and $k = i$, each term contains a factor $D^k h(z)$ with $k \in \{0, 1, \ldots, i-1\}$; such a term is $0(z^{(P+1)(m-i+1)})$.

Suppose that $j_1 = j_2 = \ldots = j_k = 1$ and $k = i$. We can write

$$D^i F_1(\bar{h}(z)) \,.\, (D\bar{h}(z), \ldots, D\bar{h}(z))$$

as a sum of terms containing $Dh(z)$ as a factor, except the terms of the form

$$\frac{\partial^i F_1}{\partial z^i}(\bar{h}(z)) = \left(0, y \frac{\partial^i}{\partial z^i}(1 + z^P \alpha(x, y, z))\right) + 0(z^\infty)$$

which is $0(z^{(P+1)m})$ if we replace (x, y) by $h(z)$

d) $\underline{\| DF_1(\bar{h}(z)) \,.\, \bar{h}^{(i)}(z) \|}$

We have:

$$DF_1(x, y, z) = D(F_{x,\varepsilon}, F_{y,\varepsilon})(x, y, z) = \begin{bmatrix} e^a & 0 & 0 \\ 0 & 1 + z^P\alpha(x, y, z) & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ y\dfrac{\partial}{\partial x}(1 + z^P\alpha(x, y, z)) & y\dfrac{\partial}{\partial y}(1 + z^P\alpha(x, y, z)) & y\dfrac{\partial}{\partial z}(1 + z^P\alpha(x, y, z)) \end{bmatrix}$$

$$+ 0(z^\infty).$$

So if we write $h = (h_1, h_2)$ we get, since (x, y) must be replaced by $h(z)$:

$$\| DF_1(\bar{h}(z)) \,.\, \bar{h}^{(i)}(z) \| = \| (e^a h_1^{(i)}(z), (1 + z^P\alpha(x, y, z)) h_2^{(i)}(z))$$
$$+ 0(z^{(P+1)m}) \| \leq (1 + z^P a_2) \| h^{(i)}(z) \| + 0(z^{(P+1)m}) \quad \text{for small } z$$
$$\leq (1 + z^P a_2) z^{(P+1)(m-i)} + 0(z^{(P+1)m})$$

e) Finally

We end at:

$$\begin{aligned}\| \mathrm{H}^{(i)}(\mathrm{Z}) \| &\leq ((1 + z^{\mathrm{P}} a_2) z^{(\mathrm{P}+1)(m-i)} + 0(z^{(\mathrm{P}+1)(m-i+1)})(1 - i\mathrm{Q} z^{\mathrm{Q}-1} b_2 + 0(z^{\mathrm{Q}})) \\ &\leq (1 + z^{\mathrm{P}} a_2) z^{(\mathrm{P}+1)(m-i)} + 0(z^{(\mathrm{P}+1)(m-i+1)}) \\ &\leq (1 + z^{\mathrm{P}} a_2 + 0(z^{\mathrm{P}+1})) z^{(\mathrm{P}+1)(m-i)} \\ &\leq (1 + z^{\mathrm{P}} a_2 + 0(z^{\mathrm{P}+1})) \mathrm{Z}^{(\mathrm{P}+1)(m-i)} (1 - (\mathrm{P}+1)(m-i)^{\mathrm{Q}-1} b_2 + 0(z^{\mathrm{Q}})) \\ &\leq \mathrm{Z}^{(\mathrm{P}+1)(m-i)}\end{aligned}$$

because $\mathrm{P} < \mathrm{Q} - 1$ and $a_2 < 0$. □

(2.3.7) LEMMA. — If $h\colon [0, \delta] \to \overline{\mathrm{B}}(0, \mu)$ is invariant under $\mathrm{F}_{\varepsilon_m}$, if $h(0) = 0$, if $\sup_{z \in]0,\delta]} \| h'(z) \| < +\infty$ and if h is C^∞ on $]0, \delta]$ then h is C^∞ on $[0, \delta]$ and $j_\infty h(0) = 0$.

Proof. — For all $z \in]0, \delta]$ we consider the sequence $(\mathrm{F}^i_{\varepsilon_m}(h(z), z))_{i \in \mathbb{N}}$. The invariance of the graph of h implies that we can write

$$\mathrm{F}^i_{\varepsilon_m}(h(z), z) = (h(z_i(z)), z_i(z))$$

for all $i \in \mathbb{N}$, with in particular $z_0(z) = z$.

If $z_a \in]0, \delta]$ and $z_b = z_1(z_a)$ then each $z \in]0, z_b]$ is of the form $z_i(\bar{z})$ for some $\bar{z} \in]z_b, z_a]$.

In order to prove the lemma it hence suffices to show the following: for all $j, s \in \mathbb{N}$ there exist $z_a \in]0, \delta]$ and $\mathrm{K}_{j,s} > 0$ such that for all $z_0 \in [z_b, z_a]$ and all $i \in \mathbb{N}$:

$$\| h^{(j)}(z_i(z_0)) \| \leq \mathrm{K}_{j,s}(z_i(z_0))^s ;$$

we can abbreviate this by writing $h^{(j)}(z_i(z_0)) = 0((z_i(z_0))^s)$. Let us write shortly $z_i = z_i(z_0)$. Let $j, s \in \mathbb{N}$.

From the recurrence formula

$$h(z_i) = \mathrm{F}_{x,\varepsilon_m}(h(z_{i-1}), z_{i-1})$$

for all $i \geq 1$ and from the estimates on $\mathrm{F}_{\varepsilon_m}$ we derive the existence of $z_a > 0$ and $\mathrm{B}_1 > 0$ such that if $z_0 \leq z_a$ then for all $i \geq 1$:

$$\begin{aligned}\| h(z_i) \| &\leq \max \{ e^a, 1 + z^{\mathrm{P}} a_2 \} \| h(z_{i-1}) \| + \mathrm{B}_1 z_{i-1}^{\mathrm{Q}+s} \\ &\leq (1 + z^{\mathrm{P}} a_2) \| h(z_{i-1}) \| + \mathrm{B}_1 z_{i-1}^{\mathrm{Q}+s}\end{aligned}$$

for small z_a since $a < 0$.

Applying this successively we get

$$\| h(z_i) \| \leq \prod_{k=0}^{i-1} (1 + z_k^{\mathrm{P}} a_2) \| h(z_0) \| + \mathrm{B}_1 \sum_{j=0}^{i-1} \prod_{k=j+1}^{i-1} (1 + z_k^{\mathrm{P}} a_2) z_j^{\mathrm{Q}+s} .$$

On the other hand

$$z_i \geq z_j \prod_{k=j}^{i-1} (1 + z_k^{Q-1} b_1)$$

for $0 \leq j \leq i-1$. So

$$\frac{\| h(z_i) \|}{z_i^s} \leq \frac{\| h(z_0) \|}{z_0^s} \prod_{k=0}^{i-1} \frac{(1 + z_k^P a_2)}{(1 + z_k^{Q-1} b_1)^s}$$
$$+ \frac{1}{z_i^s} B_1 \sum_{j=0}^{i-1} \prod_{k=j}^{i-1} \frac{(1 + z_k^P a_2)}{(1 + z_k^{Q-1} b_1)^s} \frac{1}{(1 + z_j^P a_2)} z_j^Q z_i^s .$$

If

$$z_a \leq \left(\frac{a_2}{s b_1} \right)^{\frac{1}{P-Q-1}}$$

then for all $k \in \mathbb{N}$:

$$\frac{1 + z_k^P a_2}{(1 + z_k^{Q-1} b_1)^s} \leq 1$$

(remember: $P < Q - 1$).

Thus in that case

$$\frac{\| h(z_i) \|}{z_i^s} \leq \frac{\| h(z_0) \|}{z_0^s} + B_1 \sum_{j=0}^{i-1} \frac{1}{1 + z_j^P a_2} z_j^Q .$$

Choosing

$$z_a \leq \left(\frac{-1}{2a_2} \right)^{\frac{1}{P}}$$

we have for all $j \in \mathbb{N}$:

$$\frac{1}{1 + z_j^P a_2} \leq 2 .$$

We can find a constant L_s such that for all $z \in [z_b, z_a]$: $\dfrac{\| h(z) \|}{z^s} \leq L_s$. So for all $i \in \mathbb{N}$:

$$\frac{\| h(z_i) \|}{z_i^s} \leq L_s + 2B_1 \sum_{j=0}^{i-1} z_j^Q .$$

From $z_{j+1} \leq z_j(1 + b_2 z_j^{Q-1})$ we derive $-b_2 z_j^Q \leq z_j - z_{j+1}$ so

$$\sum_{j=0}^{i-1} (-b_2) z_j^Q \leq z_0 - z_i \leq z_a \qquad \text{for all } i \geq 1 .$$

We obtain:

$$\frac{\| h(z_i) \|}{z_i^s} \leq L_s + \frac{2B_1 z_a}{-b_2}$$

for all $i \in \mathbb{N}$. Thus the case $j = 0$ is done.

Now by induction on j we prove that $h^{(j)}(z_i) = 0(z_i^s)$ for all $s \in \mathbb{N}$. We differentiate the equality

$$h(z_i) = F_{x,\varepsilon_m}(\bar{h}(z_{i-1}))$$

j times (where $\bar{h}(z) = (h(z), z)$). Proceeding for that like in the proof of lemma (2.3.6) we find for the left hand side:

$$\begin{aligned} h^{(j)}(z_i) . [(F_z \circ \bar{h})'(z_{i-1})]^j & \\ + \sum_{k=1}^{j-1} \sum_{(*)} & D^k h(z_i) . (D^{j_1}(F_z \circ \bar{h})(z_{i-1}), \ldots, D^{j_k}(F_z \circ \bar{h})(z_{i-1})) \\ = & \, h^{(j)}(z_i) [(F_z \circ \bar{h})(z_{i-1})]^j \\ + & \sum_{k=1}^{j-1} h^{(k)}(z_i) . A_k(h(z_{i-1}), \ldots, h^{(j)}(z_{i-1}), z_{i-1}) \end{aligned}$$

where A_k is some C^∞ function; it is important to observe that A_k is linear with respect to the variable $h^{(j)}(z_{i-1})$. For the right hand side we get:

$$DF_{x,\varepsilon_m}(\bar{h}(z_{i-1})) . \bar{h}^{(j)}(z_{i-1}) + \sum_{k=2}^{j} \sum_{(*)} D^k F_{x,\varepsilon_m}(\bar{h}(z_{i-1})) . (\bar{h}^{(j_1)}(z_{i-1}), \ldots, \bar{h}^{(j_k)}(z_{i-1}))$$

where the terms in the last summation never contain $h^{(j)}(z_{i-1})$. Since $\sup_{z \in]0,\delta]} \| h'(z) \| < +\infty$ we see that

$$| (F_z \circ \bar{h})'(z_{i-1}) |^{-j} \leq 1 + 0(z_{i-1}^{Q-1}).$$

For the other terms we proceed exactly like in the proof of lemma (2.3.6) but collect the terms containing $h^{(j)}(z_{i-1})$, which *a priori* might be unbounded; those terms contain a factor which is, by the induction hypothesis, an $0(z_{i-1}^N)$ for any $N \in \mathbb{N}$.

Also

$$\| DF_{x,\varepsilon_m}(\bar{h}(z_{i-1})) . \bar{h}^{(j)}(z_{i-1}) \| \leq (1 + z_{i-1}^P a_2) \| h^{(j)}(z_{i-1}) \| + 0(\| h(z_{i-1}) \|)$$

So we find that for all $N \in \mathbb{N}$:

$$\| h^{(j)}(z_i) \| \leqslant (1 + z_{i-1}^P a_2 + 0(z_{i-1}^{P+1})) \| h^{(j)}(z_{i-1}) \| + 0(z_{i-1}^N).$$

Choose $N = Q + s$; then we can find constants $a_j < 0$ and $B_j > 0$ such that

$$\| h^{(j)}(z_i) \| \leq (1 + z_{i-1}^P a_j) \| h^{(j)}(z_{i-1}) \| + B_j z_{i-1}^{Q+s}.$$

This estimate is of identical type as the one we started from when solving the case $j = 0$. Now we may go on exactly in the same way in order to obtain the desired result. □

(2.3.8). LEMMA. — There exists a neighbourhood V of (0, 0, 0) such that all orbits in V tend to (0, 0, 0) and those outside $\mathbb{R}^2 \times \{0\}$ are graphs of maps h satisfying the hypothesis of lemma (2.3.7).

Proof. — By means of the obtained C^∞ invariant graph $\{(h(z), z) \mid z \in [0, \delta]\}$ we define the C^∞ coordinate change

$$G: \begin{cases} (\bar{x}, \bar{y}) = (x, y) - h(z) \\ \bar{z} = z \end{cases}$$

Put $\tilde{X} = G_* X$ and write $\tilde{X} = (\tilde{X}_x, \tilde{X}_y, \tilde{X}_z)$. Since $\tilde{X}$ leaves the z-axis invariant we observe that

$$(\tilde{X}_x, \tilde{X}_y)(\bar{x}, \bar{y}, \bar{z}) = 0(\| (\bar{x}, \bar{y}) \|)$$

From the expression of X we calculate that, for the standard inner product on $\mathbb{R}^2$:

$$\begin{aligned} \langle (X_x, X_y)(x, y, z), (x, y) \rangle &= ax^2 + z^P f(x, y, z) y^2 + 0(z^\infty) \\ &= ax^2 + z^P (f(0, 0, 0) + \text{higher order}) y^2 + 0(z^\infty) \,. \end{aligned}$$

Because G does not alter ∞ jets in (0, 0, 0) we can write for $\tilde{X}$:

$$\begin{aligned} \langle (\tilde{X}_x, \tilde{X}_y)(\bar{x}, \bar{y}, \bar{z}), (\bar{x}, \bar{y},) \rangle = a\bar{x}^2 + \bar{z}^P (f(0, 0, 0) + \text{higher order}) \bar{y}^2 \\ + 0(\bar{z}^\infty) 0(\| (\bar{x}, \bar{y}) \|) \end{aligned}$$

Let us write again (x, y, z) instead of $(\bar{x}, \bar{y}, \bar{z})$. Since $f(0, 0, 0) = b < 0$ and since $a < 0$, we can find a neighbourhood $\tilde{V}$ of (0, 0, 0) and a constant $c > 0$ such that on $\tilde{V}$:

$$\langle (\tilde{X}_x, \tilde{X}_y)(x, y, z), (x, y) \rangle \leq - c z^P \| (x, y) \|^2 \,.$$

From this the result is easily derived. □

Lemmas used in the proof of proposition (2.3.1), case II.

(2.3.9) LEMMA. — For each $m \in \mathbb{N}$, $m \geq 1$, there exists an $\varepsilon > 0$ such that $\Gamma(F_\varepsilon^m) \subset F_\varepsilon^m$.

Proof. — The differences with the proof of lemma (2.3.6) are:

$$\begin{aligned} \| H(Z) \| &\leq (1 + z^P a_2) z^{(P+1)m} + 0(z^\infty) \\ &\leq (1 + z^P a_2) Z^{(P+1)m} + 0(z^\infty) \\ &\leq Z^{(P+1)m} \quad \text{for small } z \,; \end{aligned}$$

for b):

$$| (F_z \circ \bar{h})'(z) |^{-i} \leq 1$$

for e):

$$\begin{aligned}\| H^{(i)}(Z) \| &\leq (1 + z^P a_2) z^{(P+1)(m-i)} + 0(z^{(P+1)(m-i+1)}) \\ &\leq (1 + z^P a_2 + 0(z^{P+1})) Z^{(P+1)(m-i)} \\ &\leq Z^{(P+1)(m-i)} \quad \text{for small } z. \qquad \square\end{aligned}$$

(2.3.10) LEMMA. — There exist $\varepsilon, \mu > 0$ such that if $(x_i, y_i, z_i)_{i \in \mathbb{N}}$ is a sequence in $\overline{B}(0, \mu) \times [0, \varepsilon]$ with $\lim_{i \to \infty} (x_i, y_i, z_i) = (0, 0, 0)$ and

$$F(x_i, y_i, z_i) = (x_{i-1}, y_{i-1}, z_{i-1}) \, (i \geq 1)$$

then this sequence must lie on the graph of h_Q.

Proof. — With the C^Q coordinate change

$$G: \begin{cases} (\bar{x}, \bar{y}) = (x, y) - h_Q(z) \\ \bar{z} = z \end{cases}$$

we obtain for $\tilde{F} = (\tilde{F}_x, \tilde{F}_y, \tilde{F}_z) := G_* F$ that

$$\| (\tilde{F}_x, \tilde{F}_y)(\bar{x}, \bar{y}, \bar{z}) \| \leq (1 + z^P \tilde{a}_2) \| (\bar{x}, \bar{y}) \|$$

for some $\tilde{a}_2 < 0$ on some neighbourhood of $(0, 0, 0)$.

Denote the transformed sequence $(\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)_{i \in \mathbb{N}}$.

We have $\| (\tilde{x}_i, \tilde{y}_i) \| \leq (\tilde{x}_{i+1}, \tilde{y}_{i+1}) \|$ if ε is small. So then for all i: $\| (\tilde{x}_i, \tilde{y}_i) \| \leq \| (\tilde{x}_{i+1}, \tilde{y}_{i+1}) \| \leq \ldots \leq \| (\tilde{x}_{i+n}, \tilde{y}_{i+n}) \| \to 0$ for $n \to \infty$. Thus $\tilde{x}_i = \tilde{y}_i = 0$, that is: the sequence lies on the $\tilde{z}$-axis, which is the transformed of the graph of h_Q by G. $\square$

(2.3.11) LEMMA. — There exists a neighbourhood V of $(0, 0, 0)$ on which all orbits of X starting outside the invariant graph leave V for $t \to -\infty$.

Proof. — Precisely like in lemma (2.3.8) we obtain for some $B > 0$ and on some neighbourhood $\tilde{V}$:

$$\langle (\tilde{X}_x, \tilde{X}_y)(x, y, z), (x, y) \rangle \leq -z^P \| (x, y) \|^2 .$$

From this we see that the function $V: \mathbb{R}^3 \to \mathbb{R}: (x, y, z) \to \| (x, y) \|^2$ is a Lyapunov function for $\tilde{X}$ in the region $z > 0$, whence the result. $\square$

Lemma used in the proof of proposition (2.3.1), case III.

(2.3.12) LEMMA. — For small $\mu, \varepsilon > 0$ and $h \in F^1_{0,\varepsilon}$ the map (Y, Z): $[-\mu, \mu] \times [0, \varepsilon] \to \mathbb{R}^2 : (y, z) \to F_2(h(y, z), y, z)$ is a diffeomorphism onto (at least) $[-\mu, \mu] \times [0, \varepsilon]$.

Proof. — We have

$F_2(h(y,z),y,z)=((1+z^P\alpha(h(y,z),y,z))y,\,z+z^Qg(h(y,z),y,z))+R_\infty(h(y,z),y,z)$

where R_∞ is ∞ flat along the z-plane and has zero z-component.
So, in a short notation:

$$D(Y,Z)(y,z)=\begin{bmatrix}1+z^P\alpha+z^P(\partial_1\alpha.\partial_1 h+\partial_2\alpha)y & y(Pz^{P-1}\alpha+z^P\partial_1\alpha.\partial_2 h+z^P\partial_3\alpha)\\ z^Q(\partial_1 g.\partial_1 h+\partial_2 g) & 1+Qz^{Q-1}g+z^Q(\partial_1 g.\partial_2 h+\partial_3 g)\end{bmatrix}$$
$$+\ [\partial_1 R_\infty.\partial_1 h+\partial_2 R_\infty \qquad \partial_1 R_\infty.\partial_2 h+\partial_3 R_\infty].$$

We have: $D(Y,Z)(0,0)=$ Identity. Because of the inverse function theorem and because for all $h\in F^1_{0,\varepsilon}$: $\|Dh(y,z)\|\le 1$ there exist $\mu,\varepsilon>0$ such that for all $h\in F^1_{0,\varepsilon}$ the map (Y,Z) is a diffeomorphism on $[-\mu,\mu]\times[0,\varepsilon]$.

Put $R_\infty=(R_{\infty,y},0)$. If ε is small enough we have for all $y\in[-\mu,\mu]$:

$$\varepsilon+\varepsilon^Q g(h(y,\varepsilon),y,\varepsilon)>\varepsilon$$

and for all $z\in[0,\varepsilon]$:

$$(1+z^P\alpha(h(\mu,z),\mu,z))\mu+R_{\infty,y}(h(\mu,z),\mu,z)\ge\mu$$
$$(1+z^P\alpha(h(-\mu,z),-\mu,z))(-\mu)+R_{\infty,y}(h(-\mu,z),-\mu,z)\le-\mu\,.$$

As $(Y,Z)([-\mu,\mu]\times[0,\varepsilon])$ is simply connected, this implies that

$$(Y,Z)([-\mu,\mu]\times[0,\varepsilon])\supset[-\mu,\mu]\times[0,\varepsilon]$$

(2.3.13) LEMMA. — There exists a $\mu>0$ such that for all $m\in\mathbb{N}$, $m\ge 1$ there exists an $\varepsilon>0$ such that $\Gamma(F^m_\varepsilon)\subset F^m_\varepsilon$.

Proof. — We choose μ such that lemma (2.3.12) holds.

Let $h\in F^m_\varepsilon$, $H=\Gamma h$. For $i=0$ and $(y,z)=(y,z)(Y,Z)$ we have:

$$\begin{aligned}\|H(Y,Z)\|&=\|e^a h(y,z)\|+0(z^\infty)\\&\le e^a z^m+0(z^\infty)\\&\le z^m\\&\le Z^m\end{aligned}$$

for small ε.

Suppose, by induction on i, that

$$\|D^jH(Y,Z)\|\le Z^{m-j}$$

for all $j\in\{0,1,\ldots,i-1\}$ and all $(Y,Z)\in[-\mu,\mu]\times[0,\varepsilon]$, $i\ge1$. For brevity put

$$\begin{aligned}\bar h\colon[-\mu,\mu]\times[0,\varepsilon]&\to\mathbb{R}^3\\(y,z)&\to(h(y,z),y,z)\end{aligned}$$

and put also $F_1=F_x$, $F_2=(F_y,F_z)$. If we differentiate the equality

$H \circ F_2 \circ \bar{h} = F_1 \circ \bar{h}$ i times then we obtain in the same way as in lemma (2.3.6) that

$$\begin{aligned} \| D^i H(Y, Z) \| \leq & \Big[DF_1(\bar{h}(y, z)) . D^i \bar{h}(y, z) \\ & + \sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(y, z)) . (D^{j_1} \bar{h}(y, z), \ldots, D^{j_k} \bar{h}(y, z)) \\ & + \sum_{k=1}^{i-1} D^k H(Y, Z) . A_k(h(y, z), \ldots, D^i h(y, z), y, z)) \Big] \\ & \times \| (D(F_2 \circ \bar{h})(y, z))^{-1} \|^i . \end{aligned}$$

We estimate as follows:

a) From the expression for $D(F_2 \circ \bar{h})(y, z) = D(Y, Z)(y, z)$ obtained lemma (2.3.12) we see that we can write

$$D(F_2 \circ \bar{h})(y, z) = \text{Identity} + z M_h(y, z)$$

where $M_h(y, z)$ is some matrix with $M_h(0, 0) = 0$; this because $Q - 1 > P \geq 2$: see the rescaling construction.

Moreover we can choose $\mu, \varepsilon > 0$ such that for all $m \in \mathbb{N}$, $m \geq 1$, all $h \in F_\varepsilon^m$ and all $(y, z) \in [-\mu, \mu] \times [0, \varepsilon]$:

$$\| M_h(y, z) \| \leq 1 .$$

Then

$$\| (D(F_2 \circ \bar{h})(y, z))^{-1} \| \leq 1 + 0(z)$$

b) $$\left\| \sum_{k=1}^{i-1} D^k H(Y, Z) . A_k(h(y, z), \ldots, D^i h(y, z), y, z) \right\| = 0(Z^{m-i+1})$$

because of the induction hypothesis and because of the uniform bounds on the $\| A_k(h(y, z), \ldots, y, z) \|$.

c) $$\left\| \sum_{k=2}^{i} \sum_{(*)} D^k F_1(\bar{h}(y, z)) . (D^{j_1} \bar{h}(y, z), \ldots, D^{j_k} \bar{h}(y, z)) \right\| = 0(z^{m-i+1})$$

because here the same reasoning as in lemma (2.3.6) applies and because here in particular

$$\frac{\partial^i F_1}{\partial y^q \partial z^{i-q}} (h(y, z), y, z) = 0(z^\infty)$$

for each $q \in \{0, \ldots, i\}$

d) From

$$DF_1(x, y, z) = [e^a \quad 0 \quad 0] + 0(z^\infty)$$

we obtain

$$\| DF_1(\bar{h}(y, z)).D^i\bar{h}(y, z) \| = \| e^a D^i h(y, z) + 0(z^\infty) \| \leq e^a z^{m-i} + 0(z^\infty).$$

e) Finally:

$$\| D^i H(Y, Z) \| \leq (e^a + 0(z)) z^{m-i} (1 + 0(z))^i ;$$

If ε is small we have for all $i \in \{ 0, \ldots, m \}$:

$$(e^a + 0(z))(1 + 0(z))^i \leq 1 .$$

Hence

$$\| D^i H(Y, Z) \| \leq z^{m-i} \leq Z^{m-i}. \quad \square$$

(2.3.14) LEMMA. — If ε is small and if (x_i, y_i, z_i) is a sequence in $[-\mu, \mu] \times [-\mu, \mu] \times [0, \varepsilon]$ with $\lim_{i\to\infty} (x_i, z_i) = (0, 0)$ and

$$F(x_i, y_i, z_i) = (x_{i-1}, y_{i-1}, z_{i-1}) \, (i \geq 1)$$

then this sequence must lie on the graph of h_Q.

Proof. — Similar to the proof of lemma (2.3.10) with this time the coordinate change

$$G: \begin{cases} \bar{x} = x - h_Q(y, z) \\ \bar{y} = y \\ \bar{z} = z. \quad \square \end{cases}$$

Lemmas used in the proof of proposition (*2.3.1*), *case IV*.

(2.3.15) LEMMA. — For sufficiently small $\varepsilon > 0$ and for $h \in F^1_{0,\varepsilon}$ the map

$$\begin{aligned} (X, Z) \colon [-\mu, \mu] \times [0, \varepsilon] &\to \mathbb{R}^2 \\ (x, z) &\to (F_x(x, h(x, z), z), F_z(x, h)((x, z), z)) \end{aligned}$$

is a diffeomorphism onto at least $[-\mu, \mu] \times [0, \varepsilon]$.

Proof. — We have $(X, Z)(x, z) = (e^a x, z + z^Q g(x, h(x, z), z)) + R_\infty(x, h(x, z), z)$ so, in a short notation:

$$D(X, Z)(x, z) = \begin{bmatrix} e^a & 0 \\ z^Q(\partial_1 g + \partial_2 g . \partial_1 h) & 1 + Qz^{Q-1} g + z^Q(\partial_2 g . \partial_2 h + \partial_3 g) \end{bmatrix} + \begin{bmatrix} \partial_1 R_\infty + \partial_2 R_\infty . \partial_1 h & \partial_2 R_\infty . \partial_2 h + \partial_3 R_\infty \\ 0 & 0 \end{bmatrix}.$$

As $\| Dh(x, z) \| \leq 1$ for $h \in F^1_{0,\varepsilon}$ remark that the last matrix is an $0(z^\infty)$ independent of h.

We have: $D(X, Z)(0,0) = \begin{bmatrix} e^a & 0 \\ 0 & 1 \end{bmatrix}$. Because of the foregoing remark and the inverse function theorem there exist small $\varepsilon, \mu > 0$ independent on $h \in F^1_{0,\varepsilon}$ such that $(X, Z)|_{[-\mu,\mu]\times[0,\varepsilon]}$ is a diffeomorphism.

For all $x \in [-\mu, \mu]$: $\varepsilon + \varepsilon^Q g(x, h(x, \varepsilon), \varepsilon) \geq \varepsilon$ if ε is small because of the estimates on g on V.

Also for all $z \in [0, \varepsilon]$ and ε small $e^a\mu + R_\infty(\mu, h(\mu, z), z) \geq \mu$ and $e^a(-\mu) + R_\infty(-\mu, h(-\mu, z), z) \leq -\mu$.

Hence, since $(X, Z)([-\mu, \mu] \times [0, \varepsilon])$ is simply connected,

$$(X, Z)([-\mu, \mu] \times [0, \varepsilon]) \supset [-\mu, \mu] \times [0, \varepsilon]. \qquad \square$$

(2.3.16) LEMMA. — There exists a $\mu > 0$ such that for each $m \in \mathbb{N}$, $m \geq 1$, there exists an $\varepsilon > 0$ such that $\Gamma(F^m_\varepsilon) \subset F^m_\varepsilon$.

Proof. — Copying the scheme of lemma (2.3.6) this time one has:

$$\begin{aligned} \| H(X, Z) \| &\leq (1 + a_2 z^P) z^{(P+1)m} + 0(z^\infty) \\ &\leq Z^{(P+1)m} \end{aligned}$$

We put $F = (F_x, F_y, F_z)$, $F_1 = F_y$, $F_2 = (F_x, F_z)$.

a) We can write:

$$D(F_2 \circ \bar{h})(x, z) = \begin{bmatrix} e^a & 0 \\ 0 & 1 + Qz^{Q-1}g \end{bmatrix} + 0(z^Q)$$

hence

$$\begin{aligned} \| (D(F_2 \circ \bar{h})(x, z))^{-1} \| &= \left\| \begin{bmatrix} e^a & 0 \\ 0 & 1 - Qz^{Q-1}g \end{bmatrix} + 0(z^Q) \right\| \\ &\leq 1 - Qz^{Q-1}b_1 + 0(z^Q) \end{aligned}$$

b) and *c*): the same

d) here:

$$\begin{aligned} DF_1(x, y, z) = {} & [0 \quad 1 + z^P\alpha \quad 0] \\ & + y\left[\frac{\partial}{\partial x}(1 + z^P\alpha) \quad \frac{\partial}{\partial y}(1 + z^P\alpha) \quad \frac{\partial}{\partial z}(1 + z^P\alpha)\right] \\ & + 0(z^\infty) \end{aligned}$$

so

$$\| DF_1(\bar{h}(x, z)) . D^i\bar{h}(x, z) \| \leq (1 + z^P a_2) z^{(P+1)(m-i)} + 0(z^{(P+1)m})$$

e)
$$\begin{aligned} \| D^i H(X, Z) \| &\leq ((1 + a_2 z^P) z^{(P+1)(m-i)} + 0(z^{(P+1)(m-i+1)})) \\ &\quad \times (1 - Qz^{Q-1}b_1)^i \\ &\leq Z^{(P+1)(m-i)} \end{aligned}$$

for small z since $a_2 < 0$ and $b_1 > 0$. $\square$

(2.3.17) LEMMA. — If ε is small and if $(x_i, y_i, z_i)_{i\in\mathbb{N}}$ is a sequence in $[-\mu, \mu] \times [0, \varepsilon]$ with $\lim_{i\to\infty} (y_i, z_i) = (0, 0)$ and $F(x_i, y_i, z_i) = (x_{i-1}, y_{i-1}, z_{-1})$ $(i \geq 1)$ then this sequence must lie on the graph of h_Q.

Proof. — Analogous to the proof of lemma (2.3.10). □

(2.3.18) LEMMA. — There exists a neighbourhood V of $(0, 0, 0)$ on which all orbits of X starting outside the invariant surface leave V for $t \to -\infty$.

Proof. — After the coordinate change

$$G: \begin{cases} \bar{x} = x \\ \bar{y} = y - h(x, z) \\ \bar{z} = z \end{cases}$$

we obtain a vector field $\tilde{X} := G_* X$ whose y and z components are of the form (write again x, y, z instead of $\bar{x}, \bar{y}, \bar{z}$):

$$\tilde{X}_y(x, y, z) = z^P \tilde{f}(x, y, z) y$$
$$\tilde{X}_z(x, y, z) = z^Q \tilde{\gamma}(x, y, z)$$

where $\tilde{f}(0, 0, 0) < 0$ and $\tilde{\gamma}(0, 0, 0) > 0$.
Take a bounded neighbourhood V of $(0, 0, 0)$ and constants $\tilde{a}_2, \tilde{b}_1, \tilde{b}_2 > 0$ such that on V:

$$\tilde{X}_y(x, y, z) \leq -\tilde{a}_2 z^P y$$
$$0 < \tilde{b}_1 \leq \tilde{\gamma}(x, y, z) \leq \tilde{b}_2 .$$

Now the lemma easily follows. □
This completes the proof of proposition (2.3.1).

(2.4) ALL THE EIGENVALUES ARE ZERO.

Here we will meet all situations II.A, II.B, II.C of the main theorem.
Since we work with germs X of flatness zero, that is: $j_1X(0) \neq 0$, and since the z-axis must be formally invariant under X, we can assume, up to a linear change of coordinates preserving $\mathbb{R}^2 \times \{0\}$ and $\{0\}^2 \times \mathbb{R}$, that the 1-jet is $y\dfrac{\partial}{\partial x}$; equivalently: the matrix A introduced in the first lines of this chapter IV is:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $c_0^1 = 0$.

We will try to reduce, in some sense, this case to one of the previous cases in propositions (1.1), (2.1.1), (2.1.3), (2.2.1) and (2.3.1). With « reducing » we mean: to deform the vector field by means of (sometimes degenerate) coordinate changes which do not alter the nature of results such as for example « having ∞ contact », « being a cone of finite contact », etc. In fact all the cited cases will occur.

(2.4.1) PROPOSITION. — Suppose $X \in G^3$ satisfies

i) the z-axis $\{0\}^2 \times \mathbb{R}$ is formally invariant under X

ii) $j_1 X(0) = y \dfrac{\partial}{\partial x}$

iii) X is non-flat along the z-axis

then

a) there exists a C^∞ germ $h:([0, \infty[, 0) \to \mathbb{R}^2$ whose graph (germ) $\{(h(z), z) \mid z \in [0, \infty[\}$ is invariant under X and with $j_\infty h(0) = 0$

b) there exists a cone K of finite contact around $\{0\}^2 \times [0, \infty[$ such that in K we have situation II.A, II.B or II.C of the main theorem.

Proof. — Applying lemma (II.3.2.4) we may assume that the z-component of X is of the form

$$X_z(x, y, z) = z^Q \gamma(x, y, z)$$

with $\gamma(0, 0, 0) \neq 0$. Since $j_1 X(0) = y \dfrac{\partial}{\partial x}$ we can write X in the following form:

$$\begin{aligned} X(x, y, z) = {} & [f_1(z)x + (1 + f_2(z))y]\frac{\partial}{\partial x} + [g_1(z)x + g_2(z)y]\frac{\partial}{\partial y} \\ & + h_1(x, y, z)\frac{\partial}{\partial x} + h_2(x, y, z)\frac{\partial}{\partial y} + z^Q\gamma(x, y, z)\frac{\partial}{\partial z} \\ & + S_\infty(x, y, z) \end{aligned}$$

where f_1, f_2, g_1, g_2, h_1, h_2, γ and S_∞ are C^∞ germs and

i) $f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0$

ii) $h_1(0, 0, z) = h_2(0, 0, z) = 0$

iii) $\dfrac{\partial h_1}{\partial x}(0, 0, z) = \dfrac{\partial h_1}{\partial y}(0, 0, z) = \dfrac{\partial h_2}{\partial x}(0, 0, z) = \dfrac{\partial h_2}{\partial y}(0, 0, z) = 0$

iv) $\gamma(0, 0, 0) \neq 0$

v) $Q \geq 2$

vi) S_∞ is a germ of a vector field with zero $\dfrac{\partial}{\partial z}$-component and which is ∞ flat along $\mathbb{R}^2 \times \{0\}$.

Let me explain this a little bit. First we collected all terms linear in x and y; property *ii*) together with *vi*) reflects that the z-axis is formally invariant; property *iii*) simply means that h_1 and h_2 don't contain terms linear in x and y; property *vi*) indicates that we « absorbed » the $0(z^\infty)$ terms of the $\frac{\partial}{\partial z}$ component of X in γ.

Allthough it is not crucial, we can spare some ink and calculations if we observe that we may assume that $f_2(z) \equiv 0$. Because if not, we replace X by

$$\frac{1}{1+f_2(z)}X;$$

this germ is C^∞ equivalent with X by means of the identity map. We want to put the matrix

$$M(z) = \begin{bmatrix} f_1(z) & 1 \\ g_1(z) & g_2(z) \end{bmatrix}$$

in a more handy form by means of a coordinate change.

Let $T = f_1 + g_2$ denote the trace of this matrix. This trace will play an important role in the sequel.

If we put

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(f_1 - g_2) & 1 \end{bmatrix}$$

then one easily checks that

$$LML^{-1} = \begin{bmatrix} \frac{1}{2}T & 1 \\ \frac{1}{4}(f_1-g_2)^2+g_1 & \frac{1}{2}T \end{bmatrix}$$

Note, by the way, that $\frac{1}{4}(f_1 - g_2)^2 + g_1$ is $\frac{1}{4}$ times the discriminant of the characteristic equation of M. All this suggests the following coordinate change:

$$\phi: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0): (x, y, z) \to \left(x, \frac{1}{2}(f_1(z) - g_2(z))x + y, z\right).$$

As

$$D\phi(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}(f_1(z)-g_2(z)) & 1 & \frac{1}{2}(f_1'(z)-g_2'(z))x \\ 0 & 0 & 1 \end{bmatrix}$$

we obtain from a straight forward calculation that the new vector field in a point $(x, y', z) = \phi(x, y, z)$ can be written in the following from:

$$\begin{aligned}\phi_*X(x, y', z) &= D\phi(x, y, z).X(x, y, z)\\ &= \left[\frac{1}{2}T(z)x+y'\right]\frac{\partial}{\partial x} + \left[\frac{1}{4}(f_1(z)-g_2(z))^2+g_1(z))x+\frac{1}{2}T(z)y'\right]\frac{\partial}{\partial y'}\\ &+\frac{1}{2}(f_1'(z)-g_2'(z))X_z(x, y'-\frac{1}{2}(f_1(z)-g_2(z))x, z)x\frac{\partial}{\partial y'}\\ &+\tilde{h}_1(x, y', z)\frac{\partial}{\partial x}+\tilde{h}_2(x, y', z)\frac{\partial}{\partial y'}+z^Q\tilde{\gamma}(x, y', z)\frac{\partial}{\partial z}\\ &+\tilde{S}_\infty(x, y', z)\end{aligned}$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{\gamma}$, $\tilde{S}_\infty$ satisfy analogous properties as *ii*), *iii*), *iv*) and *vi*) above. For brevity we write

$$g(z) = \frac{1}{4}(f_1(z)-g_2(z))^2+g_1(z)+\frac{1}{2}(f_1'(z)-g_2'(z))X_z(0, 0, z).$$

Then we can write down ϕ_*X in the form (new $\tilde{h}_2$):

$$\begin{aligned}\phi_*X(x, y', z) = &\left[\frac{1}{2}T(z)x + y'\right]\frac{\partial}{\partial x} + \left[g(z)x+\frac{1}{2}T(z)y'\right]\frac{\partial}{\partial y'}\\ &+\tilde{h}_1(x, y', z)\frac{\partial}{\partial x}+\tilde{h}_2(x, y', z)\frac{\partial}{\partial y'}\\ &+z^Q\tilde{\gamma}(x, y', z)\frac{\partial}{\partial z}+\tilde{S}_\infty(x, y', z).\end{aligned}$$

For reasons which will become clear in a moment we perform a rescaling $z = u^2$ of the z-axis, just like in the proof of proposition (2.3.1) case III, by means of the map

$$R: \mathbb{R}^2 \times]0, \infty[\to \mathbb{R}^2 \times]0, \infty[: (x, y', u) \to (x, y', u^2).$$

Remember that we always restrict our attention to the upper halfspace.

Calculating straightforward we find that $R_*X' = \phi_*X$ if and only if

$$\begin{aligned}X'(x, y', u) = &\left[\frac{1}{2}T(u^2)x + y'\right]\frac{\partial}{\partial x} + \left[g(u^2)x+\frac{1}{2}T(u^2)y'\right]\frac{\partial}{\partial y'}\\ &+\tilde{h}_1(x, y', u^2)\frac{\partial}{\partial x}+\tilde{h}_2(x, y', u^2)\frac{\partial}{\partial y'}\\ &+\frac{1}{2}u^{2Q-1}\tilde{\gamma}(x, y', u^2)\frac{\partial}{\partial u}+\tilde{S}_\infty(x, y', u^2).\end{aligned}$$

Let us simplify the notations by writing again X, y, z, h_1, h_2, Q, γ, S_∞ instead of respectively X', y', u, $\tilde{h}_1 \circ R$, $\tilde{h}_2 \circ R$, $2Q - 1$, $\frac{1}{2}\tilde{\gamma} \circ R$, $\tilde{S}_\infty \circ R$; let us also put $V(u) = T(u^2)$. Then we obtain something of the form:

$$\begin{aligned} X(x, y, z) = &\left[\frac{1}{2}V(z)x + y\right]\frac{\partial}{\partial x} + \left[g(z^2)x + \frac{1}{2}V(z)y\right]\frac{\partial}{\partial y} \\ &+ h_1(x, y, z)\frac{\partial}{\partial x} + h_2(x, y, z)\frac{\partial}{\partial y} \\ &+ z^Q\gamma(x, y, z)\frac{\partial}{\partial z} + S_\infty(x, y, z) \end{aligned}$$

where, for the sake of clearness, the following properties hold:

i) $V(0) = g(0) = 0$

ii) $h_1(0, 0, z) = h_2(0, 0, z) = 0$

iii) $\frac{\partial h_1}{\partial x}(0, 0, z) = \frac{\partial h_1}{\partial y}(0,0,z) = \frac{\partial h_2}{\partial x}(0, 0, z) = \frac{\partial h_2}{\partial y}(0, 0, z) = 0$

iv) $\gamma(0, 0, 0) \neq 0$

v) $Q \geq 2$

vi) S_∞ is a vector field with zero $\frac{\partial}{\partial z}$-component and which is ∞ flat along $\mathbb{R}^2 \times \{0\}$.

Let me explain why it is no restriction to assume that

$$j_{Q-1}V(0) \neq 0 .$$

We can perform a blowing up idea similar to the one in the proof of proposition (1.1) as follows. Blowing up X n times gives a vector field of the form

$$\begin{aligned} \tilde{X}^n(x, y, z) \doteq &\left[\left(\frac{1}{2}V(z)x - nz^{Q-1}\gamma(xz^n, yz^n, z)\right)x + y\right]\frac{\partial}{\partial x} \\ &+ \left[g(z^2)x + \left(\frac{1}{2}V(z) - nz^{Q-1}\gamma(xz^n, yz^n, z)\right)y\right]\frac{\partial}{\partial y} \\ &+ \tilde{h}_1(x, y, z)\frac{\partial}{\partial x} + \tilde{h}_2(x, y, z)\frac{\partial}{\partial y} + z^Q\gamma(xz^n, yz^n, z)\frac{\partial}{\partial z} \\ &+ \tilde{S}_\infty(x, y, z) \end{aligned}$$

for some $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{S}_\infty$ satisfying similar properties as *ii)*, *iii)* and *vi)* above. The fact that $\gamma(0, 0, 0) \neq 0$ should explain our assumption $j_{Q-1}V(0) \neq 0$, which we take for granted from now on. So there exist $P_1 \in \{1, \ldots, Q-1\}$ and $a_1 \neq 0$ such that

$$V(z) = a_1 z^{P_1} + O(z^{P_1+1}) .$$

The « strongest » term in the expression of X is of course $y\frac{\partial}{\partial x}$. We want to « weaken » it with respect to the other entries in the matrix

$$\begin{bmatrix} \frac{1}{2}V(z) & 1 \\ g(z^2) & \frac{1}{2}V(z) \end{bmatrix}$$

This situation is comparable with the problem: diminish the entries « 1 » emerging in the Jordan normal form of a square matrix. Inspiring ourselves on this, we consider for each $P \in \mathbb{N}$ the coordinate change

$$\alpha_P : \mathbb{R}^2 \times]0, \infty[\to \mathbb{R}^2 \times]0, \infty[: (x, y, z) \to \left(x, \frac{y}{z^P}, z\right).$$

P will be chosen in a moment, according to the occuring cases. In fact α_P is a sort of partial blowing up.

If $X = X_x \frac{\partial}{\partial x} + X_y \frac{\partial}{\partial y} + X_z \frac{\partial}{\partial z}$ then in a point $(x, y', z) = \alpha_P(x, y, z)$ we have

$$\begin{aligned} \alpha_{P*}X(x, y', z) &= D\alpha_P(x, y, z).X(x, y, z) \\ &= X_x(x, y'z^P, z)\frac{\partial}{\partial x} \\ &+ \left[\frac{1}{z^P}X_y(x, y'z^P, z) - P\frac{y'}{z}X_z(x, y'z^P, z)\right]\frac{\partial}{\partial y'} \\ &+ X_3(x, y'z^P, z)\frac{\partial}{\partial z}. \end{aligned}$$

For our vector field here this gives:

$$\begin{aligned} \alpha_{P*}X(x, y', z) = &\left[\frac{1}{2}V(z)x + y'z^P\right]\frac{\partial}{\partial x} \\ &+ \left[\frac{1}{z^P}g(z^2)x + \left(\frac{1}{2}V(z) - Pz^{Q-1}\gamma(x, y'z^P, z)\right)y'\right]\frac{\partial}{\partial y} \\ &+ h_1(x, y'z^P, z)\frac{\partial}{\partial x} + \frac{1}{z^P}h_2(x, y'z^P, z)\frac{\partial}{\partial y} \\ &+ z^Q\gamma(x, y'z^P, z)\frac{\partial}{\partial z} + \tilde{S}_\infty(x, y', z) \end{aligned}$$

for some $\tilde{S}_\infty$ satisfying property *vi*) above.

Intermezzo.

Let me indicate here why it was necessary to rescale the z-axis. Take the example:

$$\begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}.$$

Since the only allowed choices for P are here $P = 0$ and $P = 1$, we cannot weaken the entry « 1 » without creating a similar problem. However after the rescaling we have

$$\begin{bmatrix} 0 & 1 \\ z^2 & 0 \end{bmatrix}$$

and by taking $P = 1$ we are lead to the matrix

$$\begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix}.$$

End of the intermezzo.

We distinguish the cases $j_{P_1}g(0) = 0$ and $j_{P_1}g(0) \neq 0$.

Case $j_{P_1}g(0) = 0$.

Choose $P = P_1$. We distinguish $P_1 < Q - 1$ and $P_1 = Q - 1$.

Subcase $P_1 < Q - 1$.

Here we have (write again x, y, z instead of x, y', z)

$$\begin{aligned} \alpha_{P*}X(x, y, z) = & \left[\left(\frac{1}{2}a_1 z^P + O(z^{P+1})\right)x + z^P y\right]\frac{\partial}{\partial x} \\ & + \left[\frac{1}{z^P}\, g(z^2)x + \left(\frac{1}{2}a_1 z^P + O(z^{P+1})\right)y\right]\frac{\partial}{\partial y} \\ & + h_1(x, yz^P, z)\frac{\partial}{\partial x} + \frac{1}{z^P}h_2(x, yz^P, z)\frac{\partial}{\partial y} \\ & + z^Q\gamma(x, yz^P, z)\frac{\partial}{\partial z} + \tilde{S}_\infty(x, y, z). \end{aligned}$$

Since $j_P g(0) = 0$ we may write that

$$g(z^2) = O(z^{2P+1}).$$

We devide the vector field $\alpha_{P*}X$ by z^P. We can write:

$$\begin{aligned}\frac{1}{z^P}\alpha_{P*}X(x, y, z) &= \left[\left(\frac{1}{2}a_1 + 0(z)\right)x + y\right]\frac{\partial}{\partial x}\\ &+ \left[0(z) + \left(\frac{1}{2}a_1 + 0(z)\right)y\right]\frac{\partial}{\partial y}\\ &+ \frac{1}{z^P}h_1(x, yz^P, z)\frac{\partial}{\partial x} + \frac{1}{z^{2P}}h_2(x, yz^P, z)\frac{\partial}{\partial y}\\ &+ z^{Q-P}\gamma(x, yz^P, z)\frac{\partial}{\partial z} + \frac{1}{z^P}\tilde{S}_\infty(x, y, z).\end{aligned}$$

In order to get rid of the possible unboundedness of the terms $\frac{1}{z^P}h_1(x, yz^P, z)$ and $\frac{1}{z^{2P}}h_2(x, yz^P, z)$ it suffices to blow up the latter vector field 2P times (without dividing of course); indeed: thanks to the property *iii*) above we have

$$h_1(x, yz^P, z) = 0(\|x, y)\|^2)$$
$$h_2(x, yz^P, z) = 0(\|(x, y)\|^2);$$

if we blow up 2P times, the formulas in (II.3.2.3) imply that there appears a factor z^{2P}.

This blowing up construction does not alter the 1-jet in 0 of $\frac{1}{z^P}\alpha_{P*}X$. This 1-jet is:

$$\left(\frac{1}{2}a_1x + y\right)\frac{\partial}{\partial x} + \frac{1}{2}a_1 y\frac{\partial}{\partial y}.$$

Since $a_1 \neq 0$, we clearly have a vector field satisfying the assumptions of proposition (2.1.3). The conclusions of proposition (2.1.3) remain valid for our original vector field because of the following reasons:

a) $\alpha_P^{-1}(\{(h(z), z) \mid z \in]0, \infty[\}) \cup \{(0, 0, 0)\}$ is the germ of the graph of a C^∞ map ∞ tangent to the z-axis in 0, as well as

$$R(\{(h(z), z) \mid z \in]0, \infty[\}) \cup \{(0, 0, 0)\} \quad \text{since} \quad j_\infty h(0) = 0;$$

b) if we have a cone K of the form

$$K = \{(x, y, z) \in \mathbb{R}^2 \times [0, \infty[\mid x^2 + y^2 \leq (h(z))^2\}$$

then α_P^{-1} transforms it into

$$\left\{(x, y, z) \in \mathbb{R}^2 \times \left[0, \infty\right[\,\middle|\, \frac{x^2}{(h(z))^2} + \frac{y^2}{(h(z)z^P)^2} \leq 1\right\}.$$

This set contains the cone

$$K' = \{(x, y, z) \in \mathbb{R}^2 \times [0, \infty[\mid x^2 + y^2 \leq (h(z)z^P)^2\}$$

in which we have situation II.A or II.B of the main theorem according to the case;
finally the rescaling R transforms this last cone into

$$R(K') = \{ (x, y, z) \in \mathbb{R}^2 \times [0, \infty[\mid x^2 + y^2 \leq (h(\sqrt{z}))^2 z^P \},$$

which contains a C^∞ cone of finite contact.

Subcase $P_1 = Q - 1$.

With a same construction as in the subcase above we obtain a vector field $\frac{1}{z^P} \alpha_{P*}X(x, y, z)$ with this time a $\frac{\partial}{\partial z}$-component $z\gamma(x, yz^P, z)$. After blowing up 2P times (in order to get rid of unbounded terms) we obtain a vector field satisfying the assumptions of proposition (1.1).

We can conclude just like in the subcase above.

Case $j_{P_1} g(0) \neq 0$.

There exist $P_2 \in \{ 1, \ldots, P_1 \}$ and $a_2 \neq 0$ such that

$$g(z^2) = a_2 z^{2P_2} + 0(z^{2P_2+2}).$$

Choose $P = P_2$. We distinguish three subcases: $P_2 < P_1 \leq Q - 1$, $P_2 = P_1 < Q - 1$ or $P_2 = P_1 = Q - 1$.

Subcase $P_2 < P_1 \leq Q - 1$.

We have

$$\begin{aligned} \alpha_{P*}X(x, y, z) = {} & [0(z^{P+1})x + z^P y] \frac{\partial}{\partial x} \\ & + [(a_2 z^P + 0(z^{P+2}))x + 0(z^{P+1})y] \frac{\partial}{\partial y} \\ & + h_1(x, yz^P, z) \frac{\partial}{\partial x} + \frac{1}{z^P} h_2(x, yz^P, z) \frac{\partial}{\partial y} \\ & + z^Q \gamma(x, yz^P, z) \frac{\partial}{\partial z} + \tilde{S}_\infty(x, y, z) \end{aligned}$$

We devide this vector field by z^P and obtain:

$$\begin{aligned} \frac{1}{z^P} \alpha_{P*}X(x, y, z) = {} & [0(z)x + y] \frac{\partial}{\partial x} + [(a_2 + 0(z))x + 0(z)y] \frac{\partial}{\partial y} \\ & + \frac{1}{z^P} h_1(x, yz^P, z) \frac{\partial}{\partial x} + \frac{1}{z^{2P}} h_2(x, yz^P, z) \frac{\partial}{\partial y} \\ & + z^{Q-P} \gamma(x, yz^P, z) \frac{\partial}{\partial z} + \frac{1}{z^P} \tilde{S}_\infty(x, y, z). \end{aligned}$$

Again we can get rid of the unboundedness of some terms by blowing up

2P times, just like above. Denote the result of this blowing up construction by Y.

The 1-jet of Y is

$$y\frac{\partial}{\partial x} + a_2 x\frac{\partial}{\partial y}.$$

If $a_2 < 0$, then the eigenvalues are $i\sqrt{|a_2|}$; $-i\sqrt{|a_2|}$ and 0. Hence we can apply proposition (2.2.1) to obtain the situations II.A or II.B of the main theorem. The same remarks *a*) and *b*) in « case $j_{P_1}g(0) = 0$ » hold.

If $a_2 > 0$, then the eigenvalues are $\sqrt{a_2}$, $-\sqrt{a_2}$, 0. So up to a linear change of coordinates preserving $\mathbb{R}^2 \times \{0\}$ and $\{0\}^2 \times \mathbb{R}$, Y satisfies the assumptions of proposition (2.1.1). Blowing down this situation, we still have a C^∞ cone K of finite contact and a 2-dimensional C^0 subcone S like in situation II.C of the main theorem. Next $\alpha_P^{-1}(K)$ contains a C^∞ cone K′ of finite contact just like in the foregoing case; observe also that α_P^{-1} preserves the property « having ∞ contact with the z-axis »; $S' := \alpha_P^{-1}(S)\cap K'$ is a C^0 2-dimensional subcone of K′; the rescaling R also preserves the property « having ∞ contact with the z-axis in 0 »; R(K′) contains a C^∞ cone of finite contact and R(S′) is a C^0 2-dimensional subcone of R(K′). All this shows that, for our original vector field, we are in situation II.C of the main theorem.

Subcase $P_2 = P_1 < Q - 1$.

We have

$$\begin{aligned}\alpha_{P*}X(x, y, z) = {} & \left[\left(\frac{1}{2}a_1 z^P + 0(z^{P+1})\right)x + z^P y\right]\frac{\partial}{\partial x} \\ & + \left[(a_2 z^P + 0(z^{P+1}))x + \left(\frac{1}{2}a_1 z^P + 0(z^{P+1})\right)y\right]\frac{\partial}{\partial y} \\ & + h_1(x, yz^P, z)\frac{\partial}{\partial x} + \frac{1}{z^P}h_2(x, yz^P, z)\frac{\partial}{\partial y} \\ & + z^Q\gamma(x, yz^P, z)\frac{\partial}{\partial z} + \tilde{S}_\infty(x, y, z)\end{aligned}$$

Dividing this vector field by z^P we obtain:

$$\begin{aligned}\frac{1}{z^P}\alpha_{P*}X(x, y, z) = {} & \left[\left(\frac{1}{2}a_1 + 0(z)\right)x + y\right]\frac{\partial}{\partial x} \\ & + \left[(a_2 + 0(z))x + \left(\frac{1}{2}a_1 + 0(z)\right)y\right]\frac{\partial}{\partial y} \\ & + \frac{1}{z^P}h_1(x, yz^P, z)\frac{\partial}{\partial x} + \frac{1}{z^{2P}}h_2(x, yz^P, z)\frac{\partial}{\partial y} \\ & + z^{Q-P}\gamma(x, yz^P, z)\frac{\partial}{\partial z} + \frac{1}{z^P}\tilde{S}_\infty(x, y, z).\end{aligned}$$

Blow this up 2P times to get rid of the unbounded terms. The resulting vector field has a 1-jet

$$\left(\frac{1}{2}a_1 x + y\right)\frac{\partial}{\partial x} + \left(a_2 x + \frac{1}{2}a_1 y\right)\frac{\partial}{\partial y}.$$

Since the trace of the matrix

$$\begin{bmatrix} \frac{1}{2}a_1 & 1 \\ a_2 & \frac{1}{2}a_1 \end{bmatrix}$$

is a_1 ($\neq 0$) we have at least one nonzero eigenvalue. Hence we can apply, according to the case, propositions (2.1.1), (2.1.3), (2.2.1) or (2.3.1). Next we can conclude just like in the previous subcase.

Subcase $P_2 = P_1 = Q - 1$.

With a same construction as above we obtain a vector field $\frac{1}{z^P}\alpha_{P*}X$ with a $\frac{\partial}{\partial z}$-component $z\gamma(x, yz^P, z)$.

So we can apply proposition (1.1). □

§ 3. **Proof of the main theorem (I.2.1) with exception of the C° result.**

This is merely a summary of all the foregoing.

Take a maximal directed sequence of blowing ups of X along D (definition III.1.3). If this sequence is finite, then apply proposition (III.1.4) to obtain situation I. If this sequence is infinite, then, using proposition (III.1.5), D is formally invariant under X. Theorem (III.2.1) implies that, after a finite number of blowing ups, we are led to a vector field (germ) of flatness zero. Now the theorem follows from propositions (1.1), (2.1.1), (2.1.3), (2.2.1), (2.3.1) and (2.4.1), since these contain all the possible cases of flatness zero and since we assume that X is non-flat along D. □

V. PROOF OF THE C° RESULT IN THE MAIN THEOREM (I.2.1)

§ 1. **Definitions and notations.**

We want to provide « universal models » for the situations II.A, II.B and II.C obtained in the main theorem (I.2.1). For that purpose we introduce:

(1.1) DEFINITION. — The following germs of vector fields

$$S_A := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$$

$$S_B := -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}$$

$$S_C := -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}$$

are called the *standard models* for situation II.A resp. II.B resp. II.C of the main theorem (I.2.1).

(1.2) DEFINITION. — The germ of the set

$$K_S := \{ (x, y, z) \mid x^2 + y^2 \leq z^2, z \geq 0 \}$$

is called the *standard cone* around the z-axis.

§ 2. The C^0 result.

Let us state here in a more precise way the assertation announced in the main theorem (I.2.1).

(2.1) PROPOSITION. — Let $X \in G^3$, let D be a direction such that X leaves D formally invariant and such that X is non-flat along D.

Then there exists a C^∞ cone K of finite contact around D, a C^0 cone K′ containing K such that the germ $X \mid K'$ is C^0 equivalent with either $S_A \mid K_S$ or $S_B \mid K_S$ or $S_C \mid K_S$, according to the fact that X is in situation II.A resp. II.B resp. II.C of the main theorem (I.2.1).

Moreover in situation II.B we have C^0 conjugacy.

Proof. — We may assume that D is the z-axis (see part III for comments on this).

From part IV we know that, after a finite number of blowing ups, after possibly a rescaling of the z-axis, after possibly a partial blowing up and after putting in normal form, we are always lead to a vector field Y of one of the following types: (change the sign of X if necessary; with « flat terms » we mean terms ∞ flat along the $z = 0$ plane)

1) the eigenvalue of Y along the z-axis is < 0 and the restriction of Y to the $z = 0$ plane is a hyperbolic expansion;

2) $$Y = (-a + f_1(x, y, z))x \frac{\partial}{\partial x} + (b + f_2(x, y, z))y \frac{\partial}{\partial y} + z^Q \gamma(x, y, z) \frac{\partial}{\partial z} + \text{flat terms}$$

with $a > 0$, $b > 0$, $f_1(0, 0, 0) = f_2(0, 0, 0) = 0$, $\gamma(0, 0, 0) < 0$;

3) the restriction of Y to the $z = 0$ plane is a hyperbolic expansion and the z-component of Y is $z^Q \gamma(x, y, z)$ with $\gamma(0, 0, 0) < 0$, $Q \geq 2$;

4) the restriction of Y to the $z = 0$ plane is a hyperbolic contraction and the z-component of Y is $z^Q \gamma(x, y, z)$ with $\gamma(0, 0, 0) < 0$, $Q \geq 2$;

5) $$Y = (\lambda + f(x^2 + y^2, z))\left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right) + z^P g(x, y, z)\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) + z^Q \gamma(x, y, z) \frac{\partial}{\partial z} + \text{flat terms}$$

with $f(0, 0) = 0$, $g(0, 0, 0) > 0$, $\gamma(0, 0, 0) < 0$;

6) Y has the same form as in 5 but with this time $g(0, 0, 0) < 0$;

7) $Y = ax \frac{\partial}{\partial x} + z^P f(x, y, z)y \frac{\partial}{\partial y} + z^Q \gamma(x, y, z) \frac{\partial}{\partial z}$ + flat terms with $a > 0$, $f(0, 0, 0) > 0$, $\gamma(0, 0, 0) < 0$;

8) Y has the same form as in 7 with this time $a < 0$, $f(0, 0, 0) < 0$, $\gamma(0, 0, 0) < 0$;

9) Y has the same form as in 7 with this time $a < 0$, $f(0, 0, 0) > 0$, $\gamma(0, 0, 0) < 0$;

10) $Y = z^P f(x, y, z)x \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y} + z^Q \gamma(x, y, z) \frac{\partial}{\partial z}$ + flat terms with $f(0, 0, 0) < 0$, $a > 0$, $\gamma(0, 0, 0) < 0$ (*nota bene:* we have interchanged the role of x and y compared with the situation in proposition (IV.2.3.1)).

One has the following table:

Type	Situation in the main theorem
1, 3, 5, 7	II.A
4, 6, 8	II.B
2, 9, 10	II.C

Note that the operations (blowing up, rescaling, etc.) are homeomorphisms of $\mathbb{R}^2 \times\,]0, \infty[$ with the property that a sequence tending to the $z = 0$ plane is transformed into a sequence tending to the $z = 0$ plane.

Roughly spoken, the idea is to construct in the origin a local C^0 equivalence of $Y|_{\mathbb{R}^2\times]0,\infty[}$ with the corresponding standard model blown up once. Moreover we take care that the homeomorphism, realizing the C^0 equivalence, transforms sequences tending to the $z = 0$ plane into sequences tending to the $z = 0$ plane. When returning back the whole way this will assure the continuity in $(0, 0, 0)$ of the desired C^0 equivalence for our original vector field X. We assume that for each considered situation there has been blown up at least once.

Situation II.A.

We blow up once the standard model S_A and get

$$\tilde{S}_A = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$$

This blowing up transforms the standard cone $K_S\backslash\{0\}$ into the (full) cylinder $\{(x, y, z) \mid x^2 + y^2 \leq 1, z > 0\}$.

By flattening out just like in lemmas (IV.2.3.8), (IV.2.3.10) we can assume that the z-axis is invariant under Y.

We can find a small cylinder $V_{\mu,\delta} := \overline{B}(0, \mu) \times]0, \delta]$ around the z-axis such that $\{(x, y) \mid x^2 + y^2 = \mu^2\} \times]0, \delta]$ is transversal to the orbits of Y: for types 1 and 3 this is clear from the hyperbolicity (provided decent coordinates are chosen); for types 5 and 7 we consider the function $G(x, y, z) = x^2 + y^2$ and observe that for type 5:

$\langle \nabla G(x, y, z), Y(x, y, z)\rangle = 2(x^2 + y^2)z^P g(x, y, z) +$ flat terms and for type 7:
$\langle \nabla G(x, y, z), Y(x, y, z)\rangle = 2ax^2 + 2z^P f(x, y, z)y^2 +$ flat terms;

in both cases we have $\langle \nabla G(x, y, z), Y(x, y, z)\rangle > 0$ on $V_{\mu,\delta}$ provided μ and δ are small; hence the orbits of Y are transversal to the level surfaces of G in $V_{\mu,\delta}$.

Moreover, if δ is small, then the orbits in $V_{\mu,\delta}$ are also transversal to the planes $z = z_0$, $0 < z_0 \leq \delta$. Now it is easy to construct a C^0 equivalence between $Y|_{V_{\mu,\delta}}$ and $\tilde{S}_A|_{\{x^2+y^2\leq 1 \text{ and } 0<z\leq\delta\}}$; the homeomorphism h can be chosen such that

$$h(\{(x, y, z) \mid x^2 + y^2 = \mu^2 \quad\text{and}\quad 0 < z \leq \delta\}) = \{(x, y, z) \mid x^2 + y^2 = 1, 0 < z \leq \delta\}.$$

To see this we proceed as follows. Take the homeomorphism

$$h: \{(x, y, z) \mid x^2 + y^2 = \mu^2 \quad\text{and}\quad 0 < z \leq \delta\} \to \{(x, y, z) \mid x^2 + y^2 = 1 \quad\text{and}\quad 0 < z \leq \delta\}$$
$$(x, y, z) \to \left(\frac{1}{\mu}x, \frac{1}{\mu}y, z\right).$$

Extend h to $V_{\mu,\delta}$ as follows: if $(x_0, y_0, z_0) \in V_{\mu,\delta} \setminus z$-axis then its positive orbit for Y intersects $\{(x, y, z) \mid x^2 + y^2 = \mu^2, 0 < z \leq \delta\}$ a first time in a point, say, (x_1, y_1, z_1); define $h(x_0, y_0, z_0)$ to be the intersection of the negative orbit of $h(x_1, y_1, z_1)$ for $\tilde{S}_A$ with the plane $z = z_0$; finally define $h(0, 0, z_0) = (0, 0, z_0)$ for $0 < z_0 \leq \delta$.

Situation II.B.

We blow up the standard model S_B and get

$$\tilde{S}_B = -x(1-z)\frac{\partial}{\partial x} - y(1-z)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}.$$

This vector field is transversal to the halfsphere.

$$H_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4 \quad \text{and} \quad z > 0\}$$

On the other hand we can find a small halfsphere

$$H_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 = \delta^2 \quad \text{and} \quad z > 0\}$$

transversal to the orbits of Y: for type 4 this clear from the hyperbolicity and from $\gamma(0, 0, 0) < 0$ (provided decent coordinates are chosen); for types 6 and 8 we consider the function $G(x, y, z) = x^2 + y^2 + z^2$ and observe that for type 6:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle = 2z^P(x^2 + y^2)g(x, y, z) + 2z^{Q+1}\gamma(x, y, z)$$

+ flat terms and for type 8:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle = 2ax^2 + 2z^P f(x, y, z)y^2 + 2z^{Q+1}\gamma(x, y, z)$$

+ flat terms;

in both cases $\langle \nabla G(x, y, z), Y(x, y, z) \rangle < 0$ on a small set

$$V_\delta := \{(x, y, z) \mid x^2 + y^2 + z^2 \leq \delta^2 \quad \text{and} \quad z > 0\};$$

hence the orbits are transversal to the level surfaces of G in V_δ.

Consider the homeomorphism

$$h: H_1 \to H_2 : (x, y, z) \to \frac{2}{\delta}(x, y, z).$$

Extend h to V_δ as follows. Let ϕ_Y and $\phi_{\tilde{S}_B}$ denote the flows of Y resp. $\tilde{S}_B$. For each $(x, y, z) \in V_\delta$ there exists a $t \geq 0$ such that $\phi_Y(-t, (x, y, z)) \in H_1$. We force the conjugacy to be true on V_δ by defining

$$h(x, y, z) = \phi_{\tilde{S}_B}(t, h(\phi_Y(t, (x, y, z)))).$$

We check the required property for h. Let $(x_i, y_i, z_i)_{i \in \mathbb{N}}$ be a sequence in V_δ tending to the $z = 0$ plane (that is: $\lim_{i \to \infty} z_i = 0$). Let $(t_i)_{i \in \mathbb{N}}$ denote the sequence of « times » such that $\phi_Y(-t_i, (x_i, y_i, z_i)) \in H_1$. Suppose by contradiction that the z-components of a subsequence $(h(x_{i_k}, y_{i_k}, z_{i_k}))_{k \in \mathbb{N}}$ would stay

$\geq M > 0$ for some constant M. This implies that $\sup_{k \in \mathbb{N}} t_{i_k} < +\infty$. Hence the z-components of $\phi_Y(-t_{i_k}, (x_{i_k}, y_{i_k}, z_{i_k}))$ tend to zero. A contradiction. So we have a decent conjugacy h between $Y|_{V_\delta}$ and

$$\tilde{S}_B \,|_{\{(x,y,z) | x^2+y^2+z^2 \leq 4 \text{ and } z>0\}} \,.$$

The preimage $h^{-1}(\{(x, y, z) \,|\, x^2 + y^2 \leq 1 \text{ and } z > 0\})$ still contains a cylinder around the z-axis in V_δ. Returning this situation the whole way back to our original vector field X this gives us the desired cones and conjugacy.

Situation II.C.

We blow up the standard model S_C and get

$$\tilde{S}_C = -x(1-z)\frac{\partial}{\partial x} + y(1+z)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}.$$

Let $G : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \to -x^2 + y^2 - z^2$. We have:

$$\langle \nabla G(x, y, z), \tilde{S}_C(x, y, z) \rangle = 2x^2(1-z) + 2y^2(1+z) + 2z^3 \,.$$

Hence in the region $z < 1$ the level surfaces of g are transversal to the orbits of $\tilde{S}_C$.

Take $0 < \delta_2 < 1$. Take $\alpha_2 > 0$ so small that the intersections of the level surface $G^{-1}(-\delta_2)$ with the planes $y = \alpha_2$ and $y = -\alpha_2$ are circles C_2 resp. D_2 inside the region $z < 1$. See figure 2 from now on.

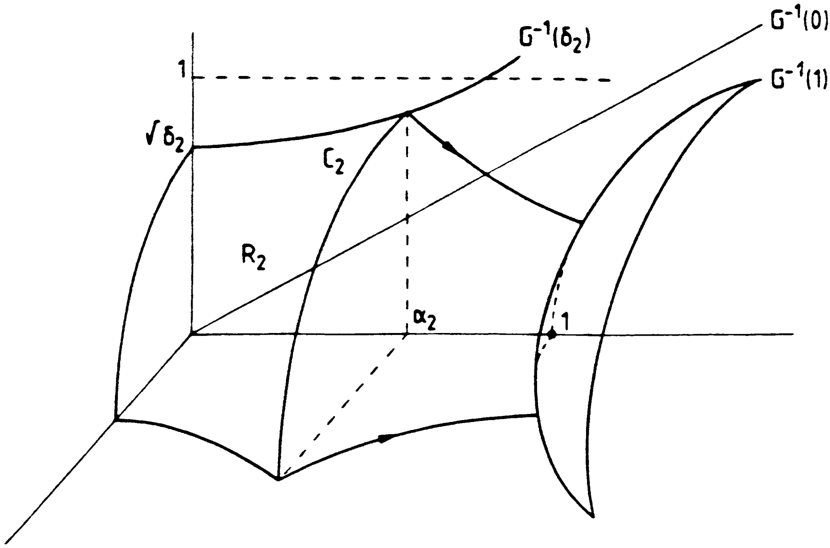

FIG. 2. — Sketch of the first octant.

We also consider the level surface $G^{-1}(1)$; this is a two-leaved hyperboloid. Next we consider the surface obtained by letting flow all the points of the circle C_2 until they hit the level surface $G^{-1}(1)$. We do the same thing for the circle D_2. In this way we enclosed in $\mathbb{R}^2 \times]0, \infty[$ a region R_2.

By flattening out we may assume that the $y = 0$ plane is invariant under Y.

We want to make an analogous construction as above for Y. We have for type 2:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle = 2(a - f_1(x, y, z))x^2 + 2(b + f_2(x, y, z))y^2 - 2z^{Q+1}\gamma(x, y, z) + \text{flat terms}$$

and for type 9:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle = -2ax^2 + 2z^P f(x, y, z)y^2 - 2z^{Q+1}\gamma(x, y, z) + \text{flat terms}$$

and for type 10:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle = -2x^2 z^P f(x, y, z) + 2ay^2 - 2z^{Q+1}\gamma(x, y, z) + \text{flat terms}.$$

So for all three types we can find a neighbourhood W of $(0, 0, 0)$ such that in $V := W \cap (\mathbb{R}^2 \times]0, \infty[)$:

$$\langle \nabla G(x, y, z), Y(x, y, z) \rangle > 0.$$

Hence on V the orbits of Y are transversal to the level surfaces of G. In V we want to make a miniature version of the construction for $\tilde{S}_C$.

For small $\delta_1 > 0$ we can make the following construction inside V. Consider the level surface $G^{-1}(-\delta_1)$. The intersections of it with the planes $y = \frac{\delta_1}{\delta_2}\alpha_2$ and $y = -\frac{\delta_1}{\delta_2}\alpha_2$ are circles C_1 resp. D_1. Consider also the surface obtained by letting flow all the points of the circle C_1 until they hit the level surface $G^{-1}\left(\frac{\delta_1}{\delta_2}\right)$. We do the same thing for the circle D_1.

In this way we enclosed a region R_1 in V.

Now we try to make an equivalence between $Y|_{R_1}$ and $\tilde{S}_C|_{R_2}$. First we define the homeomorphism

$$h: G^{-1}(-\delta_1) \cap \left\{ (x, y, z) \mid -\frac{\delta_1}{\delta_2}\alpha_2 \leq y \leq \frac{\delta_1}{\delta_2}\alpha_2 \right\}$$
$$\to G^{-1}(-\delta_2) \cap \{ (x, y, z) \mid -\alpha_2 \leq y \leq \alpha_2 \} :$$
$$(x, y, z) \to \frac{\delta_2}{\delta_1}(x, y, z).$$

Second we extend h to R_1 as follows. Require that a level surface $G^{-1}(a_1)$

$\left(a_1 \in \left[-\delta_1, \frac{\delta_1}{\delta_2}\right]\right)$ is mapped onto the level surface $G^{-1}\left(\frac{\delta_2}{\delta_1} a_1\right)$; for $(x, y, z) \in R_1$ lying in a level surface $G^{-1}(a_1)$ we consider the point (x^1, y^1, z^1) where the negative integral curve of Y through (x, y, z) hits for the first time the level surface $G^{-1}(-\delta_1)$; we define $h(x, y, z)$ to be the intersection of $G^{-1}\left(\frac{\delta_2}{\delta_1} a_1\right)$ with the positive integral curve of $\tilde{S}_C$ through $h(x^1, y^1, z^1)$.

We check the required property for h. Suppose that $(x_i, y_i, z_i)_{i \in \mathbb{N}}$ is a sequence in R_1 with $\lim_{i \to \infty} z_i = 0$.

Let (x_i^1, y_i^1, z_i^1) be the point where the negative integral curve of Y through (x_i, y_i, z_i) hits for the first time the level surface $G^{-1}(-\delta_1)$. Since the sequence (x_i^1, y_i^1, z_i^1) tends to the $y = 0$ plane, we get that $(h(x_i, y_i, z_i))_{i \in \mathbb{N}}$ tends to the $z = 0$ plane. □

(2.2) REMARK. — We have used some ideas comparable to those in [Cam].

VI. SOME EXAMPLES, COUNTEREXAMPLES AND SOME QUESTIONS

§ 1. A counterexample and some questions.

One might pose the question whether every germ in $0 \in \mathbb{R}^3$ of a C^∞ vector field satisfying a Łojasiewicz inequality (see definition I.1.17) possesses a C^∞ one-dimensional invariant manifold, or equivalently, whether it possesses an integral curve tending to 0 (in positive or negative time) in a C^∞ way (by *tending to* 0 *in a* C^r *way* we mean: if we add the origin to the integral curve, we obtain a C^r invariant manifold). The answer is no. We give an example of a germ for which no integral curve can tend to zero in a C^2 way; the germ has nonzero 1-jet.

(1.1) EXAMPLE. — Let $X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}$. Observe that X satisfies a Łojasiewicz inequality. No orbit of X can tend to 0 in a C^2 way.

Proof. — *a*) Blow up X in the x-direction:

$$\tilde{X}^x = xy\frac{\partial}{\partial x} + (z - y^2)\frac{\partial}{\partial y} + (x - zy)\frac{\partial}{\partial z}.$$

Observe that

$$\tilde{X}^x|_{x=0} = (z - y^2)\frac{\partial}{\partial y} - zy\frac{\partial}{\partial z}$$

has just one singularity in (0, 0). Blow up $\tilde{X}^x$ in the x-direction:

$$\tilde{\tilde{X}}^x = x^2y\frac{\partial}{\partial x} + (z - 2xy^2)\frac{\partial}{\partial y} + (1 - 2xyz)\frac{\partial}{\partial z}$$

Observe that

$$\tilde{\tilde{X}}^x|_{x=0} = z\frac{\partial}{\partial y} + 1.\frac{\partial}{\partial z}$$

has no singularities.

b) Blow up X in the y-direction:

$$\tilde{X}^y = (1 - xz)\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + (x^2y - z^2)\frac{\partial}{\partial z}.$$

Observe that

$$\tilde{X}^y|_{y=0} = (1 - xz)\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z}$$

has no singularities.

c) Blow up X in the z-direction:

$$\tilde{X}^z = (y - x^3z)\frac{\partial}{\partial x} + (1 - x^2yz)\frac{\partial}{\partial y} + x^2z^2\frac{\partial}{\partial z}.$$

Observe that

$$\tilde{X}^z|_{z=0} = y\frac{\partial}{\partial x} + 1\frac{\partial}{\partial y}$$

has no singularities.

Now we are able to describe the vector field $\tilde{X}$ on $S^2 \times \mathbb{R}$ obtained by blowing up X spherically (see for example II.§ 1 or [Ta, Du2] for definition and construction).

The only singularities of $\tilde{X}|_{S^2 \times \{0\}}$ are (1, 0, 0) and (− 1, 0, 0). If an orbit θ_1 of X tends to 0 in a C^2 way, then $\tilde{X}$ must have an orbit θ_2 which tends in a C^1 way to (1, 0, 0) or (−1, 0, 0) since these are the only singularities on $S^2 \times \{0\}$. If we blow up $\tilde{X}$ in (1, 0, 0) or (− 1, 0, 0) we don't have a singularities any more, except in the « corner »: see figure 3.

This contradicts the fact that θ_2 tends C^1 to (1, 0, 0) or (−1, 0, 0). □

(1.2) REMARK. — The vector field X in example (1.1) must have an orbit in the first octant $\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0\}$ tending to 0 in a C^0 way. One can see this as follows.

Denote $V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid (x = 0 \text{ or } y = 0 \text{ or } z = 0) \text{ and } x \geq 0 \text{ and } y \geq 0 \text{ and } z \geq 0 \text{ and } x + y + z \leq 1\}$ and $V_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } y \geq 0 \text{ and } z \geq 0 \text{ and } x + y + z = 1\}$ and $V_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } y \geq 0 \text{ and } z \geq 0\}$. V_3 is the first octant. See figure 4.

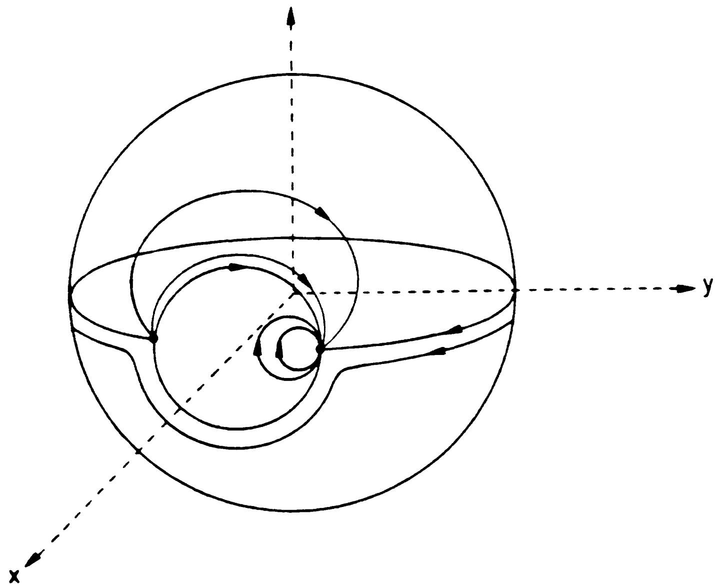

FIG. 3. — Blow up $\overline{X}$ in (1, 0, 0).

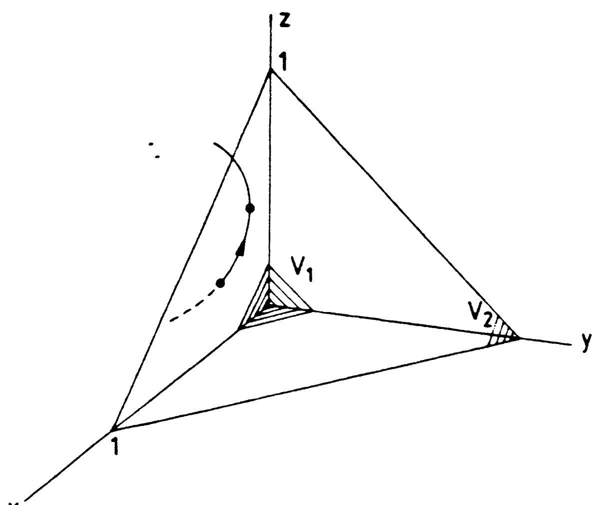

FIG. 4.

Each point of $V_1 \setminus \{0\}$ enters the first octant V_3 and never leaves it for $t \to +\infty$. This follows immediately from the expression of X. Consider the function $G(x, y, z) = x + y + z$. Since

$$\langle \nabla G(x, y, z), X(x, y, z) \rangle = y + z + x^2$$

we see that G is a Lyapunov function for X in the region $V_3 \setminus \{0\}$. Hence the positive orbit of a point of $V_1 \setminus \{0\}$ intersects V_2 once. Conversely, consider the negative orbit $\{\phi_X(t, v) \mid t \leq 0\}$ of a point $v \in V_2$. Put

$$T(v) = \inf\{t \leq 0 \mid \phi_X(t, v) \in V_3\}.$$

For each $t \geq T(v)$ there exists exactly one $\lambda_v(t) \in [0, 1]$ such that

$$\phi_X(t, v) \in G^{-1}(\lambda_v(t)); \quad \text{for} \quad T(v) = -\infty \text{ take } \lambda_v(-\infty) = 0.$$

The function $\lambda_v : [T(v), 0] \to [0, 1]$ is strictly increasing and continuous. If $T(v) = -\infty$, then necessarily $\lim_{t \to -\infty} \phi_X(t, v) = 0$. If $T(v) > -\infty$ then $\phi_X(t, v)$ leaves V_3 in negative time in some point, say,

$$h(v) \in (V_1 \setminus \{0\}) \cap G^{-1}(\lambda_v(T(v))).$$

In this way we have constructed a continuous map $h : V_2 \to V_1$ sending a point $v \in V_2$ to the point of V_1 where the negative orbit through v hits V_1 or to 0 if $\lim_{t \to -\infty} \phi_X(t, v) = 0$. Certainly $h(V_2) \supset V_1 \setminus \{0\}$ since the positive orbit of each point of $V_1 \setminus \{0\}$ hits V_2. Since V_2 is compact and since $V_1 \setminus \{0\}$ is not compact, necessarily $h(V_2) = V_1$. Hence there must be a point in V_2 tending to 0 for $t \to -\infty$.

(1.3) QUESTIONS. — I don't know whether this vector field X possesses an orbit tending in a C^1 way to 0.

I generalize this question: does there exist a germ in $0 \in \mathbb{R}^3$ of a C^∞ vector field satisfying a Łojasiewicz inequality without an orbit tending to 0 in a C^0 way? in a C^1 way? (in positive or negative time).

§ 2. Examples of vector fields having a C^∞ one dimensional invariant manifold.

A very general observation from the chapters III and IV is:

(2.1) CONSEQUENCE OF THE PROOF OF THE MAIN THEOREM. — If $X \in G^3$ leaves a direction D formally invariant (that is: invariant by the ∞ jet) and if X is non-flat along D then there exists a C^∞ one-dimensional invariant manifold ∞ tangent to D. □

If we consider germs in $0 \in \mathbb{R}^3$ of vector fields satisfying a Łojasiewicz inequality, it hence suffices to look for conditions which guarentee the existence of a formally invariant direction. In general it may be a difficult task to investigate whether a given vector field has a formally invariant direction. Some tools for this can be: apply the normal form theorem (which we will do in some examples hereafter); or: try to blow up the vector field until you find a singularity having a formally invariant direction by the normal form theorem. An important result in this sense is:

(2.2) THEOREM [B. D.]. — If $X \in G^3$ satisfies a Łojasiewicz inequality and if $A := DX(0)$ has eigenvalues $0, i\lambda, -i\lambda$ $(\lambda \in \mathbb{R} \setminus \{0\})$ then X has a C^∞

one-dimensional invariant manifold D tangent in 0 to the rotation axis of e^{tA}, $t \neq 0$. Moreover there exists a cone of finite contact around D in which we have situation II.A or II.B of the main theorem (I.2.1).

In the same spirit we can apply the normal form theorem to other types of singularities with nonzero 1-jet. For many cases the result is well known, but let us list them for the sake of completeness:

(2.3) MORE EXAMPLES. — Let $X \in G^3$, $X_1 = DX(0)$.

nr.	Jordan form of the matrix of X_1	Condition (sufficient) (*r is an integer*)	There exists a C^∞ one dimensional invariant manifold tangent to:
1	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$	$\nu \neq 0$ and $(\forall r \geq 2 : \mu r - \lambda \neq 0$ and $\nu r - \mu \neq 0)$	z-axis
2		$\lambda \neq 0$, $\mu \neq 0$, $\nu = 0$ and Łojasiewicz	
3		$\mu \neq 0$ and $(\forall r \geq 2 : \mu r - \lambda \neq 0$ and $\mu r - \nu \neq 0)$	y-axis
4		$\lambda \neq 0$, $\mu = 0$, $\nu \neq 0$ and Łojasiewicz	
5		$\lambda \neq 0$ and $(\forall r \geq 2 : \lambda r - \mu \neq 0$ and $\lambda r - \nu \neq 0)$	x-axis
6		$\lambda = 0$, $\mu \neq 0$, $\nu \neq 0$ and Łojasiewicz	
7	$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{bmatrix}$	$\nu \neq 0$ and $(\forall r \geq 2 : \nu r - \lambda \neq 0)$	z-axis
8		$\lambda \neq 0$, $\nu = 0$ and Łojasiewicz	
9		$\lambda \neq 0$ and $(\forall r \geq 2 : \lambda r - \nu \neq 0)$	x-axis
10	$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$	$\lambda \neq 0$	x-axis
11	$\begin{bmatrix} \mu & \beta & 0 \\ -\beta & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$	$\nu \neq 0$ and $(\forall r \geq 2 : \nu r - \mu \neq 0)$	z-axis
12		$\beta \neq 0$ and $\nu = 0$ and Łojasiewicz	

Proof. — Denote

$$\begin{aligned}
X_D &= \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y} + \nu z \frac{\partial}{\partial z} \\
X_R &= \beta y \frac{\partial}{\partial x} - \beta x \frac{\partial}{\partial y} \\
X_{S_1} &= y \frac{\partial}{\partial x} \\
X_{S_2} &= z \frac{\partial}{\partial y} \\
X_{S_3} &= y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}
\end{aligned}$$

where λ, μ, ν, β are real constants (zero is allowed). Put

$$\begin{aligned}
\tilde{D} &= [X_D, -]_h \\
\tilde{R} &= [X_R, -]_h \\
\tilde{S}_1 &= [X_{S_1}, -]_h \\
\tilde{S}_2 &= [X_{S_2}, -]_h \\
\tilde{S}_3 &= [X_{S_3}, -]_h
\end{aligned}$$

where $h \geq 2$ (see formulation of normal form theorem (I.1.21)). Let us also denote, for $h \geq 2$, $0 \leq j \leq h$, $0 \leq i \leq j$:

$$\begin{aligned}
e_{ij1} &= x^i y^{j-i} z^{h-j} \frac{\partial}{\partial x} \\
e_{ij2} &= x^i y^{j-i} z^{h-j} \frac{\partial}{\partial y} \\
e_{ij3} &= x^i y^{j-i} z^{h-j} \frac{\partial}{\partial z}.
\end{aligned}$$

This is a basis for H^h.

After a trivial calculation one finds that

$$\tilde{D}(e_{ijk}) = c_{ijk} e_{ijk}$$

where

$$\begin{aligned}
c_{ij1} &= \lambda(i-1) + \mu(j-i) + \nu(h-j) \\
c_{ij2} &= \lambda i + \mu(j-i-1) + \nu(h-j) \\
c_{ij3} &= \lambda i + \mu(j-i) + \nu(h-j-1).
\end{aligned}$$

So $\tilde{D}$ has a diagonal matrix with respect to this basis.

On H^h we put the standard inner product with respect to this basis.

For example 1 we see that $c_{001} \neq 0$, $c_{002} \neq 0$, so $e_{001}, e_{002} \in \operatorname{Im} \tilde{D}$. Take as complementary space $(\operatorname{Im} D)^{\perp}$; an element $\Sigma a_{ijk} e_{ijk}$ of $(\operatorname{Im} \tilde{D})^{\perp}$ cannot contain the terms $e_{001} = z^h \dfrac{\partial}{\partial x}$ nor $e_{002} = z^h \dfrac{\partial}{\partial y}$ since

$$0 = \langle \Sigma a_{ijk} e_{ijk}, e_{001} \rangle = a_{001} \quad \text{and} \quad 0 = \langle \Sigma a_{ijk} e_{ijk}, e_{002} \rangle = a_{002} .$$

Hence in the normal form the z-axis is formally invariant.

For example 2 the same reasoning applies; for example 5 we see that $e_{hh2}, e_{hh3} \in \operatorname{Im} \tilde{D}$ and we can follow an analoguous reasoning. A similar method works for examples 3, 4, 6.

Next we calculate that

$$\begin{aligned} \tilde{S}_1(e_{ij1}) &= i e_{i-1,j+2,1} \\ \tilde{S}_1(e_{ij2}) &= -\, e_{ij1} + i e_{i-1,j+2,2} \\ \tilde{S}_1(e_{ij3}) &= i e_{i-1,j+2,3} \end{aligned}$$

so $\tilde{S}_1$ has a matrix of the form

$$\begin{bmatrix} A & -\mathrm{Id} & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$$

where A is an upper triangle matrix with zero diagonal elements (so A is nilpotent). Then one easily calculates that $\tilde{S}_1$ must be nilpotent.

So for examples 7, 8, 9 we can write

$$[X_1, -]_h = \tilde{D} + \tilde{S}_1$$

where $\tilde{D}$ is semi-simple and $\tilde{S}_1$ is nilpotent.

The fact that $\operatorname{Im} \tilde{D} \subset \operatorname{Im} [X_1 -]_h$ implies, in the same way as above, the results claimed in examples 7, 8 and 9.

Concerning example 10 we find that the matrix of $\tilde{S}_2$ is of the form

$$\begin{bmatrix} B & 0 & 0 \\ 0 & B & -\mathrm{Id} \\ 0 & 0 & B \end{bmatrix}$$

where B is an upper triangle matrix with zero diagonal elements. So $\tilde{S}_3$ has a matrix

$$\begin{bmatrix} A+B & -\mathrm{Id} & 0 \\ 0 & A+B & -\mathrm{Id} \\ 0 & 0 & A+B \end{bmatrix}$$

and one easily calculates that $\tilde{S}_3$ must be nilpotent.

As $[X_1, -]_h = \tilde{D} + \tilde{S}_3$ again $\operatorname{Im} \tilde{D} \subset \operatorname{Im} [X_1, -]_h$ and the same reasoning as above can be made.

For examples 11 and 12 we find that

$$(\tilde{D} + \tilde{R})\left(\frac{1}{-\mu+vh} e_{001} + \beta e_{002}\right) = e_{001}$$

$$(\tilde{D} + \tilde{R})\left(-\beta e_{001} + \frac{1}{-\mu+vh} e_{002}\right) = e_{002}$$

so $e_{001}, e_{002} \in \operatorname{Im} [X_1, -]_h$ and we can reason like before. □

(2.4) REMARK. — In case $j_1X(0) = 0$ there is, as far as I know, very little known on normal forms. But as already said, blowing up may sometimes help. This is the case for a C^∞ gradient vector field $X = \operatorname{grad} f$: F. Takens showed me, using a blowing up argument, that if $j_\infty f(0) \neq 0$ then X has a C^∞ invariant manifold, in all (finite) dimensions.

In example (1.1) 0 was not an isolated zero of the first nonvanishing jet. Even the assumption that 0 is an isolated zero of the first nonvanishing jet (which implies that the radial eigenvalue in a singularity of $\overline{X}|_{S^2\times\{0\}}$ is nonzero) is not enough to have an invariant direction:

(2.5) EXAMPLE. — Let

$$X = (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + (2xy + x^3)\frac{\partial}{\partial} + 2xz\frac{\partial}{\partial z}.$$

Observe that 0 is an isolated zero of $j_2X(0)$. No orbit of X can tend to 0 in a C^2 way

Proof. — *a*) Blow up X in the x-direction:

$$\overline{X}^x = \frac{1}{x}\tilde{X}^x = (x - y^2x - z^2x)\frac{\partial}{\partial x} + (y + x + y^3 + yz^2)\frac{\partial}{\partial y} + (z + zy^2 + z^3)\frac{\partial}{\partial z}.$$

Observe that

$$\overline{X}^x|_{x=0} = (y + y^3 + yz^2)\frac{\partial}{\partial y} + (z + zy^2 + z^3)\frac{\partial}{\partial z}$$

has just one singularity in (0, 0).

Blow up $\overline{X}^x$ in the x-direction:

$$\overline{\overline{X}}^{xx}(x - y^2x^3 - z^2x^3)\frac{\partial}{\partial x} + (1 + 2y^3x^2 + 2x^2yz^2)\frac{\partial}{\partial y} + (2x^2y^2z + 2x^2z^3)\frac{\partial}{\partial z}$$

and observe that

$$\overline{\overline{X}}^{xx}|_{x=0} = 1\frac{\partial}{\partial y}$$

has no singularities.

b) Blow up X in the y-direction:

$$\overline{X}^y = (-x^2-1-z^2-x^4y)\frac{\partial}{\partial x} + (2xy+x^3y^2)\frac{\partial}{\partial y} - x^3yz\frac{\partial}{\partial z}$$

and observe that

$$\overline{X}^y\,|_{y=0} = (-\,x^2 - 1 - z^2)\frac{\partial}{\partial x}$$

has no singularities.

c) Blow up X in the z-direction:

$$\overline{X}^z = (-\,x^2 - y^2 - 1)\frac{\partial}{\partial x} + x^3z\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z}$$

and observe that

$$\overline{X}^z\,|_{z=0} = (-\,x^2 - y^2 - 1)\frac{\partial}{\partial x}$$

has no singularities.

Now we can conclude just like in example (1.1). □

(2.6) REMARK. — This example has orbits tending to 0 in a C^1 way. This can be seen from the expression of $\overline{X}^x$: $j_1\overline{X}(0)$ is a hyperbolic expansion.

(2.7) SOME FINAL REMARKS. — *a*) Concerning the main theorem (I.2.1): when X is analytic, it is not necessary that (one of) the obtained invariant manifold(s) is also analytic, as was pointed out to me by the referee: for $X = x\frac{\partial}{\partial x} + (y - z^2)\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$ all the invariant directions tangent to the z-axis have an ∞ jet in 0 of the form

$$x = 0 \qquad \text{and} \qquad y = \sum_{n>0} n!\, z^{n+1}$$

b) Concerning proposition (V.2.1) about C^0 equivalence with standard models: I presume (but cannot prove) that « C^0 equivalence » can be replaced by « C^0 conjugacy ».

REFERENCES

[Ar] V. ARNOLD, *Équations différentielles ordinaires*, Éditions Mir, Moscou, 1974.

[A. M.] R. ABRAHAM and J. E. MARSDEN, *Foundations of Mechanics*. Benjamin, London.

[A. R.] R. ABRAHAM and J. ROBBIN, *Transversal Mappings and Flows*. Appendix by Kelley, Benjamin, New York, 1967.

[B. D.] P. BONCKAERT and F. DUMORTIER, Smooth Invariant Curves for Germs of Vector Fields in $\mathbb{R}^3$ whose Linear Part Generates a Rotation, to appear in *Journal of Differential Equations*.

[Be] G. R. BELITSKII, Equivalence and Normal Forms of Germs of Smooth Mappings. *Russian Math. Surveys*, t. **33**, 1978, p. 107-177.

[Br] H. W. BROER, Formal Normal Form Theorems for Vector Fields and some Consequences in the Volume Preserving Case, in: Dynamical Systems and Turbulence, Warwick 1980, *Lecture Notes in Math.*, t. **898**, 1981.

[Cam] M. I. T. CAMACHO, *A contribution to the Topological Classification of Homogeneous Vector Fields in* $\mathbb{R}^3$ (preprint).

[Car] J. CARR, *Applications of Centre Manifold Theory*, Springer Verlag, New York, 1981.

[Di] J. DIEUDONNÉ, *Foundations of Modern Analysis*, Academic Press, New York, 1969.

[Du$_1$] F. DUMORTIER, Singularities of Vector Fields on the Plane. *Journal of Diff. Eq.*, t. **23**, 1977, p. 53-106.

[Du$_2$] F. DUMORTIER, Singularities of Vector Fields. *Monografias de Matematica*, t. **32**, IMPA, Rio de Janeiro, 1978.

[D. R. R.] F. DUMORTIER, P. R. RODRIGUES and R. ROUSSARIE, Germs of Diffemorphisms in the Plane. *Lecture Notes in Math.*, t. **902**, 1981.

[Go] R. E. GOMORY, Trajectories tending to a Critical Point in 3-space. *Annals of Math.*, t. **61**, 1955, p. 140-153.

[Gu] J. GUCKENHEIMER and P. HOLMES, *Nonlinear oscillations, Dynamical Systems and Bifurcations*, Springer-Verlag, New York, 1983.

[Ha] P. HARTMAN, *Ordinary Differential Equations*. Wiley and Sons, New York.

[H. P. S.] M. W. HIRSCH, C. C. PUGH and M. SHUB, Invariant Manifolds. *Lecture Notes in Math.*, t. **583**, 1977.

[Ir] M. C. IRWIN, *Smooth Dynamical Systems*, Academic Press, New York, 1980.

[Ke] A. KELLEY, The Stable, Center-Stable, Center, Center-Unstable, Unstable Manifolds. *Journal of Diff. Eq.*, t. **3**, 1967, p. 546-570.

[M. M.] J. E. MARSDEN and M. MCCRACKEN, *The Hopf Bifurcation and Its Applications*, Springer-Verlag, New York, 1976.

[Na] N. NARASIMHAM, *Analysis on Real and Complex Manifolds*, North-Holland, Amsterdam, 1968.

[N. S.] V. V. NEMYTSKII and V. V. STEPANOV, *Qualitative theory of Differential Equations*. Princeton, Princeton University Press, 1960.

[P. M.] J. PALIS and W. DE MELO, *Geometric Theory of Dynamical Systems*, Springer-Verlag, New York, 1982.

[Sh] D. S. SHAFER, Singularities of Gradient Vectorfields in $\mathbb{R}^3$. *Journal of Diff. Eq.*, t. **47**, 1983, p. 317-326.

[Sm] D. SMART, *Fixed point theorems*. Cambridge University Press, 1974.

[Ta] F. TAKENS, Singularities of Vector Fields. *Publ. Math. IHES*, t. **43**, 1974, p. 47-100.

[We] J. C. WELLS, Invariant Manifolds of Non-Linear Operators. *Pacific Journal of Math.*, t. **62**, 1976, p. 285-293.

[Ya] S. YAMAMURO, Differential calculus in topological linear spaces. *Lecture Notes in Math.*, t. **374**, 1974.

(Manuscrit reçu le 11 octobre 1984)