ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 3, nº 1 (1986), p. 1-53
<http://www.numdam.org/item?id=AIHPC_1986__3_1_1_0>

Ann. Inst. Henri Poincaré,
Vol. 3, n° 1, 1986, p. 1-53. *Analyse non linéaire*

A nonlinear scattering problem

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ABSTRACT. — We consider neutral nearly diagonal n-dimensional systems of the form $\varepsilon^2 y' = ix\Lambda(x)y + g(x, \varepsilon, y)$. We study the propagation of solutions from $x = -\infty$ to $x = +\infty$ past the complete degeneracy of the linearized problem at $x = 0$. Under several conditions on Λ and g we show that for small c in $\mathbb{C}^n$ there exists a global solution having the form $y = \left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} c$ near $x = -\infty$ and $y = \left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} S(\varepsilon, c)$ near $x = +\infty$. Here $S(\varepsilon, c) \in \mathbb{C}^n$ is the *scattering function.* Our main result is an asymptotic formula for $S(\varepsilon, c)$. We show that if $g(0, 0, y) = 0$ and $g = \Sigma g_{jk}(x, \varepsilon)y_j y_k + O(|y|^3)$ then

$$S(\varepsilon, c) = c + \Sigma c_j c_k (2\pi i \{ \Lambda(0) - (\lambda_j(0) + \lambda_k(0))I \})^{-1/2} \frac{\partial g_{jk}}{\partial \varepsilon}(0, 0) + O(|c|^3) + O(\varepsilon).$$

To establish this formula we use the Kolmogorov-Arnold-Moser method and the Moser-Jacobowitz approximation method to obtain *a priori* estimates for solutions. These *a priori* estimates provide a rigorous justification for our calculation of explicit asymptotic formulas by a technique of matched asymptotic expansions.

RÉSUMÉ. — Nous considérons des systèmes différentiels neutres à n dimensions presque diagonaux de la forme $\varepsilon^2 y' = ix\Lambda(x)y + g(x, \varepsilon, y)$.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - Vol. 3, 0294-1449
86/01/1/53/$ 7.30/© Gauthier-Villars

Nous étudions la propagation des solutions depuis $x = -\infty$ jusqu'à $x = +\infty$ à travers la dégénérescence complète du problème linéarisé à l'origine. Moyennant diverses conditions sur Λ et g nous montrons que, pour c assez petit dans $\mathbb{C}^n$, il existe une solution globale de la forme $y = \left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} c$ au voisinage de $x = -\infty$, et $y = \left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} S(\varepsilon, c)$ au voisinage de $x = +\infty$. Ici $S(\varepsilon, c)$ est la fonction de scattering.

Notre résultat principal est une formule asymptotique pour $S(\varepsilon, c)$. Pour l'établir, nous utilisons la méthode de Kolmogorov-Arnold-Moser et la formule d'approximation de Moser-Jacobowitz pour obtenir des estimations *a priori*, qui nous permettent de justifier rigoureusement le calcul des formules asymptotiques par une technique de comparaison.

Mots-clés : Matched asymptotic expansions, scattering function, Kolmogorov-Arnold-Moser method.

PART 1.

Asymptotic Theorems.

PART 2.

Linearization with derivative estimates.

PART I

ASYMPTOTIC THEOREMS

1. Introduction.

Our purpose is to determine the asymptotic behavior of certain systems $\varepsilon^2 y' = f(x, \varepsilon, y)$; $f(x, \varepsilon, 0) = 0$, near the zero solution in some cases in which the linearized problem is neutral and degenerate. We ask for results which are global in x, asymptotic as $\varepsilon \to 0^+$, and which do not require that f be holomorphic in x so that we can study the effect of smooth nonlinearities with compact x-support. We obtain results of two kinds. First are definite computational procedures for sinning boldly with the formal apparatus of perturbation theory to obtain explicit asymptotic formulas for solutions. Our main tool here is a technique of matched expansions. Second are existence theorems which vindicate the formal calculations. Here we need the Kolmogorov-Arnold-Moser technique. We proceed by supposing that f is holomorphic in y (but not x) and studying the asymptotics of the linearization problem that is, of reducing the nonlinear equation to its linear part by a change of dependent variable given by a convergent power series in y. Hartman [*1*] and Wasow [*2*] give accounts of this classical method, the latter giving special emphasis to asymptotic questions. In this paper we study a problem which in fact lies beyond a straightforward application of this method. Brjuno [*3*] gives a more recent account of the linearization problem together with counterexamples which show the unreliability of purely formal reasoning in this regime.

We consider nearly diagonal n-dimensional systems of the form

$$\varepsilon^2 y' = ix\Lambda(x)y + g(x, \varepsilon, y) \tag{1.1}$$

where $g(x, \varepsilon, o) = g(o, o, y) = 0$, $g_y(x, \varepsilon, o) = 0$, g is a smooth function of x and ε holomorphic at $y = 0$, and Λ is a smooth real diagonal matrix valued function satisfying certain nondegeneracy conditions to be stated below. The salient features here are the neutral behavior of the linearized problem and its complete degeneracy at $x = 0$. In addition we suppose that g vanishes for large $|x|$. We can then formulate the following *scattering problem*. Consider the solution which has the form $\left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} c$ for large negative x. If this solution exists for all x then for large positive x it must have the form $\left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds \right\} S(\varepsilon, c)$ where $S(\varepsilon, c)$ is a new constant vector. The mapping $c \rightarrow \mathrm{S}(\varepsilon, c)$ is the *scattering function* which measures the impact of the perturbation g on the propagation of solutions from $x = -\infty$ to $x = +\infty$. By the scattering problem we understand: find asymptotic formulas for the scattering function.

We have arranged our exposition in three stages of increasing difficulty and technical complexity. In Sections 1-5 we describe the problem, give motivating examples, discuss the chief difficulties, and state our main results (in Section 4). Sections 6-11 present a second level. In Sections 6 and 7 we formulate and state without proof the asymptotic *a priori* estimates necessary to prove our main results. In Sections 8-11 we show how these estimates can be used to derive our asymptotic theorems. With them our method of matched formal expansions, which often has only heuristic significance, becomes a rigorous deduction of asymptotic formulas. The balance of the paper exposes the third level of difficulty. In it we prove our *a priori* estimates using the KAM technique [*4*] combined with the approximation methods of Jacobowitz [*5*].

2. **An elementary example.**

The following explicitly solvable problem motivates and also delimits our results. Consider

$$\varepsilon^2 y' = imx^{m-1}y = g(x, \varepsilon)y^2 \tag{2.1}$$

where g is a smooth function vanishing for large $|x|$. The solution reducing to $c \exp \frac{i}{\varepsilon^2} x^m$ near $x = -\infty$ is

$$y(x) = c \exp \frac{i}{\varepsilon^2} x^m \left\{ 1 - \frac{c}{\varepsilon^2} \int_{-\infty}^{x} g(s, \varepsilon) \exp \frac{i}{\varepsilon^2} s^m ds \right\}^{-1}. \tag{2.2}$$

A simple computation shows that if $m > 1$

$$y(0) \sim c \left\{ 1 - c\left[g(o, o)\alpha_m \varepsilon^{-2+\frac{2}{m}} + g_\varepsilon(o, o)\beta_m \varepsilon^{-1+\frac{2}{m}} + g_x(o, o)\gamma_m \varepsilon^{-2+\frac{4}{m}} + \dots \right] \right\}^{-1}$$

where α_m, β_m, γ_m are certain nonzero constants. This formula shows that y will not in general be holomorphic in c on a domain independent of ε unless we require that $g(o, o) = 0$ and restrict the degeneracy of the linearized problem at $x = 0$ by demanding that $m = 2$. In this case the solution (2.2) can be given in the factored form

$$y = \mathrm{P}\left(x, \varepsilon, \exp i \frac{x^2}{\varepsilon^2} \mathrm{S}(x, \varepsilon, c) \right) \tag{2.3a}$$

where

$$(2.3b)\quad \mathrm{P}(x,\varepsilon,w)=\begin{cases} w\left(1-\dfrac{w}{\varepsilon^2}\displaystyle\int_{-\infty}^{x} g(s,\varepsilon)\exp\dfrac{i}{\varepsilon^2}(s^2-x^2)ds\right)^{-1} & x<0\\ w\left(1+\dfrac{w}{\varepsilon^2}\displaystyle\int_{x}^{\infty} g(s,\varepsilon)\exp\dfrac{i}{\varepsilon^2}(s^2-x^2)ds\right)^{-1} & x>0\end{cases}$$

and

$$(2.3c)\quad \mathrm{S}(x,\varepsilon,c)=\begin{cases} c & x<0\\ c\left(1-\dfrac{c}{\varepsilon^2}\displaystyle\int_{-\infty}^{\infty} g(s,\varepsilon)\exp\dfrac{is^2}{\varepsilon^2}ds\right)^{-1} & x>0\end{cases}$$

This formula has the curious feature that it expresses the holomorphic function y as the composition of the discontinuous functions P and S. However it has the virtue that P and S have simple, regular asymptotic properties as $\varepsilon \to 0^+$. This follows easily from the condition $g(o, o) = 0$ and elementary properties of the integrals in (2.3b) and (2.3c). For example some computation shows

$$\mathrm{S}(0^+, \varepsilon, c) = c + \sqrt{\pi e^{\frac{\pi i}{4}}}\, g_\varepsilon(o, o)c^2 + 0(\varepsilon^2) + 0(c^3).$$

To explain the meaning of the factorization (2.3) for our subsequent analysis we distinguish functions of x according to their rapidity of variation, that is, according to the way in which their x-derivatives are unbounded in the parameter ε. Further simple calculations with (2.3) then reveal the following.

1) The solution $\exp\dfrac{ix^2}{\varepsilon^2}$ of the linearized equation is rapidly varying at a rate $0(\varepsilon^{-2})$.

2) In contrast, the function P exhibits slower variation, $0(\varepsilon^{-1})$ at worst, if we agree to use one sided derivatives at 0. Thus the most rapid variation of y is accounted for in (2.3) entirely through dependence on the solution of the linearized problem.

3) The function P is a generalized (discontinuous at 0) solution of the partial differential equation $\varepsilon^2 \mathrm{P}_x + ix(w\mathrm{P}_w - \mathrm{P}) = g\mathrm{P}^2$ which (by remark 2) satisfies the qualitative subsidiary condition of non-rapid variation. Such solutions are far from unique but we show later that a properly formulated version of this condition uniquely determines the asymptotic properties of P.

4) Dependence of y on the data c is entirely through the piecewise constant scattering function which is holomorphic in c on a neighborhood of $c = 0$ independent of ε.

If we bear these four properties in mind we can give another description of this paper. It is devoted to obtaining analogous representations of solutions for a class of general systems.

3. **Hypotheses.**

We suppose that the system

$$\varepsilon^2 y' = ix\Lambda(x)y + g(x, \varepsilon, y) \tag{1.1}$$

satisfies the following.

H.1. *Regularity conditions.*

On $(-\infty, \infty) \times [0, \varepsilon_0] \times \{ \mid y \mid \leq r_0 \} g$ is jointly infinitely differentiable in (x, ε) and holomorphic in y. The matrix Λ is real, diagonal and infinitely differentiable.

H.2. *Eigenvalue conditions.*

On the x-support of g, for each j and each n-tuple of nonnegative integers $(m_1, \dots, m_n)$ with $\Sigma m_i \geq 2$, the diagonal elements λ_i of Λ satisfy

a) $$\lambda_j > 0$$

b) $$\lambda_j = \Sigma m_i \lambda_i \neq 0 .$$

H.3. *Small perturbation conditions.*

The perturbation g vanishes for large $|x|$ and satisfies

$$g(o, o, y) = g(x, \varepsilon, o) = 0, \qquad g_y(x, \varepsilon, o) = 0 .$$

We remark that in the case that the λ_i's are constant the eigenvalue conditions H.2 are familiar sufficient conditions for linearizing a vector field (Hartman [*1*]). It is also possible in this case to relax H.2*a* at the cost of profound complications. Likewise in our problem this raises new difficulties which we avoid in the present investigation. The eigenvalue conditions ensure that the linearized problem is nondegenerate except at $x = 0$ where it is instead completely degenerate. However we emphasize that even if we restrict our results to a closed x-interval not containing 0 (in which case the factor x can be absorbed into Λ without altering the eigenvalue conditions, and the condition $g(o, o, y) = 0$ is vacuous) we obtain results about a delicate problem, namely

$$\varepsilon_1 y' = i\Lambda(x)y + g(x, \varepsilon_1, y) \qquad g = 0(\mid y \mid^2), \varepsilon_1 = \varepsilon^2 . \tag{3.1}$$

The ε-asymptotics of this problem also demand the full power of the KAM method and are not covered by classical techniques such as those appearing in Wasow's treatise [*2*]. We therefore also give a result (Theorem 3 below) about this problem as a simple byproduct of our work.

4. Solution of the scattering problem.

By a *linearizing function* we mean a piecewise solution $P(x, \varepsilon, w)$ of the *linearizing problem*

$$\varepsilon^2 P_x - ix\Lambda P + ixP_w\Lambda w = g(x, \varepsilon, P)$$
$$P|_{w=0} = P_w|_{w=0} = 0. \tag{4.1}$$

If $P = w + Q$ we call Q a *linearizing perturbation.* An invertible solution of this equation defines a change of variable $y = P(x, \varepsilon, w)$ which linearizes problem (1.1). The following theorems justify a qualified reliance upon the far simpler problem of finding formal ε-power series solutions of (4.1) (strictly speaking, solutions of the equation resulting when g is replaced by its formal ε-y-Taylor series at $\varepsilon = 0$, $y = 0$). Specifically we use the following method of *matched formal expansions.*

By a *formal solution* of the linearizing problem we mean a formal ε-power series solution $\sum_{k=0}^{\infty} \varepsilon^k P_k(x, w)$ of (4.1). We will find that this solution is unique but fails, in general, to be defined at $x = 0$. In the technique of *matched formal solution* we augment the previous procedure to obtain formal results at $x = 0$ in the following way. Let

$$x = \varepsilon s$$
$$P(x, \varepsilon, w) = p(s, \varepsilon, w) \tag{4.2}$$

Then

$$p_s - is\Lambda(\varepsilon s)p + isp_w\Lambda(\varepsilon s)w = \varepsilon^{-1}g(\varepsilon s, \varepsilon, p) \tag{4.3}$$

$$p|_{w=0} = 0, \qquad p_w|_{w=0} = 0. \tag{4.4}$$

We note that our hypothesis $g(o, o, w) = 0$ ensures that this problem is regular in ε. Formal ε-power series solutions of (4.3) are determined by a recursive set of equations with *polynomial* data. We will show that there are solutions $\Sigma\varepsilon^k p_k(s, w)$ in which $p_k = 0(1 + |s|)^k$ on the real s-axis. We will also show that these conditions uniquely specify $\tilde{p}$. The *matched formal solution* is then the pair $(\tilde{P}, \tilde{p})$.

The main content of the following theorems is that the above formal procedure can be carried through and that it actually yields asymptotic formulas for a linearizing transformation.

THEOREM 1 (Factorization of solutions). — Suppose the system $\varepsilon^2 y' = ix\Lambda y + g$ satisfies conditions H.1-H.3. Then for small $c \in \mathbb{C}^n$ there exist solutions of the form $y = P\left(x, \varepsilon, \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds S(x, \varepsilon, c)\right)$.

The functions P and S are holomorphic at $0 \in \mathbb{C}^n$ and have uniform asymptotic expansions given by the unique matched formal solution $(\tilde{\mathrm{P}}, \tilde{p})$ of the linearizing problem according to

$$(4.5) \qquad \mathrm{P}(x, \varepsilon, w) \sim \begin{cases} \tilde{\mathrm{P}}(x, \varepsilon, w) & a\varepsilon^{\delta} \leq |x| \\ \tilde{p}(s, \varepsilon, w) & |s| \leq a\varepsilon^{-1+\delta} \end{cases}$$

for any $0 < \delta < 1$ and

$$(4.6) \qquad \mathrm{S}(x, \varepsilon, c) \sim \begin{cases} c & x < 0 \\ \tilde{p}^{-1}(0^+, \varepsilon, \tilde{p}(0^-, \varepsilon, c)) & x > 0 \end{cases}$$

By carrying out the calculations described in Theorem 1 we obtain a solution of the scattering problem. This we express in terms of the w-series expansion of the perturbation $g(x, \varepsilon, w) = \sum_q g_q(x, \varepsilon) w^q$ where q indicates a multi-index of integers $q = (q_1, \ldots, q_n)$, $w^q = w_1^{q_1} \ldots w_n^{q_n}$, and $|q| = \sum_{j=1}^{n} q_j$.

THEOREM 2 (Scattering formula). — Suppose the system $\varepsilon^2 y' = ix\Lambda y + g$ satisfies conditions H.1-H.3. Then for small $c \in \mathbb{C}^n$ the solution which has the form $\left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s) ds \right\} c$ near $x = -\infty$ exists for all x and has the form $\left\{ \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s) ds \right\} \mathrm{S}(\varepsilon, c)$ near $x = +\infty$ where

$$(4.7) \qquad \mathrm{S}(\varepsilon, c) = c + \sum_{|q|=2} c^q \left\{ 2\pi i \left(\Lambda(0) - \sum_{j=1}^{n} q_j \lambda_j(0) \mathrm{I} \right) \right\}^{-\frac{1}{2}} \frac{\partial g_q}{\partial \varepsilon}(0, 0) + 0(|c|^3) + 0(\varepsilon).$$

Finally our analysis justifies the simplest methods in the nondegenerate case.

THEOREM 3 (Parametric asymptotic linearization of neutral systems). — Suppose the system $\varepsilon y' = i\Lambda(x) y + g(x, \varepsilon, y)$ $g = 0(|y|^2)$, has smooth data on $|x| \leq x_0$ and satisfies the eigenvalue condition. Then for small $c \in \mathbb{C}^n$ there exist solutions of the form $y = \mathrm{P}\left(x, \varepsilon, \exp \frac{i}{\varepsilon} \int_0^x s\Lambda(s) ds\, c\right)$. The function $\mathrm{P}(x, \varepsilon, w)$ is holomorphic at $w = 0$ and has a uniform asymptotic expansion given by $\tilde{\mathrm{P}}$, the unique formal solution of the associated problem

$$(4.8) \qquad \begin{aligned} \varepsilon \mathrm{P}_x - i\Lambda(x)\mathrm{P} + i\mathrm{P}_w \Lambda(x) w &= g(x, \varepsilon, \mathrm{P}) \\ \mathrm{P}|_{w=0} = 0, \qquad \mathrm{P}_w|_{w=0} &= 0. \end{aligned}$$

5. A remark on the role of power series expansions.

We call special attention to the fact that Theorems 1-3 make no direct mention of convergent w-power series expansions $P(x, \varepsilon, w) = w + \sum_{|q| \geq 2} w^q P_q(x, \varepsilon)$ for the linearizing function P. Existence of this expansion together with asymptotic expansions for its coefficients are consequences of our conclusions. However a converse implication is not usually true. The existence of a convergent w-series expansion for $P(x, \varepsilon, w)$ together with asymptotic formulas for the $P_q(x, \varepsilon)$ does not entail asymptotic information about the sum unless the asymptotic formulas are uniform in the multi-index q. For example the series $G(\varepsilon, w) = \sum_{k=0}^{\infty} w^k (1 + e^k \varepsilon)^{-1}$ is uniformly convergent for $|w| \leq 1 - \delta$. But even for $w = \frac{1}{e}$ we have

$$G\left(\varepsilon, \frac{1}{e}\right) = \sum_{k=0}^{\infty} \frac{e^{-k}}{1 + e^k \varepsilon} \sim \int_0^{\infty} \frac{e^{-x} dx}{(1 + \varepsilon e^x)} = \int_1^{\infty} \frac{dt}{t^2(1 + \varepsilon t)} = 1 + \varepsilon \ln\left(\frac{\varepsilon}{1 + \varepsilon}\right).$$

Here the limiting behavior of the sum simply cannot be described by ε-power series even though the behavior of each summand can be. Now it happens that in our investigation the uniformity in q of the asymptotic behavior of $P_q(x, \varepsilon)$ is an exceedingly subtle problem. It is one of the difficulties that we overcome with the KAM method. This explains why, although we use w-power series expansion to compute individual terms in the matched formal expansions $(\tilde{P}, \tilde{p})$, we never use the w-expansion as a direct analytical tool for solving the linearizing problem itself.

These difficulties have been sometimes overlooked in the literature. The main theorem on solution of nonlinear equations with a small parameter in Wasows' book [*2*], Theorem 36.2, falls short in this way. Although this theorem is correct as stated it does not strictly contain asymptotic information about solutions of the nonlinear equation to which it refers.

6. A measure of perturbation strength.

We now introduce a collection of norms which measure the strength of the perturbation $g(x, \varepsilon, w)$. We study perturbations more general than these described in conditions H.1 and H.3 above because this added generality is essential for our proofs. We consider perturbations which are piecewise smooth in x with possible discontinuities at $x = 0$. In the following formulas we suppose that suprema over function values range over both right and left limits at 0.

Let $g(x, \varepsilon, w) = \sum_{|q| \geq 2} g_q(x, \varepsilon) w^q$ where $q = (q_1, \ldots, q_n)$, $w^q = w_1^{q_1} \ldots w_n^{q_n}$ and $|q| = q_1 + \ldots + q_n$. Also let $|w| = \max_{1 \leq i \leq n} |w_i|$. Then to each real $r > 0$ and each convex function $\phi : [0, \infty) \to (-\infty, \infty)$ we associate the norms (possibly infinite) defined by

$$(6.1) \quad \begin{aligned} a)\ \| g \|_{\phi,r}^0 &= \sup_{x,q,l} r^{|q|} \left| D_x^l g_q(x, \varepsilon) \right| e^{-\phi(l)} \\ b)\ \| g \|_{\phi,r} &= \sup_{x,q,l} r^{|q|} \left\{ (|x| + \varepsilon) D_x \right\}^l g_q(x, \varepsilon) \,|\, e^{-\phi(l)} . \end{aligned}$$

We also define the corresponding unit balls by

$$(6.2) \quad \begin{aligned} B_{\phi,r}^0 &= \{ g \mid \| g \|_{\phi,r}^0 < 1 \} \\ B_{\phi,r} &= \{ g \mid \| g \|_{\phi,r} < 1 \} . \end{aligned}$$

The ball $B_{\phi,r}^0$ consists of perturbations $g(x, \varepsilon, w)$ which are holomorphic for $|w| < r$ with a sequence of progressively higher x-derivatives growing no more rapidly than $e^{\phi(l)}$. The ball $B_{\phi,r}$ is similar but embodies estimates which allow non uniformities in x and ε (of the kind appearing in the example of Section 2). We make exacting use of the spaces $B_{\phi,r}$ in our existence arguments. However for the derivation of asymptotic results less precision is required and it suffices to consider the spaces

$$(6.3) \quad \begin{aligned} B_r^0 &= \bigcup_{\phi} B_{\phi,r}^0 \\ B_r &= \bigcup_{\phi} B_{\phi,r} . \end{aligned}$$

Our analysis requires perturbations which are $0(|x| + \varepsilon)$ in the following sense.

H.3'. *Alternate small perturbation condition.*

For some $r > 0$

$$g \in (|x| + \varepsilon) B_r .$$

However we prefer the simpler hypotheses H.3 for the statement of our theorems.

We emphasize that the condition $g \in B_r$ is not numerical in character. Each ball $B_{r,\phi}$ embodies numerical derivative estimates. But the union $B_r = \bigcup_{\phi} B_{r,\phi}$ instead embodies the *existence* of ε-independent estimates. We refer to the containment of a linearizing perturbation in some B_r or B_r^0 as an *asymptotic a priori estimate.* The term « asymptotic » indicates the non-numerical nature of the relation. The term « *a priori* » is generally used for estimates which are established prior to existence. In this investigation the basic existence problem is to find a linearization which depends regularly on the singular parameter ε. We will see that we find linearizing perturbations in B_r by powerful existence arguments which nevertheless entirely preceed the resolution of this more delicate existential question. It is with respect to this latter asymptotic problem that our estimates are « *a priori* ».

The following shows that our hypotheses H.1 and H.3 imply that each perturbation g belongs to some B_r.

PROPOSITION 6.1. — If g satisfies H.1, $g=0(|w|^2)$, and g vanishes for large x, then $g\in B_r^0$ for some $r>0$. If also $g(o,o,w)=0$ then $g\in(|x|+\varepsilon)B_r$.

Proof. — Let $\hat{g}(s,\varepsilon,w)$ be the x-Fourier transform of $g(x,\varepsilon,w)$. Since g has compact support, $\hat{g}$ is rapidly decreasing as a function of s. Moreover if $g=\Sigma g_q(x,\varepsilon)w^q$ for $|w|\le r$ then $\hat{g}=\Sigma\hat{g}_q w^q$. Since $|s|^\alpha\hat{g}$ is holomorphic in $|w|\le r$ and continuous in ε for $0\le\varepsilon\le\varepsilon_0$ this implies $|s|^\alpha r^{|q|}|\hat{g}_q|\le M_\alpha$ for $\alpha\ge 0$. Define

$$(6.4)\qquad \phi(a)=\frac{1}{2\pi}\sup_{|q|,\varepsilon} r^{|q|}\int_{-\infty}^{\infty}|s|^\alpha|\hat{g}_q(s,\varepsilon)|\,ds$$

We claim that $\log\phi$ is a convex function of x. For

$$\log\int_{-\infty}^{\infty}|s|\frac{\alpha+\alpha'}{2}|\hat{g}_q(s,\varepsilon)|\,ds\le\log\left(\int_{-\infty}^{\infty}|s|^\alpha|\hat{g}_q|\,ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|s|^{\alpha'}|\hat{g}_q|\,ds\right)^{\frac{1}{2}}$$

$$\le\frac{1}{2}\log\left(\int_{-\infty}^{\infty}|s|^\alpha|\hat{g}_q|\,ds\right)$$

$$+\frac{1}{2}\log\int_{-\infty}^{\infty}s^{\alpha'}|\hat{g}_q|\,ds.$$

Thus $\log\int_{-\infty}^{\infty}|s|^\alpha|\hat{g}_q(s,\varepsilon)|\,ds$ is a convex function of α. Moreover

$$r^q\int_{-\infty}^{\infty}|s^\alpha|\,\hat{g}_q(s,\varepsilon)ds\le 2|x_0|M_\alpha$$

if we suppose that g vanishes outside $[-x_0,x_0]$. Hence $\log\phi(\alpha)$ is the supremum of a pointwise upper bounded family of convex functions and is therefore also convex.

We next show that $g\in B_{\phi,r}^0\subset B_r^0$. For $r^{|q|}D_x^k g_q(x,\varepsilon)=\dfrac{r^{|q|}}{2\pi}\displaystyle\int_{-\infty}^{\infty}e^{ixs}(is)^k\hat{g}_q(s,\varepsilon)ds.$

Hence $r^{|q|}|D_x^k g_q(x,\varepsilon)|\le\dfrac{r^{|q|}}{2\pi}\displaystyle\int_{-\infty}^{\infty}|s|^k|\hat{g}_q(s,\varepsilon)|\,ds\le\phi(k)$. Thus $g\in B_{\phi,r}^0$. If in addition $g(o, o, w) = 0$ then g has the form $xh(x, \varepsilon, w) + \varepsilon k(x, \varepsilon, w)$. Hence $\frac{1}{|x|+\varepsilon} g = \frac{x}{|x|+\varepsilon} h + \frac{\varepsilon}{|x|+\varepsilon} k = \operatorname{sgn} x\, h + \frac{\varepsilon}{|x|+\varepsilon} (k\text{-}\operatorname{sgn} x\ h)$. By the argument above we can suppose that h and k belong to B_r^0. We next show that h, $k \in B_{\phi_1,r}$ for some ϕ_1. Computing recursively we find

$$\left((x+\varepsilon)\frac{d}{dx}\right)^k h = \sum_{j=1}^{k} m_{k_j}(x+\varepsilon)^j \left(\frac{d}{dx}\right)^j h$$

where $m_{0,0} = 1$ and $m_{k+1,j} = m_{kj-1} = j_{m_{k,j}}$. These inequalities imply $m_{k,j} \leq k\,!$ (a very crude estimate but sufficient for our purposes). It follows easily that $r^{|q|} \left| \left[(|x|+\varepsilon)\frac{d}{dx} \right]^k h_p(x, \varepsilon) \right| \leq (k+1)\,!\,(|x_0|+1)^k \max_{1 \leq j \leq k} e^{\phi(j)}$. Now we can suppose ϕ is an increasing function so that

$$r^{|q|} \left| \left[(|x|+\varepsilon)\frac{d}{dx} \right]^k h_p(x, \varepsilon) \right| \leq (k+1)\,!\,(|x_0| + 1)^k e^{\phi(k)}.$$

This shows $h \in B_{\phi_1,r}$ where $\phi_1(\alpha) = \phi(\alpha) + (\alpha+1)\log(\alpha+1) + \alpha \log(|x_0|+1)$. Similarly $k - h \in B_{\phi_1,r}$. Finally an application of the Leibniz rule for the derivation $(|x|+\varepsilon)\frac{d}{dx}$ to the product $\frac{\varepsilon}{|x|+\varepsilon}(h-k)$ implies

$$r^{|q|} \left| \left[(|x|+\varepsilon)\frac{d}{dx} \right]^k \frac{1}{|x|+\varepsilon} g \right| \leq \phi_1(k) + 2\sum_{j=0}^{k} \binom{k}{j} e^{\phi_1(k)} \leq 2^{k+1} e^{\phi_1(k)}.$$

Hence if $\phi_2(\alpha) = \phi_1(\alpha) + (k+1)\log 2$ then $(|x| + \varepsilon)^{-1} g \in B_{\phi_2,r} \in B_r$.

We make two comments about the preceeding proof. First our hypotheses permit the definition of derivatives $\left(\frac{d}{dx}\right)^\alpha g$ for real $\alpha \geq 0$ by means of the Fourier transform. Accordingly we invariably suppose that the function $\phi(k)$ describing the growth of successive x-derivatives is a convex function of a *real* argument as in formula (6.4). For convenience we also define $\phi(x) = +\infty$ for $x < 0$. Secondly we call attention to the tedious character of the preceding calculations with elementary derivative formulas. Since we must perform numerous far more complicated estimations of derivative growth rate, an essential part of our subsequent analysis will be an effective and systematic technique for doing this.

7. **Statement of *a priori* estimates.**

In this section we state our main analytic results in the form required to prove the asymptotic conclusions of Theorems 1-3. The proofs of these results are far more technical than their applications. We give the latter first. We believe that our method of separating these is itself a valuable tool which can be applied to many other problems. We have therefore tried to give a reasonable account of our asymptotic reasoning to a reader who is willing to take the results of this section for granted.

We first state the main linearization theorem.

PROPOSITION 7.1 (Linearization with derivative estimates). — Suppose $g \in \delta(|x|+\varepsilon)B_r$ ($g \in \delta B_r^0$). Then for each $r' \leq r$ and for δ sufficiently small there is a linearizing transformation $w + Q$ for Problem (1.1) (Problem 3.1) satisfying $Q \in \delta B_{r'}$ ($Q \in \delta B_{r'}^0$).

We also require similar results for certain reduced linearizing problems in the « outer scale », $|x| \geq x_1 > 0$ and the « inner scale » $x = \varepsilon s$.

PROPOSITION 7.2A. — Suppose $g \in \delta B_r^0$. Then for each $r' < r$ and for δ sufficiently small there is a unique solution Q of $-i\Lambda Q + iQ_w\Lambda w = g(x, \varepsilon, w+Q)$ satisfying $Q \in \delta B_{r'}^0$.

The following proposition provides solutions to the reduced linearizing problem in the inner scale $x = \varepsilon s$.

PROPOSITION 7.2B. — Suppose $g(s, w)$ is holomorphic for $|w| < r$ and $|\operatorname{Im}\log(1+s)| < \theta < \frac{\pi}{2}$ and satisfies $|(1+s)^{-1}g| \leq \delta$. Then for δ sufficiently small the problem

$$Q_s - is\Lambda(0)Q + isQ_w\Lambda(0)w = sh(w+Q) + k(w+Q)$$

has a unique bounded holomorphic solution on the domain $|w| < r' < r$, $|\operatorname{Im}\log(1+s)| < \theta' < \theta$ satisfying $|Q| \leq M(r', \theta')\delta |w|^2$.

We also require similar (but much easier) results about linearized forms of the preceedings propositions.

PROPOSITION 7.3. — Suppose $g \in (|x| + \varepsilon)B_r$. Then the linear problem $\varepsilon^2 Q_x - ix\Lambda Q + ixQ_w\Lambda w = g(x, \varepsilon, w)$ has a solution in $B_{r'}$ for any $r' < r$.

PROPOSITION 7.4 A. — Suppose $g \in B_r$ ($g \in B_r^0$). Then

$$-i\Lambda Q + iQ_w\Lambda w = g(x, \varepsilon, w)$$

has a unique solution in $B_{r'}$ for any $r' < r$.

PROPOSITION 7.4 B. — Suppose $g(s, w)$ is holomorphic for $|w| < r$ and $-\theta < \operatorname{Im}\log(1+s) < \theta, 0 < \theta < \frac{\pi}{2}$. Suppose $|g| \leq (1+|s|)^{m+1}|w|^2$. Then given $r' < r$ the problem

$$Q_s - is\Lambda(0)Q + isQ_w\Lambda(0)w = g(s, w)$$

has a unique solution Q satisfying $|Q| < M(1+|s|)^m |w|^2$ for $|w| < r'$, $-\theta < \operatorname{Im}\log|1+s| < \theta$.

In our reasoning we operate on B_r with the common operations of analysis. To express compositions of small perturbations of the identity map we find the following notation for composition convenient (Sternberg [4]).

DEFINITION 7.5. — $Q \circ R(w) = R(w) + Q(w + R(w))$.

DEFINITION 7.6. — If $Q \circ R = 0$ then we write $R = Q^{\#}$ and we call R the *quasi-inverse* of Q.

The operations $\circ$ and $\#$ are local in character and in general do not carry $B_r \times B_r$ or B_r into B_r. The following proposition lists some closure properties of the family $\{B_r\}_{r>0}$ as substitutes.

PROPOSITION 7.7 (Closure properties of δB_r). If g, h belong to δB_r then

a) $g + h \in \delta B_r$

b) $g_w, (|x| + \varepsilon)g_x \in \delta B_{r'}$ for $r' < r$

Moreover if δ is sufficiently small then the following belong to $\delta B_{r'}$.

c) $(I + g_w)^{-1}h$

d) $g(x, \varepsilon, w+h) - g(x, \varepsilon, w)$ and $\dfrac{1}{\varepsilon}\,[g(x, \varepsilon, w+\varepsilon h) - g(x, \varepsilon, w)]$

e) $g \circ h$

f) $g^{\#}$.

We summarize the conclusions of this section broadly. They show that our linear operations lead from data in B_r to results in $B_{r'}$ for any $r' < r$. Moreover the same is true for our nonlinear operations if the operands are small.

8. **Proof of Theorem 3.**

We begin with the proof of Theorem 3 which shows our reasoning in a relatively uncluttered form. We subdivide the proof into two parts. The first shows the existence of an asymptotic linearization; the second shows its uniqueness.

PROPOSITION 8.1. — Assume the hypotheses of Theorem 3. Then problem (3.1) has a linearizing transformation $w+Q$ where $Q \in B_r^0$ for some $r > 0$. Also Q has a uniform asymptotic expansion $Q(x, \varepsilon, w) \sim \sum_{k=0}^{\infty} \varepsilon^k Q_k(x, w)$

with coefficients $Q_k \in B_r^0$. This expansion can be x- or w-differentiated term by term any number of times.

Proof. — Suppose $g \in B_s^0$. We observe that the change of variable $w \to \delta w$ replaces $g(x, \varepsilon, w)$ by $\delta^{-1} g(x, \varepsilon, \delta w)$ in the linearizing problem (4.8). Since $g = 0(|w|^2)$ we have $\delta^{-1} g(x, \varepsilon, \delta w) \in \delta B_{s/\delta}^0 \subset \delta B_1^0$ if $\delta \leq s$. Thus at the cost of shrinking the w-domain we can suppose that $g \in \delta B_1^0$. (However it is vital that such shrinkage be controlled in later steps.) Then Propositions (7.1) and (7.2) provide a solution $w + Q$ of the linearizing problem (4.8) and a solution $w + Q_0$ of the corresponding reduced problem $- i\Lambda Q_0 + Q_{0w}\Lambda w = g(x, o, w + Q_0)$, where $Q, Q_0 \in \delta B_{1/2}^0$ if δ is small. Also by Proposition 7.7 we can suppose that $Q_0^\# \circ Q$, $(Q_0^\# \circ Q)_x$, $Q_0^\#$, $h = (I + Q_{0w})^{-1} \left\{ \frac{1}{\varepsilon} [g(x, \varepsilon, w) - g(x, o, w)] - Q_{0x} \right\}$ and $h(x, \varepsilon, w + Q_0^\# \circ Q)$ all belong to $\delta B_{1/2}^0$ if δ is small.

Let $\varepsilon R = Q_0^\# \circ Q$, that is $Q = Q_0 \circ \varepsilon R$. Then R satisfies

$$\varepsilon R_x - i\Lambda R + iR_w\Lambda w = h(x, \varepsilon, w + \varepsilon R) \tag{8.1}$$

where h is defined above and $h \in \delta B_{1/2}^0$. We show that $R \in \delta B_{1/3}^0$ by an argument which exploits the fact that R is the « compositional » remainder of a solution Q satisfying the *a priori* asymptotic estimates $Q \in \delta B_{1/2}^0$. Equation (8.1) implies that R satisfies the *linear* problem

$$- i\Lambda R + iR_w\Lambda w = h(x, \varepsilon, w + Q_0^\# \circ Q) - (Q_0^\# \circ Q)_x \tag{8.2}$$

with data in $\delta B_{1/2}^0$. Unlike (8.1) this problem has a *unique* solution in $\delta B_{1/3}^0$ which must be R. Thus we find $Q = Q_0 \circ \varepsilon R$ where Q, Q_0 and R belong to $\delta B_{1/3}^0$. This implies that $Q = Q_0 + \varepsilon R_1$ where

$R_1 = R + \frac{1}{\varepsilon} \{ Q_0(x, \varepsilon, w + \varepsilon R) - Q_0(x, \varepsilon, w) \}$ belongs to $\delta B_{1/4}^0$ for sufficiently small ε. This establishes the lowest order finite asymptotic expansion of Q with remainder, $Q = Q_0 + \varepsilon R_1$.

Obtaining higher order finite expansions with remainder is easier (the reasoning illustrates further use of our *a priori* estimates). Let $\mathscr{L}(g)$ be the unique solution of $- i\Lambda Q + iQ_w\Lambda w = g(x, \varepsilon, w)$, $Q = 0(|w|^2)$. Then equation 8.1 can be transformed into

$$R = \mathscr{L}(h(x, \varepsilon, w + \varepsilon R) - \varepsilon R_x)$$

Since $R \in \delta B_{1/3}^0$ this implies immediately that $R = \mathscr{L}(h(x, o, w)) + \varepsilon R'$ $\left(\text{where } R' = \mathscr{L}\left\{ \frac{1}{\varepsilon} [h(x, \varepsilon, w + \varepsilon R) - h(x, o, w)] - R_x \right\}\right)$. Moreover, simple iteration leads to higher order formulas with terms in B_r^0 for any $r < \frac{1}{3}$ if ε is sufficiently small. For example

$$\begin{aligned} R &= \mathscr{L}\left\{ h(x, \varepsilon, w + \varepsilon\mathscr{L}(h(x, \varepsilon, w + \varepsilon R) - \varepsilon\mathscr{L}R_r)) \right. \\ &\qquad \left. - \varepsilon\frac{d}{dx}(\mathscr{L}(h(x, \varepsilon, w + \varepsilon R) - \varepsilon R_x)) \right\} \\ &= \mathscr{L}(h(x, o, w)) + \varepsilon\left\{ \mathscr{L}(h_\varepsilon(x, o, w) + h_w(x, o, w)\mathscr{L}(h(x, o, w))) \right. \\ &\qquad \left. - \mathscr{L}^2(h_x(x, o, w)) \right\} + 0(\varepsilon^2). \end{aligned}$$

Thus the functional equation (8.1) and the estimate $R \in B_{1/3}^0$ together imply that R has an asymptotic expansion for, say, $|w| < \frac{1}{4}$. The same will be true of the composition $Q = Q_0 \circ \varepsilon R$ for ε sufficiently small and $|w| < \frac{1}{5}$. This establishes the proposition.

We next show that the expansion of Proposition 8.1 is the unique formal solution of the linearizing problem (4.8). This fact is well known and is, so to speak, the basis of the formal utility here of perturbation methods (see Wasow [2], p. 218-219). We give an alternate proof.

PROPOSITION 8.2. — The formal series of Proposition 8.1 is the unique formal solution of the linearizing problem (4.8).

Proof. — Let $w+\tilde{Q}$ be the given formal solution. Suppose $w+\tilde{Q}'$ is another. Then $\tilde{R}=\tilde{Q}^{\#} \circ Q'$ is a formal solution of $\varepsilon\tilde{R}_x - i\Lambda\tilde{R} + i\tilde{R}_w\Lambda w = 0$. Hence $\tilde{R} = -\varepsilon\mathscr{L}(\tilde{R}_x)$ where $\mathscr{L}$ is the linear solution operator of $-i\Lambda Q + Q_N\Lambda w = g$. Iteration gives $\tilde{R} = -\varepsilon^N\mathscr{L}^N D_x^N\tilde{R}$ which shows $\tilde{R} = 0$ on any domain $|w| < r' < r$, supposing that the terms of $\tilde{R}$ belong to B_r.

Propositions 8.1 and 8.2 together establish Theorem 3. We remark that all our derivations of asymptotic expansions are variants of the argument of 8.2 which appears here in its purest form.

9. A proof of asymptotic uniqueness.

Our proof of Theorem 1 will use the ideas of Section 8 but will require separate arguments in the inner scale $|x| \leq a\varepsilon^\delta$ and the outer scale $|x| \geq a\varepsilon^\delta$. A new difficulty arises in the inner scale where the reduced linearizing equation (set $\varepsilon = 0$ in problem 4.3) no longer has a unique solution. We also note that in the proof of Proposition 8.1 in the previous section we estimated the remainder R by alternately representing it as the unique solution of equation (8.2). Our next result provides a substitute for this uniqueness. We show that our qualitative estimates $Q \in \delta B_r$ are sufficient to determine uniquely the asymptotic properties of solutions of certain problems.

PROPOSITION 9.1. — Suppose Q and Q′ are solutions of

$$\varepsilon^2 Q_x - ix\Lambda Q + ixQ_w \Lambda w = \begin{cases} g(x, \varepsilon, w + Q) \\ \text{or} \\ g(x, \varepsilon, w) \end{cases}$$

belonging to B_r. Then $Q' \sim Q$ uniformly in some domain $|w| \leq r' < r$.

Proof. — Let $R = \begin{cases} Q^{\#} \circ Q' \\ \text{or} \\ Q' - Q \end{cases}$. Then in either case R satisfies

$$\varepsilon^2 R_x - ix\Lambda R + ixR_w \Lambda w = 0$$

and by choosing r' sufficiently small we can suppose $R \in B_{r'}$. For $x \geq 0$ this implies that R can be represented in the form

$$R = \exp^{\frac{i}{\varepsilon^2}} \int_{ac^\delta}^{x} s\Lambda(s)ds\, c\left(\varepsilon, \exp\left\{ \frac{-i}{\varepsilon^2} \int_{a\varepsilon^\delta}^{x} \delta\Lambda(s)ds \right\} w\right)$$

where $c(\varepsilon, w)$ is holomorphic for $|w| < r'$ and $c(\varepsilon, w) = R(a\varepsilon^\delta, \varepsilon, w)$. (R also has a similar representation for $x \leq 0$.)

The idea of the proof is that $R \in B_{r'}$ implies $\left(\frac{d}{dx}\right)^k R\,|_{x=a\varepsilon^\delta} = 0(\varepsilon^{-k\delta})$, while the preceeding representation implies $\left(\frac{d}{dx}\right)^k R\,|_{x=a\varepsilon^\delta} = 0(c\varepsilon^{-(2-\delta)k})$. These estimates are compatible only if $c = 0(\varepsilon^{2(1-\delta)k})$, that is, if $c \sim 0$. To make this precise we again use $\mathscr{L}$, the linear solution operator of the problem

$$-i\Lambda R + R_w \Lambda w = h\,; \qquad R = 0(|w|^2)\,.$$

Then $R = \frac{-\varepsilon^2}{x(|x|+\varepsilon)} \mathscr{L}((|x|+\varepsilon)R_x)$. Since (by Proposition 7.7) $(|x|+\varepsilon)R_x \in B_{r'}$, this representation immediately shows that $c(\varepsilon, w) = R\,|_{x=a\varepsilon^\delta} = 0\varepsilon^{2(1-\delta)}$. Moreover, iterating the representation gives

$$\begin{aligned} R &= \frac{\varepsilon^2}{x(|x|+\varepsilon)} \mathscr{L}\left((|x|+\varepsilon)\left\{\frac{\varepsilon^2}{x(|x|+\varepsilon)} \mathscr{L}(|x|+\varepsilon)R_x\right\}_x\right) \\ &= \varepsilon^4\left(\frac{1}{x^2(|x|+\varepsilon)^2}\{\mathscr{L}(|x|+\varepsilon)D_x\mathscr{L}\{(|x|+\varepsilon)D_xR\}\}\right. \\ &\quad \left. - \left(\frac{1}{x^3(|x|+\varepsilon)} + \frac{\operatorname{sgn} x}{x(|x|+\varepsilon)^3}\right)\mathscr{L}^2\{(|x|+\varepsilon)D_xR\}\right) \end{aligned}$$

Again setting $x = a\varepsilon^{\delta}$ we find $c(\varepsilon, w) = 0(\varepsilon^{4(1-\delta)})$. An elementary induction argument shows similarly that $c(\varepsilon, w) = 0(\varepsilon^{2N(1-\delta)})$ for each N, establishing the proposition.

10. **Proof of Theorem 1.**

We roughly follow the pattern of Section 8. The following is a two-scale analog of Proposition 8.1.

PROPOSITION 10.1. — Assume the hypotheses of Theorem 1. Then Problem (1.1) has a linearizing transformation $w + Q$ where $Q \in B_r$ for some $r > 0$. Moreover, Q has outer and inner expansions

$$(10.1)\quad Q \sim \begin{cases} \displaystyle\sum_{i,j\geq 0} R_k(x, w)\varepsilon^k & a\varepsilon^{1-\delta} \leq |x| \\ \displaystyle\sum_{i,j\geq 0} S_k\left(\frac{x}{\varepsilon}, w\right)\varepsilon^k & |x| \leq a^{\delta}. \end{cases}$$

uniformly valid for any $0 < \delta < 1$ where $x^k R_k$ and $(|x| + \varepsilon)^{-k} S_k$ belong to B_r. These expansions can be x- or w-differentiated any number of times.

Proof. — We begin with the outer expansion. As in the proof of Proposition 8.1 we can suppose $g \in \delta B_{2r}$ by a preliminary shrinking of the w-domain. Since $g(o, o, w) = 0$ we have $g = xh + \varepsilon k$ where h, $k \in \delta B_{2r}$. Let $Q_0(x, w)$ be the solution of $-i\Lambda Q_0 + iQ_{0w}\Lambda = h(x, o, w + Q_0)$ in $\delta B_{\frac{3}{2}r}$ provided by Proposition 7.2 if δ is small. Let $Q \in \delta B_{\frac{3}{2}r}$ be the linearizing perturbation given by Proposition 7.1 and let $Q = Q_0 \circ U$ where also $U \in \delta B_{\frac{3}{2}r}$. Then

$$(10.2)\qquad \varepsilon^2 U_x - ix\Lambda U + ixU_w\Lambda w = \varepsilon f(x, \varepsilon, w + U)$$

where

$$f = (I + Q_{0w})^{-1}\left\{ k(x, \varepsilon, w + Q_0) + \frac{x}{\varepsilon}\,[h(x, \varepsilon, w + Q_0) - h(x, o, w + Q_0] + \varepsilon Q_{0x} \right\}$$

For small δ we can also suppose that $f \in \delta B_{\frac{3}{2}r}$. Moreover, g has a uniform asymptotic expansion in ε since k and h do. In a word we have normalized the problem to the case in which the perturbation has the form εf.

We now establish that Q_0 is the leading term of the outer expansion.

Let $U = \frac{\varepsilon}{x} V$. Then

$$V = \mathscr{L}\{ f(x, \varepsilon, w + U) - \varepsilon U_x \}.$$

This shows that for δ sufficiently small (so that the composition $f(x, \varepsilon, w + U)$ can be estimated) we have $V \in \delta B_{\frac{4}{3}r}$. Hence $Q = Q_0 \circ \frac{\varepsilon}{x} V = Q_0 + 0\left(\frac{\varepsilon}{x}\right)$.

To obtain higher expansions we replace U by $\frac{\varepsilon}{x} V$ in the previous relation obtaining

$$V = \mathscr{L}\left\{ f\left(x, \varepsilon, w + \frac{\varepsilon}{x} V\right) + \frac{\varepsilon^2}{x^2}(V - xV_x) \right\}.$$

From the standpoint of the outer expansion the occurrences of V in the right hand side are all small. Once again iteration can be seen to yield finite expansions with remainder of any order for V. Hence we omit further details. Finally the composition $Q = Q_0 \circ \frac{\varepsilon}{x} V$ also has an outer expansion.

To obtain the inner expansion we return to the original linearization problem (4.1) and its rescaled form (4.3). Its derivation is slightly more complex. In the preceeding argument x played the role of an inert parameter because the reduced problem (set $\varepsilon = 0$ in (4.1)) is no longer a differential equation in x. However here the reduced problem (set $\varepsilon = 0$ in (4.3)) has the same character as the full problem. We therefore require a somewhat more elaborate argument although the general idea is the same.

Since the first transformation $Q = Q_0 \circ U$ is valid in both scales it will suffice to establish an inner expansion for the solution $U = Q_0^{\#} \circ Q$ of (10.2). Let U_0 be a solution of the leading part of (10.2) in the inner scale

$$\varepsilon^2 U_{0x} - ix\Lambda(0)U_0 + ixU_{0w}\Lambda(0)w = \varepsilon f(o, o, w + U_0). \tag{10.3}$$

We can suppose that $U_0 \in \delta B_{\frac{4}{3}r}$ for small δ. Let $U = U_0 \circ R_1$. Then

$$\varepsilon^2 R_{1x} - ix\Lambda R_1 + ixR_{1w}\Lambda w = \varepsilon f_1(x, \varepsilon, w + R_1) \tag{10.4}$$

where

$$f_1 = (I + U_{0w})^{-1}\Big\{ f(x, \varepsilon, w + U_0) - f(o, o, w + U_0) + \frac{ix}{\varepsilon}(\Lambda(x) - \Lambda(0))U_0 - \frac{ix}{\varepsilon} U_{0w}(\Lambda(x) - \Lambda(0)) \Big\}.$$

Since $f_1(o, o, w) = 0$ we can suppose $f_1 \in (|x| + \varepsilon)B_{\frac{5}{4}r}$ for small δ. (This is the last shrinkage of δ. We prevent further shrinkage of the w-domain by choosing ε sufficiently small.) By Proposition 7.1, for ε sufficiently small, problem (10.4) has a solution $Q'_1 \in \varepsilon B_{\frac{6}{5}r}$. However since the solution to this problem is not unique it need not be true that $R'_1 = R_1$. We therefore appeal to the asymptotic uniqueness result, Proposition 9.1, which shows that, as elements of $B_{\frac{7}{6}r}$, $R_1 \sim R'_1$. It follows that $R_1 \in \varepsilon B_{\frac{7}{6}r}$. Let $R_1 = \varepsilon T_1$. Then

$$(10.5) \qquad \varepsilon^2 T_{1x} - ix\Lambda T_1 + ixT_{1w}\Lambda w = f_1(x, \varepsilon, w + \varepsilon T_1).$$

This is an equation in which the occurrence of T_1 in the right hand side is small and which therefore can be used for an iterative derivation of finite asymptotic expansions with remainder. To elaborate this procedure it is convenient to introduce the « inner » variable $s = \dfrac{x}{\varepsilon}$ into (10.5) and to separate out the « intermediate » and « slow » x dependences of f_1 in the following way. As a function of s, our solution U_0 of 10.3 satisfies $\dfrac{dU_0}{ds} - is\Lambda(0)U_0 + isU_{0w}\Lambda(0)w = f(o, o, w + U_0)$. By Proposition 7.2 B we can choose U_0 to be of the form $U_0(s, w)$. The perturbation f, of (10.5) can then be expressed

$$\begin{aligned} f_1 &= \{ I + U_{0w}(s, w) \}^{-1} \{ f(x, \varepsilon, w + U_0(s, w)) - f(0, 0, w + U_0(s, w) \} \\ &+ is(\Lambda(x) - \Lambda(0))U_0(s, w) - isU_{0w}(\Lambda(x) - \Lambda(0)) \\ &= F_1(s, x, c, w). \end{aligned}$$

We can then express (10.5) in the form

$$(10.6) \quad T_{1s} - is\Lambda(0)T_1 + isT_{1w}\Lambda(0)w = F_1(s, \varepsilon s, \varepsilon, w + \varepsilon T_1) + \varepsilon s^2 k(\varepsilon s)T_1$$

where $\varepsilon s^2 k(\varepsilon s)T_1 = \dfrac{ix}{\varepsilon}(\Lambda(x) - \Lambda(0)T_1 - \dfrac{ix}{\varepsilon}T_{1w}(\Lambda(x) - \Lambda(0))w$. If we regard the right hand side as a given element of $(|x| + \varepsilon)B_{\frac{8}{7}r}$ then we have a linear problem which by Proposition 7.3 has a solution $T'_1 \in B_{\frac{9}{8}r}$ which we indicate by

$$T'_1 = \mathscr{L}_1 \{ F_1(s, \varepsilon s, \varepsilon, w + \varepsilon T_1) + \varepsilon s^2 k(\varepsilon s)T_1 \}.$$

We cannot assert that $T'_1 = T_1$, but by Proposition 9.1 $T'_1 \sim T_1$ as elements of $B_{\frac{10}{9}r}$. Hence

$$(10.7) \qquad T_1 = \mathscr{L}_1 \{ F_1(s, \varepsilon s, \varepsilon, w + \varepsilon T_1) + \varepsilon s^2 K(\varepsilon s)T_1 \} + Z$$

where $Z \sim 0$. This relation implies immediately that

$$T_1 = \mathscr{L}_1(I_1(s, o, o, w)) + 0(|x| + \varepsilon).$$

Similarly iteration gives higher order finite expansions with remainder of the form

$$T_1 = \sum_{k=0}^{N} \varepsilon^k T_1(k) + \varepsilon^{N+1} R_{N+1}$$

where $\varepsilon^k T^{(k)} \in (|x| + \varepsilon)^k B_{\frac{11}{10}r}$ and $\varepsilon^{N+1} R_{N+1} \in (|x| + \varepsilon)^{N+1} B_{\frac{11}{10}r}$. The expansion $T_1 \sim \sum_{k=0}^{\infty} \varepsilon^k T_1^{(k)}$ thus is an asymptotic expansion in the inner region $|x| \leq a\varepsilon^\delta$ where $(|x| + \varepsilon) = 0(\varepsilon^\delta)$. It is not however strictly the desired inner expansion since it is obtained by using a uniform solution operator $\mathscr{L}_1$ provided by Proposition 7.1. This $\mathscr{L}_1$ is not the simpler holomorphic solution operator of the reduced problem in the inner scale analysed in Proposition 7.4.B. For this reason the functions $T_1^{(k)}$ are, in principle, functions of (s, ε, w) rather than (s, w) as required in our inner expansion. However the $T_1^{(k)}$ are solutions of the recursive system of linear equations obtained by expanding (10.6) in powers of ε. These problems have polynomial data, and Proposition 7.4 B ensures their recursive solubility for functions $T_1^{(k)}(s, w)$ obtained by separately solving the system on the complex domains $-\theta < \operatorname{Im}\log(1+s) < \theta$ and $-\theta < \operatorname{Im}\log(1-s) < \theta$, $|w| \leq \frac{8}{7}r$ and piecing together at $s=0$ their restrictions to the real s-axis. These solutions $T_1^k\left(\frac{x}{\varepsilon}, w\right)$ satisfy our derivative estimates and by Proposition 9.1 we have $T_1^{(k)}(x, \varepsilon, w) \sim T_1^{(k)}\left(\frac{x}{\varepsilon}, w\right)$ as elements of $B_{\frac{12}{11}r}$. Hence

$$T_1 \sim \sum_{k=0}^{\infty} \varepsilon^k T_1^{(k)}\left(\frac{x}{\varepsilon}, w\right)$$

which gives us the required inner expansion.

Finally, the composition $Q = Q_0 \circ U_0 \circ \varepsilon T_1$ has a similar expansion with terms in B_r which we obtain by combining the above expansion of T_1 with Taylor expansions of $Q_0(\varepsilon s, w)$ in both arguments and of $U_0(s, w)$ in its second argument.

This establishes the existence of a matched asymptotic expansion (10.1). To establish its identity with the matched formal solution $(\tilde{P}, \tilde{p})$ we observe that the reasoning of Proposition (8.2) applies separately in each scale and we easily obtain the corresponding result here which, as a parallel to Proposition 8.2, we state as:

PROPOSITION 10.2. — The matched expansion (10.1) is the unique matched formal solution of the linearizing problem (4.1).

Propositions 10.1 and 10.2 together establish relation (4.5) of Theorem 1. To obtain (4.6) we observe that if $y = z + Q(x, \varepsilon, z)$, then z is a solution of $\varepsilon^2 z' = ix\Lambda(x)Z$ with a possible discontinuity at $x = 0$. Suppose for $x < 0$

$$z = \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds\, c\,.$$

In any case for $x > 0$ z is given by

$$z = \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds\, z(0^+, \varepsilon)\,.$$

But y is an ordinary solution of the full problem, that is, y is continuous at zero. Hence

$$\tilde{p}(0^+, \varepsilon, z(0^+, \varepsilon)) \sim \tilde{p}(0^-, \varepsilon, c)$$

Solving for $z(0^+, \varepsilon)$ and using

$$y = P\left(x, \varepsilon, \exp \frac{i}{\varepsilon^2} \int_0^x s\Lambda(s)ds\, z(0^+, \varepsilon)\right) \qquad \text{for} \qquad x > 0$$

gives us formula (4.6). This completes the proof of Theorem 1.

This result fully justifies more conventional calculations with formal series to which we now turn.

11. **Proof of Theorem 2.**

By Theorem 1 there is a linearizing transformation $w + Q$ holomorphic at $w = 0$ with a matched asymptotic expansion. Let

$$Q(x, \varepsilon, w) = \sum_{|q|=2} Q_q(x, \varepsilon)w^q + 0(w^3)\,.$$

Then the Q_q must satisfy

$$\varepsilon^2 Q_{qx} - ix\{\Lambda(x) - \Sigma q_i \lambda_i(x)I\}Q_q = g_q(x, \varepsilon)$$

for $|q| = 2$. Moreover the Q_q's inherit matched asymptotic expansions from Q. For example it follows that for $|x| \geq a\varepsilon^\delta$ we have

$$Q_q = \frac{1}{ix}\{\Sigma q_i \lambda_i I - \Lambda\}^{-1} g_q + 0\left(\frac{\varepsilon}{x}\right)$$

Similarly for $|x| \leq a\varepsilon^{\delta}$ we have

$$Q_q = \frac{1}{i}\{\Sigma q_j \lambda_j(0) - \Lambda(0)\}^{-1} g_{qx}(0,0)$$
$$- \int_s^{\infty} \exp i \frac{(s^2 - \sigma^2)}{2} \{\Lambda(0) - \Sigma q_j \lambda_j(0)\} d\sigma g_{q\varepsilon}(0,0)$$
$$+ 0(|x| + \varepsilon)$$

$x > 0$ and

$$Q_q = \frac{1}{i}\{\Sigma q_j \lambda_j(0) - \Lambda(0)\} g_{qx}(0,0)$$
$$+ \int_{-\infty}^{s} \exp i \frac{(s^2 - \sigma^2)}{2} (\Lambda(0) - \Sigma q_j \lambda_j(0)) d\sigma + 0(|x| + \varepsilon)$$

for $x < 0$.

The scattering function S is determined by

$$c + Q(0^-, \varepsilon, c) = S + Q(0^+, \varepsilon, S)$$

which implies

$$S = c + Q(0^-, \varepsilon, c) - Q(0^+, \varepsilon, c) + 0(|c|^3).$$

Combining this with the preceeding formulas for Q we find

$$S = c + \sum_{|q|=2} \int_{-\infty}^{\infty} \exp \frac{-i\sigma^2}{2} \{\Lambda(0) - \Sigma q_j \lambda_j(0)\} d\sigma c^q$$
$$+ 0(|c|^3) + 0(\varepsilon).$$

Evaluating the integrals explicitly we obtain (4.7).

PART II

LINEARIZATION WITH DERIVATIVE ESTIMATES

12. A sketch of some analytical methods.

The balance of our analysis is essentially devoted to proving Proposition 7.1 which establishes the existence of a linearizing perturbation satisfying our derivative estimates. Most of the other propositions of Section 7 will appear in the course of proving this main analytical result. We shall be operating with the collection of spaces $\{B_{\phi,r}\}$ keeping much more careful account of ϕ and r than was necessary in our asymptotic analysis. We find a powerful tool for manipulating the necessary derivative estimates here by combining the ideas of Jacobowitz [5] with some simple resources from the theory of convexity in a form which the second author has used previously in [*6*].

The method of Jacobowitz is to represent differentiable functions g on a real domain Ω by a sequence $g^{(N)}$ of functions holomorphic on nested complex domains Ω_N in such a way that the convergence of $g^{(N)}$ to g as Ω_N converges to Ω accurately reflects the differentiability properties of g. For holomorphic functions h the problem of obtaining derivative estimates is simple since the Cauchy integral formula gives estimates in terms of $\sup |h|$ for all derivatives of h on a slightly smaller domain. In the next section we show how to obtain from data g in $B_{\phi,r}$ a sequence of approximations $g^{(N)}$ holomorphic in x so that the rate of convergence of the $g^{(N)}$ is governed by the convex conjugate function or Young transform of the convex function ϕ. This is defined by

$$\phi^*(t) = \max_u \{ tu - \phi(u) \} \tag{12.0}$$

We show roughly that if $g \in B_{\phi,r}$ then we can find $g^{(N)}$ and nested domains Ω_N so that on Ω_{N+1} we have $g^{(N)} - g^{(N+1)} \approx \exp - \phi^*(N)$. Conversely we find that if $h^{(N)}$ is holomorphic on Ω_N and $h^{(N+1)} - h^{(N)} \approx e^{-\phi^*(N)}$ on Ω_{N+1} then $h^{(N)} - h \in B_{\phi,r}$. We thus have a *duality* between smoothness properties and approximability properties corresponding to duality of conjugate convex functions. Here we require the simple fact that our growth moduli are closed convex functions [7] so that invariably we have $\phi^{**} = \phi$.

Our main use of this technique is to solve

$$\varepsilon^2 Q_x - ix\Lambda(x)Q + ixQ_w\Lambda(x)w = g(x, \varepsilon, w + Q) \tag{12.1}$$

by choosing a sequence of approximating problems

$$(12.2)_N \quad \varepsilon^2 Q_x^{(N)} - ix\Lambda^{(N)}(x)Q^{(N)} + ixQ_w^{(N)}\Lambda^{(N)}(x)w = g^{(N)}(x, \varepsilon, w + Q^{(N)})$$

with data holomorphic on an $x - w$ domain Ω_N (which we do not describe yet). We then solve $(12.2)_N$ on a smaller domain Ω'_N by the KAM method, obtaining a sequence $Q^{(N)}$ of holomorphic approximate linearizing perturbations. Finally we deduce the differentiability properties of $Q = \lim Q^{(N)}$ from the convergence properties of the sequence $Q^{(N)}$ and Ω'_N. A notable advantage of this procedure is that it permits us to use the rather arduous KAM procedure only in the simple case of holomorphic data.

We have relied on Sternberg's account of the KAM method [*4*], specifically his lucid and technically complete exposition of holomorphic problems admitting the action of a group nucleus. However his treatment of problems with C^∞ data is now technically obsolete. In any case since no general theorems obtained by this method appear to come within light-years of our higly idiosyncratic application we are forced to give a self-contained analysis. This proceeds along the following lines.

In Section 15 we solve the linear problem

$$\varepsilon^2 Q_{0x}^{(N)} - ix\Lambda^{(N)}(x)Q_0^{(N)} + ixQ_{0w}^{(N)}\Lambda^{(N)}(x)w = g^{(N)}(x, \varepsilon, w)$$

with holomorphic data. This problem is solvable by quadratures but demands some delicate analysis of paths of integration in the complex x-plane to obtain a solution $Q_0^{(N)}$ which depends regularly on ε. A most characteristic feature here is that $Q_0^{(N)} \approx D_x g^{(N)}$. We give up an x-derivative in passing from $g^{(N)}$ to $Q_0^{(N)}$ in exchange for estimating $Q_0^{(N)}$ uniformly in the singular parameter ε. This is the famous « loss of derivatives » phenomenon which precludes the use of ordinary successive approximation methods in solving the full linearization problem. This difficulty requires Kolmogorov's idea of quadratic convergence which we next describe.

In Section 17 we introduce the change of variable $Q^{(N)} = Q_0^{(N)} \circ R_1^{(N)}$ into $(12.2)_N$ obtaining

$$\varepsilon^2 R_{1x}^{(N)} - ix\Lambda^{(N)}(x)R_1^{(N)} + ixR_{1w}^{(N)}\Lambda^{(N)}(x)w = g_1^{(N)}(x, \varepsilon, w + R_1)$$

where $g_1^{(N)} = (I + Q_{0w}^{(N)})^{-1}\{g^{(N)}(x, \varepsilon, w + Q_0^{(N)}) - g^{(N)}(x, \varepsilon, w)\}$. We thus have a problem of precisely the form $(12.2)_N$ in which $g_1^{(N)} \approx g_w^{(N)}Q_0^{(N)} \approx g_w^{(N)}D_x g^{(N)}$ if $Q_0^{(N)}$ is small. Since $g^{(N)}$ is holomorphic in (x, w) we have the vital *quadratic* estimate $g_1^{(N)} \approx (g^{(N)})^2$ on a suitably smaller (x, w) domain. Now let $Q_1^{(N)}$ be a solution of

$$\varepsilon^2 Q_{1x}^{(N)} - ix\Lambda_{(x)}^{(N)}Q_1^{(N)} + ixQ_{1w}^{(N)}\Lambda^{(N)}(x)w = g_1^{(N)}(x, \varepsilon, w).$$

Let $R_1^{(N)} = Q_1^{(N)} \circ R_2^{(N)}$, etc. We obtain a sequence $Q_0^{(N)}, Q_1^{(N)}, \ldots$ of successive approximate linearizing perturbations which (because of quadratic convergence) converge rapidly to zero on some fixed smaller (x, w) domain Ω'_N and there give a linearizing perturbation in the form of an infinite composition

$$Q^{(N)} = Q_0^{(N)} \circ Q_1^{(N)} \circ Q_2^{(N)} \ldots$$

In Section 19 we conclude our basic existence argument by showing that the $Q^{(N)}$ can be found converging to a linearizing perturbation Q belonging to some $B_{\phi,r}$.

13. **Piecewise holomorphic approximation in B_r.**

In characterizing functions by membership in some $B_{r,\phi}$ we find it necessary to avoid the use of small ϕ's which can impose very subtle conditions on g. (e. g. $\phi(k) = k$ implies g is an entire function of s of exponential type) to which our methods are insensitive. Since $\phi' \geq 0$ implies $B_{\phi,r} \subset B_{\phi',r}$ we can always choose a larger ϕ if we please. Actually a decisive condition (Carleman [8]) is that $\phi(k)$ should grow rapidly enough so that $\sum^{\infty} e \dfrac{-\phi(k)}{k} < \infty$. This ensures the existence of partitions of unity obeying the corresponding derivative estimates. However we easily pass over these refined questions by agreeing to use only ϕ's which permit partitions of unity and which even satisfy $\phi(k)/k \log k \to \infty$ as $k \to \infty$.

Now suppose that g belongs to some $B_{\phi,r}$ and vanishes for $|x| \geq x_0$. The norm $\| g \|_{\phi,r}$ defined by (6.1.b) can be expressed in terms of the variable

$$z = \log\left(1 + \frac{|x|}{\varepsilon}\right) \operatorname{sgn} x \tag{13.1}$$

in the simple form

$$\| g \|_{\phi,r} = \sup_{z,q,l} r^{|q|} \, | D_z^l g_q | \exp - \phi(l)$$

Let $g_+ = g$ if $z \geq 0$ and $g_+ = 0$ if $z < 0$. Let $g - g_+ = g_-$. We assume (choosing ϕ larger if necessary) that g_- (that g_+) has a smooth extension to $(-\infty, \infty)$ vanishing identically for $z \geq 1$ (for $z \leq -1$) and satisfying the same bounds. Then

$$\| g \|_{\phi,r} = \sup_{z,q,l} r^{|q|} \exp - \phi(l) \max \{ | D_z^l g_- |, | D_z^l g_+ | \} \tag{13.2}$$

In terms of (13.2) we now construct a pair of sequences of holomorphic functions $g_{\pm}^{(N)}$ converging to $g_{\pm}$. The functions $g_{+}^{(N)}$ will be holomorphic on domains of the form $\Omega_{N,r'}$ where $r' < r$ and

$$\Omega_N = \left\{ x \mid -e^{-N} \leq \operatorname{Re} z \leq \log\left(1 + \frac{x_0 + e^{-N}}{\varepsilon}\right), \operatorname{Im} z \leq e^{-N} \right\}$$

$$\Omega_{N,r} = \Omega_N \times \{ | w | \leq r \}$$

The functions $g_{-}^{(N)}$ will be holomorphic on $-\Omega_{N,r'}$. The domains Ω_N are simply rectangles in the z-plane

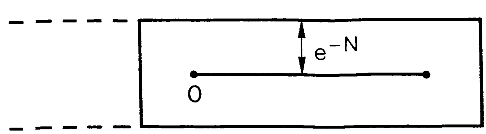

FIG. 13 A.

corresponding to truncated sectors

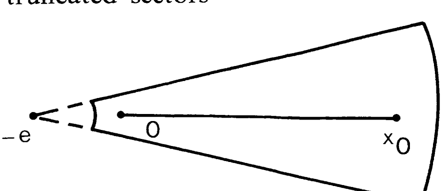

FIG. 13 B.

in the x-plane. As $N \to \infty$ these latter domains converge to the interval $[0, x_0]$. Our construction of the approximating sequences appears in the proof of the following result.

PROPOSITION 13.1 (Conjugate duality between smoothness and holomorphic approximability). — Suppose $g \in B_{\phi,r}$ and satisfies (13.2) where $\phi(k)/k \log k \to \infty$ as $k \to \infty$. Then there exist sequences $g_\pm^{(N)}$ of approximations to $g_\pm$ which are holomorphic on $\pm \Omega_{N,r'}$ for any $r' < r$ and there satisfy estimates of the form

$$(13.3) \qquad \sup_{\pm \Omega_{N,r'}} | g_\pm^{(N)} | < M\left(1 - \frac{r'}{r}\right)^{-n}$$

(where n is the dimension of the system).

$$(13.4) \qquad \sup_{\pm \Omega_{N+1,r'}} | g_\pm^{(N+1)} - g_\pm^{(N)} | < M\left(1 - \frac{r'}{r}\right)^{-n} \exp - \phi^*(N-1) + N$$

Conversely if the $h_\pm^{(N)}$ are sequences of functions holomorphic on $\pm \Omega_{N,r}$, $| h_\pm^{(0)} | \leq 1$ and

$$\sup_{+\Omega_{N,r}} | h_\pm^{(N+1)} - h_\pm^{(N)} | < \exp - \phi^*(N)$$

then the limits $h_\pm = \lim_{N\to\infty} h_\pm^{(N)}$ exist and satisfy

$$(13.5) \qquad (r')^{|q|} | D_z^l h_{\pm q} | \leq M(\delta) \exp (1 + \delta)\phi(l + 1)$$

for any $r' < r$ and any $\delta > 0$.

Proof. — Let $\hat{g}_\pm$ be the z-Fourier transform of the smooth extensions of $g_\pm$ appearing in (13.2). Let $0 \leq \theta(z) \leq 1$ be a smooth function vanishing identically for $| z | \geq 1$ and identically 1 for $| z | \leq e^{-1}$. We choose

$$(13.6) \qquad g_\pm^{(N)}(z, w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(e^{-N}\sigma) g_\pm(\sigma, w) \exp iz\sigma d\sigma .$$

Then it is easy to verify that

$$r^{|q|}(g_{\pm,q}^{(N+1)} - g_{\pm,q}^{(N)})$$
$$= \frac{r^{|q|}}{2\pi} \iint_{e^{N-1} \leq |\sigma| \leq e^{N+1}} \{ \theta(e^{-N-1}\sigma) - \theta(e^{-N}\sigma) \} g_{\pm,q}(s) \exp i(z - s)\sigma ds d\sigma$$
$$= \frac{r^{|q|}}{2\pi} \iint_{e^{N-1} \leq |\sigma| \leq e^{N+1}} \frac{1}{1+(z-\zeta)^2} (1 - D_\sigma^2) \{ [\theta(e^{-N-1})\sigma) - \theta(e^{-N}\sigma)]$$
$$. (i\sigma)^{-l} \} D_\zeta^l g_\pm \exp i\sigma(z - \zeta) d\zeta d\sigma .$$

For $|\operatorname{Im} z| \le e^{-N-1}$ we can estimate this by

$$r^{|q|}\,|(g_{\pm,q}^{(N+1)} - g_{\pm,q}^{(N)})| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\zeta}{1+(z-\zeta)^2}\int_{e^{N-1}\le|\sigma|\le e^{N+1}} |(1-D_\sigma^2)[(e^{-N-1}\sigma) - \theta(e^{-N}\sigma)]\sigma^{-l}|\,d\sigma.\exp\phi(l)$$
$$\le M\exp\phi(l)-(N-1)l+N \le M\exp -\phi^*(N-1)+N.$$

Hence on $\Omega_{N+1,r'}$, where $r' < r$, we can estimate

$$|g_\pm^{(N+1)} - g_\pm^{(N)}| \le \left|\sum_{|q|}(g_{\pm,q}^{(N+1)})w^q\right| \le M\exp\{-\phi^*(N-1)+N\}.\sum_{|q|}\left(\frac{r'}{r}\right)^{|q|}$$
$$\le M\left(1-\frac{r'}{r}\right)^{-n}\exp -\phi^*(N-1)+N$$

which proves (13.4).

Similarly on $\Omega_{0,r'}$ we can estimate

$$r^{|q|}g_{\pm,q}^{(0)}(z) = \frac{r^{|q|}}{2\pi}\iint_{e^{-1}\le|\sigma|\le 1}\frac{1}{1+(z-\zeta)^2}(1-D_\sigma^2)\theta(\sigma)g_{\pm,q}(\zeta,n)\exp i\sigma(z-\zeta)d\zeta d\sigma$$

by

$$r^{|q|}\,|g_{\pm,q}^{(0)}| \le \frac{1}{2\pi}(1-e^{-1})\sup|(1-D_\sigma^2)\theta_r|\,e \le M$$

This implies that on $\Omega_{0,r'}$ we have

$$|g_\pm^{(0)}| \le M\left(1-\frac{r'}{r}\right)^{-n}.$$

Hence on $\Omega_{N,r'}$

$$|g_\pm^{(N)}| \le |g_\pm^{(0)}| + \sum_{j=0}^{N-1}|g_\pm^{(j+1)} - g_\pm^{(j)}|$$
$$\le M\left(1-\frac{r'}{r}\right)^{-n}\sum_{j=0}^{N}\exp -\phi^*(j-1)+j$$
$$\le M\left(1-\frac{r'}{r}\right)^{-n}\sum_{j=0}^{N}e^{-j+2}\exp -\phi^*(j-1)+2(j-1)$$
$$\le M\left(1-\frac{r'}{r}\right)^{-n}\exp +\phi(2)\sum_{j=0}^{\infty}e^{2-j}$$
$$\le M'\left(1-\frac{r'}{r}\right)^{-n}$$

This establishes 13.3.

For the converse part of the argument if $h^{(N)}$ satisfy $|h^{(0)}| \leq 1$ and $|h^{(N+1)} - h^{(N)}| \leq e^{-\phi^*(N)}$ then we can estimate $D_z^l(h_q^{(N+1)} - h_q^{(N)})$ on $\Omega_{N+2,r'}$ by the Cauchy integral formula. Since each z in the projection of $\Omega_{N+2,r'}$ is the center of a disc of radius e^{-N-2} contained in the projection of $\Omega_{N+1,r'}$ we have

$$\begin{aligned}\sup_{q,\Omega_{N+2,r'}} (r')^{|q|} \,|\, D_z^l(h_q^{(N+1)} - h_q^{(N)}) \,| &\leq \frac{l\,!}{(e^{-N-2})^l} \exp - \phi^*(N) \\ &\leq l\,!\, e^{2l} \exp - \phi^*(N) + lN \\ &\leq e^{-N} l\,! \exp - \phi^*(N) + (l+1)N + 2l \\ &\leq Me^{-N} \exp \phi(l+1) + 2l + l \log l\,.\end{aligned}$$

Since $l \log l = 0(\phi(l))$ we can estimate this more crudely as

$$M(\delta)e^{-N} \exp (1 + \delta)\phi(l + 1)\,.$$

This combined with the easily obtained similar estimates for $h^{(0)}$

$$|\, D_z^l h^{(0)} \,| \leq l\,!\, e^l \leq M(\delta) \exp (1 + \delta)\phi(l + 1)$$

implies that the restrictions of $h^{(N)}$ to the positive real axis converge geometrically in the norm $\|\ \|_{\phi',r'}$ specified by $\phi'(l) = (1 + \delta)\phi(l + 1)$.

We remark that the shifts of argument $l \to l + 1$, $N \to N - 1$ and the factor $1 + \delta$ in our conclusions produce « loss of derivatives » so that at our level of formulation we have only an approximate duality. We have not attempted to minimize loss of derivatives although for other applications it would be valuable to do so.

14. Closure properties of the family $\{ B_r \}$.

We now use Proposition 13.1 of the previous section to establish closure properties of the kind given in Proposition 7.7. We obtain most of these properties from the following simple principle.

PROPOSITION 14.1. — Let F be a mapping from $\pm\,\Omega_{N,r}$ (from $\pm\,\Omega_{N,r} \times \pm\,\Omega_{N,r}$) into $\pm\,\Omega_{N+1,r'}$ for $r' < r$ satisfying $F(0)=0$ $(F(0,0)=0)$ which is Lipschitzian in the sense that

$$\sup_{\pm\Omega_{N+1,r'}} |\, F(g_1) - F(g_2) \,| \leq M \sup_{\pm\Omega_{N,r}} |\, g_1 - g_2 \,|$$

$$.\,\Big(\sup_{\pm\Omega_{N+1,r'}} |\, F(g_1, g_2) - F(g_3, g_4) \,| \leq M \sup_{\pm\Omega_{N,r}} (\,|\, g_1 - g_3 \,|, |\, g_2 - g_4 \,|\,)\Big)$$

for arguments g_i satisfying $\sup_{\pm\Omega_{N,r}} |\, g_i \,| \leq 1$. Then F induces a mapping from B_r (from $B_r \times B_r$) into $B_{r'}$.

Proof. — Suppose $g \in B_{\phi,r}$. Let $g_{\pm}^{(N)}$ be the sequence of approximations of Proposition 13.1. Then for $r' < r'' < r$

$$\sup_{\pm\Omega_{N,r''}} | g_{\pm}^{(N+1)} - g_{\pm}^{(N)} | \leq \left(1 - \frac{r''}{r}\right)^{-n} \exp - \phi^*(N-1) + N + c .$$

This implies that

$$\sup_{\pm\Omega_{N+1,r'}} | F(g_{\pm}^{(N+1)}) - F(g_{\pm}^{(N)}) | \leq \left(1 - \frac{r''}{r}\right)^{-n} \exp - \phi^*(N-1) + N + c'$$

and $| F(g_{\pm}^{(0)}) | \leq \left(1 - \frac{r''}{r}\right)^{-n} \exp c'$. Hence by the converse part of Proposition 13.1 the restrictions of $F(g_{\pm}^{(N)})$ to $z \geq 0$ converge to functions $f_{\pm}$ satisfying the estimates of $\| \ \|_{\phi',r'}$ where

$$\phi' = (1 + \delta)(\phi^*(N-1) + N + c')^* = (1 + \delta)\phi(l + 1) + c_1 l + c_2 .$$

We define $F(g)$ to be $f_{\pm}$ for $\pm z \geq 0$. The case of the binary mapping is similar.

Parts (*a*)-(*e*) of Proposition 7.7 follow easily from this. To establish part (*f*) suppose $g \in B_{r,\phi}$ and choose $r'' < r' < r$. Let $g_{\pm}^{(N)}$ be the approximations to g given by Proposition 13.1. Then finding $(\mu g)^*$ is equivalent to finding μh where h is the solution of $g(w + \mu h(w)) + h(w) = 0$. For μ sufficiently small (depending only on the common upper bound for the $| g_{\pm}^{(N)} |$) the approximating problems $g^{(N)}(w + h(w)) + h(w) = 0$ have solutions $h_{\pm}^{(N)}$ holomorphic in $\Omega_{N,r'}$ satisfying estimates of the form (13.3) and (13.4). By the converse part of Proposition 13.1 these converge to $h_{\pm}$ in $B_{r'',\phi'}$ for a suitably larger ϕ'. Then $(\mu g)^{\#} = \mu h_{\pm}$ for $\pm \times \geq 0$ establishes (*f*).

15. The linear problem with holomorphic data.

We next confront a main difficulty: to solve the linear equation

$$\varepsilon^2 Q_{0x} - ix\Lambda(x)Q_0 + ixQ_{0w}\Lambda(g) = \varepsilon g(x, \varepsilon, w)$$

supposing that Λ and g are holomorphic in x on a narrow domain about the interval $[0, x_0]$. Formally all solutions can be obtained by w-series expansion and quadratures. However it requires delicate analysis to show that any one of these solutions has the properties we require. In particular this solution must involve only a small shrinkage in the domain of analyticity in passing from g to Q. The following family of *exponential horns* $\mathscr{E}_x$ is a suitable family of x-domains for our purposes. Our reasons for this unusual choice will appear soon.

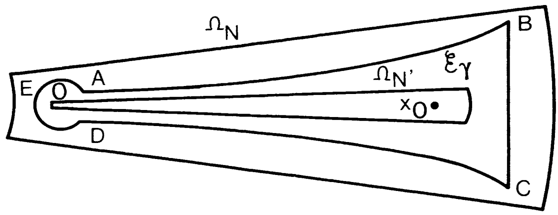

FIG. 15 A. — An exponential horn $\mathscr{E}_\gamma$ between Ω_N and $\Omega_{N'}$.

The curve AED is an arc of the circle $|x| = \varepsilon e^{-\gamma}$. Curves AB and DC have the parametric form

$$x(\rho) = \rho \pm ie^{-\gamma}\left(\frac{\rho}{x_0 + e^{-\gamma}}\right)\exp E(\rho - x_0 - e^{-\gamma}) \tag{15.1}$$

where E is a large parameter. A property of $\mathscr{E}_\gamma$ which is essential for our purposes is illustrated in Figure 15.A, namely that the domains $\mathscr{E}_\gamma$ can be estimated from above and below by domains Ω_N of the kind appearing in our method of holomorphic approximation in Section 13. This section is devoted to proving the following result.

PROPOSITION 15.1 (Solution of the linearized problem on an exponential horn). — Suppose $\Lambda(x)$ is holomorphic for $|\operatorname{Re} x| \le x_0 + e^{-\gamma}$, $|\operatorname{Im} x| \le e^{-\gamma}$ and satisfies $\sup_{x,j}(|\lambda_j|, |\lambda'_j|) < M$ and $|\lambda_j - \Sigma q_k\lambda_k| \ge \mu|q|$ for $|q| \ge 2$. Suppose g is holomorphic on $\mathscr{E}_\gamma \times (|w| \le r)$. Then there exist constants $\gamma_0(\mu, M)$ and $C_0(\mu, M)$ such that for $\gamma > \gamma_0$ the problem

$$\varepsilon^2 Q_{0x} - ix\Lambda(x)Q_0 + ixQ_{0w}\Lambda(x)w = \varepsilon g(x, \varepsilon, w)$$
$$Q_0 = 0(|w|^2)$$

has a solution satisfying

$$\sup_{\mathscr{E}_{\gamma'} \times (|w| < r')} |Q_0| \le C_0 \frac{e^{\gamma'}}{\gamma' - \gamma}\left(1 - \frac{r'}{r}\right)^{-n} \sup_{\mathscr{E}_\gamma \times (|w| \le r)} |(x + \varepsilon)^{-1} g|$$

for any $\gamma' > \gamma$ and $r' < r$.

Proof. — We have the following explicit general solution. Let

$$g(x, \varepsilon, w) = \Sigma g_q(x, \varepsilon)w^q \qquad \text{and} \qquad \mu_{jq} = \lambda_j - \Sigma q_k\lambda_k.$$

Let $\Gamma_{jq}(x)$ be rectifiable paths in $\mathscr{E}_\gamma$ terminating at x. Then

$$Q_0(x, \varepsilon, w) = \sum_{|q| \ge 2} w^q \frac{1}{\varepsilon}\int_{\Gamma_{jq}(x)} \operatorname{diag}\left[\exp \frac{i}{\varepsilon^2}\int_\zeta^x \sigma\mu_{jq}(\sigma)d\sigma\right] g_q(\zeta, \varepsilon)d\zeta \tag{15.2}$$

is a solution for $|w| \le r' < r$. The basic problem here is to find paths $\Gamma_{jq}(x)$

along which the integrals can be bounded and which cover the exponential horn $\mathscr{E}_\gamma$ suitably for each (j, q). It is in constructing such paths that we indicate our rather unusual choice of the exponential horn as a solution domain. We require some lemmas which like Proposition 15.1 establish certain conclusions for sufficiently large γ, that is, for sufficiently narrow horns.

We introduce for each (j, q) the auxiliary variables

$$\tau_{jq}(x) = \left\{ \frac{2}{\mu_{jq}(0)} \int_0^x \sigma\mu_{jq}(\sigma)d\sigma \right\}^{1/2}. \tag{15.3}$$

We will show that there exists $\gamma_1(\mu, \mathrm{M})$ such that if $\gamma > \gamma_1$ then the τ_{jq} are holomorphic and univalent for $|\operatorname{Im} x| < e^{-\gamma}$. Assuming this for the moment let

$$\tau_{jq}^{-1}(t) = \zeta_{jq}(t). \tag{15.4}$$

We can then express (15.2) componentwise (indexed by j) as

$$\mathrm{Q}_{0j}(x, \varepsilon, w) = \sum_{|q|\geq 2} w^q \frac{1}{\varepsilon} \int_{\Gamma'_{jq}} \left[\exp \frac{\mu_{jq}(0)}{2\varepsilon^2}(\tau_{jq}^2(x) - t^2) \right] \cdot g_{jq}(\zeta_{jq}(t)) \cdot \frac{d\zeta_{jq}(t)}{dt} dt \tag{15.5}$$

where $\Gamma'_{jq}(x) = \tau_{jq}(\Gamma_{jq}(x))$. To carry out our estimates in this parameterization we require lemmas for establishing the collective univalence of the mappings τ_{jq} of (15.2).

LEMMA 15.2. — Suppose $h(x)$ is holomorphic on the rectangle $|\operatorname{Im} x| \leq e^{-\gamma}$, $|\operatorname{Re} x| \leq x_0 + e^{-\gamma}$ where $|h(x)| \geq m > 0$ and $|h|, |h'| \leq \mathrm{M}$ and h is positive on the real axis. Then there exists a $\gamma_1(m, \mathrm{M})$ such that $\int_0^x \zeta h(\zeta)d\zeta$ is univalent provided $\gamma > \gamma_1$.

Proof. — We show that on the rectangle of the form indicated

$$\left| \int_{x_1}^{x_2} h(\zeta)d\zeta \right| \geq \frac{m}{2}|x_1 - x_2|$$

which implies univalence. Let $x = \rho + i\sigma$. Then

$$\begin{aligned} \int_{x_1}^{x_2} h(\zeta)d\zeta &= \int_{\rho_1+i\sigma_1}^{\rho_2+i\sigma_2} h(\zeta)d\zeta \\ &= \int_{\rho_1}^{\rho_2} h(\rho)d\rho + i\int_{\sigma_1}^{\sigma_2} h(\rho_2)d\sigma \\ &+ \int_{\rho_1}^{\rho_2} [h(\rho + i\sigma_1) - h(\rho)]d\rho + i\int_{\sigma_1}^{\sigma_2} [h(\rho_2 + i\sigma) - h(\rho_2)]ds. \end{aligned}$$

Since h is real on the real axis and $|h| \geq m$, for $\gamma > \gamma_1$ we can estimate

$$\left|\int_{x_1}^{x_2} h(\zeta)d\zeta\right| \geq m[(\rho_1-\rho_2)^2+(\sigma_1-\sigma_2)^2]^{1/2}-|\rho_2-\rho_1|e^{-\gamma_1}\mathrm{M} - |\sigma_1-\sigma_2|e^{-\gamma_1}\mathrm{M}$$
$$\geq m\left(1-2\frac{\mathrm{M}}{m}e^{-\gamma_1}\right)|x_1-x_2|.$$

Choosing γ_1 so that $\left(1-\frac{2\mathrm{M}}{m}e^{-\gamma_1}\right) > \frac{1}{2}$ we obtain the required inequality.

LEMMA 15.3. — Assuming the hypotheses of Lemma 15.2 γ_1 can be choosen so that $\left|\int_0^x \zeta h(\zeta)ds\right| \geq \frac{m}{4}|x|^2$.

Proof. — Since the rectangle is convex we can parameterize $\int_0^x \zeta h(\zeta)d\zeta$ in the form $x^2\int_0^1 th(xt)dt = x^2\int_0^1 th(\rho t+i\sigma t)dt$. Using the estimates $|h| \geq m, |h'| \leq \mathrm{M}$, for $\gamma \geq \gamma_1$ we find that

$$\left(\int_0^x \zeta h(\zeta)d\zeta\right) \geq |x|^2\int_0^1 t\{m-\mathrm{M}e^{-\gamma_1}\}\,dt.$$

Our previous choice of γ_1 gives $\frac{\mathrm{M}}{m}e^{-\gamma_1} < \frac{1}{2}$ which implies the conclusion.

LEMMA 15.4. — There exists $\gamma_2(\mu, \mathrm{M})$ such that for $\gamma > \gamma_2$ each mapping $x \to \tau_{jq}(x)$ is univalent on $\mathscr{E}_\gamma$. Moreover there is a bound $\mathrm{C}_2(\mu, \mathrm{M})$ for the moduli of the first two derivatives of the inverse mapping $t \to \zeta_{jq}(t)$ on $\tau_{jq}(\mathscr{E}_\gamma)$.

Proof. — For

$$|\operatorname{Im} x| < e^{-\gamma}, \quad |\operatorname{Re} x| \leq x_0 + e^{-\gamma},$$
$$\max|\mu_{j,q}| \leq (|q|+1)\max_{|x|\leq x_0+e^{-\mathrm{N}}}|\lambda_j(x)| = (|q|+1)\mathrm{M}.$$

Also by hypothesis $\min|\mu_{j,q}| \geq |q|\mu$. By Lemma 15.3 we can choose γ_1 so that for $\gamma > \gamma_1$ we have $\left|\int_0^x \zeta\mu_{j,q}(\zeta)d\zeta\right| \geq \frac{\mu}{4}|q||x|^2$. Then

$$x^{-1}\tau_{jq}(x) = \left\{\frac{1}{2x^2\mu_{jq}(0)}\int_0^x \zeta\mu_{jq}(\zeta)d\zeta\right\}^{1/2}$$

can be estimated from above and below by

$$\left(\frac{\frac{\mu}{4}|q|}{M(|q|+1)}\right)^{1/2} \leq |x^{-1}\tau_{jq}(x)| \leq \left(\frac{M(|q|+1)}{\mu|q|}\right)^{1/2}$$

or more crudely by

$$\left(\frac{\mu}{6M}\right)^{1/2} \leq |x^{-1}\tau_{jq}(x)| \leq \left(\frac{3}{2}\frac{M}{\mu}\right)^{1/2}.$$

Hence for $\gamma > \gamma_1$ we can estimate $\tau'_{jq}(x) = \dfrac{x\mu_{jq}(x)}{\mu_{jq}(o)\tau_{jq}(x)}$ from above and below by

$$\left(\frac{2}{3}\frac{\mu}{M}\right)^{1/2} \frac{\mu|q|}{M(|q|+1)} \leq |\tau'_{jq}(x)| \leq \frac{M(|q|+1)}{\mu|q|}\left(\frac{6M}{\mu}\right)^{1/2}$$

or

$$\left(\frac{2}{3}\frac{\mu}{M}\right)^{3/2} \leq |\tau'_{jq}(x)| \leq \frac{3}{2}\sqrt{6}\left(\frac{M}{\mu}\right)^{3/2}.$$

By Lemma 15.2 there exists a γ_2 such that for $\gamma > \gamma_2$ the mapping $\tau_{j,q}$ is univalent. The lower bound $\left(\frac{2}{3}\frac{\mu}{M}\right)^{3/2}$ for $\tau'_{j,q}$ gives an upper bound for $\zeta'_{jq} = \dfrac{1}{\tau'_{jq}}$. Similarly $\zeta''_{jq} = \dfrac{-\tau''_{jq}}{\tau'_{jq}}$ and some calculation shows that

$$\begin{aligned}
\tau''_{jq}/\tau'_{jq} &= \frac{d}{dt}\log\frac{x\mu_{jq}}{\mu_{jq}(o)\tau_{jq}} = \frac{\mu'_{jq}}{\mu_{jq}} + \frac{1}{x} - \frac{\tau'_{jq}}{\tau_{jq}} \\
&= \frac{\mu'_{jq}}{\mu_{jq}} + \frac{2\tau^2_{jq} - x(\tau^2_{jq})'}{2x\tau^2_{jq}} = \frac{\mu'_{jq}}{\mu_{jq}} - \frac{\frac{1}{x^3}\int_0^x \zeta\mu'_{jq}(\zeta)d\zeta}{\mu_{jq}(o)(x^{-1}\tau_{jq})^2} \\
&\leq \frac{2M}{\mu} + \frac{2M}{\mu}\left(\frac{6M}{\mu}\right)^2
\end{aligned}$$

giving us an upper bound for ζ''_{jq} also. This completes the proof of the lemma.

This lemma allows us to parameterize each curve Γ_{jq} (yet to be determined) by its image in the t-plane under the mapping $t = \tau_{jq}(x)$. To determine Q_0 (given by 15.1) it is usual to seek paths of integration along which the exponents $\dfrac{i}{\varepsilon^2}\displaystyle\int_\zeta^x \zeta\mu_{jq}(\zeta)ds$ in this formula have preponderantly negative real parts. The geometry of our problem precludes this. Instead we rely upon paths along much of which these integrals are purely imaginary.

However if $x = 0(\varepsilon)$ and $\zeta = 0(\varepsilon)$ these integrals are bounded in ε. This permits us to choose the innermost portions of our paths freely (and accounts for the more or less arbitrarily chosen inner circular portion of our exponential horn). The most decisive property of this domain is described in the following lemma.

LEMMA 15.5. — There exist a constant $E(\mu, M)$ in the parameterization (15.1) of the curves AB and DC and a $\gamma_0(\mu, M)$ such that for $\gamma > \gamma_0$ each image of these curves and the vertical segment CB under a mapping $\tau_{j,q}$ is transverse to the family of hyperbolas, $\operatorname{Im} t^2 = \text{constant}$.

Proof. — Let $\eta(x) = \mu_{jq}^{-1}(0)\mu_{jq}(x)$. Then our hypotheses imply

$$\frac{\mu|q|}{M(|q|+1)} \leq |\eta(x)| \leq \frac{M(|q|+1)}{\mu|q|} \quad \text{and} \quad |\eta'(x)| \leq \frac{M(|q|+1)}{\mu|q|}$$

so that $\frac{\mu}{2M} \leq |\eta| \leq 2\frac{M}{\mu}$ and $|\eta'| \leq \frac{2M}{\mu}$. For the image of the parametric curve $x(\rho) = \rho + i\sigma(\rho)$ under $\tau_{jq}(x) = \left(\frac{2}{\mu_{jq}(o)}\int_0^x \zeta\mu_{jq}(\zeta)d\zeta\right)^{1/2}$ to be transversal to $\operatorname{Im} t^2 = \text{constant}$, it suffices that

$$\operatorname{Im}\frac{d}{d\rho}\tau_{jq}^2(x(\rho)) > 0 \qquad \text{or} \qquad \operatorname{Im}(1 + i\sigma')(\rho + i\sigma)k(\rho + i\sigma) > 0.$$

Estimating $|\eta(\rho + i\sigma) - \eta(\rho)| \leq \frac{2M}{\mu}\sigma$ we find that it suffices that (for $\rho \geq 0$, $\sigma \geq 0$, $\sigma' \geq 0$) $\eta(\rho)(\rho\sigma' + \sigma) \geq \frac{2M}{\mu}\sigma|1+i\sigma'|\,|\rho + i\sigma|$ or even $\eta(\rho)(\rho\sigma' + \sigma) \geq \frac{2M}{\mu}(\rho + \sigma)(1 + \sigma')$. Since $|\eta| \geq \frac{\mu}{2M}$, this will be a consequence of $\left\{\rho - \left(\frac{2M}{\mu}\right)^2(\rho + \sigma)\sigma\right\}\sigma' \geq \left(\frac{2M}{\mu.}\right)^2(\rho + \sigma)\sigma$. In the parameterization (15.1) we have $\sigma \leq e^{-\gamma}\frac{\rho}{x_0 + e^{-\gamma}} \leq 1$ so it suffices that

$$\rho\left(1 - \left(\frac{2M}{\mu}\right)^2(x_0 + 1)\frac{e^{-\gamma}}{x_0}\right)\sigma' \geq \left(\frac{2M}{\mu}\right)^2(x_0 + 2)\sigma.$$

Choosing γ_0 so large that $\left(\frac{2M}{\mu}\right)^2(x_0+1)\frac{e^{-\gamma_0}}{x_0} < \frac{1}{2}$ we find that it suffices that

$$\rho\sigma' \geq \frac{8M^2}{\mu^2}\left(\frac{x_0 + 1}{x_0}\right)\sigma.$$

To check this it suffices to substitute $\sigma = \rho e^{E\rho}$ yielding $1 + E\rho > \frac{8M^2}{\mu^2}\left(\frac{x_0+1}{x_0}\right)\rho$, whence $E = \frac{8M^2}{\mu^2}\left(\frac{x_0 + 1}{x_0}\right)$ is a suitable choice of E. Transversality of the image of CB for large γ is much easier to check along similar lines and we omit the verification.

We now construct curves $\Gamma_{jq}(x)$ on $\mathscr{E}_\gamma$ by utilizing properties of the image $\tau_{jq}(\mathscr{E}_\gamma)$. (See figure 15 B.)

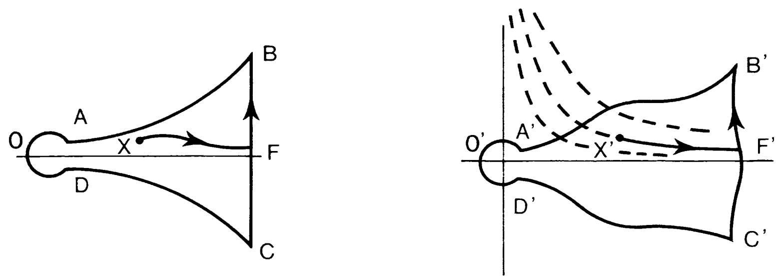

FIG. 15 B.

The paths in the t-plane are of two kinds. If x lies inside the curve OABCDO then we follow the level curve $\operatorname{Im} \tau_{jq}^2 = \text{constant}$ through x until it intersects the segment BC. This is possible since the hyperbola $\operatorname{Im} t^2 = \operatorname{Im} \tau_{jp}^2(x)$ does not intersect the real axis at all and can intersect the curve O′B′ (or O′C′) only once (by the global monotonicity of $\operatorname{Im} t^2$ along O′B′ established in the previous lemma). Thus the hyperbola must intersect B′C′. Since B′C′ is transverse to the hyperbolas, $\operatorname{Im} t^2$ is monotonic along B′C′. We complete $\Gamma_{jq}(x)$ by following CB to either B or C in the direction along which the imaginary part of $\int_0^x \zeta \mu_{jq}(\zeta) d\zeta = 2\mu_{jq}(o)t^2$ is decreasing. If x lies outside OABCDO we joint x to the origin by a line segment and then complete the curve as in the previous case. This completes our definition of the solution operator given by equation (15.2). It remains to verify that the resulting Q_0 satisfies the conclusions of Proposition 15.1. We must estimate integrals of the form

$$(15.6) \qquad I_{jq} = \frac{1}{\varepsilon}\int_{\Gamma_{jq}(x)} \exp - \frac{i}{\varepsilon^2}\int_\zeta^x \sigma \mu_{jq}(\sigma) d\sigma \,.\, g_{jq}(\zeta) d\zeta \,.$$

LEMMA 15.6. — *For $\gamma > \gamma_0(\mu, M)$ the integral I_{jq} defined by equation* (15.6) *satisfies* $| I_{jq} | \leq C_1(\mu, M) \sup_{E_\gamma} (| g_{jq}(x) |, | g'_{jq}(x) |)$.

Proof. — Depending on the location of x in E_γ the curve $\Gamma_{jq}(x)$ consists of one, two, or three different pieces: Γ_1, a segment through the origin of length less than $\varepsilon e^{-\gamma}$; Γ_2, a vertical segment on the right edge of E_γ; and Γ_3, a level curve of $\operatorname{Im} \int_0^x \sigma \mu_{jq}(\sigma) d\sigma$. We let I_1, I_2, and I_3 be the integrals along these subcurves respectively. I_1 is present only if $x = \varepsilon s$ where $|s| \leq e^{-\gamma}$. In this case

$$I_1 = \frac{1}{\varepsilon} \int_0^{\varepsilon s} \exp - \frac{i}{\varepsilon^2} \int_\zeta^{\varepsilon s} \sigma \mu_{jq}(\sigma) d\sigma . g_{jq}(\zeta) d\zeta$$

$$= \int_0^s \exp - i \int_\zeta^s \sigma \mu_{jq}(\varepsilon\sigma) d\sigma . g_{jq}(\varepsilon\zeta) ds$$

which can be bounded by

$$|I_1| \leq e^{-\gamma} + e^{-2\gamma} M \sup_{E_\gamma} |g_{jq}| .$$

We next estimate I_2. Suppose x lies on Γ_3 and $\mu_{jq}(x_0) < 0$. In this case the level curve of $\operatorname{Im} \int_0^\zeta \sigma \mu_{jq}(\sigma) d\sigma$ through x meets Γ_2 in a point x'. We estimate

$$I_2 = \frac{1}{\varepsilon} \int_{x'}^{B} \exp - \frac{i}{\varepsilon^2} \int_\zeta^x \sigma \mu_{jq}(\sigma) d\sigma g_{jq}(\zeta) d\zeta$$

very crudely $\left(\text{using } \operatorname{Re} i \int_{x'}^x \sigma \mu_{jq}(\sigma) d\sigma = 0\right)$ to obtain

$$|I_2| \leq \sup_{E_\gamma} |g_{jq}| \frac{1}{\varepsilon} \int_{x'}^{B} \exp - \operatorname{Re} \frac{i}{\varepsilon^2} \int_\zeta^B \sigma \mu_{jq}(\sigma) d\sigma \, |d\zeta|$$

Now parameterize the segment $x'B$ by $\zeta = x_0 + e^{-\gamma}(1 + it)$ and estimate $|\mu_{jq}(x_0 + e^{-\gamma}(1 + it)) - \mu_{jq}(x_0)| \leq M e^{-\gamma}(|q| + 1)$. Then for large γ we find

$$|I_2| \leq \sup_{E_\gamma} |g_{jq}| e^{-\gamma} \frac{1}{\varepsilon} \int_0^1 \exp - \frac{\mu}{4\varepsilon^2}(1 - t^2) dt$$

$$\leq \sup_{E_\gamma} |g_{jq}| e^{-\gamma} \frac{1}{\varepsilon} \int_0^1 \exp - \frac{\mu}{2\varepsilon^2}(1 - t) dt$$

$$\leq \frac{2\varepsilon}{\mu} e^{-\gamma} \sup_{\mathscr{E}_\gamma} |g_{jq}| .$$

In case $x \in \Gamma_1$ the estimation of I_2 requires a simple combination of the two preceeding arguments which we omit.

The integral I_3 has an oscillatory kernel and requires more delicate reasoning. Again we suppose $x \in \Gamma_3$. We use the parameterization in the t-plane

$$I_3 = \frac{1}{\varepsilon}\int_{\tau_{jq}(x')}^{\tau_{jq}(x)} \exp - \frac{i}{2\varepsilon^2}\mu_{jq}(0)(\tau_{jq}^2(x) - \tau^2) g_{jq}(\zeta_{jq}(\tau))\zeta'_{jq}(\tau)d\tau\,.$$

Let

$$k_3(t) = \frac{1}{\varepsilon}\int_{\tau_{jq}(x')}^{t} \exp - \frac{i}{\varepsilon^2}\mu_{jq}(0)(\tau_{jq}^2(x) - \theta^2)d\theta\,.$$

Then

(15.7)
$$\begin{aligned} I_3 &= k_3(\tau_{jq}(x))g_{jq}(x)\zeta'_{jq}(\tau_{jq}(x)) \\ &\quad - \int_{\tau_{jq}(x')}^{\tau_{jq}(x)} k_3(\tau)(g_{jq}(\zeta_{jq}(\tau))\zeta_{jq}(\tau))'d\tau\,. \end{aligned}$$

To estimate this we must estimate k_3. Let $t = \rho_1 + is_1\ \tau_{jq}(x) = \rho_0 + i\sigma_0$, $\theta = \rho + i\sigma$. Then, since $\operatorname{Im}\tau_{jq}^2 = \operatorname{Im}\theta^2 = C$ along Γ_3, we have $\sigma = \dfrac{c}{\rho}$ and

$$k_3 = \frac{1}{\varepsilon}\int_{\rho_0}^{\rho 1} \exp \frac{i}{2\varepsilon^2}\mu_{jq}(0)\left(\rho_0^2 - \rho^2 - \frac{c^2}{\rho_0^2} + \frac{c^2}{\rho^2}\right)\left(1 - \frac{ic}{\rho^2}\right)d\rho\,.$$

It therefore suffices to bound the integral

$$J_3 = \frac{1}{\varepsilon}\int_{\rho_0}^{\rho_1} \exp - \frac{i}{2\varepsilon^2}\mu_{jq}(0)\left(\rho^2 - \frac{c^2}{\rho^2}\right)\left(1 - \frac{ic}{\rho^2}\right)d\rho\,.$$

Suppose $c > 0$. Then letting $\rho = c^{1/2}t$ and $\varepsilon' = \dfrac{\varepsilon}{c''^2}$ we find

$$J_3 = \frac{1}{\varepsilon'}\int_{t_0}^{t_1} \exp - \frac{i\mu}{2\varepsilon'^2}\left(t^2 - \frac{1}{t^2}\right)\left(1 - \frac{i}{t}\right)dt\,.$$

It suffices here to consider $t_0 = 1$ since J_3 can be represented as the sum of two such integrals. Suppose $t_1 > 1$. Let $u^2 = \left(t^2 - \dfrac{1}{t^2}\right)$. Then

$$J_3 = \frac{1}{\varepsilon'}\int_0^{u_1} \exp\left\{ - \frac{i\mu}{2\varepsilon'^2}u^2 \right\} \frac{(u^4 - 1)^{1/2}(u^2 - i)}{u^4 + 1}du$$

This integral is easily estimated by the method of stationary phase. We find

$$J_3 \sim \sqrt{\frac{2\pi}{i\mu}}\,.$$

In case $\operatorname{Im} t^2 = c < 0$ or $c = 0$ the argument is similar and we omit further details.

Returning to Equation (15.7) and using Lemma 15.4 which ensures that $|\zeta'_{jq}|$ and $|\zeta''_{jq}|$ have bounds on $\tau_{jq}(\mathscr{E}_\gamma)$ we conclude that

$$|I_3| \leq C_1(\mu, M) \sup_{E_\gamma} (|g_{jq}(x)|, |g'_{jq}(x)|)\,.$$

Since we now have estimates of this form for I_1, I_2 and I_3 the conclusion of the lemma follows.

We now have the resources to finish the proof of Proposition 15.1. Estimating the integrals in (15.2) by Lemma 15.6 we find that for $r' < r$

$$|Q_0| \leq C_1 \Sigma (r')^{|q|} \sup_{E_\gamma} (|g_{jq}(x)|, |g'_{jq}(x)|)$$

On $E_{\gamma'}$ where $\gamma' > \gamma$ we can estimate $g'_{jq}(x)$ in terms of g_{jq} by Cauchy's estimate. It is easy to see that the distance between $x \in E_{\gamma'}$ and ∂E_γ can be estimated from below by $\varepsilon e^{-\gamma'}(\gamma' - \gamma)(x + \varepsilon)$. Hence on $E_{\gamma'}$ we can estimate $g'_{jq}(x)$ using

$$g'_{jq}(x) = \frac{1}{2\pi i} \int_{|x-t| = \rho(x)} g_{jq}(t) \frac{dt}{t - x}.$$

where $\rho(x) = ce^{-\gamma'}(\gamma' - \gamma) | x + \varepsilon |$. Then

$$\begin{aligned} | g'_{jq}(x) | &\leq \frac{1}{2\pi} \int_{|t-x| = \rho(x)} \left| \frac{g_{jq}(t)}{t + \varepsilon} \right| \frac{|x| + \rho(x)}{\rho(x)} | dt | \\ &\leq \sup_{E_{\gamma'}} \left| \frac{g_{jq}(t)}{t + \varepsilon} \right| \sup_{E_{\gamma'}} \left(\frac{|x|}{\rho(x)} + 1 \right) \\ &\leq C' \frac{\varepsilon^\gamma}{\gamma' - \gamma} \sup_{E_{\gamma'}} \left| \frac{g_{jq}(t)}{t + \varepsilon} \right|. \end{aligned}$$

Finally, representing $g_{jq}(x, \varepsilon)$ as $\frac{1}{(2\pi i)^n} \int_{|z|=r} g(x, \varepsilon, z) z^{-q} dz_1 \ldots dz_n$, we find

$$| Q_0 | \leq C_0 \frac{e^{\gamma'}}{\gamma' - \gamma} \sum_q \left(\frac{r'}{r} \right)^{|q|} \sup_{E_{\gamma'} \times \{|w| \leq r'\}} \left| \frac{g}{x + \varepsilon} \right|$$

This completes our solutions of the linear problem.

16. Subsidiary linear problems.

We require results similar to that of the previous section for the related problems appearing in Propositions 7.1 to 7.4 B. First, by restricting Proposition 16.1 to an interval not containing $x = 0$ we obtain a result about the simpler nondegenerate problem

(16.1) $\quad \varepsilon Q_{0x} - i\Lambda(x) Q_0 + i Q_{0w} \Lambda(x) w = g(x, \varepsilon, w); \; Q_0 = 0(|w|^2)$

on domains of the form $\mathscr{F}_\gamma \times \{ |w| \leq r \}$ where $\mathscr{F}_\gamma$ is the frustrum of an exponential horn as shown in Figure 16 A.

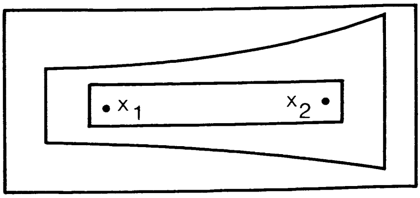

FIG. 16 A.

A frustrum $\mathcal{F}_\gamma$ between two rectangles surrounding $[x_1, x_2]$ bounded by the curves

$$(16.2)\quad x(\rho) = \rho \pm \frac{i\rho - x_1 - e^{-\gamma}}{x_2 - x_1 - 2e^{-\gamma}} \exp\{-\gamma + \mathrm{L}(\rho - x_1 - e^{-\gamma})\}$$

and vertical segments at $x_1 - e^{-\gamma}$ and $x_2 + e^{-\gamma}$. In this case the natural domains entering into our approximation methods are rectangles rather than the truncated sectors of Figure 15 A. We have also illustrated these in Figure 16 A.

PROPOSITION 16.1. — Suppose $\Lambda(x)$ is holomorphic for

$$|\operatorname{Im} x| \le e^{-\gamma}, \qquad x_1 - e^{-\gamma} \le |\operatorname{Re} x| \le x_2 + e^{-\gamma}$$

and satisfies $\sup_{\chi,j}(|\lambda_j(x)|, |\lambda_j'(x)|) < \mathrm{M}$ and $|\lambda_j - \Sigma\lambda_k q_k| \ge \mu|q|$ for $|q| \ge 2$. Suppose g is holomorphic on $\mathcal{F}_\gamma \times \{|w| \le r\}$. Then there exist constants $\gamma_0(\mu, \mathrm{M})$ and $\mathrm{C}_0(\mu, \mathrm{M})$ such that for $\gamma > \gamma_0$ the problem (16.1) has a solution Q_0 satisfying

$$\sup_{\mathcal{F}_{\gamma'} \times \{|w| \le r'\}} |\mathrm{Q}_0| \le \mathrm{C}_0 \frac{\varepsilon^{\gamma'}}{\gamma' - \gamma}\left(1 - \frac{r'}{r}\right)^{-n} \sup_{\mathcal{F}_\gamma \times |w| \le r} |g|,$$

for any $\gamma' > \gamma$ and $r' < r$.

Proof. — This is an immediate consequence of Proposition 16.1. We next consider the reduced nondegenerate problem.

$$(16.3)\qquad -i\Lambda(x)\mathrm{Q}_0 + i\mathrm{Q}_0\Lambda(x)w = g(x, \varepsilon, w); \qquad \mathrm{Q}_0 = 0(|w|^2).$$

PROPOSITION 16.2. — Suppose $\Lambda(x)$ is holomorphic on

$$\mathrm{R}_\gamma = \{|\operatorname{Im} x| \le e^{-\gamma}, x_1 - e^{-\gamma} \le |\operatorname{Re} x| \le x_2 + e^{\pm\gamma}\}$$

and satisfies $\sup_{\chi,j} |\lambda_j(x)| < \mathrm{M}$ and $|\lambda_j - \Sigma\lambda_k q_k| \ge \mu|q|$ for $|q| \ge 2$. Suppose g is holomorphic on $R_\gamma \times \{ | w | \leq r \}$. Then for $r' < r$ the problem (16.3) has a solution Q_0 satisfying

$$\sup_{R_\gamma \times \{|w| \leq r'\}} | Q_0 | \leq C_0(M, \mu)\left(1 - \frac{r'}{r}\right)^{-n} \sup_{R_\gamma \times \{|w| \leq r'\}} | g |$$

Proof. — This follows easily from the explicit series solution

$$Q_0(x, \varepsilon, w) = \sum_{|q| \geq 2} w^q (\text{diag}\, [\mu_{jq}])^{-1} g_q(x, \varepsilon).$$

Finally we must analyse the reduced problem in the inner scale on the unbounded truncated sector $\mathcal{S}_\gamma = \{ | 1+s | \geq e^{-\gamma}, | \arg(1+s) | \leq e^{-\gamma} \}$. We consider the problem

(16.4) $\quad Q_{0s} - is\Lambda(0)Q_0 + isQ_{0w}\Lambda(0)w = s^m g(s, w); \qquad Q_0 = 0(| w |^2)$

on this domain. This has much in common with the full problem but the presence of $\Lambda(0)$ rather than $\Lambda(x)$ simplifies the analysis greatly.

PROPOSITION 16.3. — Suppose $| \lambda_j(0) - \Sigma q_k \lambda_k(o) | \geq \mu | q |$ for $| q | \geq 2$ and $\max | \lambda_j(o) | = M$. Suppose $g(s, w)$ is bounded and holomorphic on $\mathcal{S}_\gamma \times \{ | w | \leq r \}$. Then for $\gamma' > \gamma$ and $r < r'$ Problem (16.4) has a solution Q_0 satisfying

$$\sup_{\mathcal{S}_{\gamma'}} | (1+s)^{-m+1} Q_{0jq} | \leq C_0(\mu, M) e^{-m\gamma'} (\gamma' - \gamma)^{-m} \sup_{\mathcal{S}_\gamma} | g_{jq} |$$

Proof. — The estimate is obtained by induction. For $m = 0$ let

$$Q_0(s, w) = - \sum_{|q| \geq 2} w^q \int_s^\infty \text{diag}_j \left[\exp \frac{i\mu_{jq}(o)}{2\varepsilon^2}(s^2 - \sigma^2) \right] g_q(\sigma) d\sigma$$

where integration is taken along the part of the hyperbola $\text{Im}\, \sigma^2 = \text{Im}\, s^2$ asymptotic to the real s-axis if s is in the right half plane, and otherwise is the segment joining s to the origin plus the positive s-axis.

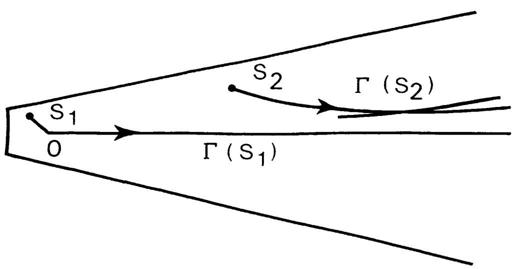

FIG. 16 B.

These two choices are described in Figure 16 B. But in the proof of Proposition (15.1) we estimated the integrals

$$K_{jq}(s) = \int_s^\infty \exp \frac{i\mu_{jq}(o)}{2\varepsilon^2}(s^2 - \sigma^2)ds$$

along these paths. The argument given there carries over almost verbatim to establish the conclusion in this case.

Now suppose $m > 0$. Let

$$R_0 = \sum_{|q| \geq 2} w^q \operatorname{diag}\,[i\mu_{jq}(0)]^{-1} g_q(s, w).$$

Then R_0 satisfies

$$-i\Lambda(o)R_0 + iR_{0w}w = g(s, w).$$

Let $Q_0 = S^{m-1}R_0 + U_0$. Then U_0 satisfies

$$U_{0s} - is\Lambda(o)U_0 + isU_{0w}\Lambda(o)w = -s^{m-1}R_{0x} - (m-1)s^{m-2}R_0.$$

The functions R_{0jq} and R_{0jqx} are bounded on $\mathscr{S}_{\gamma''}$ for $\gamma < \gamma'' < \gamma'$ by $C_1 \dfrac{e^{\gamma''}}{\gamma'' - \gamma} \sup_{\mathscr{S}_\gamma} |g_{jq}|$. Now U_0 is the sum of solutions of two equations of the form (16.4). Hence by the induction hypothesis we have the existence of a U_0 satisfying

$$\sup_{\mathscr{S}_{\gamma'}} |(1+s)^{-m+2}U_{0jq}| \leq C_2 \frac{e^{(m-1)\gamma'}}{(\gamma' - \gamma'')^{m-1}} \frac{e^{\gamma''}}{\gamma'' - \gamma} \sup_{\mathscr{S}_\gamma} |g_{jq}|$$

Choosing $\gamma'' = \dfrac{\gamma + \gamma'}{2}$ we easily find that the inequality

$$|(1+s)^{-m+1}Q_{0jq}| \leq |(1+s)^{-1}(1+s)^{-m+2}U_0| + \left| \left(\frac{s}{1+s}\right)^{m-1} R_0 \right|$$

leads to the required estimate.

The most important cases of this result are $m = 0$ and 1.

17. **The nonlinear problem with holomorphic data.**

We next parlay the result of Section 15 on the linear problem into a similar result for the corresponding nonlinear problem. The following proposition differs from Proposition 15.1 chiefly in the requirement that if the x-domain is narrow and if we wish a small loss in the radius of the w-domain then the perturbation must be very small. We will overcome this limitation later by using the approximation methods of Section 13.

PROPOSITION 17.1 (Linearization on an exponential horn). — Suppose Λ is holomorphic on $|\operatorname{Re} x| \leq x_0 + e^{-\gamma}$, $|\operatorname{Im} x| \leq e^{-\gamma}$ and satisfies

$|\lambda_j|, |\lambda'_j| \leq M$ and $|\lambda_j - \Sigma q_k \lambda_k| \geq \mu |q|$ for $|q| \geq 2$. Then there exists $\gamma^*(\mu, M)$, $\Delta^*(\mu, M)$ and $c^*(\mu, M)$ such that $\gamma > \gamma^*$, $\Delta < \Delta^*$ and

$$\sup_{\mathscr{E}_\gamma \times \{|w| \leq r\}} |(x+\varepsilon)^{-1} g| \leq c^* e^{-\gamma}(r - r')^{n+1}\Delta$$

imply the existence of a linearizing perturbation Q satisfying $|Q|$,

$$|Q_w| \leq \Delta \quad \text{on} \quad \mathscr{E}_{\gamma+1} \times \{|w| \leq r'\}.$$

Proof. — We index our data to show a recursive argument. Let $g_0 = g$, $\gamma_0 = \gamma$, $r_0 = r$ and suppose

$$\sup_{\mathscr{E}_{\gamma_0} \times \{|w| \leq r_0\}} |(x+\varepsilon)^{-1} g_0| \leq e^{-\gamma_0}(r_0 - r')^{n+1}\Delta_0 .$$

Then $\varepsilon^2 Q_{0x} - ix\Lambda Q_0 + ixQ_{0w}\Lambda_w = g_0(x, \varepsilon, w)$ has a solution satisfying

(17.1)
$$\sup_{\mathscr{E}_{\gamma_1} \times \left\{|w| \leq \frac{r_1 + r_0}{2}\right\}} |Q_0| \leq 2^n C_0(\mu, M) e^{\gamma_1 - \gamma_0}(\gamma_1 - \gamma_0)^{-1}(r_0 - r')^{n+1}(r_0 - r_1)^{-n}\Delta_0$$

whatever $\gamma_1 > \gamma_0$ and $r_1 < r_0$ provided γ_0 is large (by Proposition 15.1). Let $Q = Q_0 \circ R_1$. Then R_1 satisfies

$$\varepsilon^2 R_{1x} - ix\Lambda R_1 + ixR_{1w}\Lambda w = g_1(x, \varepsilon, w + R_1)$$

where $g_1 = (I + Q_{0w})^{-1}\{ g_0(x, \varepsilon, w + Q_0) - g_0(x, \varepsilon, w)\}$. Formally this is the original problem with g_0 replaced by g_1. We now determine relations among the parameters which suffice to give the recursive estimate

$$\sup_{\mathscr{E}_{\gamma_1} \times \{|w| \leq r_1\}} |(x+\varepsilon)^{-1} g_1| \leq e^{-\gamma_1}(r_1 - r')^{n+1}\Delta_1 .$$

First g_1 is defined on this domain if $|Q_0| \leq r_0 - r_1$ and $|Q_{0w}| \leq \frac{1}{2}$. Bounding $|Q_0|$ by (17.1) and estimating $|Q_{0w}|$ on the small domain $|w| \leq r_1$ by the Cauchy formula, we find that both of these estimates are consequences of

(17.2) $$2^{n+1} n C_0 e^{\gamma_1 - \gamma_0}(\gamma_1 - \gamma_0)^{-1}(r_0 - r')^{n+1}(r_0 - r_1)^{-n-1}\Delta_0 \leq \frac{1}{2}.$$

Moreover we then have on $\mathscr{E}_{\gamma_1} \times \{|w| \leq r_1\}$

$$\begin{aligned}
|(x+\varepsilon)^{-1} g_1| &\leq \frac{1}{1 - \frac{1}{2}} \int_0^1 |(x+\varepsilon)^{-1} g_{0w}(x, \varepsilon, w + tQ_0) Q_0| \, dt \\
&\leq 2 \sup_{|w| \leq \frac{r_0 + r_1}{2}} |(x+\varepsilon)^{-1} g_{0w}| \sup_{|w| \leq r_1} |Q_0| \\
&\leq \frac{4}{r_0 - r_1} \sup_{|w| \leq r_0} |(x+\varepsilon)^{-1} g_0| \sup_{|w| \leq r_1} |Q_0| \\
&\leq e^{-\gamma_1}(r_1 - r')^{n+1}\{ 2^{n+2} C_0 e^{2(\gamma_1 - \gamma_0)}(r_0 - r')^{2n+2}(\gamma_1 - \gamma_0)^{-1} \\
&\quad \cdot (r_0 - r_1)^{-n-1}(r_1 - r')^{-n-1}\Delta_0^2 \}.
\end{aligned}$$

The bracketed expression gives an estimate for Δ_1. We note that it embodies the crucial quadratic dependence $\Delta_1 \approx \Delta_0^2$.

We now show that it is possible to choose Δ_k, γ_k and r_k so that

$$a) \quad \Delta_k \to 0, \quad r_k \to r_\infty > \frac{r'+r}{2}, \quad \gamma_k \to \gamma_\infty < \gamma_0 + 1,$$

$$b) \quad \Delta_{k+1} \geq 2^{n+2} C_0 e^{2(\gamma_{k+1}-\gamma_k)} (\gamma_{k+1} - \gamma_k)^{-1} \cdot (r_k - r')^{2n+2} (r_k - r_{k+1})^{-n-1} (r_{k+1} - r')^{-n-1} \Delta_k^2$$

$$c) \quad 2^{n+1} n C_0 e^{(\gamma_{k+1}-\gamma_k)} (\gamma_{k+1} - \gamma_k)^{-1} (r_k - r')^{-n-1} (r_k - r_{k+1})^{n+1} \Delta_k \leq \frac{1}{2}.$$

This will imply that we have a recursive scheme for determining a sequence $Q_0, Q_1, \ldots$, which, provided Δ_k tends to 0 rapidly enough, gives us a linearizing perturbation Q on the domain $\mathscr{E}_{\gamma+1} \times \{ |w| \leq r' \}$ in the form of a convergent infinite composition

$$Q = Q_0 \circ Q_1 \circ Q_2 \circ \ldots .$$

Let $\gamma_k = \gamma_0 + \frac{1}{2} - 2^{-k-1}$. Then $\gamma_k \to \gamma_0 + \frac{1}{2}$. Let $\Delta_k^{1/2} = (r_k - r_{k+1})^{n+1}$ $(r - r')^{-n-1}$. Then, provided $r_k \geq \frac{r+r'}{2}$ (17.3) $b)$ and $c)$ are consequences of relations of the form

$$(17.4) \qquad \begin{aligned} &a) \quad \Delta_{k+1} = C_1(\mu, M) 2^k \Delta_k^{3/2} \\ &b) \quad \Delta_k \leq C_2(\mu, M) 2^{-2k}. \end{aligned}$$

where $C_i > 0$. The equation 17.4 a, has the explicit solution

$$\Delta_k = C_1^{-2} 2^{-2k-4} (16 C_1^2 \Delta_0)^{\left(\frac{3}{2}\right)^k}$$

which implies

$$\Delta_k \leq C_1^{-2} (16 C_1^2 \Delta_0)^{\left(\frac{3}{2}\right)^k}.$$

Let $\Delta_0 = \frac{1}{16 C_1^2} e^{-l(\mu, M)}$ for notational convenience. Then $\Delta_k \leq C_3 e^{-l\left(\frac{3}{2}\right)^k}$.

If l is large enough this implies (17.4 b). To complete the inductive scheme it remains to show $r_k \geq \frac{r+r'}{2}$ if $l(\mu, M)$ is large. But $r_{k-1} - r_k = (r-r') \Delta_k^{\frac{1}{2(n+1)}}$. Hence

$$r_k \geq r - \sum_{k=0}^{\infty} (r_k - r_{k+1}) \geq r - (r-r') C_3 \sum_{k=0}^{\infty} e^{-\frac{l}{n+1}\left(\frac{3}{2}\right)^k}.$$

Choosing $l(\mu, M)$ so large that $C_3 \sum_{k=0}^{\infty} e^{-\frac{l}{2(n+1)}\left(\frac{3}{2}\right)^k} < \frac{1}{2}$ we obtain $r_k \geq \frac{r+r'}{2}$.

This ensures that a sequence $Q_0, Q_1, \ldots$ is defined satisfying

$$\sup_{\mathcal{E}_{\gamma k} \times \left\{ |w| \leq \frac{r_k + r_{k+1}}{2} \right\}} | Q_k | \leq r_k - r_{k+1} \leq C_3(r - r')e^{\frac{l}{2(n+1)}\left(\frac{3}{2}\right)^k}.$$

(17.5) and

$$\sup_{\mathcal{E}_{\gamma k} \times \{|w| \leq r_{k+1}\}} | Q_{kw} | \leq \frac{1}{2}.$$

We next show that these estimates ensure the uniform convergence of the infinite composition $Q_0 \circ Q_1 \ldots$ to a holomorphic function Q on $\mathcal{E}_{\gamma+1} \times \{ | w | \leq r' \}$. Let $Q^{(k)} = Q_0 \circ Q_1 \circ \ldots \circ Q_k$. We first let $M_k = \sup_{\mathcal{E}_{\gamma k} \times \{|w| \leq r_{k+1}\}} | Q_w^{(k)} |$ recursively. First $M_0 = \sup_{\mathcal{E}_{\gamma 0} \times \{|w| \leq r_1\}} | Q_{0w} | \leq \frac{1}{2}$.

Moreover

$$Q^{(k+1)} = Q^{(k)} \circ Q_{k+1} = Q^{(k)}(w + Q_{k+1}) + Q_{k+1}.$$

Hence

$$\sup_{\mathcal{E}_{\gamma k+1} \times \{|w| \leq r_{k+2}\}} | Q_w^{(k+1)} | \leq \sup_{\mathcal{E}_{\gamma k} \times \{|w| \leq r_{k+1}\}} | Q_w^{(k)} | (1 + \sup_{\mathcal{E}_{\gamma k+1} \times \{|w| \leq r_{k+2}\}} |Q_{k+1,w}|) + \sup_{\mathcal{E}_{\gamma k+1} \times \{|w| \leq r_{k+2}\}} | Q_{k+1,w} |$$

which implies

$$M_{k+1} \leq \frac{3}{2} M_k + \frac{1}{2}.$$

Hence

$$M_k < \left(\frac{3}{2}\right)^k - 1.$$

We can estimate $| Q^{(k+1)} - Q^{(k)} |$ on $\mathcal{E}_{\gamma+\frac{1}{2}} \times \left\{ | w | \leq \frac{r' + r}{2} \right\}$ by

$$\begin{aligned} | Q^{(k+1)} - Q^{(k)} | &\leq | Q^{(k)}(w + Q_{k+1}) - Q^{(k)}_{(w)} + Q_{k+1} | \\ &\leq \left\{ \left(\frac{3}{2}\right)^k - 1 + 1 \right\} \sup | Q_{k+1} | \\ &\leq \left(\frac{3}{2}\right)^k C_3(r - r')e^{-\frac{l}{2(n+1)}\left(\frac{3}{2}\right)^k}. \end{aligned}$$

Similarly on $\mathcal{E}_{\gamma+1} \times \{ | w | < r' \}$ we have

$$| Q_w^{(k+1)} - Q_w^k | \leq 2\left(\frac{3}{2}\right)^k C_3 e^{-\frac{l}{2(n+1)}\left(\frac{3}{2}\right)^k}.$$

This implies the convergence of $Q^{(k)}$ together with all its partial derivatives on the smaller domain $\mathcal{E}_{\gamma+1} \times \{ | w | < r' \}$ to a holomorphic function Q.

Since $\Delta_0 = \dfrac{1}{16C_1^2(\mu, M)} e^{-l(\mu, M)}$, if we choose c^* so small that

$$c_0 \Sigma 2\left(\frac{3}{2}\right)^k C_3 e^{\frac{-l}{2(n+1)}\left(\frac{3}{2}\right)^k} \leq \Delta_0 \quad \text{we have} \quad |Q_{0w}| \leq \Delta_0 .$$

It remains to show that Q actually satisfies the linearizing equation. This is a consequence of the relation

$$(17.6) \qquad \varepsilon^2 Q_x^{(k+1)} - ix\Lambda Q^{(k+1)} + ixQ_w^{(k+1)}\Lambda w = g(x, \varepsilon, w + Q^{(k)})$$

and the convergence of $Q^{(k)}$, $Q_x^{(k)}$, $Q_w^{(k)}$. To establish (17.6) with a minimum of computation denote it briefly by $L_\varepsilon Q^{(k+1)} = g(w + Q^{(k)})$. We have repeatedly used the fact that the change of variable $Q = Q' \circ R$ in $L_\varepsilon Q = g(w + Q)$ leads to $L_\varepsilon(R) = g_1(w + R)$ where

$$g_1 = (I + Q'_{0w})^{-1} \{ g(w + Q') - L_\varepsilon Q' \} .$$

Denote g_1 by $Q' \times g$. The action $Q' \times$ on g is induced by a local group of coordinate changes and is therefore associative in the sense that $Q_1 \times (Q_0 \times g) = (Q_0 \circ Q_1) \times g$ (this is unpleasant to verify by direct calculation). The relations $L_\varepsilon R = Q' \times g(w)$ and $L_\varepsilon(Q' \circ R) = g(w + Q')$ then express the same relation in different coordinates. Now Q_{k+1} is exactly the solution of

$$L_\varepsilon Q_{k+1} = Q_k \times Q_{k-1} \times \ldots \times Q_0 \times g(w)$$

or

$$L_\varepsilon Q_{k+1} = (Q_0 \circ Q_1 \circ Q_2 \ldots \circ Q_i) \times g(w)$$
$$= Q^{(k)} \times g(w).$$

Hence

$$L_\varepsilon(Q^{(k)} \circ Q^{(k+1)}) = g(w + Q^{(k)})$$

or

$$L_\varepsilon Q^{(k+1)} = g(w + Q^{(k)}) .$$

This concludes the proof.

18. **Subsidiary nonlinear problems.**

The following results correspond to the Propositions of Section 16.

PROPOSITION 18.1. — Assume the hypotheses of Proposition 16.1. Then there exist $\gamma^*(\mu, M)$, $c^*(\mu, M)$ and $\Delta^*(\mu, M)$ such that if $\gamma > \gamma^*$, $\Delta < \Delta^*$ and $\sup_{\mathcal{F}_\gamma \times \{|w| \leq r\}} |g| \leq c^* e^{-\gamma}(r - r')^{n+1}\Delta$ then

$$\varepsilon Q_x - i\Lambda(x)Q + iQ_0\Lambda w = g(x, \varepsilon, w + Q) ; \qquad Q = 0(|w|^2)$$

has a solution satisfying $|Q|$, $|Q_w| < \Delta$ on $\mathcal{F}_{\gamma+1} \times \{ |w| < r' \}$.

Proof. — This is a consequence of Proposition 17.1.

PROPOSITION 18.2. — Assume the hypotheses of Proposition 16.2. Then there exist $\gamma^*(\mu, M)$, $c^*(\mu, M)$ and $\Delta^*(\mu, M)$ such that if $\gamma > \gamma^*$, $\Delta < \Delta^*$ and $\sup_{R_\gamma \times \{|w| \leq r\}} |g| \leq c^*(r - r')^{n+1}\Delta$ then

$$- i\Lambda Q + iQ_w\Lambda_w = g(x, \varepsilon, w + Q)\,; \qquad Q = 0(|w|^2)$$

has a solution satisfying $|Q|, |Q_{0w}| \leq \Delta$ on $R_{\gamma+1} \times \{|w| \leq r'\}$.

Proof. — The proof of Proposition 17.1 carries over almost verbatim except that estimates for the solution of the linearized problem come from Proposition 16.2. These contain no factor e^γ resulting from the loss of x-derivative and permit a stronger conclusion here, namely that the requisite smallness of g does not depend on the width of the x-domain.

PROPOSITION 18.3. — Assume the hypotheses of Proposition 16.3. Suppose $g(s, w)$ is holomorphic on $\mathscr{S}_\gamma \times \{|w| \leq r\}$. Then there exist $\gamma^*(\mu, M)$, $c^*(\mu, M)$ and $\Delta^*(\mu, M)$ such that if $\sup|(1+s)^{-1}g| \leq c^*e^\gamma(r-r')^{n+1}\Delta$ then $Q_s - is\Lambda(o)Q + isQ_w\Lambda(o)s = g(s, w+Q)$; $Q = 0(|w|^2)$ has a solution satisfying $|(1+s)^{-1}Q|, |(1+s)^{-1}Q_w| < \Delta$ on $\mathscr{S}_{\gamma+1} \times \{|w| \leq r'\}$.

Proof. — The proof is nearly that of Proposition 17.1 expressed in terms of the variable $x = \varepsilon s$. The only difference is the occurrence of the unbounded domain $\mathscr{S}_\gamma$. This is however entirely accounted for in the solution of the linearized problem in Proposition 16.3. The balance of the proof is the same and we omit further details.

19. **Linearization in B_r.**

We complete our analysis by proving the results of Section 7 along the lines sketched in the remarks of Section 12. We articulate as lemmas several steps in the proof of the main result, Proposition 7.1. For these we assume the hypotheses of the proposition.

LEMMA 19.1. — Suppose $g \in \delta(|x|+\varepsilon)B_{r,\phi_1}$ and $|D_x^k\Lambda| \leq e^{\phi_2(k)}$. Then for any $r' < r$ there exist holomorphic approximations $g^{(N)}$ and $\Lambda^{(N)}$ to g and Λ satisfying

$$(19.1) \qquad \begin{aligned} &|(x + \varepsilon)^{-1}g^{(N)}| \leq C_1(r - r')^{-n}\delta \\ &|(x + \varepsilon)^{-1}(g^{(N)} - g^{(N+1)})| \leq C_1(r - r')^{-n}\delta \exp - \phi^*(N) \end{aligned}$$

on

$$\Omega_N \times \left\{ |w| \leq \frac{r + r'}{2} \right\}$$

and

$$(19.2) \qquad \begin{aligned} &|\Lambda^{(N)}|, |D_x\Lambda^{(N)}| \leq M_1 \\ &|\Lambda^{(N)} - \Lambda^{(N-1)}| \leq M_1 e^{-\phi^*(N)} \end{aligned}$$

on $|\operatorname{Im} x| \leq e^{-N}$, where ϕ is a convex function satisfying. $\phi^{(k)}/k \log k \to \infty$ as $k \to \infty$.

Proof. — Let $\phi_3(k) = \max \{ \phi_1(k), \phi_2(k+1) \}$. Then by Proposition 13.1 we can find approximations $g^{(N)}$ to g and $w^{(N)}$ to $D_x\Lambda$ of the required kind with $\phi^*(N) = \phi_3^*(N-1) + N = \{ \phi_3(K+1) + K + 1 \}^*(N)$. The approximations $\Lambda^{(N)}$ to Λ given by $\Lambda^{(N)} = \Lambda(o) + \int_0^x w^{(N)}(\xi) ds$ will satisfy similar estimates.

LEMMA 19.3. — For N sufficiently large, depending on $\mu = \inf |q|^{-1} |\mu_{jq}|$ and $M = \sup(|\lambda_j|, |\lambda'_j|)$, the elements of $\Lambda^{(N)}$ satisfy

$$|\lambda_j^{(N)} - \Sigma q_k \lambda_k^{(N)}| \geq \frac{\mu}{2} |q| \quad \text{on} \quad R_n.$$

Proof. — This follows from the uniform convergence of $\lambda_j^{(N)}$ to λ_j on the real interval $[-x_0, x_0]$ and the bound $|\lambda^{(N)}| \leq C_1$ (from (19.2) of Lemma 19.2) on the narrow domain R_N about this interval.

LEMMA 19.4. — There exists an $N_1(\mu, M)$ such that $\Omega_{N+2N_1} \subset \mathscr{E}_{N+N_1} \subset \Omega_N$.

Proof. — The parameter E of Equation (15.1), by Lemma 15.5, depends only on μ and M. The existence of an N_1 yielding the configuration of Figure 15 A is simple to check.

REMARK 19.5. — Because of estimate (19.2) and the lower bound of Lemma 19.3 we choose the parameter E determining the family of exponential horns to be $E\left(\frac{\mu}{2}, M_1\right)$. It is this family which appears in all subsequent arguments.

LEMMA 19.6. — There exist N_1^*, c_1^* and Δ_1^* such that for $N > N_1^*$ and $\Delta < \Delta_1^*$, each of the problems

$$L^{(N)}Q = \varepsilon^2 Q_x - ix\Lambda^{(N)}(x)Q + ixQ_w\Lambda^{(N)}w = h(x, \varepsilon, w + Q)$$

has a solution Q satisfying $|Q|, |Q_w| < \Delta$ on $\mathscr{E}_{N+1} \times \{ |w| \leq r' \}$ provided h is holomorphic on $\mathscr{E}_N \times \{ |w| \leq r \}$ and there satisfies

$$c^* e^{-N}(r - r')^{n+1} |(x + \varepsilon)^{-1} h| \leq \Delta.$$

Proof. — Here also by (19.2) and Lemma 19.3 we have collective estimates for the parameters determining the applicability of Proposition 17.1 to each of these problems.

We now prove Proposition 7.1 by recursively solving the sequence of problems $L^{(N)}(Q^{(N)}) = g^{(N)}(w + Q^{(N)})$. We begin by directly solving the problem

$$(19.3)_{N_0} \qquad L^{(N_0)}(Q^{(N_0)}) = g^{(N_0)}(w + Q^{(N_0)})$$

for a suitable choice of N_0. For higher indices $N > N_0$ we seek solutions in the factored form

$$(19.4)_N \qquad Q^{(N)} = Q^{(N-1)} \circ Q_N .$$

The compositional remainder Q_N must then satisfy

$$(19.5)_N \qquad L^{(N)}(Q_N) = g_N(w + Q_N)$$

where

$$(19.6)_N \quad g_N = (I + Q_w^{N-1})^{-1} \{ g^{(N)}(w + Q^{(N-1)}) - g^{(N-1)}(w + Q^{(N-1)}) - (L^{(N)} - L^{(N-1)})Q^{(N-1)} \}.$$

By Lemmas 19.1 and 19.4, for N sufficiently large we have the estimates

$$(19.7)_N \qquad |(x + \varepsilon)^{-1} | g^{(N)} - g^{(N-1)}) | \leq C_1(r - r)^{-n}\delta e^{-\phi^*(N)}$$

and

$$(19.8)_N \qquad | \Lambda^{(N)} - \Lambda^{(N-1)} | \leq C_1 e^{-\phi^*(N)}$$

on

$$\mathscr{E}_{N+N_0} \times \left\{ |w| \leq \frac{r + r'}{2} \right\}.$$

We also suppose that N_0 is so large that for $N \geq N_0$ the conclusion of Lemma 19.6 holds. We now attempt to solve $(19.3)_{N_0}$-$(19.6)_N$ recursively by choosing N_0 and δ suitably. Let

$$r_{N_0} = \frac{3r' + r}{4}, \quad r_{N-1} - r_N = \frac{r - r'}{2^{N-N_0+4}}, \quad r_\infty = \frac{7r' + r}{8} > r'$$

$$A_N = \sup_{\mathscr{E}_{N_0+N} \times \{|w| \leq r_N\}} \{ | Q^{(N)} |, | Q_w^{(N)} | \}$$

$$(19.9)_N \quad G_N = \sup_{\mathscr{E}_{N_0+N} \times \left\{ |w| \leq \frac{r_{N-1} + r_N}{2} \right\}} | (x + \varepsilon)^{-1} g_N |$$

$$\Delta_N = \sup_{\mathscr{E}_{N_0+N} \times \{|w| \leq r_{N+1}\}} (| Q_N |, | Q_{Nw} |)$$

Then, if $A_N \leq \frac{r - r'}{4}$, and $A_N \leq \frac{1}{2}$, we can estimate $G_{N+1} = \sup | (x + \varepsilon)^{-1} g_{N+1} |$ using $(19.6)_N$-$(19.8)_N$ on $\mathscr{E}_{N+1+N_0} \times \left\{ |w| \leq \frac{r_N + r_{N+1}}{2} \right\}$

$$G_{N+1} \leq 2 \left\{ C_1(r - r')^{-n}\delta e^{-\phi^*(N)} + 2 \sup_{N+1+N_0} \frac{x}{x + \varepsilon} C_1 e^{-\phi^*(N)} A_N \right\}.$$

These inequalities have the form

$$(19.10)\qquad A_N \leq a$$
$$G_{N+1} \leq e^{-\phi^*(N)}\{ b\delta + cA_N \}.$$

Next, we can use this estimate for g_{N+1} to obtain an estimate for Q_{N+1} as the solution of $(19.5)_{N+1}$ given by Lemma 19.6 provided

$$C_1^* e^{N_0+N}\left\{\frac{r_{N+1}-r_N}{2}\right\}^{-n-1} G_{N+1} \leq \Delta_{N+1}$$

(and $\Delta_{N+1} \leq \Delta_1^*$). This inequality has the form

$$(19.11)\qquad G_{N+1} \leq de^{-\alpha N}\Delta_{N+1}.$$

Finally the explicit form of $(19.4)_{N+1}$, namely

$$Q^{(N+1)} = Q^{(N)}(w + Q_{N+1}) + Q_{N+1}$$

leads to the estimate

$$A_{N+1} \leq \sup(A_N + \Delta_{N+1}, A_N(1 + \Delta_{N+1}) + \Delta_{N+1})$$

provided $\Delta_{N+1} \leq e^{-\beta N} \leq r_N - r_{N+1}$

$$(19.12)\qquad A_{N+1} < A_N \leq (1 + A_N)\Delta_{N+1}.$$

To summarize, if we choose δ and N_0 so that the following hold, then the quantities A'_N, G'_N, Δ'_N will majorize the quantities defined in (19.9) :

i) $A'_N \leq a$

ii) $G'_{N+1} = e^{-\phi^*(N)}\{ b\delta + cA'_N \}$

iii) $G'_{N+1} = de^{-\alpha N}\Delta'_{N+1}$

iv) $\Delta'_{N+1} \leq e^{-\beta N}, \qquad \Delta_{N+1} \leq \Delta_1^*$

v) $A'_{N+1} - A'_N \leq (1 + A'_N)\Delta'_N.$

(Here a, b, c, d, α and β are nonnegative quantities depending on μ, M, r' with specific properties that no longer concern us.

Combining *ii*), *iii*), and *v*, we obtain

$$(19.13)\qquad A'_{N+1} - A'_N = e^{-\phi^*(N)+\alpha N}(1 + A'_N)\left(\frac{b}{d}\delta + \frac{c}{d}\delta A'_N\right).$$

We now choose N_0 so large that

$$e^{-\phi^*(N)+\alpha N}\max\left(\frac{b}{d}, \frac{c}{d}\right) \leq \frac{1}{2}e^{-N}.$$

This is possible since, using $\phi^{**} = \phi$, we have

$$\begin{aligned} e^{-\phi^*(N)+\alpha N} &= e^{-2N-(\phi^*(N)-(\alpha+2)N)} \\ &\leq e^{-2N+\phi(\alpha+2)} \\ &= 0(e^{-N}). \end{aligned}$$

The sequence defined by (19.13) will then be majorized by solutions of the simpler scheme

$$A''_{N+1} - A''_N = \frac{1}{2} e^{-N}(1 + A''_N)(\delta + A''_N) \tag{19.14}$$

Moreover, now that N_0 has been fixed, we can apply Lemma 19.6 to the problem $L^{(N_0)}(Q^{(N_0)}) = g^{(N_0)}$ to conclude that for small δ we have $A_{N_0} \leq K\delta$. We therefore add to (19.14) the initial condition

$$A''_{N_0} = K\delta . \tag{19.15}$$

It is now easy to prove by induction that $A''_{N+1} - A''_N \leq Ae^{-N}\delta$ for some A. For if $A''_{j+1} - A''_j \leq Ae^{-N}$ for $j < N$, then $A''_N \leq k\delta + \frac{1}{1-e^{-1}} A\delta$ and $A''_{N+1} - A''_N \leq \frac{1}{2}\left(1+k+\frac{1}{1-e^{-1}}A\right)\left(1+K\delta+\frac{1}{1-e^{-1}}A\delta\right)e^{-N\delta}$. Thus induction is justified if

$$\frac{1}{2}\left(1 + K + \frac{1}{1 - e^{-1}} A\right)\left(1 + K\delta + \frac{1}{1 - e^{-1}} A\delta\right) < A .$$

But this inequality is true for large A and small δ. Hence we conclude that

$$A_N \leq \left(K + \frac{A}{1 - e^{-1}}\right)\delta .$$

This implies, using *iii*) and *iv*) above, that

$$\Delta_{N+1} \leq Ce^{-\phi^*(N) + \alpha N}\delta .$$

Then *iv*) is a consequence of

$$Ce^{-\phi^*(N) + \alpha N}\delta \leq \min \{ (r - r')e^{-(N - N_0 + 4) \log 2}, \Delta_1^* \}$$

which is true for small δ. We also estimate

$$\begin{aligned}\sup_{\mathscr{E}_{N_0 - N} \times \{|w| \leq r_{N+1}\}} | Q^{(N+1)} - Q^{(N)} | &= \sup | Q^{(N)}(w + Q_{N+1}) - Q^{(N)}(w) + Q_{N+1} | \\ &\leq (A_N + 1)\Delta_{N+1} \\ &\leq C'e^{-\phi^*(N) + \alpha N}\delta .\end{aligned}$$

By Lemma 19.4, $\Omega_{N+2N_0} \subset \mathscr{E}_{N_0+N}$. Hence these estimates imply

$$\sup_{\Omega_{N-2N_0} \times \{|w| \leq r'\}} | Q^{(N+1)} - Q^{(N)} | \leq C'\delta e^{-\phi^*(N) + \alpha N}$$

or for $N \geq 2N_0$

$$(19.6) \quad \sup_{\Omega_N \times \{|w| \leq r'\}} |Q^{(N-2N_0+1)} - Q^{(N-2N_0)}| \leq C'\delta e^{-\phi^*(N-2N_0)+\alpha(N-2N_0)}.$$

The converse portion of Proposition (13.1) now implies that the sequence $Q^{(N)}$ converges on $[0, x_0]$ to a function in B'_r. Moreover the convergence of $Q_x^{(N)}$, Q_w^N, $g^{(N)}$ to Q_x, Q_w, g ensures that the limit satisfies $L(Q) = g$ on $[0, x_0]$.

Similarly we can construct a linearizing perturbation on $[-x_0, 0]$. Piecing these together we obtain a linearizing perturbation on the interval $[-x_0, x_0]$ containing the support of g. This concludes the proof of Proposition 7.1.

A proof of Proposition 7.2 A can be given along the same lines but is much easier. However this result is basically the known result that the vector field $i\Lambda + g$ (in which x and ε merely appears as parameters) can be linearized if Λ satisfies the eigenvalue conditions. We therefore omit the proof.

PROPOSITION 7.2B. — Since it deals with holomorphic data, is a consequence of Proposition 18.2.

To prove Proposition 7.3 we use a simpler variant of the proof of Proposition 7.1 in which the operation 0 is replaced by +. Briefly we consider $L(Q) = g(w)$ as the limit of holomorphic problems $L^{(N)}(Q^{(N)}) = g^{(N)}(w)$. Letting $Q^{(N+1)} = Q^{(N)} + Q_{N+1}$ we find

$$L^{(N+1)}(Q_{N+1}) = g^{(N+1)} - g^{(N)} - (L^{(N+1)} - L^{(N)})Q^{(N)}.$$

A much simpler version of the recursive estimation scheme in the above proof shows that for some large α and N_0 we have

$$Q_{N+1} = Q^{(N+1)} - Q^{(N)} \sim e^{-\phi^*(N)+\alpha N} \quad \text{on} \quad \Omega_{N+N_0} \times \{|w| \leq r'\}.$$

The converse part of our approximation scheme then shows that the $Q^{(N)}$ converge to an element of $B_{r'}$.

The conclusions of Proposition 7.4A can be derived from the explicit solution $Q = \Sigma w^q \{\Lambda - (\Sigma q_i{}^\lambda{}_i)I\}^{-1} g_q$ of $-i\Lambda Q + iQ_w\Lambda_w = g(x, \varepsilon.w)$. Proposition 7.4B is a consequence of Proposition 16.3. Finally we recall that the closure properties of $\{B_r\}$ were established in Section 14.

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(Manuscrit reçu le 16 juillet 1984)

(révisé le 15 avril 1985)