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Lagrangian embeddings and critical point theory

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ABSTRACT. — We derive a lower bound for the number of intersection points of an exact Lagrangian embedding of a compact manifold into its cotangent bundle with the zero section. To do this the intersection problem is converted into the problem of finding solutions of a Hamiltonian system satisfying canonical boundary conditions. The dynamical problem is then solved by global variational methods on a Hilbert manifold.

Key-words: Lagrangian Intersection theory, symplectic geometry, Hamiltonian systems, variational methods, strongly indefinite functionals.

RÉSUMÉ. — Soit M une variété différentiable, compacte et connexe, $T^*M \to M$ son fibré cotangent, $\sigma : M \to T^*M$ la section nulle, $\varphi : M \to T^*M$ un plongement lagrangien. Cet article démontre que $\varphi(M) \cap \sigma(M)$ contient du moins $c(M)$ points, où $c(M)$ est la catégorie cohomologique de M. Dans le cas $M = T^n$, tore à n dimensions, ce résultat avait été conjecturé par Arnold et démontré par M. Chaperon.

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I. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

I.1. Lagrangian embeddings and Hamiltonian systems.

Let M be a compact connected differentiable manifold and $\tau_M^*: T^*M \to M$ its cotangent bundle. Denote by $s: M \to T^*M$ the zero section and let $\Sigma = s(M)$. On T^*M there exists a unique 1-form λ, called the Liouville-form, such that $\beta^*\lambda = \beta$ for all 1-forms β on M considered as maps $M \to T^*M$. The associated 2-form $\omega = d\lambda$ is called the canonical symplectic form on T^*M. An immersion $\phi: M \to T^*M$ is called « Lagrangian » if $\phi^*\omega = 0$. If $\phi^*\lambda$ is exact we call it « exact Lagrangian ».

DEFINITION 1. — A Lagrangian embedding $\phi: M \to T^*M$ is called « nice » if

(*i*) ϕ is exact.

(*ii*) There exists a differentiable map $\tilde{\phi}: [0,1] \times M \to T^*M$ such that $\tilde{\phi}(\{0\} \times \Sigma) = \Sigma$, $\tilde{\phi}(1, .) = \phi$ and $\tilde{\phi}(t, .)$ is a Lagrangian embedding for all $t \in [0,1]$.

In order to give the statement of the main result we use the notion of cohomological category.

DEFINITION 2. — The cohomological category of a topological space X, denoted by $c(X)$ is the maximal number k such that there exists a ring R and cohomology classes $x_j \in H^{n(j)}(X, R)$, $n(j) \geq 1$, for $j = 1, \ldots, k-1$, such that

$$x_1 \cup x_2 \cup \ldots \cup x_{k-1} \neq 0 .$$

If there exist arbitrarily long products of the above type one puts $c(X) = \infty$, and if $X = \emptyset$ one define $c(X) = 0$.

The main result is the following:

THEOREM 1. — Let $\phi : M \to T^*M$ be a nice Lagrangian embedding. Then $\phi(M) \cap \Sigma$ contains at least $c(M)$ points.

Let us state a theorem which implies Theorem 1. Denote by

$$h^* : [0,1] \times T^*M \to \mathbb{R}$$

a smooth map with compact support and let $h_t^* = h^*(t, .)$. We introduce the associated (exact) Hamiltonian vectorfield X_t by

$$i_{X_t}\omega = dh_t^*$$

and study the time-dependent Hamiltonian system with boundary conditions

$$\text{(HS)} \quad \dot{x} = X_t(x), \qquad x(0), \qquad x(1) \in \Sigma .$$

We have

THEOREM 2. — Let h^* be as described above. Then (HS) possesses at least $c(M)$ different solutions.

That Theorem 2 implies Theorem 1 was observed by M. Chaperon, who proved Theorem 2 for the special case $M = T^n$, which has been conjectured by Arnold [*2*]. The more general statement in Theorem 1 had been conjectured by M. Chaperon [*5*]. In order to reduce Theorem 1 to Theorem 2 we need the following lemma due to Chaperon ([*5*, 0.4.2 Theorem or (for the case $M = T^n$) 6, Lemme 2]).

LEMMA. — Let $\phi : M \to T^*M$ be a nice Lagrangian embedding. Then there exists a smooth map $h^* : [0,1] \times T^*M \to \mathbb{R}$ with compact support, such that the points in $\phi(M) \cap \Sigma$ are in bijective correspondance to the solutions of (HS), where X_t is the time-dependent Hamiltonian vectorfield associated to h^*.

I.2. Sketch of the proof.

Following the ideas of M. Chaperon we reduce Theorem 1 to Theorem 2. The problem to find solutions of (HS) is variational (the well-known degenerate classical variational principle). In order to give a good variational formulation fix a Riemannian metric $\langle .,. \rangle$ on M. Using the canonical map $TM \to T^*M : x \to \langle x, . \rangle$ the symplectic structure on T^*M induces oneon TM. Therefore, without loss of generality, we may assume that $\dot{x} = X_t(x)$ is a Hamiltonian system on TM defined by a smooth map with compact support, say $h : [0,1] \times TM \to \mathbb{R}$.

Denote by Λ the Hilbert manifold consisting of absolutely continuous curves $q : [0,1] \to M$ with square-integrable derivatives. Moreover denote by $L\Lambda$ the vector bundle over Λ consisting of L^2-sections along H^1-curves. Let $\pi : L\Lambda \to \Lambda$ be the canonical projection. Define a C^1-map $\Psi_\infty : L\Lambda \to \mathbb{R}$ by

$$\Psi_\infty(x) = \int_0^1 \langle \dot{q}, x \rangle \, dt - \int_0^1 h_t(x(t))dt$$
$$=: \Psi(x) - \alpha_\infty(x)$$

where $q = \pi x$. The solutions of (HS) are exactly the critical points of Ψ_∞. α_∞ is a bounded perturbation of Ψ. So one might expect that the behaviour of Ψ_∞ is similar to that of Ψ as far as critical points are concerned. The first indication that this is true is the fact that Ψ and Ψ_∞ satisfy the so-called Palais-Smale condition. On the other hand Ψ is of class C^∞ and the linearisation at a critical point has an infinite positive and negative Morse-index. This implies that passing a critical level one has to attach infinite-dimensional cells which are invisible from the topological point of view. Clearly this will cause some difficulties. This difficulty with Morse theory or more generally with variational techniques for Hamiltonian systems is of course not new and well-known. One should mention here that in the framework of convex Hamiltonian systems a Morse theory, due to I. Ekeland [*21*], exists, which however cannot be applied in our case.

Now let us have a look at Ψ. Its critical points are exactly the constant maps $t \to O_m \in TM$, $m \in M$, where O_m denotes the zero-element in T_mM. In the following we shall identify M with the zero section in TM and TM with the constant curves in $L\Lambda$. Define the sets

$$S_1 = \{ \dot{q} \mid q \in \Lambda \}$$
$$S_2 = \{ -\dot{q} \mid q \in \Lambda \}.$$

We have

$$\Psi(S_1 \backslash M) \subset (0, +\infty)$$
$$\Psi(S^2 \backslash M) \subset (-\infty, 0)$$
$$\Psi(M) = 0$$
$$S_1 \cap S_2 = M.$$

Moreover $\Psi(\dot{q}) \to +\infty$ and $\Psi(-\dot{q}) \to -\infty$ as $\int |\dot{q}|^2 \to +\infty$. This shows that we have a « hyperbolic structure ». A feature already clearly exhibited in the seminal paper by Conley and Zehnder [*7*], where for the first time a global problem of symplectic geometry on a manifold was solved by means of a classical variational principle in the loop space over the manifold. This hyperbolic structure will be preserved under the perturbation α_∞. The natural procedure to find critical points of Ψ_∞ would be to apply the minus-gradient flow (which we assume to exist) $L\Lambda \times \mathbb{R} \to L\Lambda : (x, t) \to x * t$ to the set S_2 and to show that the infinite-dimensional intersection problem $S_2 * t \cap S_1 \neq \emptyset$ has a solution for every $t \geq 0$. Then the number c given by

$$+\infty > c := \lim_{t \to +\infty} \sup \Psi_\infty(S_2 * t) \geq \inf \Psi_\infty(S_1) > -\infty$$

would be a critical level, since the Palais-Smale condition holds. A more sophisticated procedure, taking into account the size of the intersection, would give at least $c(M)$ critical points.

What we have just described will be in fact the underlying idea of our procedure. However, there are several underlying difficulties. The study of the intersection problem $S_2 * t \cap S_1 \neq \emptyset$ leads to a fixed point problem. In order to show the existence of fixed points one needs topological tools, which however are only applicable if some form of compactness is available in the problem, for example if the fixed point set is compact. Unfortunately, this cannot be shown. The reason for this is the fact that the gradient of α_∞ is not small as far as compactness is concerned: the vertical component of the gradient will not be compact (in local coordinates). To avoid this difficulty one approximates α_∞ by functionals α_n, which have in some sense a compact gradient. This approximation will be carried out in III.4 and IV. Instead of Ψ_∞ one studies the functionals $\Psi_n = \Psi - \alpha_n$. The question of course is how good is the behaviour of Ψ_∞ described by the behaviour of the family of functionals (Ψ_n). Here, an abstract critical point theorem proved in chapter II. (Theorem 3) reduces the study of Ψ_∞ essentially to the study of a single Ψ_n. Having this abstract result we apply the procedure outlined already for Ψ_∞ to the functional Ψ_n (for some n large). It turns out that the corresponding intersection problem $S_2 * t \cap S_1 \neq \emptyset$ can be studied by converting it to a fixed point problem for a fibre-preserving map in some infinite-dimensional vector-bundle over M, where the maps in the fibres are compact. Hence the topological machinery is applicable. In order to carry out the conversion intersection problem $\leftrightarrow$ fixed point problem for a compact map, one derives a representation for the flow associated to Ψ_n which relates it in some sense to the flow of the unperturbed problem. The representation for the flow and some compactness estimates will be derived in chapter V. Our procedure shows as in [7] quite clearly, in fact in contrast to the coercive closed geodesic problem, that for the variational problems for Hamiltonian systems only the topology of the underlying manifold itself is reflected in the topology of the critical points. It also shows that these can be found by studying the gradientflow in relation to the hyperbolic flow of the unperturbed problem.

I.3. Concluding remarks.

The first who employed global variational methods to solve global problems in symplectic geometry were Conley and Zehnder [*7*], in their astounding solution of Arnold's Conjecture on the number of fixed points for symplectic self-maps of Tori. Their method was adapted by M. Chaperon [*5*], to solve our Theorem 2 for the case $M = T^n$ and to prove Theorem 1 for T^n. Motivated by [*7*] Weinstein proved Theorem 2 for all compact manifolds M, requireing however the C^1-smallness of h^* [*19*]. From that point of view we find in fact not a new phenomenon, but we remove this smallness-condition imposed by Weinstein. Other related results are concerned with fixed point theorems for symplectic maps on compact manifolds. For example, Fortune and Weinstein [*12*], show that a symplectic map $P^n\mathbb{C} \to P^n\mathbb{C}$ homologous to the identity has at least $n + 1$ fixed points. Floer [*11*], recently proved that a symplectic map $M \to M$ homologous to the identity, where M is a compact Kohler manifold with a vanishing second homotopy group, Abelian Holonomy, and non-positive sectional curvature, has at least $c(M)$ fixed points. Fortune and Weinstein extend Conley and Zehnder's idea and lift the problem into an Euclidean space invariant under symmetries. Then they use different methods in the spirit of [*4*] [*14-15*]. Floer carries out a nonlinear variant of the Liapunov-Schmidt reduction. The obtained finite-dimensional problem is then solved in the spirit of [*7*].

Now a few remarks concerning the method employed to solve Theorem 2. The first who used the classical variational principle to study Hamiltonian systems in the large was P. Rabinowitz [*20*]. His ideas were later on abstracted, extended and simplified (see for example [*3*] [*4*] [*14*] [*15*]). Our approach here is motivated by the results in [*3*] [*15*]. In fact, as in this papers we attack the variational problem without carrying out a finite-dimensional reduction, except at the very end where we have to carry out some reduction—not in the variational problem, but—in the fixed point problem representing the intersection problem for the sets S_1 and S_2. This is in contrast to the reductions which have been carried out in [*5*] [*7*] [*11*].

II. AN ABSTRACT CRITICAL POINT THEOREM

We shall give an abstract result concerned with the behaviour of a functional and certain approximations.

DEFINITION 3. — Let $(L, (.,.))$ be a connected metrically complete

Hilbert manifold and $\Psi_\infty \in C^1(L, \mathbb{R})$. An $\mathbb{N}$-family for Ψ_∞ is a sequence of maps $\Psi_n : L \to \mathbb{R}$, $n \in \mathbb{N}$, such that

(*i*) $\Psi_n \in C^\infty(L, \mathbb{R})$ for all $n \in \mathbb{N}$.

(*ii*) For all sequences $(x_k) \subset L$, $x_k \to x$, and $(n_k) \subset \mathbb{N}_\infty = \mathbb{N} \cup (\infty)$, $n_k \to +\infty$, we have $\Psi_{n_k}(x_k) \to \Psi_\infty(x)$. Moreover, $\Psi'_\infty(x) = 0$ implies $\| \Psi'_{n_k}(x_k) \| \to 0$ and $\| \Psi'_{n_k}(x_k) \| \to 0$ implies $\Psi'_\infty(x) = 0$.

(*iii*) If for some sequences $(x_k) \subset L$, $(n_k) \subset \mathbb{N}_\infty$, $n_{k+1} \geq n_k$, we have $\| \Psi'_{n_k}(x_k) \| \to 0$ and $\Psi_{n_k}(x_k) \to d$, then (x_k) is precompact.

In the above definition of course Ψ'_n denotes the gradient and L is equipped with the metric $d_L : L \times L \to \mathbb{R}$ derived from $(.,.)$ in the usual way.

Let (Ψ_n) be a $\mathbb{N}$-family for Ψ_∞ and let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth map such that

. β is monotone decreasing (not necessarily strict).

. $\beta(s) = 1$ for all $s \leq 1$ and $\beta(s) = s^{-1}$ for all $s \geq 2$.

We introduce vectorfields $G_n : L \to TL$ by

$$G_n(x) = -\beta(\| \Psi'_n(x) \|)\Psi'_n(x).$$

Clearly the G_n are smooth. Since $\| G_n(x) \| \leq 2$ we have global existence for the corresponding flows $\phi_n : L \times \mathbb{R} \to L$ defined by

$$\phi'_{n,x} = G_n(\phi_{n,x}), \qquad \phi_{n,x}(0) = x.$$

In order to simplify the notation we shall write

$$x *_n s := \phi_n(x, s).$$

Denote by CL the set consisting of all closed subsets of L.

DEFINITION 4. — The Lyusternik-Schnirelman category on L is a map

$$\text{cat} : L \to \mathbb{N}_{0,\infty} = \mathbb{N}_\infty \cup \{0\}$$

satisfying the following

(*i*) $\text{cat}(\emptyset) = 0$.

(*ii*) $\text{cat}(D) = (k \in \mathbb{N}$ if there exist k open sets $U_1, \ldots, U_k$ in L, each contractible to a point in L, such that their union covers D, and D cannot be covered by a collection of $k-1$ contractible (in L) open sets.

(*iii*) $\text{cat}(D) = \infty$ if there exists no finite open covering as above.

One calls cat (D) the Lyusternik-Schnirelman category of the set D in L. We have the following result.

THEOREM 3. — Assume $\Psi_\infty \in C^1(L, \mathbb{R})$ and (Ψ_n) is a $\mathbb{N}$-family for Ψ_∞. Suppose S_1 and S_2 are closed subsets of L such that for some number $d \in \mathbb{R}$, $d > 0$

$$-d \leq \inf \Psi_n(S_1) \leq \sup \Psi_n(S_2) \leq d$$

for all $n \in \mathbb{N}$. Define maps $i_n : \mathrm{CL} \to \mathbb{N}_{0,\infty}$ by $(n \in \mathbb{N})$

$$i_n(\mathrm{D}) = \inf_{t \in \mathbb{R}}, \operatorname{cat}((\mathrm{D} *_n t) \cap \mathrm{S}_1)$$

and assume

$$i_n(\mathrm{S}_2) \geq \mathrm{N} \geq 1$$

for all $n \in \mathbb{N}$ for some $\mathrm{N} \in \mathbb{N}$. Then Ψ_∞ has at least N critical points with corresponding every levels in the interval $[-d, d]$.

Index maps like the i_n were introduced by Benci [*4*], and in a weak form by the author [*14*], to overcome the difficulties of the infinite Morse-index.

Let us collect first some properties of cat and the index maps i_n. Let D, $\mathrm{E} \in \mathrm{CL}$.

. If $\mathrm{D} \subset \mathrm{E}$ then $\operatorname{cat}(\mathrm{D}) \leq \operatorname{cat}(\mathrm{E})$.

. If $\mathrm{H} : [0,1] \times \mathrm{L} \to \mathrm{L}$ is continuous and $\mathrm{H}(0, .) = \mathrm{Id}$ and $\mathrm{H}(t, .)$ is a homeomorphism for all $t \in [0,1]$ then $\operatorname{cat}(\mathrm{H}(\{t\} \times \mathrm{D})) = \operatorname{cat}(\mathrm{D})$ for all $t \in [0,1]$.

. If $\operatorname{cat}(\mathrm{D}) \geq 2$ then D contains infinitely many points.

. If D is compact then cat (D) is finite and there exists an open neighbourhood U of D such that $\operatorname{cat}(\mathrm{cl}(\mathrm{U})) = \operatorname{cat}(\mathrm{D})$.

. If $\mathrm{D} \subset \mathrm{E}$ then $i_n(\mathrm{D}) \leq i_n(\mathrm{E})$.

. $i_n(\mathrm{D} \cup \mathrm{E}) \leq i_n(\mathrm{D}) + \operatorname{cat}(\mathrm{E})$.

. $i_n(\mathrm{D} *_n s) \geq i_n(\mathrm{D})$ for all $s \in \mathbb{R}^+$.

The above properties are trivial consequences of the definitions of cat and i_n.

Now define for $n \in \mathrm{N}$ and $j \in \{1, \ldots, \mathrm{N}\}$.

$$c_j(n) = \inf_{\mathrm{D} \in \mathrm{CL}, i_n(\mathrm{D}) \geq j} \sup \Psi_n(\mathrm{D}). \tag{1}$$

Let us show

$$-d \leq c_1(n) \leq \ldots \leq c_{\mathrm{N}}(n) \leq d \tag{2}$$

for all $n \in \mathbb{N}$. If $\mathrm{D} \in \mathrm{CL}$ and $i_n(\mathrm{D}) \geq 1$ we infer from the definition of the category that $\mathrm{D} \cap \mathrm{S}_1 \neq 0$. This implies

$$\sup \Psi_n(\mathrm{D}) \geq \inf \Psi_n(\mathrm{S}_1) \geq -d \tag{3}$$

for all $n \in \mathbb{N}$. On the other hand since $i_n(\mathrm{S}_2) \geq \mathrm{N} \geq 1$ we conclude for $n \in \mathbb{N}$ and $j \in \{1, \ldots, \mathrm{N}\}$

$$c_j(n) \leq \sup \Psi_n(\mathrm{S}_2) \leq d. \tag{4}$$

Hence combining (3) and (4) and using the monotonicity of i_n we find (2). Eventually dropping some of the Ψ_n and making some renumbering we may assume that $\lim_{n \to +\infty} c_j(n) =: c_j$ exists for all $j \in \{1, \ldots, \mathrm{N}\}$. Clearly we must have

$$-d \leq c_1 \leq c_2 \ldots \leq c_{\mathrm{N}} \leq d. \tag{5}$$

We shall show the following

(6) If $c_j = \ldots = c_{j+k}$ for some $k \in \{0, \ldots, N-1\}$ then the set $Cr = \{x \in L \mid \Psi'_\infty(x) = 0,\ \Psi_\infty(x) = c_j\}$ has at least category $k+1$ in L.

In particular this implies that the c_j are critical levels for Ψ_∞. Since the (Ψ_n) are a $\mathbb{N}$-family for Ψ_∞ condition (*iii*) implies that Cr is compact. Assume cat (Cr) $\leq k$. We shall prove (6) by deriving a contradiction. There exists an open neighbourhood U of Cr such that cat (cl(U))=cat (Cr). Clearly if Cr $= \phi$ we have U $= \phi$. Define

$$\rho = \frac{1}{2} \operatorname{dist} (\partial U, Cr)$$

and

(7) $$V = \{x \in L \mid \operatorname{dist}(x, Cr) < \rho\}$$

Then cl(V) $\subset$ U and dist $(\partial U, V) \geq \rho$.

Next we shall show

(8) There exist $\varepsilon_0 > 0$, $\tau > 0$, and $n_0 \geq 1$ such that $\| \Psi'_n(x) \| \geq \tau$ for all $x \in L \backslash V$ with $\Psi_n(x) \in [c_j - \varepsilon_0, c_j + \varepsilon_0]$ provided $n \geq n_0$ or $n = +\infty$.

Arguing indirectly we find sequences $(x_e) \subset L\backslash V$, $(n_e) \subset \mathbb{N}$, $n_e < n_{e+1}$, with

$$\| \Psi'_{n_e}(x_e) \| \leq \frac{1}{e}$$

and

$$| \Psi_{n_e}(x_e) - c_j | \leq \frac{1}{e}.$$

By (*iii*) (x_e) is precompact. Hence eventually taking a subsequence we may assume $x_e \to x \in L\backslash V$. By (*ii*) $\Psi_\infty(x) = c_j$ and $\Psi'_\infty(x) = 0$ giving a contradiction. Hence (8) must hold for all natural numbers $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Using (*iii*), eventually replacing ε_0 and τ by smaller numbers, we find that it also holds for $n = +\infty$. Now fix $\varepsilon > 0$ such that

(9) $$0 < \varepsilon < \frac{1}{8} \min \{\varepsilon_0, \tau, \tau^2, \rho\tau, \rho\tau^2\}.$$

Following the arguments given in [*22*] one easily obtains

(10) For all $x \in L\backslash U$ such that

$$\Psi_n(x) \leq c_j + \varepsilon \text{ we have } \Psi_n(x *_n 1) \leq c_j - \varepsilon.$$

Now we find $n_1 \geq n_0$ such that for all $n \geq n_1$ and all $i \in \{1, \ldots, N\}$ we have

(11) $$| c_i(n) - c_i | \leq \frac{\varepsilon}{2}$$

and

(12) $$\mathrm{Cr}_i(n) = \{ x \mid \Psi_n'(x) = 0, \Psi_n(x) = c_i(n) \}$$

is contained in V for $i = j, \ldots, j + k$.

In fact (12) follows immediately from (*ii*) and (*iii*). We find a set $D \in CL$, $i_n(D) \geqslant j + k$
and

$$\Psi_n(D) \subset (-\infty, c_{j+k}(n) + \frac{\varepsilon}{2}] \subset (-\infty, c_j + \varepsilon].$$

Define $\tilde{D} = D \backslash U$. By the previous discussion we have using (10)

$$\Psi_n(\tilde{D} *_n 1) \subset \left(-\infty, c_j - \varepsilon\right] \subset \left(-\infty, c_j(n) - \frac{\varepsilon}{2}\right].$$

By the definition of i_n and $c_j(n)$ this implies

(13) $$i_n(\tilde{D}) < j.$$

On the other hand we infer

(14) $$\begin{aligned} k + j &\leq i_n(D) \\ &= i_n(\tilde{D} \cup (\mathrm{cl}(U) \cap D)) \\ &\leq i_n(\tilde{D}) + \mathrm{cat}\,(\mathrm{cl}(U) \cap D) \\ &\leq i_n(\tilde{D}) + \mathrm{cat}\,(\mathrm{cl}(U)) \\ &= i_n(\tilde{D}) + \mathrm{cat}\,(\mathrm{Cr}) \\ &\leq i_n(\tilde{D}) + k. \end{aligned}$$

Hence

$$j \leq i_n(\tilde{D})$$

which contradicts (14). This proves (6). Now (6) implies the following: If all the c_j are different we have N different critical levels and consequently at least N different critical point. On the other hand, if two numbers c_j and $c_{j'}, j \neq j'$, coincide the corresponding critical set has at least category two and therefore contains infinitely many critical points.

III. NOTATIONS AND PRELIMINARY RESULTS

In this chapter we shall introduce the Hilbert and Banach manifolds which will be used in the proof of the main results. In general we shall use the notation in Klingenberg's closed geodesic book [*16*], we also borrow some results from A. Floer [*11*], who was the first to study a related variational problem in this general setting. Further we study certain fibre-preserving maps in vectorbundles over spaces of curves. The main part, however, will be the study of the behaviour of compact subsets of fibres under transport (for example, parallel transport). Finally, we introduce a family of smoothing operators which make the compactness concept work.

III.1. The spaces of H^1-curves and corresponding vectorbundles.

We equip the compact connected manifold M with a Riemannian metric $\langle ., . \rangle : TM \oplus TM \to \mathbb{R}$ and denote by $\tau_M, TM \to M$ the tangent bundle. Moreover, we denote by $\exp : TM \to M$ the exponential map associated to the Riemannian metric on M. By Λ we denote the set $H^1([0,1], M)$ consisting of absolutely continuous curves $q: [0,1] \to M$ such that

(1) $$E(q) := \int_0^1 |\dot{q}(t)|^2 dt < +\infty .$$

We call $E(q)$ the energy of q. Further let C be the Banach manifold consisting of all continuous maps $q : [0,1] \to M$. We equip C with the metric

$$d_C(q, \hat{q}) = \max_{t\in[0,1]} d_M(q(t), \hat{q}(t)) .$$

There is a standard metric on TM derived from $\langle ., . \rangle$ and the associated Levi-Cività connection K, turning TM into a Riemannian manifold. Namely

$$\langle a, b \rangle_{TM} = \langle Ka, Kb \rangle + \langle T\tau_M a, T\tau_M b \rangle .$$

For $q \in \Lambda$ we define

$$T_q\Lambda = \{ y \in H^1([0,1], TM) \mid \tau_M y = q \} .$$

We equip $T_q\Lambda$ with the structure of a Hilbert space by defining the inner product

$$(x, y)_\Lambda = \int_0^1 (\langle x(t), y(t) \rangle + \langle \nabla x(t), \nabla y(t) \rangle) dt .$$

Here ∇x denotes the covariant derivative along the curve q associated to K, which is almost everywhere defined. It is well-known that $T\Lambda = \cup_{q\in\Lambda} T_q\Lambda$ can be canonically identified with the tangent space of Λ [*16*] [*9*]. Therefore the notation is justified. We have a canonical embedding $p: M \to \Lambda$ by $p(m)(t) = m$ for all $t \in [0,1]$. To simplify notations we shall write m for $p(m)$ and M for $p(M)$ if there is no danger of confusion. We denote by $\pi_M : B \to M$ the pullback of the tangent bundle $\tau_\Lambda : T\Lambda \to \Lambda$ via p. π_M possesses an important smooth sub-bundle $\pi_M^0 : B^0 \to M$, where B^0 is the kernel of the smooth fibre-preserving map over M defined by

(2) $$B \to TM : x \to \int_0^1 x(t) dt .$$

Here, of course, we consider B to be $T\Lambda\,|_M$. The vector bundles π_M^0 and π_M possess smooth Riemannian metrics induced from the inner product $(.,.)_\Lambda$. We can equip $T_q\Lambda$ with a different inner product

$$(x, y) = \int_0^1 \langle x(t), y(t) \rangle \, dt. \tag{3}$$

Denoting by $L_q\Lambda$ the complection of $T_q\Lambda$ with respect to the norm $\| . \|$ associated to $(.,.)$ we obtain a Hilbert space. It is well-known that $L\Lambda = \cup_{q\in\Lambda} L_q\Lambda$ carries in a natural way the structure of a smooth vector-bundle over Λ and moreover $(.,.)$ defines a smooth Riemannian metric for the bundle $\pi : L\Lambda \to \Lambda$.

We introduce a continuous map $\| \ \|_\infty : T\Lambda \to \mathbb{R}$ by

$$\| x \|_\infty = \max_{t\in[0,1]} | x(t) |.$$

In the following we need several standard estimates (see [*16*])

$$\| x \| \leq \| x \|_\infty \leq 2 \| x \|_\Lambda \tag{4}$$

for all $x \in T\Lambda$. Moreover for all $q \in \Lambda$ we have the estimate

$$d_M(q(t), q(s)) \leq \sqrt{E(q)}\, | t - s |^{\frac{1}{2}}. \tag{5}$$

Denote by $d_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ the metric induced by the Riemannian metric $(.,.)_\Lambda : T\Lambda \oplus T\Lambda \to \mathbb{R}$. By a result in [*16*] we have the estimate

$$E(q)^{\frac{1}{2}} - E(\tilde{q})^{\frac{1}{2}} \leq d_\Lambda(q, \tilde{q}). \tag{6}$$

Using the compactness of M the Ascoli-Arzela-Theorem can be applied to conclude from (5) and (6) that the embedding $(\Lambda, d_\Lambda) \hookrightarrow (C, d_C)$ is compact. Before we give some canonical charts for the manifolds just introduced we state a simple interpolation inequality which will be very useful later on.

$$\| x \|_\infty^2 \leq \| x \|^2 + 2 \| x \| \, \| \nabla x \| \tag{7}$$

for all $x \in T\Lambda$. The proof of (7) is very simple. Fix $t_0 \in [0, 1]$ such that $| x(t_0) | = \| x \|_\infty$. Then

$$\begin{aligned} | x(t) |^2 &= | x(t_0) |^2 + 2 \int_{t_0}^t \langle x(s), \nabla x(s) \rangle \, ds \\ &\geq \| x \|_\infty^2 - 2 \| x \| \, \| \nabla x \|. \end{aligned}$$

Integrating over $[0, 1]$ gives (7).

To introduce local trivializations denote by $\exp_m : T_m M \to M$ the restriction of the exponential map. For $q \in \Lambda$ we define $\exp_q$ by

$$\exp_q : T_q\Lambda \to \Lambda : \exp_q(x)(t) = \exp_{q(t)}(x(t)).$$

This map is injective on an open neighbourhood V of $O \in T_q\Lambda$ and can be used to define the differentiable structure on Λ. Hence

$$\exp_q : V \to U \tag{8}$$

is a diffeomorphism, where U is a suitable open neighbourhood in Λ. In fact, we can take rather big neighbourhoods V. For example V can be taken of the form $\{ x \in T_q\Lambda \mid x(t) \in W \text{ for all } t \in [0, 1] \}$, where W is an open neighbourhood of the zero-section of $TM \to M$ (see [*16*]). Note that by the compact embedding $\Lambda \hookrightarrow C$ already finitely many of such « big » neighbourhoods will cover a bounded set in Λ.

For $m \in M$ and $x \in T_mM$ let

$$D\exp_m(x) : T_mM \to T_{\exp(x)}M \tag{9}$$

denote the linearisation of $\exp_m$ at x. Then with $\exp_q$ as defined above we define a map Φ_q by

$$\Phi_q : V \times L_q\Lambda \to L\Lambda_{|U} \tag{10}$$

where

$$\Phi_q(x, y)(t) = D\exp_{q(t)}(x(t)).y(t). \tag{11}$$

Similar maps give the local trivializations of the tangent bundle of Λ. In order to give local trivializations of π_M and π_M^0 let $V_m \subset T_mM$ be an open neighbourhood of zero in T_mM such that $\exp_m : V_m \to U$ establishes a diffeomorphism for a suitable $U \subset M$.

Define

$$\Phi_m : V_m \times T_m\Lambda \to B|_U \tag{12}$$

by

$$\Phi_m(a, x)(t) = D\exp_m(a).x(t). \tag{13}$$

Clearly a local trivialization in π_M^0 is given by

$$V_m \times \left\{ x \in T_m\Lambda \mid \int_0^1 x(t)dt = 0 \right\} \to B^0|_U \tag{14}$$

$$(a, x) \to D\exp_m(a).x$$

It is well-known that the bundle $\pi : L\Lambda \to \Lambda$ possesses a connection $\hat{K} : TL \to L\Lambda$ induced by the Levi-Civita connection K on τ_M [*16*] 1.3.4. Using $\hat{K}$ and π we can turn $L\Lambda$ into a Hilbert manifold by defining

$$(a, b)_L = (\hat{K}a, \hat{K}b) + (T\pi a, T\pi b)_\Lambda \tag{15}$$

for $a, b \in T_xL\Lambda$. The metric $d_L : L\Lambda \times L\Lambda \to \mathbb{R}$ induced by $(.,.)_L$ turns $L\Lambda$ into a complete metric space. Hence $(L\Lambda, (.,.)_L)$ is a metrically complete Hilbert manifold. Recall the definition of the bundle $\pi_M^0 : B^0 \to M$. Denote by B_ρ^0 for some $\rho > 0$ the subset of B^0 consisting of all $x \in B^0$ such that $\|x\|_\Lambda < \rho$ (again we consider $B \approx T\Lambda|_M$).

LEMMA 1. — There exists $\rho > 0$ such that the map

$$\Phi : B^0 \to \Lambda : \Phi(x)(t) = \exp(x(t)) \tag{16}$$

induces a diffeomorphism of B^0_ρ onto an open neighbourhood of $\Lambda_0 = p(M)$ in Λ.

Proof. — Fix $0_m \in B^0$. We show that $T\Phi(0_m) : T_{0_m}B^0 \to T_m\Lambda$ is an isomorphism. In local co-ordinates we have with

$$m = \exp_{m_0}(a)$$
$$x(t) = D\exp_{m_0}(a).\xi(t)$$

the following representation of the local representative of Φ

$$\Phi_{m_0}(a, \xi)(t) = \exp_{m_0}^{-1}(\exp(D\exp_{m_0}(a).\xi(t))).$$

Here, of course, $\Phi_{m_0} : V_{m_0} \times (B^0)_{m_0} \to T_{m_0}\Lambda$.

Since

$$\Phi_{m_0}(0, \xi)(t) = \xi(t), \ \Phi_{m_0}(a, 0)(t) = a$$

we infer

$$D\Phi_{m_0}(0, 0)(h, k)(t) = h + k(t).$$

Clearly this means that $D\Phi_{m_0}(0, 0)$ establishes an isomorphism

$$T_{m_0}M \times (B^0)_{m_0} \to T_{m_0}\Lambda.$$

Therefore, $T\Phi(0_m)$ is a diffeomorphism for all $m \in M$. By the inverse function theorem Φ establishes a diffeomorphism from an open neighbourhood of 0_m onto an open neighbourhood of $p(m)$ for all $m \in M$. By the compactness of M we find $\rho > 0$ and $U \subset \Lambda$ open, $\Lambda_0 \subset U$, such that the map

$$\Phi : B^0_\rho \to U$$

is onto and a local diffeomorphism. Therefore it is enough to show that $\Phi | B^0_\rho$ is injective in order to complete the proof of Lemma 1. By the previous discussion the number of points in $(\Phi | B^0_\rho)^{-1}(q)$ is finite for all $q \in U$ and constant. The injectivity of $\Phi | B^0_\rho$ will follow if we can show that $(\Phi | B^0_\rho)^{-1}(m) = \{ 0_m \}$. Arguing indirectly assume for some $x \neq 0_m$, $x \in B^0_\rho$

$$\exp(x(t)) = m$$

for all $t \in [0, 1]$. Then $x(t) = x_0$ for all $t \in [0, 1]$ for some constant curve in B^0. Since $\int_0^1 x_0 dt = 0_{\tilde{m}}$ for some $\tilde{m} \in M$ we must have $x_0 = 0_{\tilde{m}}$. Hence

$$m = \exp(x_0) = \exp(0_{\tilde{m}}) = \tilde{m}$$

which implies $\tilde{m} = m$ giving a contraction. □

The vectorbundle $\pi^0_M : B^0 \to M$ will be crucial in the construction of the intersection pairs.

III.2. Covariant derivatives and curvature.

Denote by $\delta : T\Lambda \to L\Lambda$ the smooth fibre preserving map defined by

$$(\delta x)(t) = \nabla x(t). \tag{1}$$

It is well-known that δ is the covariant derivative of the smooth section $\partial : \Lambda \to L\Lambda$ defined by

$$\partial q = \dot{q}, \tag{2}$$

see [16], Proposition 1.3.5.

Let G denote one of the bundles $T\Lambda$ or $L\Lambda$ over Λ. If $\xi : \Lambda \to G$ is a smooth section and $x \in T_q\Lambda$ we define the covariant derivative of ξ at $q \in \Lambda$ in the direction of x by

$$(D_x\xi)(t) = \nabla_s \xi_x(s, t)\,|_{s=0}, \tag{3}$$

where $\xi_x(s, t) = \xi\,(\exp_q(s.x))(t)$.

Clearly we have

$$(D_x\xi)(t) = (\hat{K}(T\xi.x))(t) \tag{4}$$

if ξ is a section of $L\Lambda \to \Lambda$.

Since we consider $T(T\Lambda)$ in a natural way as a subset of $T(L\Lambda)$, formula (4) remains valid for sections of $T\Lambda \to \Lambda$. Now let G_1 and G_2 denote bundles of type $L\Lambda \to \Lambda$ or $T\Lambda \to \Lambda$ and assume $\rho : G_1 \to G_2$ is a fibre-preserving map, we define the covariant derivative at q in the direction $x \in T_q\Lambda$ by

$$(D_x\rho)\xi = D_x(\rho\xi) - \rho(D_x\xi) \tag{5}$$

where ξ is a smooth section of $G_1 \to \Lambda$.

Denote by $R : TM \oplus TM \oplus TM \to TM$ the curvature tensor of $(M, \langle .,. \rangle)$. For $q \in \Lambda$, $\eta \in T_q\Lambda$ we define a map

$$R(\eta, \dot{q}) : T_q\Lambda \to L_q\Lambda \tag{6}$$

by

$$(R(\eta, \dot{q})x)(t) = R(\eta(t), \dot{q}(t))x(t). \tag{7}$$

LEMMA 2. — $D_x\delta = R(x, \dot{q})$, where $x \in T_q\Lambda$.

A proof can be found in [11].

Before we proceed further, let us note that if $\rho_1 : G_1 \to G_2$ and $\rho_2 : G_2 \to G_3$ are fibre-preserving maps, where the G_i are as before we must have

$$D_x(\rho_2\rho_1) = (D_x\rho_2)\rho_1 + \rho_2(D_x\rho_1).$$

LEMMA 3. — For all $q \in \Lambda$ δ induces a surjective Fredholm operator $\delta_q : T_q\Lambda \to L_q\Lambda$ of index $i(\delta_q) = \dim(M)$. Moreover if S is a bounded subset of Λ there exists a positive constant $c = c(S)$ such that

$$(8) \qquad \| \delta x \| \geq c \| x \|_\Lambda$$

for all $x \in T\Lambda|_S$ such that x is $(.,.)$-orthogonal to $\operatorname{kern}(\delta_q)$, where $q = \pi x$.

Proof. — Arguing indirectly we find a sequence $(x_n) \subset T\Lambda$ such that $q_n = \pi x_n \in S$ and

$$(9) \qquad \| x_n \|_\Lambda = 1, \qquad \| \delta x_n \| \leq \frac{1}{n}, \qquad x_n \perp \operatorname{kern}(\delta_{q_n}).$$

Since (q_n) is bounded we may assume, without loss of generality that

$$(10) \qquad q_n \to q_0 \quad \text{in} \quad C$$

where $q_0 \in C$. Taking a chart centred at some $q \in \Lambda$ close to q_0 we have

$$q_n = \exp_q(\xi_n)$$

for all n large enough. It is easy to see that $(\| \xi_n \|_\Lambda)$ must be bounded. Hence (ξ_n) possesses a weakly convergent subsequence in $T_q\Lambda$. This, of course, implies that q_0 is, in fact, an element in Λ. Let $q = q_0$. We have

$$x_n = \Phi_q(\xi_n, \eta_n).$$

Using the notation in [*16*], p. 7-22, (9) looks, in local co-ordinates, like

$$(G(\xi_n)(\nabla\eta_n + D_2\Theta_q(\xi_n)\eta_n + \Gamma_q(\xi_n)(\eta_n, \partial_q\xi_n)),$$
$$\nabla\eta_n + D_2\Theta_q(\xi_n)\eta_n + \Gamma_q(\xi_n)(\eta_n, \partial_q\xi_n)) \leq \frac{1}{n}.$$

Since $G(\xi_n) \to G(0) = \mathrm{Id}$, $D_2\Theta_q(\xi_n) \to 0$ and $\Gamma_q(\xi_n) \to 0$ uniformly as $n \to +\infty$ we obtain

$$(11) \qquad \lim \| \nabla\eta_n \| = 0.$$

Since $T_q\Lambda$ is compactly embedded in $L_q\Lambda$ and $\| x_n \|_\Lambda = 1$, we infer eventually taking a subsequence

$$\eta_n \to \eta \quad \text{in} \quad L_q\Lambda$$
$$\eta_n \rightharpoonup \eta \quad \text{in} \quad T_q\Lambda.$$

Hence by (11) $\nabla\eta = 0$. Therefore $\eta \in \operatorname{kern}(\delta_q) = \operatorname{kern}(\nabla)$. On the other hand, we know that

$$x_n \perp \operatorname{kern}(\delta_{q_n}).$$

Using the standard L^2-theory for ordinary differential equations we find for all $n \in \mathbb{N}$ a unique solution $\tilde{\eta}_n$ on $H^1([0,1], TM)$, $\tilde{\eta}_n(t) \in T_{q(t)}M$ such that

$$(12) \qquad \nabla\tilde{\eta}_n + D_2\Theta_q(\xi_n)\tilde{\eta}_n + \Gamma_q(\xi_n)(\tilde{\eta}_n, \partial_q\xi_n) = 0$$
$$\tilde{\eta}_n(0) = \eta(0).$$

Let $\tilde{x}_n = \Phi_q(\xi_n, \tilde{\eta}_n)$. Then $\delta \tilde{x}_n = 0$. Hence $| \tilde{x}_n(t) | = \text{const}$. Since $(\| \partial_q \xi_n \|)$ is bounded we find that

$$D_2\theta_q(\xi_n)\tilde{\eta}_n + \Gamma_q(\xi_n)(\tilde{\eta}_n, \partial_q \xi_n) \to 0$$

in $L_q\Lambda$. Hence eventually taking a subsequence of $(\tilde{\eta}_n)$ we may assume

$$\begin{aligned} \tilde{\eta}_n &\rightharpoonup \tilde{\eta} \quad \text{in} \quad T_q\Lambda \\ \eta_n &\to \tilde{\eta} \quad \text{uniformly.} \end{aligned}$$

Taking the limit in (12) gives

$$\begin{aligned} \nabla \tilde{\eta} &= 0 \\ \tilde{\eta}(0) &= \eta(0) . \end{aligned}$$

By the uniqueness of solutions for given initial value we must have $\tilde{\eta} = \eta$. We have by our assumption

$$(x_n, \tilde{x}_n) = 0 . \tag{13}$$

Hence in local co-ordinates

$$0 = (G(\xi_n)\eta_n, \tilde{\eta}_n) \to (\eta, \eta) = 1$$

giving a contradiction. □

Denote by $A : L\Lambda \to L\Lambda$ the fibre-preserving map defined by

$$A_x = (\delta_{\pi x} \mid \text{kern}\, (\delta_{\pi x})^{\perp} \cap T\Lambda)^{-1} x . \tag{14}$$

LEMMA 4. — A is smooth.

The easy proof is left to the reader.

Denote by $\| A_q \|$ the norm of the operator $A : L_q\Lambda \to L_q\Lambda$. By Lemma 3 the map

$$q \to \| A_q \|$$

maps bounded sets into bounded sets. Denote by $A^* : L\Lambda \to L\Lambda$ the L-adjoint of A, i. e.

$$(Ax, y) = (x, A^*y)$$

for all $(x, y) \in L\Lambda \oplus L\Lambda$. Clearly A^* is smooth and $\| A_q^* \| = \| A_q \|$.

LEMMA 5. — Let $x \in L_q\Lambda$ and assume

$$(x, \delta y) \leq c \| y \| \tag{15}$$

for all $y \in T_q\Lambda$ for some constant c independent of y. Then $x \in T_q\Lambda$.

This is an easy consequence of the H = W result of Serrin and Meyers [*1*].

III.3. Compactness and transport equations.

The compactness concept we shall introduce in this section is promted by the corresponding theorem by M. Riesz in L^p. The following results provide the necessary background. Given $q \in \Lambda$ we denote by

$$A^q_{s,t} : T_{q(t)}M \to T_{q(s)}M$$

for $t, s \in [0, 1]$ the parallel transport along q. Since q is of class H^1 the standard L^2-theory of the Cauchy problem $\nabla_{\dot{q}} x = 0$ then shows uniqueness and existence of $A^q_{s,t}$. We define a one parameter family $(Z(\tau))_{\tau \in [-1,1]}$ of fibre preserving maps $Z(\tau) : L\Lambda \to L\Lambda$ by

$$(1) \qquad (Z(\tau)x)(t) = \begin{cases} A^q_{t,t+\tau}x(t+\tau) & \text{if } t+\tau, t \in [0,1] \\ 0 & \text{if } t+\tau \notin [0,1], t \in [0,1] \end{cases}$$

where $x \in T_q\Lambda$. Since $\| Z(\tau)x \| \leq \| x \|$ and $T_q\Lambda$ is dense in $L_q\Lambda$ it determines a unique map $L\Lambda \to L\Lambda$. Note that

$$(2) \qquad (Z(\tau)x, y) = (x, Z(-\tau)y)$$

for all $\tau \in [-1, 1]$ and all $(x, y) \in L\Lambda \oplus L\Lambda$. In fact we calculate assuming that $\tau \geq 0$

$$\begin{aligned} (Z(\tau)x, y) &= \int_0^{t-\tau} \langle A^q_{t,t+\tau}x(t+\tau), y(t) \rangle \, dt \\ &= \int_0^{t-\tau} \langle x(t+\tau), A_{t+\tau,t}y(t) \rangle \, dt \\ &= \int_\tau^1 \langle x(t), A_{t,t-\tau}y(t-\tau) \rangle \, dt \\ &= (x, Z(-\tau)y). \end{aligned}$$

Moreover,

$$(3) \qquad Z(1)x = Z(-1)x = 0 \quad (\text{in } L_q\Lambda).$$

LEMMA 6. — For a bounded set S in $L_q\Lambda$ the following statements are equivalent

(*i*) S is precompact

(*ii*) Given any $\varepsilon > 0$ there exists $\rho \in (0, 1]$ such that $\| Z(\tau)x - x \| \leq \varepsilon$ for all $x \in S$ and $|\tau| \leq \rho$.

For the moment we shall postpone the proof. Later on we shall reduce Lemma 6 to a standard result in L^p-theory. Next we give a useful estimate relating to the $T_q\Lambda$-norm and the quantitative expression in (*ii*).

LEMMA 7. — For all $x \in T\Lambda$ we have the estimate

$$(4) \qquad \| Z(\tau)x - x \| \leq 2 \| x \|_\Lambda |\tau|^{\frac{1}{2}}.$$

Proof. — Assume first t and $t + \tau \in [0, 1]$. Since

$$\begin{aligned} |(Z(\tau)x)(t) - x(t)| &= |A^q_{t,t+\tau}x(t + \tau) - x(t)| \\ &= |x(t + \tau) - A^q_{t+\tau,t}x(t)| \end{aligned}$$

we infer

$$\begin{aligned} \frac{d}{d\tau}|(Z(\tau)x)(t) - x(t)|^2 &= 2\langle x(t + \tau) - A^q_{t+\tau,t}x(t), \nabla x(t + \tau)\rangle \\ &\leq 2|x(t + \tau) - A^q_{t+\tau,t}x(t)||\nabla x(t+\tau)| \\ &= 2|A^q_{t,t+\tau}x(t + \tau) - x(t)||\nabla x(t+\tau)| \\ &= 2|(Z(\tau)x)(t) - x(t)|\,|\nabla x(t + \tau)|. \end{aligned}$$

Hence

$$(5) \qquad \begin{aligned} |(Z(\tau)x)(t) - x(t)| &\leq \left|\int_t^{t+\tau} |\nabla x(s)|\, ds\right| \\ &\leq \|\delta x\|\,|\tau|^{\frac{1}{2}}. \end{aligned}$$

If $t + \tau \notin [0, 1]$, $t \in [0, 1]$ we have

$$(Z(\tau)x)(t) = 0.$$

Consequently

$$|(Z(\tau)x)(t) - x(t)|^2 = |x(t)|^2.$$

Hence with $A_\tau = \{ t \in [0, 1] \mid t + \tau \in [0, 1] \}$ and $B_\tau = [0, 1]\backslash A_\tau$ we infer

$$\begin{aligned} \|Z(\tau)x - x\|^2 &= \int_{A_\tau} |(Z(\tau)x)(t) - x(t)|^2 dt + \int_{B_\tau} |x(t)|^2 dt \\ &\leq \|\delta x\|^2|\tau| + \int_{B_\tau} \|x\|^2_\infty dt \\ &\leq \|\delta x\|^2|\tau| + (\|x\|^2 + 2\|x\|\,\|\delta x\|)|\tau| \\ &= (\|\delta x\| + \|x\|)^2|\tau| \\ &\leq (2\|x\|_\Lambda)^2|\tau|. \end{aligned}$$

This yields (4). □

Let $\phi \in C^1(\mathbb{R}, \Lambda)$ and $\beta : \mathbb{R} \to [-1, 1]$ smooth. Denote by ∇_ϕ, the covariant derivative along the curve ϕ in the bundle $L\Lambda \to \Lambda$ associated to $\hat{K}$. Consider the differential equation

$$(6) \qquad \nabla_{\phi'}a = \beta(s)a.$$

We denote by $C(s_1, s_0) : L_{\phi(s_0)}\Lambda \to L_{\phi(s_2)}\Lambda$ the induced linear « transport map ». Clearly we have the estimate

$$(7) \qquad \|C(s_1, s_0)\| \leq \exp(|s_1 - s_0|)$$

for the operator norm.

The following result is crucial.

PROPOSITION 1. — Let ϕ be as defined above. We have the following estimate

(8) $\| Z(\tau)C(s_1, s_0)x - C(s_1, s_0)x \| \leq \tilde{M}[\phi, s_1, s_0](\|Z(\tau)x - x\| + \|x\| \, |\tau|^{\frac{1}{2}})$

for all $x \in L_{\phi(s_0)}\Lambda$, all $\tau \in [-1, 1]$, for a suitable constant $\tilde{M}$ only depending on $(M, \langle \cdot, \cdot \rangle)$. Here $[\phi, s_1, s_0]$ denotes the number

(9) $[\phi, s_1, s_0]$
$= (1 + \max \{ \| \partial\phi(h) \| \, \| \phi'(h) \|_\infty \mid h \in [\min\{s_1, s_0\}, \max\{s_1, s_0\}] \})$
$. \exp(3 \mid s_1 - s_0 \mid)$.

Proof. — Define a map $\hat{\phi} : \mathbb{R} \times [0, 1] \to M$ by $\hat{\phi}(s, t) = \phi(s)(t)$.

First we shall prove the Proposition under the additional assumption that $\hat{\phi}$ is smooth. Denote for $t_0, t_1 \in [0, 1]$ and $s_0, s_1 \in \mathbb{R}$ by

$$A(s_0 ; t_1, t_0) : T_{\hat{\phi}(s_0, t_0)}M \to T_{\hat{\phi}(s_0, t_1)}M$$

the parallel transport along the path $t \to \hat{\phi}(s_0, t)$. Moreover denote by

$$B(t_0 ; s_1, s_0) : T_{\hat{\phi}(s_0, t_0)}M \to T_{\hat{\phi}(s_1, t_0)}M$$

the map induced by the differential equation $\nabla_{\hat{\phi}'(., t_0)}a = \beta(s)a$ where $\hat{\phi}'(., t_0)$ denotes the partial differential with respect to s. Clearly we have for $x \in T_{\phi(s_0)}\Lambda$

$$(C(s_1, s_0)x)(t) = B(t; s_1, s_0)(x(t)).$$

Moreover we compute for $x \in T_{\hat{\phi}(s_0, t)}M$ and $y \in T_{\hat{\phi}(s_0, t_0)}M$ with the abbreviations

$$a(s, t) = B(t; s, s_0)x - A(s; t, t_0)B(t_0; s, s_0)y$$
$$b(s, t) = A(s; t, t_0)B(t_0; s, s_0)y$$

the following

(10) $$\left(\frac{\partial}{\partial s} | a |^2 \right)(s, t) = 2 \langle a(s, t), \beta(s)B(t; s, s_0)x - \nabla_s b(s, t) \rangle$$

Denote by $\hat{\phi}_t$ and $\hat{\phi}_s$ the partial derivatives with respect to t and s, respectively. We infer

(11) $$\begin{aligned} \nabla_t \nabla_s b(s, t) &= \nabla_s \nabla_t b(s, t) + R(\hat{\phi}_t(s, t), \hat{\phi}_s(s, t))b(s, t) \\ &= R(\hat{\phi}_t(s, t), \hat{\phi}_s(s, t))b(s, t). \end{aligned}$$

Here R denotes the curvature tensor. Now using the variation of constant formula we deduce

(12) $$\begin{aligned} \nabla_s b(s, t) &= A(s; t, t_0)(\nabla_s b(s, t_0)) \\ &+ \left(\int_{t_0}^{t} A(s; t, h)R(\hat{\phi}_t(s, h), \hat{\phi}_s(s, h))A(s; h, t_0)dh \right) \\ &. \, B(t_0; s, s_0)y \\ &=: A(s; t, t_0)(\beta(s)B(t_0; s, s_0)y) \\ &+ K(s; t, t_0)B(t_0; s, s_0)y. \end{aligned}$$

Here

$$K(s; t, t_0) = \int_{t_0}^{t} A(s; t, h) R(\hat{\phi}_t(s, h), \hat{\phi}_s(s, h)) A(s; h, t_0) dh .$$

Now combining (10) and (12) yields

$$(13) \quad \left(\frac{\partial}{\partial s} | a |^2\right)(s, t)$$
$$= 2 \langle a(s, t) ; \beta(s) a(s, t) \rangle - 2 \langle a(s, t), K(s; t, t_0) B(t_0; s, s_0) y \rangle$$
$$= 2\beta(s) | a(s, t) |^2 - 2 \langle a(s, t), K(s; t, t_0) B(t_0; s, s_0) y \rangle$$

Now let $x \in T_{\phi(s_0)}\Lambda$. Then

$$(14) \quad (C(s, s_0)x)(t) = B(t; s, s_0)(x(t)), \quad t \in [0, 1];$$
$$(Z(\tau)C(s, s_0)x)(t) = A(s; t, t+\tau) B(t+\tau; s, s_0)(x(t+\tau)) \quad \text{if} \quad t+\tau \in [0, 1];$$
$$(Z(\tau)C(s, s_0)x)(t) = 0 \quad \text{if} \quad t+\tau \notin [0, 1].$$

Now by the previous discussion we infer if $t+\tau \in [0, 1]$

$$\frac{\partial}{\partial s} | C(s, s_0)x)(t) - (Z(\tau)C(s, s_0)x)(t) |^2$$
$$\leq 2\beta(s) | (C(s, s_0)x)(t) - (Z(\tau)C(s, s_0)x)(t) |^2$$
$$+ 2 | (C(s, s_0)x)(t) - (Z(\tau)C(s, s_0)x)(t) |$$
$$. \hat{M} \left| \int_{t+\tau}^{t} | \hat{\phi}_t(s, h) | \right| \hat{\phi}_s(s, h) | dh | | x(t + \tau) | \exp (| s - s_0 |) .$$

Here $\hat{M}$ is a constant only depending on $(M, \langle .,. \rangle)$ such that

$$| R(a, b, c) | \leq \hat{M} | a | | b | | c | \quad \text{for all} \quad (a, b, c) \in TM \oplus TM \oplus TM .$$

Further we have used that parallel transport preserves the inner product on M.

We introduce the abbreviation

$$c(s, t) = | (C(s, s_0)x)(t) - (Z(\tau)C(s, s_0)x)(t) | .$$

Hence we have provided that $t + \tau \in [0, 1]$

$$(14') \quad \left(\frac{\partial}{\partial s} | c |^2\right)(s, t)$$
$$\leq 2 | c(s, t) |^2 + 2 | c(s, t) | \hat{M} \| \phi'(s) \|_\infty \| \partial\phi(s) \| | \tau |^{\frac{1}{2}} . | x(t+\tau) | \exp (| s - s_0 |)$$
$$= 2 | c(s, t) |^2 + (2\hat{M} \exp (| s - s_0 |) | \tau |^{\frac{1}{2}} \| \partial\phi(s) \| \| \phi'(s) \|_\infty | c(s, t) | | x(t+\tau) | .$$

If $t + \tau \notin [0, 1]$ we find simply

$$(15) \quad \left(\frac{\partial}{\partial s} | c |^2\right)(s, t) = \frac{\partial}{\partial s} | (C(s, s_0)x)(t) |^2$$
$$\leq 2 | (C(s, s_0)x)(t) |^2 = 2 | c(s, t) |^2$$

combining (14) and (15) yields

$$\frac{d}{ds}\| Z(\tau)C(s, s_0)x - C(s, s_0)x \|^2 \leq 2 \| Z(\tau)C(s, s_0)x - C(s, s_0)x \|^2$$
$$+ (2\hat{M} \exp(|s-s_0|) \|\partial\phi(s)\| \|\phi'(s)\|_\infty |\tau|^{\frac{1}{2}}) \|x\| . \| Z(\tau)C(s, s_0)x - C(s, s_0)x \|.$$

Now applying Gronwall's Lemma yields for a suitable constant M^* only depending on $(M, \langle .,. \rangle)$

(16) $$(\| Z(\tau)C(s, s_0)x - C(s, s_0)x \| \leq (\| Z(\tau)x - x \| + M^* \|x\| |\tau|^{\frac{1}{2}} |s-s_0| \exp(|s-s_0|)\gamma(s, s_0)) . \exp(|s-s_0|)$$

where $\gamma(s, s_0) = \max\{ \|\partial\phi(h)\| \|\phi'(h)\|_\infty \mid h \in [\min\{s, s_0\}, \max\{s, s_0\}] \}$.

Now taking a suitable constant $\tilde{M}$ large enough only depending on $(M, \langle .,. \rangle)$ we obtain

(17) $$\| Z(\tau)C(s, s_0)x - C(s, s_0)x \| \leq \tilde{M}[\phi, s, s_0](\| Z(\tau)x - x \| + \|x\| |\tau|^{\frac{1}{2}})$$

as claimed. Next we remove the assumption that the induced map $\hat{\phi} : \mathbb{R} \times [0,1] \to M$ is smooth. Assume we can show for given s_1, s_0, say $s_0 \leq s_1$, the existence of a sequence of maps $\phi_n \in C^1(\mathbb{R}, \Lambda)$ such that

$$d_{T\Lambda}(\phi_n'(s), \phi'(s)) \leq \frac{1}{n},$$

for all $s \in [s_0, s_1]$ and that moreover the induced maps $\hat{\phi}_n : [s_0, s_1] \times [0,1] \to M$ are smooth. Clearly $[\phi_n, s_1, s_0] \to [\phi, s_1, s_0]$ as $n \to +\infty$. Moreover, given $x \in L_{\phi(s_0)}\Lambda$ we find $(x_n) \subset L\Lambda$, $x_n \in L_{\phi_n(s_0)}\Lambda$ such that $x_n \to x$. By the continuous parameter dependence theory we infer further that $C^n(s_1, s_0)x_n \to C(s_1, s_0)x$ where C^n is the family of linear maps corresponding to ϕ_n. Since for fixed $\tau \in [-1, 1]$ $Z(\tau) : L\Lambda \to L\Lambda$ is continuous (we leave the proof as an easy exercise to the reader) we infer

$$\begin{aligned} \| Z(\tau)C(s_1, s_0)x - C(s_1, s_0)x \| &= \lim \| Z(\tau)C^n(s_1, s_0)x_n - C^n(s_1, s_0)x_n \| \\ &\leq \tilde{M}[\phi, s_1, s_0](\| Z(\tau)x - x \| + \|x\| |\tau|^{\frac{1}{2}}) \end{aligned}$$

which completes the proof. So it remains to show the approximation result. □

LEMMA 8. — Given $\phi \in C^1(\mathbb{R}, \Lambda)$ and $s_0, s_1 \in \mathbb{R}$ (say $s_0 \leq s_1$) there exists a sequence $(\phi_n) \subset C^1(\mathbb{R}, \Lambda)$ such that $d_{T\Lambda}(\phi_n'(s), \phi'(s)) \leq \frac{1}{n}$ for all $s \in [s_0, s_1]$ and $\hat{\phi}_n : \mathbb{R} \times [0,1] \to M$ is smooth.

Proof. — We embed $(M, \langle .,. \rangle)$ isometrically in some $\mathbb{R}^m$ and denote the image by M. Denote by H the space $H^1([0,1], \mathbb{R}^m)$. We can consider ϕ as a map in $C^1(\mathbb{R}, H)$. Let $U \subset \mathbb{R}^m$ open, $M \subset U$ be a tubular neighbourhood of M and denote by $p : U \to M$ the projection. Let

$$H_U = \{ q \in H \mid q([0,1]) \subset U \}.$$

Then H_U is an open subset of H. Moreover p induces a smooth map $\tilde{p}$: $H_U \to \Lambda$ by $\tilde{p}(q) = p \circ q$. If we approximate in $C^1([s_0, s_1], H)$, the composition of the approximation with $\tilde{p}$ will be an approximation of ϕ with image in Λ. By the previous remark we have only to show the following: given a map $\phi \in C^1(\mathbb{R}, H)$ and numbers s_0, s_1 ; $s_0 \leq s_1, \varepsilon > 0$ there exists $\tilde{\phi} \in C^1(\mathbb{R}, H)$ such that

$$\| \phi - \tilde{\phi} \|_{C^1([s_0,s_1],H)} \leq \varepsilon$$

and $\hat{\tilde{\phi}}$ is smooth.

This approximation can be carried out using mollifiers. First we define

$$\phi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$$

by

$$\phi_1(s)(t) = \begin{bmatrix} \phi(s)(0) & t < 0 \\ \phi(s)(t) & t \in [0,1] \\ \phi(s)(1) & t > 1 \end{bmatrix}$$

Then one mollifies in the t-variable. One gets for $\varepsilon > 0$

$$\phi_2^\varepsilon \in C^1(\mathbb{R}, H)$$

by

$$\phi_2^\varepsilon(s)(t) = (\theta_\varepsilon *_t(\phi_1(s)))(t), \quad t \in [0,1]$$

where θ_ε denotes the standard mollifier say $\theta_\varepsilon(t) = \frac{1}{\varepsilon}\theta\left(\frac{t}{\varepsilon}\right)$, and

$$\theta(t) = \begin{bmatrix} C \exp\left(-\frac{1}{1-t^2}\right) & \text{if} \quad |t| < 1 \\ 0 & \text{if} \quad |t| \geq 1\,. \end{bmatrix}$$

Here $C > 0$ is a constant such that $\int_{-\infty}^{\infty} \theta = 1$. If $\varepsilon \to 0$ $\phi_2^\varepsilon \to \phi$ in C^1. Moreover $t \to \phi_2^\varepsilon(s)(t)$ is of class C^∞. Next one mollifies ϕ_2^ε in the s-variable getting $\phi^{\varepsilon,\delta} \in C^\infty(\mathbb{R}, H)$, where $\phi^{\varepsilon,\delta}(s)$ is of class C^∞. Moreover $\phi^{\varepsilon,\delta} \to \phi$ as $\varepsilon, \delta \to 0$ in C^1 on $[s_0, s_1]$.

Now note that

$$\hat{\phi}^{\varepsilon,\delta}(s,t) = \int_{-\infty}^{+\infty} \theta_\delta(s-\tau)\left(\int_{-\infty}^{+\infty} \theta_\varepsilon(t-y)\phi_2(\tau)(y)dy\right)d\tau\,.$$

Clearly $\hat{\phi}^{\varepsilon,\delta}$ is smooth. □

Proof of Lemma 6. — Since Λ is connected we find a smooth path ϕ : $[0,1] \to \Lambda$ such that $\phi(0) = m$, $\phi(s_1) = q$. Here $m \in M$ is an arbitrarily fixed element. Applying Proposition 1 to the parallel transport B : $L_m\Lambda \to L_q\Lambda$ we find a suitable constant $a > 0$ such that

$$a^{-1} \| Z(\tau)x - x \| \leq \| Z(\tau)Bx - Bx \| + \| x \| \, |\tau|^{\frac{1}{2}}$$
$$\leq a(\| Z(\tau)x - x \| + \| x \| \, |\tau|^{\frac{1}{2}})\,.$$

Hence the statements in Lemma 6 are equivalent provided they are equivalent in $L_m\Lambda$ for $B^{-1}(S)$. In $L_m\Lambda$ we have for all x

$$(Z(\tau)x)(t) = \begin{bmatrix} 0 & \text{if} \quad t + \tau \notin [0,1] \\ x(t+\tau) & \text{if} \quad t + \tau \in [0,1]. \end{bmatrix}$$

Define $\tilde{x} : \mathbb{R} \to T_m M$ by $x(s) = \tilde{x}(s)$ if $s \in [0,1]$ and 0 otherwise. By a result in [1], Theorem 2.21, we have the following equivalence for a bounded set S of $L^2(0,1;\mathbb{R}^n)$

(17) S is precompact iff for all $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\int_{-\infty}^{+\infty} |\tilde{x}(t+\tau) - \tilde{x}(t)|^2 dt \leq \varepsilon^2 \qquad \text{for all} \quad x \in S \quad \text{and} \quad |\tau| \leq \rho.$$

Taking $\mathbb{R}^n \cong T_m M$ we get a criterion on $L_m\Lambda$. A straightforward calculation shows for $t \in [-1,1]$.

$$\begin{aligned} &\| Z(\tau)x - x \|^2 + \| Z(-\tau)x - x \|^2 \\ &\qquad \leq \int_{-\infty}^{+\infty} |\tilde{x}(t+\tau) - \tilde{x}(t)|^2 dt + \int_{-\infty}^{\infty} |\tilde{x}(t-\tau) - \tilde{x}(t)|^2 dt \\ &\qquad \leq 3(\| Z(\tau)x - x \|^2 + \| Z(-\tau)x - x \|^2). \end{aligned}$$

Hence (17) implies our assertion. □

DEFINITION 5. — A subset $S \subset L\Lambda$ is called uniformly fibre-compact (*ufp c*) if the following holds:

(*i*) S is bounded.

(*ii*) Given any $\varepsilon > 0$ there exists $\rho \in (0,1]$ such that $\| Z(\tau)x - x \| \leq \varepsilon$ for all $x \in S$ and $|\tau| \leq \rho$.

If, in addition, S is closed we call S uniformly fibre-compact (*ufc*).

In the following we denote by $[\ ] : L\Lambda \to \mathbb{R}^+$ the map defined by

$$[x] = \| x \|^2 + E(\pi x). \tag{18}$$

DEFINITION 6. — A smooth map $D : L\Lambda \to L\Lambda$ is called fibre-compact if the following holds:

(*i*) D maps bounded sets into bounded sets.

(*ii*) There exists a number $\sigma \in (0,1]$ and a monotone increasing map $\mu_D : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\| Z(\tau)Dx - Dx \| \leq \mu_D([x]) |\tau|^\sigma$ for all $x \in L\Lambda$ and $\tau \in [-1,1]$.

For example the fibre-preserving map $A : L\Lambda \to L\Lambda$ is fibre compact. In fact, by Lemma 3 we find a monotone increasing map $\tilde{\mu}_A$ such that

$$\tilde{\mu}_A(\| \partial q \|) \| x \| \geq \| Ax \|_\Lambda .$$

Combining this with Lemma 7 we find for a suitable monotone map $\mu_A : \mathbb{R}^+ \to \mathbb{R}^+$

$$\| Z(\tau)Ax - Ax \| \leq \mu_A([x]) \mid \tau \mid^{\frac{1}{2}}.$$

LEMMA 9. — Assume $(q_k) \subset \Lambda$ is a sequence such that $(\partial q_k) \subset L\Lambda$ is uniformly fibre-compact. Then (q_k) is precompact.

Proof. — We have to construct a convergent subsequence of (q_k). Since (∂q_k) is bounded, we find eventually, taking a subsequence, that

$$q_k \to q \text{ in C for some } q \in \Lambda.$$

Taking local co-ordinates based at q let

$$q_k = \exp_q(\xi_k) \quad \text{and} \quad \partial q_k = \Phi_q(\xi_k, \eta_k).$$

This is well-defined provided k is large enough.
Using the notations in [*16*] we have

$$\eta_k = \nabla\xi_k + \theta_q(\xi_k).$$

Define a family of curves $\phi_k : [0, 1] \to \Lambda$ by

$$\phi_k(s) = \exp_q((1 - s)\xi_k).$$

Note that (ξ_k) is bounded in $T_q\Lambda$. So by the special form of the ϕ_k the numbers $[\phi_k, 0, 1]$ will be uniformly bounded. Moreover we must have $\xi_k \to 0$ uniformly.

We carry out the parallel transport along ϕ_k denoting by $x_k \in L_q\Lambda$ the image of ∂q_k. By Proposition 1 there exists a constant $c_2 > 0$ such that

$$\| Z(\tau)x_k - x_k \| \leq C_2(\| Z(\tau)\partial q_k - \partial q_k \| + \mid \tau \mid^{\frac{1}{2}})$$

for all $k \in \mathbb{N}$ and all $\tau \in [-, 1]$. Here, of course, the constant bounding $(\| \partial q_k \|)$ is contained in C_2. Since (∂q_k) is $(u f p\ c)$ we infer that (x_k) is precompact in $L_q\Lambda$. Eventually, taking a subsequence we may assume

$$x_k \to x \quad \text{in} \quad L_q\Lambda,$$

for some $x \in L_q\Lambda$. In local co-ordinates the transport equation is given by

$$\dot{a} = \Gamma_q((1 - s)\xi_k)(\xi_k, a)$$
$$a(0) = \eta_k.$$

Since $\xi_k \to 0$ uniformly and $s_k \geq 1$, we obtain $\| \Gamma_q((1-s)\xi_k)(\xi_k, a) \| \leq \varepsilon_k \| a \|$, where $\varepsilon_k \to 0$. For σ_k being the local representative of x_k this implies

$$\| \sigma_k - \eta_k \| \to 0.$$

Since $\sigma_k \to \sigma$, where σ is the representative of x, we infer $\eta_k \to \sigma$. Hence

$$\nabla\xi_k + \theta_q(\xi_k) \to \sigma \quad \text{in} \quad L_q\Lambda$$
$$\xi_k \to 0 \quad \text{uniformly}.$$

Therefore

$$\xi_k \to 0 \quad \text{in} \quad T_q\Lambda .$$

This implies

$$q_k \to q \quad \text{in} \quad \Lambda . \qquad \square$$

Next we study the question how uniformly fibre-precompact sets are mapped by certain operators $L\Lambda \to L\Lambda$.

LEMMA 10. — Let $f: TM \to TM$ be a continuous fibre-preserving map and assume there exists a constant $C > 0$ such that $|f(x)| \leq C(1+|x|)$. Then the map $\tilde{f}: L\Lambda \to L\Lambda$ defined by $\tilde{f}(x)(t) = f(x(t))$ is continuous and maps uniformly fibre-precompact sets into uniformly fibre-precompact sets.

Proof. — The continuity is a well-known fact from nonlinear analysis. For the second assertion note that we can restrict ourselves to a chart since a $(u f p c)$ set can be covered by finitely many « big » exponential charts. So assume

$$\pi(S) \subset \exp_q\left(\frac{1}{2}V\right),$$

where $\exp_q : V \to U$ is a local co-ordinate system on Λ. Moreover we may assume V is convex. Let $(\xi, \eta) \in V \times L_q\Lambda$ be the representative of some $x \in S$.

Let

$$S_q = \{ \eta \in L_q\Lambda \mid \exists \xi \in V \text{ such that } \Phi_q(\xi, \eta) \in S \}$$

In view of the technique used in the proofs of Lemma 6 and Lemma 9 it is clear that the set S_q is precompact. Denote by $\tilde{f}_q$ the representative of $\tilde{f}$. Then

$$\tilde{f}_q(\xi, \eta) = \Phi_q^{-1} \circ \tilde{f}(\Phi_q(\xi, \eta)) .$$

It is enough to show that $pr_2 \circ \tilde{f}(\Phi_q^{-1}(S))$ is precompact (using Proposition 1 and the previous remark). Let $(\tilde{\eta}_k) \subset pr_2 \circ \tilde{f}_q(\Phi_q^{-1}(S))$. Taking $\tilde{\eta}_n = pr_2 \circ \tilde{f}_q(\xi_n, \eta_n)$ where $\eta_n \in S_q$ we infer eventually taking a subsequence $\xi_n \to \xi$ uniformly and $\eta_n \to \eta$ in $L_q\Lambda$ for some $\xi \in V$ and $\eta \in L_q\Lambda$. Hence $\tilde{\eta}_n \to pr_2 \circ f_q(\xi, \eta)$. This proves Lemma 10. $\square$

III.4. A family of smoothing operators.

One of the difficulties in the proof of the main result is the fact—as already explained—that the vertical component of the gradient of α_∞ is not compact. The idea is now to replace α_∞ by $\alpha_n = \alpha_\infty \circ F_n$, where the F_n are the smoothing operators introduced below.

Define for all $n \in \mathbb{N}$ a fibre-preserving map $F_n : L\Lambda \to T\Lambda$ by

$$(1) \qquad \frac{1}{n}(\delta F_n x, \delta y) + (F_n x, y) = (x, y)$$

for $x \in L_q\Lambda$ and all $y \in T_q\Lambda$.

LEMMA 11. — For all $n \in \mathbb{N}$ F_n is a smooth map. Moreover we have

$$(2) \qquad (F_n x, y) = (x, F_n y) \quad \text{for all} \quad (x, y) \in L\Lambda \oplus L\Lambda .$$

If we consider F_n as a map $L\Lambda \to L\Lambda$ it is fibre-compact and we have the estimates

$$(3) \qquad \begin{aligned} &\| Z(\tau)F_n x - F_n x \| \leq 2\sqrt{n}\, \| x \| \, | \tau |^{\frac{1}{2}} \\ &\| F_n x \| \leq \| x \| . \end{aligned}$$

The proof uses Lemma 7 and is straight forward.

LEMMA 12. — If $(x_k) \subset L\Lambda$ is $(u f p\ c)$ and $(n_k) \subset \mathbb{N}$, $n_k \to +\infty$, then $(F_{n_k} x_k)$ is $(u f p\ c)$. Moreover if $x_k \to x$ then $F_{n_k} x_k \to x$ in $L\Lambda$.

Proof. — Let $q_k = \pi x_k$. Since (x_k) is bounded (q_k) is bounded. So we find finitely many exponential charts $\exp_{p_i} : V_{p_i} \to U_{p_i}$ such that V_{p_i} is convex in $T_{p_i}\Lambda$ and the family $\left(\exp_{p_i}\left(\frac{1}{2}V_{p_i}\right)\right)$ cover the whole sequence (q_k). Clearly it is enough by the previous construction to show that the subsequence of (x_k) consisting of those elements that the corresponding q_k lie in a specific chart $\exp\left(\frac{1}{2}V_{p_i}\right)$ is $(u f p\ c)$. So we may assume without loss of generality that $(q_k) \subset \exp\left(\frac{1}{2}V\right)$ for some convex open zero-neighbourhood V in $T_q\Lambda$.

We carry out the parallel transport along the curves $\phi_k : s \to \exp_q(1-s)(\xi_k)$. Note that (ξ_k) is bounded in $T_q\Lambda$. Denote the image of $F_{n_k} x_k$ by $\hat{z}_k \in L_q\Lambda$. If we can show that $(\hat{z}_k)$ is precompact we are done by Proposition 1. So we shall show that $(\hat{z}_k)$ has a convergent subsequence. Eventually taking a subsequence of (q_k) we may assume $q_k \to q_0$ in C for some $q_0 \in \Lambda$. Denoting by z_k the local representation of $F_{n_k} x_k$ we have for all $y \in T_q\Lambda$ using the notation in [*16*].

$$(4) \quad \begin{aligned} \frac{1}{n_k}\big(G(\xi_k)(\nabla z_k + D_2\theta_q(\xi_k)z_k + \Gamma_q(\xi_k)(z_k, \partial_q \xi_k)),& \\ \nabla y + D_2\theta_q(\xi_k)y + \Gamma_q(\xi_k)(y, \partial_q \xi_k)\big)& \\ + (G(\xi_k)z_k, y)& \\ = (G(\xi_k)\eta_k, y)&, \end{aligned}$$

where η_k is the local representative of x_k. We have $\xi_k \to \xi_0$ uniformly. Denote by $\hat{x}_k$ the image of x_k under the parallel transport along ϕ_k. By our assumption we may assume eventually taking a subsequence $\hat{x}_k \to \hat{x}_0$ for some $\hat{x}_0 \in L_q\Lambda$. In local coordinates

$$\hat{x}_k = a_k(1), \qquad \eta_k = a_k(0),$$

where

$$\dot{a}_k = \Gamma_q((1-s)\xi_k)(\xi_k, a_k).$$

Since $\xi_k \to \xi_0$ uniformly we infer that $\eta_k \to \eta_0$, where $\eta_0 = a_0(0)$, $\hat{x}_0 = a_0(1)$ and $\dot{a}_0 = \Gamma_q((1-s)\xi_0)(\xi_0, a_0)$. By the definition of F_n it is clear that $\| \delta F_{n_k} x_k \| \leq \sqrt{n_k} \| x_k \|$.

Hence, taking the limit in (4) for fixed $y \in T_q\Lambda$ we infer

$$\text{(5)} \qquad \begin{aligned} \lim (G(\xi_0)z_k, y) &= \lim (G(\xi_k)z_k, y) \\ &= (G(\xi_0)\eta_0, y). \end{aligned}$$

Since (z_k) is bounded and $T_q\Lambda$ dense in $L_q\Lambda$ we deduce that (5) holds for all $y \in L_q\Lambda$. This implies

$$z_k \rightharpoonup \eta_0 \quad \text{weakly in } L_q\Lambda.$$

Moreover $\liminf \| z_k \| \, \| G(\xi_0)y \| \geq (\eta_0, G(\xi_0)y)$. This implies

$$\liminf \| z_k \| \geq \| \eta_0 \|.$$

On the other hand $\| F_{n_k} x_k \| \leq \| x_k \|$. This implies

$$\limsup \| z_k \| \leq \| \eta_0 \|.$$

Therefore

$$z_k \rightharpoonup \eta_0 \quad \text{weakly in } L_q\Lambda$$
$$\| z_k \| \to \| \eta_0 \|.$$

This implies

$$z_k \to \eta_0 \quad \text{strongly in } L_q\Lambda.$$

Now

$$\hat{z}_k = b_k(1) \qquad z_k = b_k(0),$$

where

$$\dot{b}_k = \Gamma_q((1-s)\xi_k)(\xi_k, b_k).$$

Since $\xi_k \to \xi_0$ uniformly and $z_k \to \eta_0$ we obtain $\hat{z}_k \to \hat{z}_0$, where

$$\hat{z}_0 = b_0(1), \qquad \eta_0 = b_0(0)$$

and

$$\dot{b}_0 = \Gamma_q(1-s)\xi_0)(\xi_0, b_0).$$

This shows that $(\hat{z}_k)$ is precompact. Hence $(F_{n_k} x_k)$ is $(u\,f\,p\;c)$. The proof of the second part is implicitly contained in the previous arguments. The details are left to the reader. □

LEMMA 13. — For fixed $x \in L_q\Lambda$; $y, z \in T_q\Lambda$ we have the identity

$$(6) \qquad \frac{1}{n}(R(y,\dot q)F_n x, \delta z) + \frac{1}{n}(\delta F_n x, R(y,\dot q)z) + \frac{1}{n}(\delta(D_y F_n)x, \delta z) + ((D_y F_n)x, z) = 0$$

Moreover, the following estimate holds for some constant $\tilde{M}$ only depending on M

$$(7) \qquad \left(\frac{1}{n}\|(D_y F_n)x\|_\Lambda^2 + \left(1 - \frac{1}{n}\right)\|D_y F_n x\|^2\right)^{\frac{1}{2}} \le \tilde{M} n^{-\frac{1}{4}} \|x\| \; \|\partial q\| \; \|y\|_\infty$$

Proof. — (6) follows from a straightforward application of the covariant derivative D_y to the definition of F_n, (1), using Lemma 2. In order to prove (7) let $z = (D_y F_n)x$. Moreover note that there exists a constant $c_1 > 0$ only depending on M such that $|R(a,b,c)| \le c_1 |a|\,|b|\,|c|$ for all $(a,b,c) \in TM \oplus TM \oplus TM$. We obtain from (6)

$$(8) \qquad \frac{1}{n}\|z\|_\Lambda^2 + \left(1 - \frac{1}{n}\right)\|z\|^2 \le \frac{1}{n} c_1 \|y\|_\infty \|F_n x\|_\infty \|\dot q\| \, \|\delta z\| + \frac{1}{n} c_1 \|y\|_\infty \|z\|_\infty \|\delta F_n x\| \; \|\dot q\| .$$

Applying the interpolation inequality III.1 (7) to $F_n x$ and z we infer

$$(9) \qquad \|F_n x\|_\infty^2 \le \|F_n x\|^2 + 2\|F_n x\| \; \|\delta F_n x\| \le (1 + 2\sqrt{n})\|x\|^2$$

and with $$a = \left(\frac{1}{n}\|z\|_\Lambda^2 + \left(1 - \frac{1}{n}\right)\|z\|^2\right)^{\frac{1}{2}}$$

$$(10) \qquad \|z\|_\infty^2 \le \|z\|^2 + 2\|z\| \; \|\delta z\| \le a^2 + 2\sqrt{n}a^2 = (1 + 2\sqrt{n})a^2.$$

Combining (8)-(10) gives

$$\begin{aligned} a^2 &\le 2c_1 \|\dot q\| \; \|y\|_\infty \|x\| \left((1 + 2\sqrt{n})n^{-1}\right)^{\frac{1}{2}} a \\ &\le 8 c_1 n^{-\frac{1}{4}} \|\dot q\| \; \|y\|_\infty \|x\| a = \tilde{M} n^{-\frac{1}{4}} \|\dot q\| \; \|x\| \; \|y\|_\infty a . \qquad \square \end{aligned}$$

IV. CONSTRUCTION OF A ℕ-FAMILY

In this chapter we shall introduce the Ψ_n described in I.2. Here the smoothing operators F_n and the compactness concept introduced in III will be crucial.

IV.1. The set-up.

We introduce a functional $\Psi \in C^\infty(L\Lambda, \mathbb{R})$ by

(1) $$\Psi(x) = (\partial\pi x, x).$$

For $a \in T_x L\Lambda$ we compute

(2) $$d\Psi(x)a = (\partial\pi x, \hat{K}a) + (\delta(T\pi)a, x).$$

We introduce the smooth gradient vectorfield $\Psi' : L\Lambda \to TL\Lambda$ by

$$d\Psi(x)a = (\Psi'(x), a)_L.$$

Let

(3) $$\Sigma = (T\pi)\Psi'$$

and put $y = (T\pi)a$ and $z = \hat{K}a$ for some $a \in T_x L\Lambda$. We find using (2)

$$(\hat{K}\Psi'(x), z) + (\Sigma x, y) + (\delta\Sigma x, \delta y) = (\partial\pi x, z) + (\delta y, x).$$

This is true for all $y \in T_q\Lambda$ and all $z \in L_q\Lambda$.
Hence

(4) $$\partial\pi = \hat{K}\Psi'$$

and

(5) $$(\Sigma x, y) + (\delta\Sigma x, \delta y) = (x, \delta y).$$

Using the fibre preserving map $A^* : L\Lambda \to L\Lambda$ we deduce from (5)

(6) $$A^*\Sigma + \delta\Sigma = \mathrm{Id}.$$

Further (5) implies

(7) $$\|\Sigma x\|_\Lambda \leq \|x\|.$$

Define a smooth fibre-preserving map $\phi : T^*M \to TM$ by $f = \langle \phi(f), . \rangle$. Introduce a smooth map with compact support $h: [0,1] \times TM \to \mathbb{R}$ by

(8) $$h(t, x) = h^*(t, \phi^{-1}(x))$$

where h^* is the map introduced in Chapter I before Theorem 2. We define a map $\alpha_\infty : L\Lambda \to \mathbb{R}$ by

(9) $$\alpha_\infty(x) = \int_0^1 h(t, x(t))dt.$$

Then $\alpha_\infty \in C^1(L\Lambda, \mathbb{R})$, but in general it has no better regularity properties. However, if we consider $\alpha_\infty \mid T\Lambda : T\Lambda \to \mathbb{R}$ for the differentiable structure on $T\Lambda$ it is of class C^∞. We define now $\alpha_n : L\Lambda \to \mathbb{R}$ by

$$\alpha_n = \alpha_\infty \circ F_n.$$

Since $F_n : L\Lambda \to T\Lambda$ is smooth we infer $\alpha_n \in C^\infty(L\Lambda, \mathbb{R})$. Finally, define for $n \in \mathbb{N}_\infty$

$$\Psi_n = \Psi - \alpha_n. \tag{11}$$

Clearly $\Psi_n \in C^\infty(L\Lambda, \mathbb{R})$ for $n \in \mathbb{N}$. With $a \in T_x L\Lambda$ we have

$$\begin{aligned}(\alpha_n'(x), a)_L &= d\alpha_\infty(F_n x)(TF_n a)\\ &= (a_\infty'(F_n x), TF_n a)_L .\end{aligned}$$

Further

$$(\alpha_\infty'(x), a)_L = (\hat{K}H(x), \hat{K}a) + ((T\pi)H(x), (T\pi)a)$$

where $H : L\Lambda \to L\Lambda$ is defined by

$$H(x)(t) = (\text{grad } h_t)(x(t))$$

and

$$\langle \text{grad } h_t x, b \rangle_{TM} = dh_t(x) . b .$$

Hence

$$\begin{aligned}(\alpha_n'(x), a)_L &= (\hat{K}HF_n x, \hat{K}(TF_n)a) + ((T\pi)HF_n x, (T\pi)(TF_n)a)\\ &= (\hat{K}HF_n x, F_n\hat{K}a) + (\hat{K}HF_n x, (D_{(T\pi a)}F_n)x) + ((T\pi)HF_n x, (T\pi)a)\\ &= (F_n\hat{K}HF_n x, \hat{K}a) + (\hat{K}HF_n x, (D_{(T\pi a)}F_n)x) + ((T\pi)HF_n x, (T\pi)a).\end{aligned} \tag{12}$$

IV.2. **Compactness estimates.**

We shall now derive some estimates which will be used two times. In the proof that the family (Ψ_n) constructed in the previous paragraph is in fact an $\mathbb{N}$-family and in the representation result for the gradient flows.

LEMMA 14. — There exists a constant $\tilde{H}$ only depending on M and h such that

$$\begin{aligned}&\| \alpha_n'(x) \|_L \leq \tilde{H}(1 + n^{-\frac{1}{4}} \| x \| \; \| \dot{q} \|)\\ &\| \hat{K}\alpha_n'(x) \| \leq \tilde{H}\end{aligned} \tag{1}$$

for all $x \in L\Lambda$, where $q = \pi x$. The estimate is true for all $n \in \mathbb{N}_\infty$ if we put $\infty^{-\frac{1}{4}} = 0$.

Proof. — From IV.1 (12) we infer

$$\| \hat{K}\alpha_n'(x) \| = \| F_n\hat{K}HF_n x \| \leq \| \hat{K}HF_n x \| \leq \tilde{H}$$

for a suitable constant $\tilde{H} > 0$ since h has compact support. Moreover using IV.1 (12) again, combining it with Lemma 13, we infer for arbitrary $a \in T_x L\Lambda$

$$| (\alpha_n'(x), a)_L | \leq \tilde{H} \| \hat{K}a \| + \tilde{H}\tilde{M}n^{-\frac{1}{4}} \| x \| \; \| \dot{q} \| \; \| (T\pi)a \|_\infty + \tilde{H} \| (T\pi)a \| .$$

Hence, eventually taking a greater constant

$$|(\alpha_n'(x), a)_L| \leq \tilde{H}(1 + n^{-\frac{1}{4}} \|x\| \|\dot{q}\|) \|a\|_L.$$

Taking $a = \alpha_n'(x)$ yields (1). The claim for $n = \infty$ is trivial. □

For the following we define smooth maps $B^n : L\Lambda \to L\Lambda$ for $n \in \mathbb{N}$ by

$$(2) \qquad (B^n x, y) = ((T\pi)HF_n x, Ay) + (\hat{K}HF_n, (D_{Ay}F_n)x)$$

where A was defined in III.2. Denote by A^* the $L\Lambda$-adjoint of A, i.e.

$$(Ax, y) = (x, A^*y).$$

Further denote by $Q : L\Lambda \to L\Lambda$ the fibre-preserving maps which induces on each fibre the orthogonal projection onto the orthogonal space in $L\Lambda$ of kern (δ). Recalling that kern (δ) was a smooth bundle over Λ we infer that Q is smooth. However, this property is not needed for the following estimate. We have

$$(A^*x, \delta y) = (x, A\delta y) = (x, A\delta Qy) = (x, Qy) = (Qx, y).$$

Since this is true for all $y \in T_q\Lambda$ we infer that $A^*x \in T_q\Lambda$, where $q = \pi x$, by Lemma 5. Hence

$$\|\delta A^*x\| \leq \|x\|.$$

Moreover $\|A_q^*\| = \|A_q\|$. This of course implies that A^* is fibre-compact with factor $\frac{1}{2}$, i.e.

$$\|Z(\tau)A^*x - A^*x\| \leq \mu_{A^*}([x])|\tau|^{\frac{1}{2}}$$

Lemma 15. — For all $n \in \mathbb{N}$ B^n is fibre-compact. Moreover there exists a monotone increasing map $\mu_B : \mathbb{R}^+ \to \mathbb{R}^+$ independent of $n \in \mathbb{N}$ such that

$$(3) \qquad \|Z(\tau)B^n x - B^n x\| \leq \mu_B([x])|\tau|^{\frac{1}{4}}$$

for all $x \in L\Lambda$ and $\tau \in [-1, 1]$.

Proof. — In fact, using Lemma 7, Lemma 13 and Lemma 14, we infer

$$\begin{aligned}(4) \quad (Z(\tau)B^n x - B^n x, y) &= (B^n x, Z(-\tau)y - y) \\ &= ((T\pi)HF_n x, A(Z(-\tau)y - y)) \\ &\qquad + (\hat{K}HF_n x, (D_{A(Z(-\tau)y-y)}F_n)x) \\ &\leq (A^*(T\pi)HF_n x, Z(-\tau)y - y) \\ &\qquad + \|\hat{K}HF_n x\| \, \|(D_{A(Z(-\tau)y-y)}F_n)x\| \\ &\leq (Z(\tau)A^*(T\pi)HF_n x - A^*(T\pi)HF_n x, y) \\ &\qquad + \tilde{H}\tilde{M}n^{-\frac{1}{4}} \|x\| \, \|\dot{q}\| \, \|A(Z(-\tau)y - y)\|_x \\ &\leq \mu_{A^*}([T\pi)HF_n x])|\tau|^{\frac{1}{2}} \|y\| \\ &\qquad + \tilde{H}\tilde{M} \|x\| \, \|\dot{q}\| \, \|A(Z(-\tau)y - y)\|_x .\end{aligned}$$

Next we estimate

$$\begin{aligned}(A(Z(-\tau)y-y),z) &= (Z(-\tau)y-y, A^*z)\\ &= (y, Z(\tau)A^*z - A^*z)\\ &\leq \| y \| 2 \| A^*z \|_\Lambda | \tau |^{\frac{1}{2}}\\ &\leq 2 \| y \| (\| z \|^2 + \| A_q \|^2 \| z \|^2)^{\frac{1}{2}} | \tau |^{\frac{1}{2}}.\end{aligned}$$

This implies

$$(5) \qquad \| A(Z(-\tau)y - y) \| \leq 2(1 + \| A_q \|^2)^{\frac{1}{2}} \| y \| \; | \tau |^{\frac{1}{2}}.$$

Now we use the interpolation inequality III.1 (7) and find using that $|\tau| \leq 1$

$$\begin{aligned}(6) \quad & \| A(Z(-\tau)y - y) \|_\infty^2\\ &\leq \| A(Z(-\tau)y - y) \|^2 + 2 \| A(Z(-\tau)y - y) \| \; \| Z(-\tau)y - y \|\\ &\leq 4(1 + \| A_q \|^2) \| y \|^2 | \tau | + 4(1 + \| A_q \|^2)^{\frac{1}{2}} \| y \|^2 | \tau |^{\frac{1}{2}}\\ &\leq 8(1 + \| A_q \|^2) \| y \|^2 | \tau |^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$(7) \qquad \| A(Z(-\tau)y - y) \|_\infty \leq 3(1 + \| A_q \|^2)^{\frac{1}{2}} \| y \| \; | \tau |^{\frac{1}{4}}.$$

Combining (4) and (7) yields

$$\begin{aligned}(Z(\tau)B^n x - B^n z, y) \leq{} & (\mu_{A^*}([(T\pi)HF_n x]) | \tau |^{\frac{1}{2}}\\ & + 3\tilde{H}\tilde{M} \| x \| \; \| \dot{q} \| (1 + \| A_q \|^2)^{\frac{1}{2}} | \tau |^{\frac{1}{4}}) \| y \|.\end{aligned}$$

Note that the expression on the right hand side is independent of $n \in \mathbb{N}$. Now for a suitable monotone increasing map μ_B our assertion follows.

□

IV.3. Verification of $\mathbb{N}$-family properties.

The following result is one in the key steps of reducing Theorem 2 to Theorem 3.

PROPOSITION 2. — There exists $n_0 \geq 1$ such that $(\Psi_n)_{n \geq n_0}$ is an $\mathbb{N}$-family for Ψ_∞.

Proof. — We have to check (*i*)-(*iii*) of Definition 3. Clearly (*i*) holds by our construction. Next we prove (*ii*). Let $(x_k) \subset L\Lambda$ such that $x_k \to x$ and $(n_k) \subset \mathbb{N}_\infty$ with $n_k \to +\infty$. Define $F_\infty = \mathrm{Id}$. By Lemma 12 we have $F_{n_k} x_k \to x$. This implies $\alpha_{n_k}(x_k) \to \alpha_\infty(x)$.

Hence

$$\lim \Psi_{n_k}(x_k) = \Psi_\infty(x).$$

Next assume $\Psi'_\infty(x) = 0$. We have

$$\| \Psi'_{n_k}(x_k) \|_L \leq \| \Psi'_\infty(x_j) \|_L + \| \alpha'_{n_k}(x_k) - \alpha'_\infty(x_k) \|_L.$$

In order to show that the left hand side tends to zero we have only to show that the second term on the right hand side tends to zero. Using Lemma 13 and Lemma 14 we infer for a suitable constant $\tilde{\mathrm{H}}$ only depending on M and h

$$\begin{aligned}\| \alpha'_\infty(x_k)-\alpha'_{n_k}(x_k) \|_{\mathrm{L}} &\le \| \hat{\mathrm{K}}\mathrm{H}x_k - \mathrm{F}_{n_k}\hat{\mathrm{K}}\mathrm{H}\mathrm{F}_{n_k}x_k \| \\ &\quad + \| (\mathrm{T}\pi)\mathrm{H}x_k - (\mathrm{T}\pi)\mathrm{H}\mathrm{F}_{n_k}x_k \| + 2\,\tilde{\mathrm{M}}\tilde{\mathrm{H}} \| x_k \| \, \| \partial q_k \| \, n_k^{-\frac{1}{4}} \\ &= \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k .\end{aligned}$$

Since $\mathrm{F}_{n_k}x_k \to x$ clearly $\mathrm{I}_k + \mathrm{II}_k \to 0$. Since (x_k) is bounded in $\mathrm{L}\Lambda$ and $n_k \to +\infty$ we infer that $\mathrm{III}_k \to 0$. Similarly if $\| \Psi'_{n_k}(x_k) \|_{\mathrm{L}} \to 0$ we infer

$$\| \Psi'_\infty(x_k) \|_{\mathrm{L}} \le \| \Psi'_{n_k}(x_k) \|_{\mathrm{L}} + \| \alpha'_{n_k}(x_k) - \alpha'_\infty(x_k) \|_{\mathrm{L}} =: \mathrm{I}_k + \mathrm{II}_k .$$

By our assumption $\mathrm{I}_k \to 0$ and similar to the argument used before $\mathrm{II}_k \to 0$. Hence $\| \Psi'_\infty(x_k) \| \to 0$. Since $x_k \to x$ this implies $\Psi'_\infty(x) = 0$. The proof of *(ii)* is complete.

Next we show that *(iii)* holds. Let $(x_k) \subset \mathrm{L}\Lambda$ and $(n_k) \subset \mathbb{N}_\infty$, $n_{k+1} \ge n_k$, such that

$$\| \Psi'_{n_k}(x_k) \|_{\mathrm{L}} \to 0, \qquad \Psi_{n_k}(x_k) \to d .$$

With $q_k = \pi x_k$ we infer

$$\| \partial q_k - \mathrm{F}_{n_k}\hat{\mathrm{K}}\mathrm{H}\mathrm{F}_{n_k}x_k \| \to 0 \tag{8}$$

and

$$\| \Sigma x_k - (\mathrm{T}\pi)\alpha'_{n_k}(x_k) \|_\Lambda \to 0 . \tag{9}$$

Define $d_k^1 := \| \partial q_k - \mathrm{F}_{n_k}\hat{\mathrm{K}}\mathrm{H}\mathrm{F}_{n_k}x_k \|$. By Lemma 14 we have

$$\| \partial q_k \| \le d_k^1 + \tilde{\mathrm{H}} . \tag{10}$$

By Lemma 3 there exists a constant $c > 0$ only depending on $\tilde{\mathrm{H}}$ such that for all $x \in \mathrm{T}_q\Lambda$ with $\| \partial q \| \le 2\tilde{\mathrm{H}} + 1$ we have

$$\| \delta x \| \ge c \| x \|_\Lambda$$

provided $x \perp \mathrm{kern}\,(\delta)$.

We have

$$(\delta\Sigma x, \delta y) + (\Sigma x, y) = (x, \delta y) .$$

Therefore

$$\| x \|^2 = (\delta\Sigma x, x) + (\Sigma x, \mathrm{A}x) \le \| \Sigma x \|_w \| x \| + \| \Sigma x \|_\Lambda c^{-1} \| x \| .$$

Hence for a suitable constant $\hat{c} > 0$ only depending on $\tilde{\mathrm{H}}$ and therefore only depending on M and h we find

$$\begin{aligned} d_k^2 &:= \| \Sigma x_k - (\mathrm{T}\pi)\alpha'_{n_k}(x_k) \|_\Lambda \\ &\ge \| \Sigma x_k \|_\Lambda - \tilde{\mathrm{H}}(1 + n_k^{-\frac{1}{4}} \| x_k \| (\tilde{\mathrm{H}} + d_k^1)) \\ &\ge \hat{c} \| x_k \| - \mathrm{H}(1 + n_k^{-\frac{1}{4}} \| x_k \| (\tilde{\mathrm{H}} + d_k^1)) \end{aligned} \tag{11}$$

provided k is large enough since $d_k^1 \to 0$. Clearly if *a priori* $n_k \geq n_0$ for a suitable n_0 only depending on M and h (11) will imply that the sequence $(\| x_k \|)$ is bounded. For the following we fix such a n_0. Hence if $n_1 \geq n_0$ and $\| \Psi'_{n_k}(x_K) \|_L \to 0$ then $(\| \partial q_k \|^2 + \| x_k \|^2)$ is bounded. It is easy to see that this implies that (x_k) is bounded in $L\Lambda$. So for the following let $(n_k) \subset \mathbb{N}$, $n_{k+1} \geq n_k \geq n_0$, and $(x_k) \subset L\Lambda$ such that $\| \Psi'_{n_k}(x_k) \|_L \to 0$. Define

$$b_k = \Sigma x_k - (T\pi)\alpha'_{n_k}(x_k) . \tag{12}$$

We have $\| b_k \|_\Lambda \to 0$. Hence taking the inner product with Az where $z \in L_{q_k}\Lambda$ we obtain

$$\begin{aligned}(b_k, Az)_\Lambda &= (x_k, z) - ((T\pi)HF_{n_k}x_k, Az) - (\hat{K}HF_{n_k}x_k, (D_{Az}F_{n_k})x_k) \\ &= (x_k - B^{n_k}x_k, z) .\end{aligned}$$

Since $(\| \partial q_k \|)$ is bounded we find by Lemma 3 a suitable constant $c > 0$ such that

$$\| x_k - B^{n_k}x_k \| \leq c \| b_k \|_\Lambda \to 0 . \tag{13}$$

Moreover

$$\| \partial q_k - F_{n_k}\hat{K}HF_{n_k}x_k \| \to 0 . \tag{14}$$

Since (x_k) is bounded using (13) and Lemma (15) we infer that (x_k) is $(u f p c)$. If (n_k) becomes stationary for k large enough we conclude from Lemma 9 and Lemma 11 that (q_k) must be precompact. If $n_k \to +\infty$ we infer from Lemma 12 that $(F_{n_k}x_k)$ is $(u f p c)$. Hence by Lemma 10 $(\hat{K}HF_{n_k}x_k)$ is $u f p c)$. Finally that implies that $(F_{n_k}\hat{K}HF_{n_k}x_k)$ is $(u f p c)$. Hence by (14) (∂q_k) is $(u f p c)$ and by Lemma 9 (q_k) is precompact. Hence (q_k) is precompact and (x_k) $(u f p c)$. Therefore (x_k) is precompact. □

V. ESTIMATES AND REPRESENTATIONS FOR GRADIENT FLOWS

V.1. Parameterised families of transport equations and a compactness estimate.

In the following we may assume, without loss of generality, that the number n_0 introduced in Proposition 2 is equals one. We fix for the following a $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where we denote by Ψ_0 the functional Ψ. Denote by

$$L\Lambda \times \mathbb{R} \to L\Lambda : (x, s) \to x \circ s$$

the global flow corresponding to the differential equation

$$x' = -\beta(\| \Psi'_n(x) \|_L)\Psi'_n(x) =: G_n(x)$$

where β is as described in I.2. We define a smooth vector bundle $\gamma : \mathscr{L} \to \Lambda \times \Lambda$ where the fibre over $(q_2, q_1) \in \Lambda \times \Lambda$ is given by

$$\mathscr{L}_{(q_2,q_1)} = \mathscr{L}(\mathrm{L}_{q_1}\Lambda, \mathrm{L}_{q_2}\Lambda).$$

Here the right hand side denotes the Banach space of continuous linear operators $\mathrm{L}_{q_1}\Lambda \to \mathrm{L}_{q_2}\Lambda$. Further we denote by

$$\kappa : \mathrm{L}\Lambda \times \mathbb{R} \times \mathbb{R} \to \Lambda \times \Lambda$$

the smooth map defined by

$$(x, s, t) \to (\pi(x \circ s), \pi(x \circ t)). \tag{1}$$

We define a map $\mu_n : \mathrm{L}\Lambda \times \mathbb{R} \times \mathbb{R} \to \mathscr{L}$ such that $\gamma \circ \mu_n = \kappa$ by

$$(x, s_2, s_1) \to \mu_n(x, s_2, s_1), \tag{2}$$

where $\mu_n(x, s_2, s_1) \in \mathscr{L}(\mathrm{L}_{\pi(x \circ s_1)}\Lambda, \mathrm{L}_{\pi(x \circ s_2)}\Lambda)$ is defined by the Cauchy problem

$$\nabla_{\frac{d}{ds}\pi(x \circ s)} a = \beta(\| \Psi_n'(x \circ s) \|_{\mathrm{L}})a \tag{3}$$
$$a(s_1) \in \mathrm{L}_{\pi(x \circ s_1)}\Lambda \quad \text{given}$$

LEMMA 16. — μ_n is smooth.

Proof. — Let us first prove the assertion for the case that $| s_2 - s_1 |$ is small. Then the problem is localised in a co-ordinate chart.

Let $\exp_q : \mathrm{V}_q \to \mathrm{U}_q$ and $\exp_{\tilde{q}} : \mathrm{V}_{\tilde{q}} \to \mathrm{U}_{\tilde{q}}$ be co-ordinate systems such that for some $\varepsilon > 0$ we have

$$x \in \mathrm{U}_q, \quad \pi(x \circ s) \in \mathrm{U}_{\tilde{q}} \quad \text{for} \quad s \in (s_1 - \varepsilon, s_2 + \varepsilon). \tag{4}$$

We assume without loss of generality that $s_1 \leq s_2$.

Let

$$x = \Phi_q(\xi, \eta)$$
$$\pi(x \circ s) = \exp_{\tilde{q}}(b_{\tilde{q}}(s, (\xi, \eta))).$$

Define a map G by

$$\mathrm{G} : \mathrm{V}_q \times \mathrm{L}_q\Lambda \times (s_1 - \varepsilon, s_2 + \varepsilon) \times \mathscr{L}(\mathrm{L}_{\tilde{q}}\Lambda, \mathrm{L}_{\tilde{q}}\Lambda) \to \mathscr{L}(\mathrm{L}_{\tilde{q}}\Lambda, \mathrm{L}_{\tilde{q}}\Lambda)$$
$$(\xi, \eta, s, \sigma) \to \Gamma_{\tilde{q}}(b_{\tilde{q}}(s, (\xi, \eta)))\left(\frac{\partial}{\partial s} b_{\tilde{q}}(\sigma, (\xi, \eta)), s\right) + \beta(\| \Psi_n'(\Phi_q(\xi, \eta) \circ s) \|_{\mathrm{L}}\sigma.$$

Clearly G is smooth. Then (3) written in local co-ordinates corresponds to the operator differential equation

$$\sigma' = \mathrm{G}(\xi, \eta, s)\sigma \tag{4}$$
$$\sigma(s_1) = \mathrm{Id}.$$

By the continuous and differentiable-parameter dependence theory for ordinary differential equations, we infer that the map which associates to $(\xi, \eta, s, t) \in V_q \times L_q\Lambda \times (s_1 - \varepsilon, s_1 + \varepsilon) \times (s_2 - \varepsilon, s_2 - \varepsilon)$ the unique linear map $\tilde{\sigma} \in \mathscr{L}(L_q\Lambda, L_q\Lambda)$ such that $\tilde{\sigma} = \sigma(t)$, $\sigma(s) = \mathrm{Id}$ and σ satisfies (4) is a smooth assignment. Hence μ_n is smooth provided $|s_1 - s_2|$ is small. In the general case we can write $s \geq t$

$$\mu_n(x, s, t) = \mu_n(x, s_N, s_{N-1})\mu_n(x, s_{N-1}, s_{N-2}) \ldots \mu_n(x, s_1, s_0)$$

where $s = s_N > s_{N-1} > \ldots > s_1 > s_0 = t$. The previous argument applies to all $\mu_n(x, s_i, s_{i-1})$ provided $|s_i - s_{i-1}|$ is small enough. Hence $\mu_n(x, s, t)$ is smoothly depending on x, s, t in the general case. □

μ_n can be considered as a smooth parametrisation of certain linear transport equations of type (3). The importance of μ_n will become clear in the next section.

Since $\beta(\| \Psi'_n(x \circ s) \|_L) \leq 1$ for all $x \in L\Lambda$, $s \in \mathbb{R}$ we immediately obtain denoting by $\| \, . \, \| : \mathscr{L} \to \mathbb{R}^+$ the operator norm the following estimate

$$\| \mu_n(x, s_2, s_1) \| \leq \exp(|s_2 - s_1|). \tag{5}$$

Define a smooth fibre preserving map

$$D^n : L\Lambda \to L\Lambda$$

$$D^n x = -A^*\Sigma x - B^n x + F_n\hat{K}HF_n x + A^*(T\pi)\alpha'_n(x) \tag{6}$$

for $n \in \mathbb{N}$ and

$$D^0 = -A^*\Sigma$$

where B^n was introduced in IV.2 (2).

LEMMA 17. — For all $n \in \mathbb{N}_0$ there exists a monotone increasing map $\mu_{D^n} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\| Z(\tau)D^n x - D^n x \| \leq \mu_{D^n}([x]) \, |\tau|^{\frac{1}{4}} \tag{7}$$

for all $x \in L\Lambda$ and $\tau \in [-1, 1]$.

Proof. — We have using the compactness estimates for A^* (before Lemma 15, Lemmas 11, 13 and 14.

$$\begin{aligned} &\| Z(\tau)D^n x - D^n x \| \\ &\leq \| Z(\tau)A^*\Sigma x - A^*\Sigma x \| + \| Z(\tau)B^n x - B^n x \| + \| Z(\tau)F_n\hat{K}HF_n x - F_n\hat{K}HF_n x \| \\ &\qquad + \| Z(\tau)A^*(T\pi)\alpha'_n(x) - A^*(T\pi)\alpha'_n(x) \| \\ &\leq \mu_{A^*}([x]) \, |\tau|^{\frac{1}{2}} + \mu_B([x]) \, |\tau|^{\frac{1}{4}} + 2\sqrt{n}\tilde{H} \, |\tau|^{\frac{1}{2}} + \mu_{A^*}([(T\pi)\alpha'_n(x)]) \, |\tau|^{\frac{1}{2}} \\ &\leq \mu_{A^*}([x]) \, |\tau|^{\frac{1}{2}} + \mu_B([x]) \, |\tau^{\frac{1}{4}} + 2\sqrt{n}\tilde{H} \, |\tau|^{\frac{1}{2}} + \mu_{A^*}(\sigma([x])) \, |\tau|^{\frac{1}{2}} \end{aligned}$$

where $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing. Hence for suitable monotone increasing maps $\mu_{D^n} : \mathbb{R}^+ \to \mathbb{R}^+$

$$\| Z(\tau)D^n x - D^n x \| \leq \mu_{D^n}([x]) \, |\tau|^{\frac{1}{4}}$$

which is the desired result. □

V.2. The representation proposition.

Define for $n \in \mathbb{N}_0$ a map

$$E_n : L\Lambda \times \mathbb{R} \to L\Lambda$$

by

(1) $$E_n(x, s) = x *_n s - \partial\pi(x *_n s).$$

PROPOSITION 3. — For all $n \in \mathbb{N}_0$ we have the following representation for E_n

(2) $$E_n(x, s) = \mu_n(x, s, 0)E_n(x, 0) + K_n(x, s)$$

where

(3) $$K_n(x, s) = \int_0^s \mu_n(x, s, h)\beta(\| \Psi_n'(x *_n h) \|_L D^n(x *_n h)dh.$$

Moreover there exists a monotone increasing map $\nu_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(4) $$\| Z(\tau)K_n(x, s) - K_n(x, s) \| \leq \nu_n([x] + |s|)|\tau|^{\frac{1}{4}}$$

for all $x \in L\Lambda$, $s \in \mathbb{R}$, and $\tau \in [-1, 1]$.

Proof. — We calculate

(5) $$\begin{aligned} \nabla_{\frac{d}{ds}\pi(x *_n s)}(E_n(x, s)) &= \hat{K}G_n(x *_n s) - \hat{K}(T\partial)(T\pi)G_n(x *_n s) \\ &= -\beta(\| \Psi_n'(x *_n s) \|_L)(\partial\pi(x *_n s) - F_n\hat{K}HF_n(x *_n s)) \\ &\quad + \beta(\| \Psi_n'(x *_n s) \|_L)(\delta(T\pi)\Psi_0'(x *_n s) - \delta(T\pi)\alpha_n'(x *_n s)) \\ &= \beta(\| \Psi_n'(x *_n s) \|_L)(\delta\Sigma(x *_n s) - \partial\pi(x *_n s)) \\ &= \beta(\| \Psi_n'(x *_n s) \|_L)(F_n\hat{K}HF_n(x *_n s) - \delta(T\pi)\alpha_n'(x *_n s)). \end{aligned}$$

Now using IV.1 (12) and IV.2 (2) we infer

$$\begin{aligned} (\delta(T\pi)\alpha_n'(x), z) &= -((T\pi)\alpha_n'(x), Az) + ((T\pi)HF_n x, Az) + (\hat{K}HF_n x, (D_{Az}F_n)x) \\ &= -(A^*(T\pi)\alpha_n'(x), z) + (B^n x, z). \end{aligned}$$

Hence

(6) $$\begin{aligned} F_n\hat{K}HF_n(x *_n s) - \delta(T\pi)\alpha_n'(x *_n s) &= F_n\hat{K}HF_n(x *_n s) + A^*(T\pi)\alpha_n'(x *_n s) \\ &\quad - B^n(x *_n s) = D^n(x *_n s) + A^*\Sigma(x *_n s). \end{aligned}$$

Moreover

$$(\delta\Sigma x, \delta y) + (\Sigma x, y) = (x, \delta y).$$

This implies

(7) $$\delta\Sigma + A^*\Sigma = \mathrm{Id}.$$

Combining (5), (6) and (7) gives

$$(8)\quad \nabla_{\frac{d}{ds}(\pi(x *_n s))}(E_n(x, s)) = \beta(\| \Psi_n'(x *_n s) \|_L)(x *_n s - \partial\pi(x *_n s)) + \beta(\| \Psi_n'(x *_n s) \|_L)D^n(x *_n s) = \beta(\| \Psi_n'(x *_n s) \|_L)E_n(x, s) + \beta(\| \Psi_n'(x *_n s) \|_L)D^n(x *_n s).$$

Now applying the variation of constant formula to (8) using the definition of μ_n gives

$$E_n(x, s) = \mu_n(x, s, 0)E_n(x, 0) + \int_0^s \mu_n(x, s, h)\beta(\| \Psi_n'(x *_n h) \|_L)D^n(x *_n h)dh = \mu_n(x, s, 0)E_n(x, 0) + K_n(x, s).$$

This proves (2). This representation together with the fact that K., s) is uniformly fibre compact as we shall prove, will be one of the key ingredients of the proof. Next we prove (4). We have

$$(9)\quad \| Z(\tau)K(x, s) - K(x, s) \| \leq \left| \int_0^s \| Z(\tau)\mu_n(x, s, h)D^n(x *_n h) - \mu_n(x, s, h)D^n(x *_n h) \| \, dh \right| .$$

Observe that $\| G_n(x) \|_L \leq 2$. Hence

$$(10)\quad \left\| \frac{d}{ds}(\pi(x *_n s)) \right\|_\Lambda = \| (T\pi)G_n(x *_n s) \|_\Lambda \leq 2 .$$

Using (10) we can apply Proposition 1 and find for a suitable constant $\tilde{M}$ only depending on M

$$(11)\quad \| Z(\tau)\mu_n(x, s, h)D^n(x *_n h) - \mu_n(x, s, h)D^n(x *_n h) \| \leq \tilde{M}[\pi(x *_n (.)), s, h](\| Z(\tau)D^n(x *_n h) - D^n(x *_n h) \| + \| D^n(x *_n h) \| \, | \tau |^{\frac{1}{2}}).$$

Now using Lemma 17 and the fact that $Z(-1)x = 0$ (in $L\Lambda$) we infer

$$(12)\quad \| Z(\tau)\mu_n(x, s, h)D^n(x *_n h) - \mu_n(x, s, h)D^n(x *_n h) \| \leq \tilde{M}[\pi(x *_n (.)), s, h](\mu_{D^n}([x *_n h]) \, | \tau |^{\frac{1}{4}} + \mu_{D^n}([x *_n h]) \, | \tau |^{\frac{1}{2}}) \leq 2\tilde{M}[\pi(x *_n (.)), s, h]\mu_{D^n}([x *_n h]) \, | \tau |^{\frac{1}{4}}.$$

Now let us investigate the expressions $[\pi(x *_n (.)), s, h]$ and $[x *_n h]$. We have by (10)

$$\left\| \frac{d}{ds} \pi(x *_n s) \right\|_\Lambda \leq 2 .$$

Moreover

$$\frac{d}{ds} \| x *_n s \|^2 = 2(x *_n s, \hat{K}G_n(x *_n s) \leq 4 \| x *_n s \| .$$

Hence

$$(13)\quad \| x *_n s \| \leq \| x *_n t \| + 2 \, | t - s |$$

for all $t, s \in \mathbb{R}$. Further

$$\frac{d}{ds}\| \partial\pi(x *_n s) \|^2 = 2(\partial\pi(x *_n s), \delta(\mathrm{T}\pi)\mathrm{G}'_n(x *_n s)) \leq 4 \| \partial\pi(x *_n s) \|.$$

Hence

(14) $$\| \partial\pi(x *_n s) \| \leq \| \partial\pi(x *_n t) \| + 2 | t - s |$$

for all $t, s \in \mathbb{R}$. Now using the definition of $[x *_n h]$ we infer

(15) $$\begin{aligned}[x *_n h] &= \| x *_n h \|^2 + \| \partial\pi(x *_n h) \|^2 \\ &\leq (\| x \| + 2 | h |)^2 + (\| \partial\pi x \| + 2 | h |)^2 \\ &\leq 2[x] + 16 | h |^2 \\ &\leq 2[x] + 16 | s |^2 \\ &\leq 16([x] + | s |^2).\end{aligned}$$

Using the definition of $[\pi(x *_n (.)), s, h]$ we infer

(16) $$[(x *_n (.)), s, h] \leq 2 \exp(3 | s |)(1 + \| \partial\pi x \| + 2 | s |).$$

Hence combining (15), (16) and (12) we find for a suitable increasing map $\tilde{v}_n : \mathbb{R}^+ \to \mathbb{R}^+$

(17) $$\| \mathrm{Z}(\tau)\mu_n(s, x, h)\mathrm{D}^n(x *_n h) - \mu_n(s, x, h)\mathrm{D}^n(x *_n h) \| \leq \tilde{v}_n([x] + | s |) | \tau |^{\frac{1}{4}}.$$

Now we combine (9) and (17) and find

$$\begin{aligned}\| \mathrm{Z}(\tau)\mathrm{K}_n(x, s) - \mathrm{K}_n(x, s) \| &\leq \tilde{v}_n([x] + | s |) | s | \, | \tau |^{\frac{1}{4}} \\ &\leq \tilde{v}_n([x] + | s |)([x] + | s |) | \tau |^{\frac{1}{4}} \\ &=: v_n([x] + | s |) | \tau |^{\frac{1}{4}},\end{aligned}$$

completing the proof. □

VI. CONSTRUCTION OF INTERSECTION PAIRS AND COMPUTATION OF INDICES

In this chapter we shall construct the sets S_1, S_2 occurring in the abstract critical point theorem and compute the indices i_n for relevant sets

VI.1. **Intersection pairs.**

Recall the definition of the bundles $\pi_{\mathrm{M}} : \mathrm{B} \to \mathrm{M}$ and $\pi^0_{\mathrm{M}} : \mathrm{B}^0 \to \mathrm{M}$ introduced in III.1 as well as the contents of Lemma 1:

(1) $$\Phi : \mathrm{B}^0_\rho \to \mathrm{U}$$

is a diffeomorphism onto an open neighbourhood U of $\Lambda_0 = p(\mathrm{M})$, where $p : \mathrm{M} \to \Lambda$ was defined by $m \to m$.

We find a number $\varepsilon > 0$ such that the set

(2) $$\Lambda_\varepsilon = \{ q \in \Lambda \mid d_\Lambda(q, \Lambda_0) \le \varepsilon \}$$

is completely contained in U. We denote by $bd(\Lambda_\varepsilon)$ the boundary of Λ_ε:

(3) $$bd(\Lambda_\varepsilon) = \{ q \in \Lambda \mid d_\Lambda(q, \Lambda_0) = \varepsilon \}.$$

Denote by $\mathrm{L}\Lambda \times \mathbb{R} \to \mathrm{L}\Lambda : (x, s) \to x \circ s$ the flow associated to the differential equation

$$x' = -\beta(\| \Psi'(x) \|_{\mathrm{L}})\Psi'(x).$$

Define subsets of $\mathrm{L}\Lambda \times \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$, by

(4) $$\begin{aligned} \Omega &= \{ ((-\partial q) \circ t, t) \mid q \in \Lambda_\varepsilon,\ t \in \mathbb{R}^+ \} \\ \tilde{\Omega} &= \{ ((-\partial q) \circ t, t) \mid q \in bd(\Lambda_\varepsilon),\ t \in \mathbb{R}^+ \}. \end{aligned}$$

For fixed $t \in \mathbb{R}^+$ define

(4) $$\begin{aligned} \Omega_t &= \{ x \mid (x, t) \in \Omega \} \\ \tilde{\Omega}_t &= \{ x \mid (x, t) \in \tilde{\Omega} \}. \end{aligned}$$

Moreover define

(5) $$\mathrm{S}_1 = \{ \partial q \mid q \in \Lambda \}.$$

Note that

(6) $$\begin{aligned} \Omega_t \cap \mathrm{S}_1 &= p(\mathrm{M}) \quad \text{for all} \quad t \in \mathbb{R}^+ \\ \tilde{\Omega}_t \cap \mathrm{S}_1 &= \phi \quad\quad\ \text{for all} \quad t \in \mathbb{R}^+. \end{aligned}$$

(6) follows immediately from the fact that the map $t \to \Psi(x \circ t)$ is decreasing for all $x \in \mathrm{L}\Lambda$ and that $\Psi(\tilde{\Omega}_t) \subset (-\infty, 0)$, $\Psi(\mathrm{S}_1) \subset [0, +\infty)$. Moreover the set $\Lambda_0 = p(\mathrm{M})$ is the set of stationary points of the flow.

LEMMA 18. — Given any constant $c > 0$ there exists $t_0 > 0$ such that for all $t \ge t_0$ the following holds

(7) $$\begin{aligned} \Psi(\tilde{\Omega}_t) &\subset (-\infty, -c] \\ \Psi(\Omega_t) &\subset (-\infty, 0]. \end{aligned}$$

Proof. — Clearly the second statement in (7) is trivial. In order to prove the first statement concerning $\tilde{\Omega}$ we note that

$$\sup (\Psi(\tilde{\Omega}_0)) < 0.$$

In fact, arguing indirectly we find otherwise a sequence $(q_n) \subset bd(\Lambda_\varepsilon)$ such that

$$-\| \partial q_n \|^2 = \Psi(-\partial q_n) \to 0.$$

Hence (∂q_n) is $(u f p\ c)$ and therefore (q_n) is precompact. Eventually taking a subsequence we may assume $q_n \to q_0$ in Λ and $\| \partial q_0 \| = 0$. Hence $q_0 \in \Lambda_0 = p(\mathrm{M})$ giving the contradiction

$$\varepsilon = d_\Lambda(q_n, \Lambda_0) \to d_\Lambda(q_0, \Lambda_0) = 0\,.$$

Applying Proposition 2 to Ψ (consider $\Psi = \Psi_\infty$ with $h = 0$) the following compactness property holds (Palais-Smale-condition).

$$(8) \qquad \| \Psi'(x_n) \|_{\mathrm{L}} \to 0 \quad \text{and} \quad \Psi'(x_n) \to d$$

for some sequence (x_n) imply that (x_n) is precompact.

Since Ψ has only the critical level 0 we find a constant $r > 0$ such that

$$(9) \quad \| \Psi'(x) \|_{\mathrm{L}} \geq r \quad \text{for all} \quad x \in \mathrm{L}\Lambda \quad \text{such that} \quad \Psi(x) \in [-c, \sup \Psi(\tilde{\Omega}_0)]\,.$$

Hence there exists a $t_0 > 0$ such that (7) holds. □

Now let h be the Hamiltonian with compact support from Chapter IV. Since h has compact support we have for a suitable constant $c > 0$

$$(10) \qquad | \alpha_n(x) | \leq \frac{1}{3} c \quad \text{for all} \quad x \in \mathrm{L}\Lambda$$

and all $n \in \mathbb{N}$. This implies with the $t_0 > 0$ given in Lemma 18 corresponding to c

$$(11) \qquad \inf \Psi_n(\mathrm{S}_1) > \sup \Psi_n(\tilde{\Omega}_t)$$

for all $t \geq t_0$. For the following let

$$(12) \qquad \mathrm{S}_2 = \Omega_{t_0},\ \tilde{\mathrm{S}}_2 = \tilde{\Omega}_{t_0}\,.$$

In the following section we shall study the problem $(\mathrm{S}_2 *_n t) \cap \mathrm{S}_1$.

VI.2. Fixed points of fibre-preserving maps and index maps.

Fix $n \in \mathbb{N}$. We have to calculate

$$(1) \qquad \operatorname{cat}((\mathrm{S}_2 *_n s) \cap \mathrm{S}_1)$$

for all $s \in \mathbb{R}^+$ and to show that $c(\mathrm{M})$ is a lower bound for the expression on (1). This will imply that $i_n(\mathrm{S}) \geq c(\mathrm{M})$. Now consider the intersection problem

$$(2) \qquad (\mathrm{S}_2 *_n s) \cap \mathrm{S}_1 \neq \phi\,.$$

Using the fact that $s \to \Psi_n(x *_n s)$ is decreasing and the estimate (11) in IV.1 we infer

$$(3) \qquad (\tilde{\mathrm{S}}_2 *_n s) \cap \mathrm{S}_1 = \phi \qquad \forall s \in \mathbb{R}^+\,.$$

Using the maps E_0, μ_0, E_n and μ_n introduced in V.2, we define a continuous family of fibre-preserving map ($\Lambda_\varepsilon \times \mathbb{R}^+$ considered as bundle over Λ_ε).

$$\Xi_n : \Lambda_\varepsilon \times \mathbb{R}^+ \to L\Lambda \tag{4}$$

by

$$\Xi_n(q,t) = \begin{cases} \partial q - \frac{1}{2}\mu_0(-\partial q, 0, t)K_0(-\partial q, t) & \text{for } t \in [0, t_0] \\ \partial q - \frac{1}{2}\mu_0(-\partial q, 0, t_0)K_0(-\partial q, t_0) & \\ -\frac{1}{2}\mu_0(-\partial q, 0, t_0)\mu_n((-\partial q)\circ t_0, 0, t-t_0)K_n((-\partial q)\circ t_0, t-t_0) & \text{for } t > t_0 . \end{cases}$$

Clearly Ξ_n is continuous by the properties of E_0, E_n, μ_0, μ_n, K_0 and K_n. Moreover

LEMMA 19. — For all $t \in \mathbb{R}^+$ we have

$$\Xi_n(q,t) \neq 0 \quad \text{if} \quad q \in bd(\Lambda_\varepsilon) . \tag{5}$$

Moreover the following statements are equivalent

(*i*) $\qquad x *_n s \in S_1 \quad \text{where} \quad x \in S_2$

and

(*ii*) $\quad x = (-\partial q)\circ t_0 \quad$ for some $q \in \Lambda_\varepsilon$ and $\Xi_n(q, t_0 + s) = 0$.

Proof. — We infer if $x \in S_2$ and $x *_n s \in S_1$ that

$$x = (-\partial q)\circ t_0 \quad \text{for some} \quad q \in \Lambda_\varepsilon$$

and

$$E_n((-\partial q)\circ t_0, s) = 0 . \tag{6}$$

(6) is equivalent to $0 = \mu_n((-\partial q)\circ t_0, 0, s)E_n((-\partial q)\circ t_0, s)$.

Hence (6) is equivalent to

$$0 = E_n((-\partial q)\circ t_0, 0) + \mu_n((-\partial q)\circ t_0, 0, s)K_n((-\partial q)\circ t_0, s) .$$

That is

$$0 = E_0((-\partial q), t_0) + \mu_n((-\partial q)\circ t_0, 0, s)K_n((-\partial q)\circ t_0, s) .$$

Multiplying this with $\mu_0(-\partial q, 0, t_0)$ gives an equivalent equation to which the representation result Proposition 3 can be applied. In order to get (7) we use the identity $\mu_0(-\partial q, 0, t_0)^{-1} = \mu_0(-\partial q, t_0, 0)$

$$\begin{aligned} 0 = {} & -2\partial q + \mu_0(-\partial q, 0, t_0)K_0(-\partial q, t_0) \\ & + \mu_0(-\partial q, 0, t_0)\mu_n((-\partial q)\circ t_0, 0, s)K_n((-\partial q)\circ t_0, s) . \end{aligned} \tag{7}$$

However (7) is equivalent to

$$0 = \Xi_n(q, t_0 + s).$$

Hence we have proved the equivalence $(i)(=)(ii)$. (5) follows from this equivalence together with (3). □

For the following define $\sigma_n : \Lambda_\varepsilon \times \mathbb{R}^+ \to L\Lambda$ by

$$\Xi_n(q, t) = \partial q + \sigma_n(q, t). \tag{8}$$

LEMMA 20. — For all $n \in \mathbb{N}$ there exists a monotone increasing map $\kappa_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\| Z(\tau)\sigma_n(q, t) - \sigma_n(q, t) \| \leq \kappa_n(t) | \tau |^{\frac{1}{4}} \tag{9}$$

for all $t \in \mathbb{R}^+$, $\tau \in [-1, 1]$ and $q \in \Lambda_\varepsilon$.

Proof. — We have for $t \in [0, t_0]$

$$\sigma(q, t) = -\frac{1}{2}\mu_0(-\partial q, 0, t)K_0(-\partial q, t).$$

Now observe that $\| \partial q \| \leq c$ for a suitable c for all $q \in \Lambda_\varepsilon$ by the construction of Λ_ε (Recall $\Lambda_\varepsilon \subset U = \Phi(B_s^0)$).

Now combining Proposition 3 (4) and Proposition 1 we conclude that (9) holds for all $t \in [0, t_0]$ for a suitable κ_n. Similarly we can estimate σ for $t > t_0$. The details of the proof are shorter but similar to that of Proposition 3 and are left to the reader. □

Recall the definition of the bundles π_M^0, the sets Λ_ε and $bd(\Lambda_\varepsilon)$ and the contents of Lemma 1. We define an open subset V of B_ρ^0 by

$$V = \{ x \in B_\rho^0 \mid \Phi(x) \in \Lambda_\varepsilon \backslash bd(\Lambda_\varepsilon) \}. \tag{10}$$

Moreover define

$$bd(V) = \{ x \in B_\rho^0 \mid \Phi(x) \in bd(\Lambda_\varepsilon) \}.$$

Note that in fact $bd(V)$ is the boundary of V in B^0. We consider V, $\overline{V} = V \cup bd(V)$ and $bd(V)$ as bundles over M. Define $U = \Phi(B_\rho^0)$. Denote by α_1 the diffeomorphism defined by

$$B_\rho^0 \oplus B^0 \to B_\rho^0 \oplus L\Lambda|_{\Lambda_0} : (x, y) \to (x, \delta y).$$

Moreover define the map α_2 by

$$B_\rho^0 \oplus L\Lambda|_{\Lambda_0} \to L\Lambda|_U : (x, z) \to \Phi_{\pi_M^0(x)}(x, z)$$

where the Φ_m are the maps already used in III.1. Here we identify of course again B and $T\Lambda|_{\Lambda_0}$. Define $\alpha : B_\rho^0 \oplus B^0 \to L\Lambda|_U$ to be the composition of α_1 and α_2

$$\alpha = \alpha_2\alpha_1.$$

It turns out that α is a diffeomorphism. In fact, it is enough to show that α_2 is a diffeomorphism. Clearly α_2 is smooth. The inverse is the smooth map $\tilde{\alpha}_2$ defined by

$$\tilde{\alpha}_2(z) = (\Phi^{-1}(\pi(z)), \Phi_{\pi_M^0(\Phi^{-1}(\pi(z)))}(\Phi^{-1}(\pi(z)), .)^{-1} z .$$

Define a map $\tilde{\Xi}_n$ by

$$\tilde{\Xi}_n : (\overline{V}, bd(V)) \times \mathbb{R}^+ \to (B^0, \tilde{B}^0)$$

where $\tilde{B}^0$ denotes the complement of the zero section in B^0, by

$$\tilde{\Xi}_n(x, s) = pr_2 \circ \alpha^{-1} \circ \Xi_n(\Phi(x), s) .$$

Here $pr_2 : B_\rho^0 \oplus B^0 \to B^0$ denotes the projections onto the second factor. Clearly $\tilde{\Xi}_n$ is continuous. Moreover it preserves for fixed $s \in \mathbb{R}^+$ the fibre over M. In fact, if $\pi_M^0(x) = m$ we find $\pi(\Xi_n(\Phi(x), s)) = \Phi(x)$. Hence

$$\pi_M^0 \circ \alpha^{-1} \circ \Xi_n(\Phi(x), s) = \pi_M^0 \circ \Phi^{-1} \circ \Phi(x) = \pi_M^0(x) = m .$$

Further, in order to see that $bd(V) \times \mathbb{R}^+$ is mapped into $\tilde{B}^0$ we observe that $\tilde{\Xi}_n(x, s) = 0$ is equivalent to $\Xi_n(\Phi(x), s) = 0$. This implies $\Phi(x) \notin bd(\Lambda_\varepsilon)$. Hence $x \notin bd(V)$. In the following we shall relate the cohomological properties of the zero set of $\tilde{\Xi}_n(, s + t_0)$ for some fixed $s \in \mathbb{R}^+$ to the cohomological properties of the set $(S_2 *_n s) \cap S_1$. We need

LEMMA 21. — The map $\gamma : L\Lambda|_{\Lambda_\varepsilon} \to B^0$ defined by

$$\gamma(z) = pr_2 \circ \alpha^{-1}(z) \tag{11}$$

maps $(ufp\,c)$ sets into precompact sets.

Proof. — Let $D \subset L\Lambda|_{\Lambda_\varepsilon}$ be $(ufp\,c)$ and $(z_n) \subset \gamma(D)$. We have to show that (z_n) possesses a convergent subsequence. Let $(x_n) \subset D$ such that $z_n = \gamma(x_n)$. Since $\delta : B^0 \to L\Lambda|_{\Lambda_0}$ establishes a diffeomorphism it is enough to show that (y_n), $y_n = \delta z_n$, possesses a convergent subsequence. Let $q_n = \pi(x_n)$ and $m_n = \pi_M^0(z_n)$. Eventually taking a subsequence we may assume

$$m_n \to m \quad \text{in} \quad M$$
$$q_n \to q \quad \text{in} \quad C, \qquad \text{where} \quad q \in \Lambda .$$

We have

$$\alpha(\Phi^{-1}(\pi(x_n)), z_n) = x_n .$$

Then

$$\Phi_{m_n}(\Phi^{-1}(q_n), y_n) = x_n .$$

Clearly $\Phi^{-1}(q_n) \to \Phi^{-1}(q)$ uniformly for the metric d_{TM}, i.e.

$$d_{TM}(\Phi^{-1}(q_n)(t), \Phi^{-1}(q)(t)) \to 0$$

uniformly in $t \in [0,1]$. This implies that $\Phi^{-1}(q) \in (B^0)_m$. For $n \in \mathbb{N}$ we define a smooth path $a_n : [0,1] \to \Lambda$ by

$$a_n(s) = \Phi((1-s)\Phi^{-1}(q_n)).$$

We have $a_n'(s) = -\Phi_{m_n}((1-s)\Phi^{-1}(q_n), \Phi^{-1}(q_n))$. Since the parallel transport along a curve is only depending on the curve but not on its parameterisation we can apply Proposition 1 to find that the images $\tilde{x}_n$ by the parallel transport along a_n of x_n build a $(u\,f\,p\;c)$ sequence. Since $(\pi(\tilde{x}_n)) \subset \Lambda_0 = M$ (we identify Λ_0 and M, B and $T\Lambda|_M$) we infer by an easy application of Proposition 1 by the compactness of M that $(\tilde{x}_n)$ is precompact. (If $(y_n) \subset L\Lambda$ is a sequence such that $(\pi(y_n))$ converges in Λ and (y_n) is $(u\,f\,p\;c)$, then (y_n) is precompact; this is an easy exercise using Proposition 1). Hence eventually taking a subsequence we may assume

$$\tilde{x}_n \to \tilde{x} \quad \text{in} \quad L\Lambda$$

where $\tilde{x} \in L\Lambda|_M$. Let $\tilde{m} = \pi(\tilde{x})$.

We have

$$\pi(\tilde{x}_n) = \Phi(0\,.\,\Phi^{-1}(q_n)) = \pi_M^0(\Phi^{-1}(q_n)) = m_n.$$

Hence

$$\tilde{m} = \lim \pi(\tilde{x}_n) = \lim m_n = m.$$

We introduce local co-ordinates based at $m \in M$ by (for n large enough)

$$\begin{aligned} q_n &= \exp_m(\xi_n) && \xi_n \in T_m\Lambda \\ x_n &= \Phi_m(\xi_n, \eta_n) && \eta_n \in L_m\Lambda \\ \tilde{x}_n &= \Phi_m(b_n, \tilde{\eta}_n) && \tilde{\eta}_n \in L_m\Lambda \\ m_n &= \exp_m(b_n) && \\ a_n(s) &= \exp_m(\tilde{a}_n(s)). && \end{aligned}$$

Denoting by c_n the solution of

$$c_n'(s) = -\Gamma_m(\tilde{a}_n(s))(\tilde{a}_n'(s), c_n(s))$$

with

$$c_n(0) = \eta_n$$

we have

$$c_n(1) = \tilde{\eta}_n.$$

By the special form of the a_n it is clear that $\tilde{a}_n(s)$ and $\tilde{a}_n'(s)$ converge uniformly for $s \in [0,1]$ to $\tilde{a}(s)$ and $\tilde{a}'(s)$ respectively given by $a(s) = \exp_m(\tilde{a}(s))$ and

$$a(s) = \Phi((1-s)\Phi^{-1}(q)).$$

Hence $\tilde{a}(s) = (1-s)\Phi^{-1}(q)$. Since $\tilde{\eta}_n \to \tilde{\eta}$ in $L_m\Lambda$ where $\tilde{x} = \Phi_m(0, \tilde{\eta})$ we infer that η_n converges to some η in $L_m\Lambda$ where $\eta = c(0)$, $\tilde{\eta} = c(1)$ and $c'(s) = -\Gamma_m(a(s))(a'(s), c(s))$. Now

$$\Phi_{m_n}(\Phi^{-1}(q_n), y_n) = \Phi_m(\xi_n, \eta_n).$$

This gives

$$(\Phi^{-1}(q_n), y_n) = \Phi_{m_n}^{-1}\Phi_m(\xi_n, \eta_n).$$

Therefore

$$y_n = pr_2 \circ \Phi_{m_n}^{-1}\Phi_m(\xi_n, \eta_n).$$

Since $\xi_n \to \xi$ uniformly where ξ represents q and $\eta_n \to n$ in $L_m\Lambda$, $m_n \to m$ we infer $y_n \to y$ in $L\Lambda|_M$. □

LEMMA 22. — Define $\tilde{\sigma}_n : \overline{V} \times \mathbb{R}^+ \to B^0$ by

$$\tilde{\sigma}_n(x, t) = \tilde{\Xi}_n(x, t) - x. \tag{12}$$

Then $\tilde{\sigma}_n$ maps sets of the form $\overline{V} \times [0, s]$ into precompact sets.

Proof. — By the definition of $\tilde{\Xi}_n$ we have

$$\begin{aligned}\tilde{\sigma}_n(x, s) &= \tilde{\Xi}_n(x, s) - x \\ &= pr_2 \circ \alpha^{-1}(\partial\Phi(x) + \sigma_n(\Phi(x), s)) - x \\ &= pr_2 \circ \alpha^{-1}(\Phi_{\pi_B^0(x)}(x, \delta x) + \sigma_n(\Phi(x), s)) - x \\ &= pr_2 \circ \alpha^{-1} \circ \alpha(x, x) - x + pr_2 \circ \alpha^{-1} \circ \sigma_n(\Phi(x), s) \\ &= pr_2 \circ \alpha^{-1} \circ \sigma_n(\Phi(x), s) \\ &= \gamma \circ \sigma_n(\Phi(x, s).\end{aligned}$$

By Lemma 21 σ_n maps bounded sets in $\overline{V} \times \mathbb{R}^+$ into $(ufp\,c)$ sets. By Lemma 22 γ maps $(ufp\,c)$ sets into precompact sets. □

Denote by $TM \oplus i\,TM \to M$ the complexification of the tangent bundle. For $j \in \mathbb{Z}$ we define smooth maps

$$e_j : [0, 1] \to \mathbb{C}$$

by

$$e_j(t) = \begin{cases} t - \dfrac{1}{2} & \text{if } j = 0 \\ e^{2\pi i j t} & \text{if } j \neq 0. \end{cases}$$

It is well-known that the family (e_j') ($'$ denotes the derivative) is a total subset of $L^2(0, 1; \mathbb{C})$. Since all e_j have mean value 0 the span of the e_j will be a subspace of the H^1-functions with mean value zero .Since the derivative-map induces a linear isomorphism we infer that (e_j) is a total subset in $\left\{ x \in H^1([0, 1], \mathbb{C}) \,\middle|\, \int_0^1 x = 0 \right\}$. For $k \in \mathbb{N}$ we introduce smooth subbundles $\pi_k^0 : B_k^0 \to M$ of $B^0 \to M$ by

$B_k^0 = \{ x \in B^0 \mid x = \Sigma_{j=-k}^k a_j e_j,$ where $a_j \in TM \oplus i\,TM$ and $\overline{a}_j = a_{-j}$ ($\overline{a}_j$ the complex conjugate) $\}$.

Clearly the maps in B_k^0 are real valued. A local trivialisation, say centred at $m \in M$, is given by

$$V_m \times (B_k^0)_m \to B_k^0 \,|_{\exp(V_m)}$$
$$(a, x) \to \Phi_m(a, x),$$

where $V_m \subset T_m M$ is open such that the restriction of the exponential map gives a diffeomorphism onto $\exp(V_m)$. Note that

$$\Phi_m(a, \Sigma_{j=-k}^{k} a_j e_j)(t) = \Sigma_{j=-k}^{k} (D \exp_m(a).a_j)(e_j(t)).$$

Denote by $P_k : B^0 \to B_k^0$ the fibre-preserving map which projects on each fibre in B^0 orthogonally onto the corresponding fibre in B_k^0. Here, of course, « orthogonally » referes to the $T\Lambda$-inner product. (Recall that we identified $B^0 \subset B \cong T\Lambda \,|_{\hat{M}}$.) It is easy to see that P_k is smooth and that

$$P_k x \to x \quad \text{in} \quad T\Lambda \quad \text{as} \quad k \to +\infty.$$

LEMMA 23. — Given any $s_0 \in \mathbb{R}^+$ and an open neighbourhood W of the set

$$\{ x \in \overline{V} \mid \tilde{\Xi}_n(x, s_0) = 0 \} \qquad \text{(which is allowed to be empty)}$$

there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the following holds

$$\{ x \in B_k^0 \cap \overline{V} \mid P_k \tilde{\Xi}(x, s_0) = 0 \} \subset W \tag{12}$$

and

$$\{ x \in B_k^0 \cap \overline{V} \mid \text{There exists } s \in [0, s_0] \text{ with } P_k \tilde{\Xi}_n(x, s) = 0 \} \tag{13}$$

is contained in $V_k = V \cap B_k^0$.

Proof. — We carry out the proof of (13). The proof of (12) is similar and left to the reader. Arguing indirectly we find sequences $k_j \to +\infty$, $x_j \in bd(V) \cap B_{k_j}^0$ and $s_j \in [0, s_0]$ such that

$$x_j + P_{k_j} \tilde{\sigma}_n(x_j, s_j) = 0.$$

By Lemma 21 the sequence (x_j) must be precompact. Hence we may assume eventually taking a subsequence

$$s_j \to s, \qquad x_j \to x$$

where $s \in [0, s_0]$ and $x \in bd(V)$, and moreover

$$\tilde{\Xi}_n(x, s) = 0.$$

However, we know by Lemma 19 that this is impossible. This proves (13) □

LEMMA 24. — Given any open neighbourhood X of $\{ x \in V \mid \tilde{\Xi}_n(x, t_0) = 0 \}$ in V the projection map $\pi_X := \pi_M^0 \mid X : X \to M$ induces an injective map

$\pi_X^* : H(M) \to H(X)$ in cohomology. (t_0 is the number introduced in Lemma 18).

Proof. — We use an approach similar to that used by Dold in the definition of the fixed point transfer, [*8*]. We find $k \in \mathbb{N}$ such that

$$F(t) := \{ x \in V \cap B_k^0 \mid P_k \tilde{\Xi}_n(x, t) = 0 \} \quad \text{(it could be of course empty)}$$

is compact for all $t \in [0, t_0]$ and $F(t_0) \subset X$. (This can be done by Lemma 22). We have the following commutative diagram ($X_k = X \cap B_k^0$).

$$\begin{array}{ccc} X & \xleftarrow{\;j\;} & X_k \\ & {\scriptstyle \pi_X}\searrow \quad \swarrow{\scriptstyle \pi_k} & \\ & M & \end{array}$$

where j is the inclusion and π_X and π_k denote the projections induced from π_M^0. If we can show that the map π_k^* induced in cohomology is injective, it follows from $\pi_k^* = j^* \pi_X^*$, that π_X^* is injective. Hence we have reduced the infinite-dimensional problem to a finite-dimensional problem. The finite-dimensional vector-bundle $B_k^0 \to M$ is an ENR_M in the sense of Dold [*8*]. That is there are fibre-preserving maps over M:

$$\begin{array}{ccccc} B_k^0 & \xrightarrow{\;i\;} & \mathcal{O} & \xrightarrow{\;r\;} & B_k^0 \\ & {\scriptstyle \pi_k}\searrow & \downarrow{\scriptstyle \tau} & \swarrow{\scriptstyle \pi_k} & \\ & & M & & \end{array}$$

such that $ri = id$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^m \times M$ for some $m \in \mathbb{N}$. Here, of course, $\mathbb{R}^m \times M$ is considered as the trivial bundle over M : $(x, m) \to m$. Moreover τ is the induced projection. Define

$$\tilde{V}_k = r^{-1}(V_k),$$

where $V_k = V \cap B_k^0$ and maps $\tilde{f}_t$ by

$$\tilde{f}_t : \tilde{V}_k \to \mathbb{R}^m \times M : \tilde{f}_t(a) = i(-P_k \tilde{\sigma}_n(r(a), t)).$$

Moreover let

$$\tilde{F}(t) = \{ a \in \tilde{V}_k \mid \tilde{f}_t(a) = a \}.$$

If $a \in \tilde{F}(t)$ we infer

$$a = i(-P_k \tilde{\sigma}_n(r(a), t)).$$

Hence

$$r(a) = ri(-P_k \tilde{\sigma}_n(r(a), t)) = -P_k \tilde{\sigma}_n(r(a), t)$$

or equivalently

$$P_k \tilde{\Xi}_n(r(a), t) = 0.$$

Therefore

$$r(\tilde{F}(t)) \subset F(t).$$

If $x \in \mathrm{F}(t)$ we have similarly $x = -\mathrm{P}_k\tilde{\sigma}_n(x, t)$ which implies

$$i(x) = i(-\mathrm{P}_k\tilde{\sigma}_n(x, t)) = \tilde{f}_t(i(x)).$$

Therefore

$$i(\mathrm{F}(t)) \subset \tilde{\mathrm{F}}(t)$$

and in fact i and r induce homeomorphisms

$$\mathrm{F}(t) \xrightarrow{\sim} \tilde{\mathrm{F}}(t), \qquad \tilde{\mathrm{F}}(t) \xrightarrow{\sim} \mathrm{F}(t)$$

which are inverse to each other. This implies the compactness of $\tilde{\mathrm{F}}(t)$ for $t \in [0, t_0]$ and that $\tilde{\mathrm{X}}_k = r^{-1}(\mathrm{X}_k)$ is a neighbourhood of $\tilde{\mathrm{F}}(t_0)$. We have the commutative diagram

$$\begin{array}{ccc} \tilde{\mathrm{X}}_k & \xrightarrow{r} & \mathrm{X}_k \\ {\scriptstyle \tau|\tilde{\mathrm{X}}_k}\searrow & & \swarrow{\scriptstyle \pi_k} \\ & \mathrm{M} & \end{array}$$

If we can show that $(\tau \mid \tilde{\mathrm{X}}_k)^*$ is injective we infer from $(\tau \mid \tilde{\mathrm{X}}_k)^* = r^*\pi_k^*$ that π_k^* is injective. So, in order to complete the proof we show that $(\tau \mid \tilde{\mathrm{X}}_k)^*$ is injective. To simplify notation we shall write $\tau = \tau \mid \tilde{\mathrm{X}}_k$. Define

$$\tilde{\mathrm{F}} = \cup_{t\in[0,t_0]}\tilde{\mathrm{F}}(t).$$

Consider the following diagram. (We denote all inclusion maps by the letter j, $pr_1 : \mathbb{R}^m \times \mathrm{M} \to \mathbb{R}^m$ denotes the projection onto the first factor. Moreover maps denoted by « exc » are inclusion maps inducing in cohomology the natural excision isomorphism.)

$$\begin{array}{ccc} (\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \tilde{\mathrm{X}}_k & \xleftarrow{(pr_1(j-\tilde{f}_{t_0}),\, id)} & (\tilde{\mathrm{X}}_k, \tilde{\mathrm{X}}_k\backslash\tilde{\mathrm{F}}(t_0)) \\ & & \downarrow{\scriptstyle \mathrm{exc}} \\ \downarrow{\scriptstyle id \times \tau} & & (\tilde{\mathrm{V}}_k, \tilde{\mathrm{V}}_k\backslash\tilde{\mathrm{F}}(t_0)) \\ & & \uparrow \\ (\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \mathrm{M} & \xleftarrow{(pr_1(j-\tilde{f}_{t_0}),\, \tau)} & (\tilde{\mathrm{V}}_k, \tilde{\mathrm{V}}_k\backslash\tilde{\mathrm{F}}). \end{array}$$

By the homotopy-invariance of the cohomology the bottom map induces in cohomology

$$d^* = (pr_1(j - \tilde{f}_0), \tau)^*.$$

Hence we obtain the diagram

$$\begin{array}{ccc} \mathrm{H}((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \tilde{\mathrm{X}}_k) & \xrightarrow{(pr_1(j-f_{t_0}),id)^*} & \mathrm{H}(\tilde{\mathrm{X}}_k, \tilde{\mathrm{X}}_k\backslash\tilde{\mathrm{F}}(t_0)) \\ & & \uparrow\wr \\ \uparrow{\scriptstyle (id\times\tau)^*} & & \mathrm{H}(\tilde{\mathrm{V}}_k, \tilde{\mathrm{V}}_k\backslash\tilde{\mathrm{F}}(t_0)) \\ & & \downarrow{\scriptstyle j^*} \\ \mathrm{H}((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \mathrm{M}) & \xrightarrow{d^*} & \mathrm{H}(\tilde{\mathrm{V}}_k, \tilde{\mathrm{V}}_k\backslash\tilde{\mathrm{F}}) \end{array}$$

If we can show the bottom map is injective the map $(id \times \tau)^*$ must be injective.

Clearly $d\colon (\tilde{V}_k, \tilde{V}_k\backslash\tilde{F}) \to (\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M$ has the form

$$d(a, m) = (a - pr_1 \circ i(0_{r(a,m)}), m)$$

where 0_x denotes the zero element in $(B_k^0)_x$.

Denote by W a compact neighbourhood of $\tilde{F}$ such that $W \subset \tilde{V}_k$. Then the inclusion map $(W, W\backslash\tilde{F}) \to (\tilde{V}_k, \tilde{V}_k\backslash\tilde{F})$ will induce an isomorphism in cohomology. For B_ρ being the open ball of radius ρ around 0 in $\mathbb{R}^m$ we find for $\rho > 0$ large enough a continuous extension $e\colon \mathbb{R}^m \times M \to B_\rho$ of the map $W \to B_\rho : (x, m) \to pr_1 \circ i(0_{r(x,m)})$ (of course using Tietze's Theorem).

Define a map d_1 :

$$(\mathbb{R}^m \times M, (\mathbb{R}^m\backslash B_\rho) \times M) \to (\mathbb{R}^m \times M, (\mathbb{R}^m\backslash\{0\}) \times M)$$

by
$$d_1(x, m) = (x - e(x, m), m)\,.$$
Note that d_1 is homotopic to the inclusion map. In order to show that d^* is injective, consider the following diagrams where we assume that B_ρ defined above satisfies in addition $B_\rho \times M \supset \tilde{F}$.

$$\begin{array}{ccc}
(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M & \xleftarrow{\;d\;} & (\tilde{V}_k, \tilde{V}_k\backslash\tilde{F}) \\
 & {\scriptstyle d|W}\nwarrow & \uparrow{\scriptstyle \mathrm{exc}} \\
 & & (W, W\backslash\tilde{F}) \\
 & & \downarrow{\scriptstyle \mathrm{exc}} \\
 & & (\mathbb{R}^m \times M, (\mathbb{R}^m \times M)\backslash\tilde{F}) \\
 {\scriptstyle id}\nwarrow & & \uparrow{\scriptstyle j} \\
 & & (\mathbb{R}^m \times M, (\mathbb{R}^m\backslash B_\rho) \times M) \\
 & & \downarrow{\scriptstyle d_1} \\
 & & (\mathbb{R}^m \times M, (\mathbb{R}^m\backslash\{0\}) \times M) \\
 & & \| \\
 & & (\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M\,.
\end{array}$$

Note that in cohomology $d_1^* = j^*$ where j:

$$(\mathbb{R}^m \times M, \mathbb{R}^m\backslash B_\rho) \times M) \to (\mathbb{R}^m \times M, (\mathbb{R}^m\backslash\{0\}) \times M)$$

denotes the inclusion map. By the homotopy invariance and the long exact sequence j^* will be an isomorphism. Hence we obtain for some homomorphism α the following diagram in cohomology.

$$\begin{array}{ccc}
H((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M) & \xrightarrow{\;d^*\;} & H(\tilde{V}_k, \tilde{V}_k\backslash\tilde{F}) \\
{\scriptstyle id}\downarrow & \searrow{\scriptstyle (d|W)^*} & | \\
H((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M) & \xleftarrow{\;\alpha\;} & H(W, W\backslash\tilde{F}).
\end{array}$$

Hence $(d \mid W)^*$ is injective which implies the injectivity of d^*. Summing up we have shown that the map

$$H((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M) \xrightarrow{(id \times \tau)^*} H((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \tilde{X}_k)$$

is injective. Denote by $\sigma : H^j(\Gamma) \to H^{j+1}((\mathbb{R}, \mathbb{R}\backslash\{0\}) \times \Gamma)$ the natural suspension isomorphism. Clearly the map $(id \times \tau)^*$ is the map induced under m-fold suspension. Therefore we finally obtain

$$\begin{array}{ccc} H^{j+m}((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times M) & \xrightarrow{(id\times\tau)^*} & H^{j+m}((\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \times \tilde{X}_k) \\ \downarrow\wr\, \sigma^m & & \uparrow\wr\, \sigma^m \\ H^j(M) & \xrightarrow{\tau^*} & H^j(\tilde{X}_k) \end{array}$$

Hence the map τ^* is injective. That completes the proof of the Lemma. □

PROPOSITION 4. — For all $n \in \mathbb{N}$ we have

$$i_n(S_1) \geq c(M). \tag{14}$$

Proof. — We prove (14) by arguing indirectly. Assume $i_n(S_2) < c(M)$. We find $s_0 \in \mathbb{R}^+$ such that

$$\operatorname{cat}((S_2 *_n s_0) \cap S_1) = i_n(S_2).$$

Define

$$D = (S_2 *_n s_0) \cap S_1 .$$

By Lemma 19 we have with

$$E = \{ x \in \overline{V} \mid \tilde{\Xi}_n(x, t_0 + s_0) = 0 \}$$

the following relation between D and E

$$D = ((-\partial\Phi(E) \circ t_0) *_n s_0).$$

By Lemma 21 the set E is compact. Hence D is compact. Clearly

$$(D *_n (-s_0)) \circ (-t_0) = -\partial\Phi(E)$$

and

$$\operatorname{cat}(\partial\Phi(E)) = \operatorname{cat}(-\partial\Phi(E)) = \operatorname{cat}(D) < c(M).$$

We find an open neighbourhood $W \subset L\Lambda$ of $\partial\Phi(E)$ such that

$$\operatorname{cat}(cl(W)) = \operatorname{cat}(\partial\Phi(E))$$
$$\pi(cl(W)) \subset \Phi(V).$$

We shall show that the inclusion map

$$i : W \to L\Lambda$$

cannot induce an injective map $i^* : H(L\Lambda) \to H(W)$ in cohomology. Assume, arguing indirectly, it does. We find $U_1, \ldots, U_k$ open in $L\Lambda$, $k = \operatorname{cat}(cl(W))$ such that

$$\cup_{i=1}^k U_i = W$$
U_i is contractible in $L\Lambda$.

From the long exact sequence for the pair $(L\Lambda, U_i)$ we infer in dimension $j > 0$ that the induced map

$$H^j(L\Lambda, U_i) \to H^j(L\Lambda)$$

is bijective. Hence any cohomology class in $H^j(L\Lambda)$, $j \geq 1$, can be represented by some cohomology class in $H^j(L\Lambda, U_i)$. Consider the maps

$$\begin{array}{ccccc} M & \to & L\Lambda & \to & M \\ m & \to & 0_m, & x & \to \pi(x)(0) \end{array}$$

which constitutes a factorisation of the identity map of M through $L\Lambda$. Hence the second map induces an injective map in cohomology

$$H(M) \to H(L\Lambda).$$

Therefore

$$c(L\Lambda) \geq c(M).$$

Let $v_i \in H^{n(i)}(L\Lambda)$; $n(i) \geq 1$; $i = 1, \ldots, c(M) - 1$, such that

$$v := v_1 \cup \ldots \cup \mu_{c(M)-1} \neq 0.$$

Take representatives $\mu_1, \ldots, \mu_{c(M)-1}$ such that $\mu_i \in H^{n(i)}(L\Lambda, U_i)$. Then

$$\mu := \mu_1 \cup \ldots \cup \mu_{c(M)-1} \neq 0.$$

On the other hand by standard properties of the cup-product we have $\mu \in H^{\Sigma n(i)}(L\Lambda, W)$. Since the inclusion $W \to L\Lambda$ induces an injective map in cohomology we infer from the long exact sequence for the pair $(L\Lambda, W)$ that the map $H(L\Lambda, W) \to H(L\Lambda)$ is the zero map. This implies, since $\mu \to v$ that $v = 0$ which establishes a contradiction. Up to now we have shown that $i_n(S_2) < c(M)$ leads to the conclusion that there exists a certain neighbourhood W of $\partial\Phi(E)$ such that the inclusion map $W \to L\Lambda$ cannot induce an injective map in cohomology. Define a map χ by

$$\chi : \Phi^{-1}(\pi(W) \to M : \chi(x) = \pi^0_M(x).$$

Consider moreover the factorisation of the map

$$\tilde{\chi} : \Phi^{-1}(\pi(W) \to M : \tilde{\chi}(x) = \Phi(x)(0)$$

by (i denotes the inclusion, $\beta(x) = \partial\Phi(x)$).

$$\Phi^{-1}(\pi(W)) \xrightarrow{\beta} W \xrightarrow{i} L\Lambda \xrightarrow{\pi} \Lambda \xrightarrow{\alpha} M$$

where $\alpha(q) = q(0)$. Let us show that χ and $\tilde{\chi}$ are homotopic if $\varepsilon > 0$ is small enough (which we may assume, recall $\overline{V} = \Phi^{-1}(\Lambda_\varepsilon)$). Clearly if $\varepsilon > 0$ is small enough we have

$$d_M(\chi(x), \tilde{\chi}(x)) < \delta_\varepsilon$$

for all $x \in \Phi^{-1}(\pi(W))$ where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We can embed $\hat{i} : M \to \mathbb{R}^m$

for some $m \in \mathbb{N}$ and take a tubular neighbourhood of M in $\mathbb{R}^m$. Let γ denote the projection of the tubular neighbourhood onto M. Then the

$$\text{map } [0,1] \times \Phi^{-1}(\pi(\mathrm{W})) \to \mathrm{M}$$

defined by $(t, x) \to \gamma(t\hat{i}(\tilde{\chi}(x)) + (1-t)i(\tilde{\chi}(x)))$ gives a homotopy. This implies for the induced maps in cohomology

$$\chi^* = \tilde{\chi}^* = \beta^* i^* \pi^* \alpha^* .$$

If we can show that $\pi^*\alpha^*$ is an isomorphism we find since i^* is not injective that χ^* is not injective. Consider $\alpha: \Lambda \to \mathrm{M}$ and define

$$\tilde{\alpha} : [0,1] \times \Lambda \to \Lambda : \tilde{\alpha}(s, q)(t) = q(st).$$

Then $\tilde{\alpha}$ is continuous and a deformation retraction onto Λ_0, which is homeomorphic to M. Therefore α^* is an isomorphism. Next consider $\pi : \mathrm{L}\Lambda \to \Lambda$ and define

$$\tilde{\pi} : [0,1] \times \mathrm{L}\Lambda \to \mathrm{L}\Lambda : \tilde{\pi}(s, x) = sx.$$

Denote by Σ_0 the zero section of $\mathrm{L}\Lambda$. Hence the inclusion map of $\Sigma_0 \to \mathrm{L}\Lambda$ induces an isomorphism in cohomology. Since Σ_0 and Λ are naturally isomorphic we find that π^* is an isomorphism.

Therefore

$$\chi : \Phi^{-1}(\pi(\mathrm{W})) \to \mathrm{M}$$

does not induce an injective map in cohomology. This, however, contradicts Lemma 24 and Proposition 4 is proved. □

VII. PROOF OF THEOREM 2

We give now the proof of Theorem 2. Clearly it will essentially consist of collecting the results we have already proved.

In IV we have constructed a $\mathbb{N}$-family (Ψ_n) for $\Psi_\infty \in \mathrm{C}^1(\mathrm{L}\Lambda, \mathbb{R})$, where

$$\Psi_\infty(x) = (\partial\pi x, x) - \alpha_\infty(x).$$

In VI.1, 2 we have constructed the sets

$$\mathrm{S}_1 = \{\, \partial q \mid q \in \Lambda \,\}$$

and

$$\mathrm{S}_2 = \{\, (-\partial q) \circ t_0 \mid q \in \Lambda_\varepsilon \,\} = \Omega_{t_0}$$

and proved that

$$i_n(\mathrm{S}_2) \geq c(\mathrm{M})$$

for all $n \in \mathbb{N}$. By Theorem 3 the functional Ψ_∞ has at least $c(\mathrm{M})$ critical points. If x is a critical point of Ψ_∞ we have $(q = \pi x)$

$$\partial q = \hat{\mathrm{K}}\mathrm{H}(x)$$

and

$$(x, \delta y) = ((T\pi)H(x), y) \quad \text{for all} \quad y \in T_q\Lambda .$$

This implies $x \in T_q\Lambda$. Integrating by parts yields

$$-(\delta x, y) + \langle x(1), y(1) \rangle - \langle x(0), y(0) \rangle = ((T\pi)H(x), y)$$

for all $y \in T_q\Lambda$. Therefore

$$x(0) = 0_{q(0)} \quad \text{and} \quad x(1) = 0_{q(1)}$$

and

$$-\delta x = (T\pi)H(x) .$$

Denote by $\sigma : TM \to T^*M$ the natural isomorphism for Riemannian manifolds

$$\sigma(a) = \langle a, . \rangle .$$

Then the pull back $\tilde{\omega}$ of ω via σ is given by

$$\tilde{\omega}(a, b) = \langle T\pi a, Kb \rangle - \langle Ka, T\pi b \rangle$$

(see [*16*], 3.1.3). Hence we have for all $y \in T_q\Lambda$

$$\int_0^1 \omega(\dot{x}, y)dt = \int_0^1 dh_t^*(\sigma(x))(T\sigma)ydt$$

(Here we have used the definition of h). Define $\gamma : [0, 1] \to T^*M$ by $\gamma(t) = \sigma(x(t))$. We infer

$$\int_0^1 \omega(\dot{\gamma}, (T\sigma)y)dt = \int_0^1 dh_t^*(\gamma)(T\sigma)ydt .$$

That is

$$\dot{\gamma} = X_t(\gamma) .$$

Since $x(0) = 0$, $x(1) = 0$ we infer $\gamma(0), \gamma(1) \in \Sigma$. Therefore we have found $c(M)$ different solutions for $\dot{\gamma} = X_t(\gamma)$, $\gamma(0), \gamma(1) \in \Sigma$. □

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