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Heavy viable trajectories of controlled systems

by

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ABSTRACT. — We define and study the concept of *heavy* viable trajectories of a controlled system with feedbacks. Viable trajectories are trajectories satisfying at each instant given constraints on the state. The controls regulating viable trajectories evolve according a *set-valued feedback map.* Heavy viable trajectories are the ones which are associated to the controls in the feedback map whose velocity has at each instant minimal norm. We construct the differential equation governing the evolution of the controls associated to heavy viable trajectories and we prove their existence. These results are applied to exchange economies for finding heavy trajectories of a dynamical decentralized allocation mechanism explaining the evolution of prices.

Keywords: Controlled systems, differential inclusions, viable trajectories.

RÉSUMÉ. — Nous définissons le concept de trajectoire viable *lourde* d'un système contrôlé avec rétroaction. Les trajectoires viables sont celles qui obéissent à chaque instant à des contraintes pesant sur l'état du système. Les contrôles régulant les trajectoires viables évoluent selon une loi de rétroaction multivoque. Les trajectoires viables lourdes sont associées aux contrôles régulant les trajectoires viables dont la *norme de la vitesse est minimale* (parmi ceux régulant les trajectoires viables). Nous construisons les équations différentielles gouvernant les contrôles associés aux trajectoires viables lourdes et nous démontrons leur existence. Ces résultats sont appliqués aux économies d'échange pour obtenir les trajectoires viables lourdes d'un mécanisme d'allocations de ressources dynamique et décentralisé, ce qui peut expliquer l'évolution des prix.

1. INTRODUCTION

When we study the evolution of macrosystems which arise in economics and the social sciences as well as in biological evolution, we would take into account not only:

(1) our ignorance of the future environment of the system but also,
(2) the absence of determinism (including the impossibility of a comprehensive description of the dynamics of the system),
(3) our ignorance of the laws relating certain controls to the states of this system,
(4) the variety of dynamics available to the system.

We propose to translate these requirements into mathematics by means of differential inclusions, which describe how the velocity depends in a multi-valued way upon the current state of the system. Another feature of such macrosystems is that the state of the system must obey given restrictions known as *viability constraints*, which determine the viability domain; *viable trajectories* are those lying entirely within the viability domain. Finding viable trajectories of a differential inclusion provides a mechanism of selection of trajectories which, contrary to optimal control theory, does not assume implicitely

(1) the existence of a decision maker operating the controls of the system (there may be more than one decision maker in a game-theoretical setting)
(2) the availability of information (deterministic or stochastic) on the future of the system; this is necessary to define the costs associated with the trajectories
(3) that decisions (even if they are conditional) are taken once and for all at the initial time.

Viability Theorems provide necessary and sufficient conditions for the existence of at least one viable trajectory starting from any viable initial state. It also provides the *feedbacks* (concealed in both the dynamics and the viability constraints) which relate the state of the system to the controls. These feedbacks are not necessarily deterministic: they are set-valued maps associating a *subset* of controls with each state of the system. We observe that the larger these subsets of controls are, the more flexible —and, thus, the more robust—the regulation of the system will be.

Finally the third feature shared by those macrosystems is the *high inertia* of the controls which change only when the viability of the system is at stake. Associated trajectories are called *heavy viable trajectories*: they minimize at each instant the norm of the velocity of the control. We shall provide a formal definition of heavy viable trajectories, which requires an adequate concept of derivative of the set-valued feedback map.

We show that as long as the state of the system lies in the interior of the viability domain, any regulating control will work. Therefore, along a heavy trajectory, the system can maintain the control inherited from the past. (The regulatory control remains constant even though the state may evolve quite rapidly).

What happens when the state reaches the boundary of the viability domain? If the chosen velocity is « inward » in the sense that it pushes the trajectory back into the domain, then we can still keep the same regulatory control.

However, if the chosen velocity is « outward », we are in a period of crisis and must find, as slowly as possible, another regulatory control such that the new associated velocity pushes the trajectory back into the viability domain.

When this strategy for « structural change » fail, the trajectory « dies » i. e., it is no longer viable.

We shall associate with any control u a « viability cell », which is the (possibly empty) subset of viable states which can be regulated by this control. In such a viability cell, an heavy viable trajectory is governed by the constant control to which it is associated.

To pass from one cell to another requires the control to be changed. The boundaries of these cells signal the need for structural change. We mentionned biological evolution as a motivation for studying heavy viable trajectories. Paleontological concepts such as *punctuated equilibria* proposed by Elredge and Gould are consistent with the concept of heavy viable trajectories.

Indeed, for the first time, excavations at Kenya's Lake Turkana have provided clear fossil evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolutionary journey that seems to have proceeded in fits and starts. Williamson [1981] examined 3.300 fossils showing how thirteen species of molluscs changed over several million years. What the record indicated was that the animals stayed much the same for immensely long stretches of time. But twice, about 2 million years ago and then again 700.000 years ago, the pool of life seemed to explode—set off, apparently, by a drop in the lake's water level. In an instant of geologic time, as the changing lake environment allowed new types of molluscs to win the race for survival, all of the species evolved into varieties sharply different from their ancestors. That immediate forms appeared so quickly, with new species suddenly evolving in 5.000 to 50.000 years after millions of years of constancy, challenges the traditional theories of Darwin's disciples since the fossils of Lake Turkana don't record any gradual change; rather, they seem to reflect eons of stasis interrupted by brief evolutionary « revolutions ».

We shall also illustrate the behavior of heavy trajectories in the framework of a simple economic model. The viability domain is the set of allocations of available commodities among n consumers. The behavior of each consumer is described by a differential equation governing the evolution of its consumptions with respect to the evolution of prices. In other words, prices play the role of the control. This is a decentralized model, because each consumer don't need to know the set of available commodities nor the consumptions of other consumers. Hence we shall characterize the feedback law which relates the price to the consumptions of consumers and we shall deduce the differential equation governing the evolution of prices associated with heavy viable trajectories; this may describe the behavior of Adam Smith's invisible hand which sets the price summarizing enough information for allowing consumers to modify their consumptions in a decentralized way, so that, at each instant, the total consumption meets the scarcity constraints.

2. BACKGROUND NOTES

We introduce a viability domain K, which is a subset of a finite dimensional space X, a finite dimensional control space U, a set-valued map F from K to U and a continuous function f from graph (F) to X. We define the *viability problem for a controlled system with feedbacks* as follows: $\forall x_0 \in \mathrm{K}$, find $\mathrm{T} > 0$ and an absolutely continuous function $x(\cdot)$ satisfying

$$(2.1) \qquad \begin{cases} i) \text{ for almost all } t \in [0, \mathrm{T}], & x'(t) = f(x(t), u(t)) \\ ii) \text{ for almost all } t \in [0, \mathrm{T}], & u(t) \in \mathrm{F}(x(t)) \\ iii) \ x(0) = x_0 . \end{cases}$$

which are *viable* on $[0, \mathrm{T}]$ in the sense that

$$(2.2) \qquad \text{for all} \quad t \in [0, \mathrm{T}], \qquad x(t) \in \mathrm{K}.$$

By taking $\mathrm{U} = \mathrm{X}$, $f(x, u) = u$, we obtain the particular case of a viability problem for a differential inclusion

$$(2.3) \qquad x'(t) \in \mathrm{F}(x(t)), \qquad x(0) = x_0 .$$

The viability requirement (2.2) involves naturally restrictions of the dynamical system at the boundary of K. It happens that the best way to describe these conditions is to use the *contingent cone* to K at x (see Aubin-Cellina [1984], p. 176-179, for instance) defined by:

$$(2.4) \qquad \mathrm{T}_{\mathrm{K}}(x) := \left\{ v \in \mathrm{X} \mid \liminf_{h \to 0+} \frac{d_{\mathrm{K}}(x + hv)}{h} = 0 \right\}.$$

We define the *feedback map* R from K to U by

$$R(x) := \{ u \in F(x) \mid f(x, u) \in T_K(x) \}. \tag{2.5}$$

We observe that any viable trajectory of the controlled system (2.1) is a solution to the « feedback » differential inclusion

$$\begin{cases} i) \text{ for almost all } t \in [0, T], & x'(t) = f(x(t), u(t)) \\ ii) \text{ for almost all } t \in [0, T], & u(t) \in R(x(t)) \\ iii) \ x(0) = x_0 \end{cases} \tag{2.6}$$

(the initial set-valued map F is replaced by the feedback map R).

The main viability theorem (see Haddad [1981], Aubin-Cellina [1984] p. 239-240) provides necessary and sufficient conditions for the existence of viable trajectories of (2.3).

THEOREM 2.1. — We assume that

$$\begin{cases} i) \text{ K is locally compact} \\ ii) \text{ F is upper semicontinuous with nonempty convex compact values} \\ iii) \ f \text{ is continuous and is affine with respect to the control.} \end{cases} \tag{2.7}$$

Then the « first order » tangential condition

$$\forall x \in K, \qquad R(x) \neq \emptyset \tag{2.8}$$

is necessary and sufficient for the existence of a viable trajectory of the controlled system (2.1) for all $x_0 \in K$ on some interval $[0, T_{x_0}]$ ▲

As a by-product of our study of heavy viable trajectories, we shall prove the existence of viable trajectories under another set of assumptions: The convexity of the values of F will no longer be required, but we shall need the differentiability of f and F (which we shall define) and above all, we shall add to the first-order tangential condition (2.8) a « second order tangential condition » involving the derivative of the feedback map R. ■

Before defining heavy viable trajectories, we need to recall the following facts.

When K is a subset of a finite dimensional X, we can define other concepts of tangent cones, among which we mention

a) the tangent cone (introduced by Clarke [1975]):

$$C_K(x) := \left\{ v \in X \mid \lim_{\substack{h \to 0+ \\ y \to x \\ y \in K}} \frac{d_K(y + hv)}{h} = 0 \right\} \tag{2.9}$$

b) the Dubovickii-Miljutin [1963] cone:

$$D_K(x) := \{ v \in X \mid \exists \varepsilon > 0 \mid x + [0, \varepsilon](v + \varepsilon B) \subset K \}. \tag{2.10}$$

We have the following relations (see Aubin-Ekeland [1984] p. 409)

$$C_K(x) \subset T_K(x) \tag{2.11}$$

and

$$\operatorname{Int} C_K(x) \subset D_K(x) \subset \operatorname{Int} T_K(x). \tag{2.12}$$

The tangent cone is always convex. It coincides with the contingent cone when K is a smooth manifold (tangent space) or when K is convex or, more generally, when K is *soft* in the sense that

$$x \to T_K(x) \quad \text{is lower semicontinuous}. \tag{2.13}$$

(see Aubin-Clarke [1977]).

Consider now a set-valued map R from X to U and a point (x, u) of its graph. The *contingent derivative* $DR(x, u)$ is the set-valued map from X to U defined by

$$w \in DR(x, u)(v) \Leftrightarrow (v, w) \in T_{\mathrm{Graph}(R)}(x, u). \tag{2.14}$$

It is equivalent to say that

$$\liminf_{\substack{h \to 0+ \\ v' \to v}} d\left(w, \frac{R(x + hv') - u}{h} \right) = 0. \tag{2.15}$$

The contingent derivative $DR(x, u)$ is a closed process (a map whose graph is a closed cone). We say that the map F is *soft* if its graph is soft. Then $DR(x, u)$ is a closed convex process, because its graph is equal to the tangent cone to Graph (R) at (x, u). We shall say that R is *lower semicontinuously differentiable if*

$$(x, u, v) \to DR(x, u)(v) \tag{2.16}$$

is lower semicontinuous on its domain of definition.

We observe that in this case $DR(x, u)$ is also a closed convex process because property (2.16) implies that $(x, u) \to T_{\mathrm{Graph}(R)}(x, u)$ is lower semicontinuous, and thus, Graph $DR(x, u)$ is a closed convex cone.

Finally, when K is a closed subset of X, we denote by

$$m(K) := \{ u \in K \mid \| u \| = \min_{v \in K} \| v \| \} \tag{2.17}$$

the subset of elements of K with minimal norm. If F is a continuous set-valued map with closed convex images, the single-valued map $x \to m(F(x))$

is continuous. This is no longer the case when F is only upper or lower semicontinuous (with closed convex values). However,

$$(2.18) \quad \begin{cases} \text{if F is lower semicontinuous, then } x \to d(0, F(x)) \text{ is upper} \\ \text{semicontinuous.} \end{cases}$$

We refer to Aubin [1984] and Aubin-Ekeland [1984], Chapter 7, Clarke [1983] for a general presentation of nonsmooth analysis relevant to this study.

We shall denote by B the unit ball and, when K is a subset, by $B_K(x_0, \varepsilon)$ the neighborhood $K \cap (x_0 + \varepsilon B)$.

3. HEAVY VIABLE TRAJECTORIES

We consider the viability problem (2.1), (2.2) for controlled systems with feedbacks. We have seen that viable trajectories are solutions to the feedback differential inclusion (2.6). When the functions $x(\cdot)$ and $u(\cdot)$ are absolutely continuous, we can differentiate the « first order » feedback law

$$(3.1) \qquad \forall t \in [0, T], \qquad u(t) \in R(x(t))$$

and obtain the « second order » feedback law

$$(3.2) \quad \text{for almost all} \quad t \in [0, T], \qquad u'(t) \in DR(x(t), u(t))(f(x(t), u(t))).$$

We now propose to select among all regulatory controls satisfying (3.2) the ones whose velocity has a minimal norm: such trajectories seem to be present in the evolution of macrosystems arising in social, economic and biological sciences (which motivated viability theory in the first place).

DEFINITION 3.1. — We shall say that absolutely continuous functions $(x(\cdot), u(\cdot))$ form a heavy viable trajectory if it is a solution to the system of differential inclusions:

$$(3.3) \qquad \begin{cases} i) \ x' = f(x, u) \\ ii) \ u' \in m(DR(x, u)(f(x, u))) \\ iii) \ (x(0), u(0)) \quad \text{given in} \quad \text{Graph}(R) \end{cases}$$

which are viable in the sense that

$$(3.4) \qquad \forall t \in [0, T], \qquad x(t) \in K \quad \text{and} \quad u(t) \in R(x(t)).$$

We shall say that the subsets (which may be empty)

$$(3.5) \qquad C(u) := \{ x \in K \mid 0 \in DR(x, u)(f(x, u)) \}$$

are the *viability cells* of the system. ▲

We observe that along a heavy viable trajectory, a system will keep the control $u(t_0)$ as long as the state $x(t)$ remains in the viability cell $C(u(t_0))$ for $t \geqslant t_0$, because in this case inclusion (3.3) *ii)* states that $u'(t) = 0$. If not, when $x(t)$ leaves the viability cell $C(u(t_0))$ at time $\bar{t}_0$, the control starts to evolve at time $\bar{t}_0$ until the time t_1 when $x(t_1) \in C(u(t_1))$.

In the case of ordinary differential inclusions (when $U = X$ and $f(x, u) = u$), heavy viable trajectories can be written $x(t_0) + (t - t_0)x'(t_0)$ when $x(t_0) \in C(x'(t_0))$ as long as $x(t_0) + (t - t_0)x'(t_0)$ remains in $C(x'(t_0))$. In this case, the viability cells display areas of the viability domain where « linear quantitative growth » holds true.

We observe also the following inclusion

$$\{ x \in R^{-1}(u) \mid f(x, u) \in T_{R^{-1}(u)}(x) \} \subset C(u) \tag{3.6}$$

for all $u \in \mathrm{Im}(R)$.

We shall state our main existence theorem.

THEOREM 3.2. — We assume that f is C^1 in a neighborhood of Graph (F) and that

$$\text{the maps F and } T_K \text{ are soft}. \tag{3.7}$$

We posit the following « transversality condition »

$$\left\{ \begin{array}{l} \forall (x, u) \in \mathrm{Graph}(R), \quad \forall (y, z) \in X \times X, \quad \exists v \in X \quad \text{satisfying} \\ z \in f'_x(x, u)v + f'_u(x, u)DF(x, u)(v) - DT_K(x, f(x, u))(v - y). \end{array} \right. \tag{3.8}$$

Then the derivative of the feedback map R can be written

$$\left\{ \begin{array}{l} DR(x, u)(v) = \\ DF(x, u)(v) \cap f'_u(x, u)^{-1}(DT_K(x, f(x, u))(v) - f'_x(x, u)v). \end{array} \right. \tag{3.9}$$

▲

THEOREM 3.3. — Assume moreover that

$$\mathrm{Graph}(F) \quad \text{is locally compact} \tag{3.10}$$

and that

(3.11) The feedback map R is lower semicontinuously differentiable.

Then the « first order » condition

$$\forall x \in K, \qquad R(x) \neq \emptyset \tag{3.12}$$

and the « second order » condition

$$\forall (x, u) \in \mathrm{Graph}(R), \quad f(x, u) \in \mathrm{Dom}\, DF(x, u) \cap \mathrm{Dom}\, DT_K(x, f(x, u)) \tag{3.13}$$

imply the existence of heavy *viable trajectories* of the controlled system (2.1) for any initial state $x_0 \in K$ and initial control $u_0 \in R(x_0)$ if

$$\mathrm{Graph}(T_K) \quad \text{is locally compact}. \tag{3.14}$$

If the graph of T_K is not locally compact, the initial control must satisfy

$$(3.15) \qquad f(x_0, u_0) \in D_K(x_0) . \qquad \blacktriangle$$

Several comments are in order. Theorem 3.3 should be compared to Theorem 2.1. Theorem 3.3 does not involve convexity requirements, but smoothness conditions (3.7) and (3.11) and second order condition (3.13) on top of the first order condition (3.12). The solutions are more regular (the control is absolutely continuous instead of being only measurable), but there exist the restriction (3.15) on the initial state when the graph of T_K is not locally compact. Unfortunately, this happens whenever K involves inequality constraints: Take for instance $K = \mathbb{R}_+$. Then $\text{Graph}(T_K) = (\{0\} \times \mathbb{R}_+) \cup (]0, \infty[\times \mathbb{R})$ is not locally compact. But the map T_K is soft and even lower semicontinuously differentiable because

$$(3.16) \quad DT_K(x, u)(v) = \begin{cases} \mathbb{R} & \text{if } \ (x, u) \in \text{graph}(T_K), \ \text{ and } \ x > 0 \text{ or } v \geqslant 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

This crucial example shows that assumption (3.7) is not unreasonable.

Since the maps F and T_K are soft, the derivative $DR(x, u)$ defined by (3.9) is a closed convex process and the differential inclusion (3.3) *ii)* governing the evolution of the control is actually the differential equation

$$(3.17) \qquad u'(t) = m(DR(x(t), u(t))(f(x(t), u(t)))) .$$

There are no general explicit formula allowing to couch $m(DR(x, u)(f(x, u)))$ in terms of $DF(x, u)(f(x, u))$, $DT_K(x, u)(f(x, u))$ and $f'(x, u)$ by using formula (3.9). However, see Annex for some further remarks on this problem.

We can also provide sufficient conditions for the regularity assumption (3.11) to hold true. For instance, thanks to a theorem on the lower semicontinuity of the intersection of two lower semicontinuous maps (see Aubin-Cellina [1984] p. 49), conditions (3.7) and (3.11) follow from the following ones

(3.18)
- *i)* the set-valued maps F and T_K are lower semicontinuously differentiable
- *ii)* $\exists c > 0$ such that $(x, u) \to DF(x, u)(f(x, u)) \cap cB \neq \varnothing$ on some neighborhood of each point (x_0, u_0) of Graph (R)
- *iii)* $\forall (x, u) \in \text{Graph}(R)$, $\exists \gamma > 0$, $\exists \varepsilon > 0$ such that $\forall (y, v) \in B_{\text{Graph}(R)}((x, u), \varepsilon)$, $\forall z \in B(0, \gamma)$, we have $z \in f'_x(y, v)f(y, v) + f'_u(y, v)DF(y, v)(f(y, v)) - DT_K(y, f(y, v))(f(y, v))$

$\blacksquare$

We can adapt Theorem 3.3 to viability domains K which are the intersection of a subset L whose tangent cone has a locally compact graph and another subset. Namely, consider the case of a viability domain of the form:

$$(3.19) \qquad K = L \cap A^{-1}(M)$$

where A is a C^1-map from X to a finite dimensional space Y. We also assume that

$$(3.20) \quad T_K(x) = T_L(x) \cap T_{A^{-1}(M)}(x) \quad \text{and} \quad T_{A^{-1}(M)}(x) = A'(x)^{-1}T_M(Ax).$$

This holds true for instance when:

$$(3.21) \quad \begin{cases} \text{L and M are closed convex subsets and A is linear, satisfying} \\ 0 \in \operatorname{Int}(A(L) - M) \end{cases}$$

(see Aubin-Cellina [1984] p. 325) or when

$$(3.22) \quad \begin{cases} \text{L and M are soft} \\ \text{and when} \quad \forall x \in K, \qquad A'(x)T_L(x) - T_M(Ax) = X \end{cases}$$

(see Aubin-Ekeland [1984] p. 440).

In this case, the feedback map R can be written

$$(3.23) \quad R(x) := \{ u \in F(x) \mid f(x, u) \in T_L(x) \quad \text{and} \quad A'(x)f(x, u) \in T_M(Ax) \}.$$

COROLLARY 3.4. — Let us assume that (3.20) holds true and that the graphs of F and T_L are locally compact. We posit assumptions (3.7), (3.8), (3.11), (3.12) and (3.13) of Theorem 3.3.

Then for any $x_0 \in K$ and any control $u_0 \in F(x_0)$ satisfying

$$(3.24) \qquad A'(x_0)f(x_0, u_0) \in D_M(Ax_0)$$

there exist $T > 0$ and a heavy viable trajectory of the controlled system (2.1) on $[0, T]$. ▲

Proof. — We replace F by $F \cap T_L$, whose graph is locally compact and we observe that $A'(x)^{-1}D_M(Ax) \subset D_{A^{-1}(M)}(x)$. ■

Let us formulate Theorems 3.1 and 3.2 in the particular case of differential inclusions, when $U := X$ and $f(x, u) := u$.

COROLLARY 3.5. — Let us assume that the maps F and T_K are soft and satisfy the « transversality condition »

$$(3.25) \quad \begin{cases} \forall (x, u) \in \operatorname{Graph}(R), \quad \forall (y, z) \in X \times X, \quad \exists v \in X \quad \text{such that} \\ z \in DF(x, u)(v) - DT_K(x - u)(v - y). \end{cases}$$

Then

$$(3.26) \qquad DR(x, u)(v) = DF(x, u)(v) \cap DT_K(x, u)(v). \qquad ▲$$

COROLLARY 3.6. — Assume that the graph of F is locally compact and that the regularity condition (3.11) is satisfied. We posit the first and second order conditions

$$(3.27) \quad \begin{cases} i) \ \forall x \in \mathrm{K}, \quad \mathrm{R}(x) := \mathrm{F}(x) \cap \mathrm{T_K}(x) \neq \emptyset \\ ii) \ \forall (x, u) \in \mathrm{Graph}(\mathrm{R}), \quad u \in \mathrm{Dom}\, \mathrm{DF}(x, u) \cap \mathrm{Dom}\, \mathrm{DT_K}(x, u) \end{cases}$$

Then, for any $x_0 \in \mathrm{K}$ and any $u_0 \in \mathrm{F}(x_0)$ satisfying either

(3.28) $\quad u_0 \in \mathrm{T_K}(x_0)$ when the graph of $\mathrm{T_K}$ is locally compact

or

$$(3.29) \qquad u_0 \in \mathrm{D_K}(x_0)$$

there exist $\mathrm{T} > 0$ and a C^1 heavy viable trajectory of $x' \in \mathrm{F}(x)$, $x(0) = x_0$ and $x'(0) = u_0$, a solution to the second order differential equation

$$(3.30) \quad x''(t) = m(\mathrm{DR}(x(t), x'(t))(x'(t))), \qquad x(0) = x_0, \quad x'(0) = u_0 . \qquad \blacktriangle$$

4. PROOF OF THE THEOREM

We shall prove this theorem in several steps. We begin by computing the derivative of the feedback map in terms of the derivatives of f, F and $\mathrm{T_K}$.

LEMMA 4.1. — We posit assumptions (3.7) and (3.8). Then formula (3.9) holds true and $\mathrm{DR}(x, u)$ is a closed convex process. $\blacktriangle$

Proof. — We set $\phi(x, u) := (x, f(x, u))$ and we observe that

$$(4.1) \qquad \mathrm{Graph}(\mathrm{R}) = \mathrm{Graph}\,\mathrm{F} \cap \phi^{-1}(\mathrm{Graph}(\mathrm{T_K})) .$$

Therefore, we know that

$$(4.2) \quad \mathrm{T_{Graph(R)}}(x, u) \subset \mathrm{T_{Graph(F)}}(x, u) \cap \phi'(x, u)^{-1}\mathrm{T_{Graph(T_K)}}(\phi(x, u)).$$

For proving the other inclusion, we use the formula of Aubin [1984] (see also Aubin-Ekeland, [1984] p. 440) to compute the tangent cone of Graph (R). The transversality assumption (3.8) and (3.7) imply that

$$(4.3) \qquad \phi'(x, u)\mathrm{C_{Graph(F)}}(x, u) - \mathrm{C_{Graph(T_K)}}(x, u) = \mathrm{X} \times \mathrm{X} .$$

Then we deduce that

$$(4.4) \quad \mathrm{C_{Graph(F)}}(x, u) \cap \phi'(x, u)^{-1}\mathrm{C_{Graph(T_K)}}(\phi(x, u) \subset \mathrm{C_{Graph(R)}}(x, u).$$

Since the maps F and $\mathrm{T_K}$ are soft, then the tangent and contingent cones coincide. Hence inclusions (4.2), (4.4) and $\mathrm{C_{Graph(R)}}(x, u) \subset \mathrm{T_{Graph(R)}}(x, u)$ imply the equality

$$(4.5) \quad \mathrm{T_{Graph(R)}}(x, u) = \mathrm{T_{Graph(F)}}(x, u) \cap \phi'(x, u)^{-1}\mathrm{T_{Graph(T_K)}}(\phi(x, u))$$

which, obviously, implies formula (3.9) and that $(x, u) \to \text{Graph DR}(x, u)$ is a closed convex process. ■

We set

$$(4.6) \qquad G(x, u) := d(0, DR(x, u)(f(x, u)))B$$

which is the ball of radius $d(0, R(x, u)(f(x, u)))$.

LEMMA 4.2. — The trajectories of the first order system of differential inclusions

$$(4.7) \qquad \begin{cases} i) \ x'(t) = f(x(t), u(t)) \\ ii) \ u'(t) \in G(x(t), u(t)) \\ iii) \ (x(0), u(0)) = (x_0, u_0) \quad \text{given in} \quad \text{Graph}(R) \end{cases}$$

which are viable in the sense that

$$(4.8) \qquad \forall t \geqslant 0, \quad (x(t), u(t)) \in \text{Graph}(R)$$

are heavy viable trajectories of the controlled system (2.1). ▲

Proof. — Indeed, the viable trajectories of (4.7), (4.8) satisfy

$$u'(t) \in DR(x(t), u(t))(f(x(t), u(t))).$$

This inclusion and inclusion (4.7) *ii)* imply that $u'(t)$ belongs to $m(DR(x(t), u(t))(f(x(t), u(t))))$. ■

LEMMA 4.3. — Let us assume that f is continuous, that the graph of F is locally compact, that

$$(4.9) \qquad \begin{cases} (x, u) \in \text{Graph}(R) \to d(0, DR(x, u)(f(x, u))) \\ \text{is upper semicontinuous} \end{cases}$$

and that the first and second order conditions

$$(4.10) \qquad \begin{cases} i) \ \forall x \in K, \quad R(x) \neq \emptyset \\ ii) \ \forall (x, u) \in \text{Graph}(R), \quad f(x, u) \in \text{Dom DR}(x, u) \end{cases}$$

hold true.

Then for any $x_0 \in K$ and any control $u_0 \in F(x_0)$ satisfying either

$$(4.11) \qquad f(x_0, u_0) \in T_K(x_0) \quad \text{when} \quad \text{Graph}(T_K) \quad \text{is locally compact}$$

or

$$(4.12) \qquad f(x_0, u_0) \in D_K(x_0)$$

there exist $T > 0$ and a heavy viable trajectory of the controlled system (2.1) on $[0, T]$. ▲

Proof. — *a*) By Lemma 4.2, we have to prove the existence of viable trajectories to the system (4.7), (4.8). Condition (4.9) is equivalent to the upper semicontinuity of the set valued map G. Then

$$(4.13)\quad \begin{cases} (x, u) \to f(x, u) \times \mathrm{G}(x, u) \text{ is upper semicontinuous with com-} \\ \text{pact convex values.} \end{cases}$$

It is also clear that the tangential condition:

$$(4.14)\qquad (f(x, u) \times \mathrm{G}(x, u)) \cap \mathrm{T}_{\mathrm{Graph}(\mathrm{R})}(x, u) \neq \varnothing$$

is satisfied. If the subset Graph (R) were locally compact, it is sufficient to apply Theorem 2.1 to the problem (4.7)-(4.8) for solving the problem. This is possible under assumption (4.11). Unfortunately, we have observed that this is not necessarily the case. We then shall follow the method proposed by Cornet and Haddad [1983] for solving the viability problem for second order differential inclusions. The idea is to replace Graph (R) by a locally compact subset $\mathscr{K} \subset$ Graph (R) large enough for the tangential condition (4.14) to remain valid for $\mathscr{K}$. The price to pay is to choose the initial control $u_0 \in \mathrm{F}(x_0)$ such that $f(x_0, u_0)$ belongs to the tangent cone $\mathrm{D}_\mathrm{K}(x_0)$ introduced by Dubovickii-Miljutin [1963].

b) Assume now that the graph of T_K is no longer locally compact and take $f(x_0, u_0)$ in $\mathrm{D}_\mathrm{K}(x_0)$.

There exist $\varepsilon > 0$ and $\alpha > 0$ such that the compact convex subset

$$(4.15)\qquad \mathrm{K}_0 := x_0 + [0, \alpha](f(x_0, u_0) + \varepsilon\mathrm{B}) \subset \mathrm{K}.$$

We observe that the interior of K_0 is non empty and that $f(x_0, u_0)$ belongs to the interior of $\mathrm{T}_{\mathrm{K}_0}(x_0)$. Since the graph of $x \to \operatorname{Int} \mathrm{T}_{\mathrm{K}_0}(x)$ is open (see Aubin-Cellina [1984] p. 221), there exists $\delta \in]0, \varepsilon[$ such that

$$\forall x \in \mathrm{K}_0 \cap (x_0 + \delta\mathring{\mathrm{B}}), \qquad f(x_0, u_0) + \delta\mathrm{B} \subset \operatorname{Int} \mathrm{T}_{\mathrm{K}_0}(x) = \mathrm{D}_{\mathrm{K}_0}(x).$$

By the continuity of f, there exist $\rho < \delta$ and $\eta < \delta$ such that:

$$(4.16)\quad \forall x \in \mathrm{K}_0 \cap (x_0 + \rho\mathring{\mathrm{B}}),\quad \forall u \in u_0 + \eta\mathring{\mathrm{B}},\quad f(x, u) \in f(x_0, u_0) + \delta\mathring{\mathrm{B}} \subset \mathrm{D}_{\mathrm{K}_0}(x)$$

We set

$$(4.17)\qquad \mathscr{K} := \mathrm{Graph}\,(\mathrm{F}) \cap (\mathrm{K}_0 \cap (x_0 + \rho\mathring{\mathrm{B}}) \times (u_0 + \eta\mathring{\mathrm{B}}))$$

which is obviously locally compact.

Since $\mathrm{D}_{\mathrm{K}_0}(x) \subset \mathrm{T}_{\mathrm{K}_0}(x) \subset \mathrm{T}_\mathrm{K}(x)$, we deduce from (4.16) that $\mathscr{K}$ is a subset of Graph (R).

c) Let us set $v := f(x, u)$ and choose w in $m(\mathrm{DR}(x, u)(v))$. So (v, w) belongs to $f(x, u) \times \mathrm{G}(x, u)$. It is sufficient to check that

$$(4.18)\qquad \forall (x, u) \in \mathscr{K}, \qquad (v, w) \in \mathrm{T}_{\mathscr{K}}(x, u)$$

for implying the viability condition

$$(4.19) \qquad \forall (x, u) \in \mathscr{K}, \qquad (f(x, u) \times \mathrm{G}(x, u)) \cap \mathrm{T}_{\mathscr{K}}(x, u) \neq \varnothing .$$

Since $w \in \mathrm{DR}(x, u)(v)$, then $(v, w) \in \mathrm{T}_{\mathrm{Graph(R)}}(x, u) \subset \mathrm{T}_{\mathrm{Graph(F)}}(x, u)$. There exist sequences $h_n \to 0_+$, $v_n \to v$ and $w_n \to w$ such that

$$(4.20) \qquad (x + h_n v_n, u + h_n w_n) \in \mathrm{Graph}\,(\mathrm{F}) .$$

Also, for n large enough, $x + h_n v_n$ belongs to $x_0 + \rho \mathring{\mathrm{B}}$ and $u + h_n w_n$ belongs to $u_0 + \eta \mathring{\mathrm{B}}$. Since $f(x, u)$ belongs to $f(x_0, u_0) + \delta \mathring{\mathrm{B}}$, then v_n belongs to $f(x_0, u_0) + \delta \mathring{\mathrm{B}}$ for n large enough so that, by the very definition of K_0, $x_0 + h_n v_n$ belongs to K_0. Hence

$$(4.21) \qquad (x + h_n v_n, u + h_n w_n) \text{ belongs to } \mathscr{K} \text{ for large } n\text{'s}$$

and property (4.18) ensues.

d) We then apply the viability theorem: there exists a solution of the system (4.7) such that,

$$(4.22) \qquad \text{for all} \quad t \in [0, \mathrm{T}], \qquad (x(t), u(t)) \in \mathscr{K} \subset \mathrm{Graph}\,(\mathrm{R}) .$$

This is a heavy solution of the controlled system (2.1). ■

Proof of Theorems 3.2 and 3.3. — Theorem 3.2 follows from Lemma 4.1 and the Theorem 3.3 from Lemma 4.3, by observing that (4.9) follows from (3.11). ■

5. EXAMPLE: THE SMOOTH CASE

We consider now the smooth case, when

$$\mathrm{K} := \{ x \in \mathrm{X} \mid g(x) = 0 \}$$

and when F is the constant map equal to the vector space U.

THEOREM 5.1. — Let us assume that f is a C^1-function from $\mathrm{X} \times \mathrm{U}$ to X, g is a C^2-function from X to Y and that, $\forall x \in \mathrm{K}$, $g'(x) \in \mathscr{L}(\mathrm{X}, \mathrm{Y})$ is surjective. We suppose that

$$(5.1) \qquad \forall x \in \mathrm{K}, \qquad \mathrm{R}(x) := \{ u \in \mathrm{U} \mid g'(x) f(x, u) = 0 \} \neq \varnothing$$

and that

$$(5.2) \quad \forall (x, u) \in \mathrm{Graph}\,(\mathrm{R}), \qquad g'(x) f'_u(x, u) \in \mathscr{L}(\mathrm{U}, \mathrm{Y}) \quad \text{is surjective} .$$

Then, for any $x_0 \in \mathrm{K}$ and u_0 such that $g'(x_0)f(x_0, u_0) = 0$, there exists a heavy viable trajectory, a solution to the system of differential inclusions:

$$(5.3)\quad \begin{cases} i)\ x' = f(x, u) \\ ii)\ u' = -\, f'_u(x, u)^* g'(x)^* (g'(x) f'_u(x, u) f'_u(x, u)^* g'(x)^*)^{-1} \\ \qquad\qquad [g'(x) f'_x(x, u) f(x, u) + g''(x)(f(x, u), f(x, u))] \\ iii)\ (x(0), u(0)) = (x_0, u_0)\,. \end{cases}$$

▲

Proof. — In this simple case, we can compute $m(\mathrm{DR}(x, u))(v)$ explicitely. Indeed, since $\mathrm{T_K}(x) = \mathrm{Ker}\, g'(x)$, then $\mathrm{R}(x) = \{ u \in \mathrm{U} \mid g'(x)f(x, u) = 0 \}$. By setting:

$$(5.4)\qquad h(x, u) := (g(x), g'(x)f(x, u))$$

we observe that $\mathrm{Graph}\,(\mathrm{R}) = \{ (x, u) \mid h(x, u) = 0 \}$. This function is C^1 and we check easily that

$$h'(x, u)(v, w) = (g'(x)v,\ g''(x)(f(x, u), v) + g'(x)f'_x(x, u)v + g'(x)f'_u(x, u)w)\,.$$

Since both $g'(x)$ and $g'(x)f'_u(x, u)$ are surjective by assumption, then $h'(x, u)$ is surjective. Therefore, the contingent cone to Graph (R) at (x, u)—actually, its tangent space—is the set of pairs (v, w) such that $h'(x, u)(v, w) = 0$. Hence

$$(5.5)\quad \mathrm{DR}(x, u)(v) = -\, f'_u(x, u)^{-1} g'(x)^{-1} [g''(x)(f(x, u), v) + g'(x)f'_x(x, u)v]\,.$$

whenever v belongs to $\ker g$.

Now, we can compute explicitely the element of minimal norm $m(\mathrm{DR}(x, u)(v))$, which minimize the norm $\| w \|$ under the linear constraint

$$g'(x)f'_u(x, u)w = -\, g'(x)f'_x(x, u)v - g''(x)(f(x, u), v))\,.$$

Its solution is given explicitely by the right-hand side of (5.3) *ii*).

Example. — *Heavy viable trajectories on affine subspaces.*

We consider the case when $\mathrm{K} = \{ x \in \mathrm{X} \mid \mathrm{G}x = y \}$ where $\mathrm{G} \in \mathscr{L}(\mathrm{X}, \mathrm{Y})$ is surjective. We assume that

$$(5.6)\quad \begin{cases} i)\ \forall x \in \mathrm{K},\ \ \exists u \in \mathrm{U} \quad \text{such that} \quad \mathrm{G}f(x, u) = 0 \\ ii)\ \forall x \in \mathrm{K},\ \ \forall u \in \mathrm{U} \quad \text{such that} \quad \mathrm{G}f(x, u) = 0,\ \ \mathrm{G}f'_u(x, u) \\ \quad\ \text{is surjective}\,. \end{cases}$$

Then for any x_0 satisfying $\mathrm{G}x_0 = y$, u_0 satisfying $\mathrm{G}f(x_0, u_0) = 0$, there exists a heavy viable trajectory, a solution to the system of differential equations

$$(5.7)\quad \begin{cases} i)\ x' = f(x, u) \\ ii)\ u' = -\, f'_u(x, u)^* \mathrm{G}^* (\mathrm{G} f'_u(x, u) f'_u(x, u)^* \mathrm{G}^*)^{-1} \mathrm{G} f'_x(x, u) f(x, u). \end{cases}$$

When $G = g \in X^*$ ($Y = R$), assumptions (5.6) become

$$(5.8) \quad \begin{cases} i)\ \forall x \in K, \quad \exists u \in U \quad \text{such that} \quad \langle g, f(x, u) \rangle = 0 \\ ii)\ \forall x \in K, \quad \forall u \in U \quad \text{such that} \quad \langle g, f(x, u) \rangle = 0, \\ \quad \text{then} \quad f'_u(x, u)^* g \neq 0 \end{cases}$$

and equation (5.7) *ii)* becomes

$$(5.9) \qquad u' = -\frac{f'_u(x, u)^* g}{\| f'_u(x, u)^* g \|^2} \langle g, f'_x(x, u) f(x, u) \rangle .$$

Example. — *Heavy viable trajectories on the sphere.*

Let G be a symmetric positive definite linear operator from X to X* and we take

$$(5.10) \qquad K := \{ x \in X \mid \langle Gx, x \rangle = 1 \} .$$

We assume that

$$(5.11) \quad \begin{cases} i)\ \forall x \in K, \quad R(x) := \{ u \in U \mid \langle Gx, f(x, u) \rangle = 0 \} \neq \emptyset \\ ii)\ \forall (x, u) \in \mathrm{Graph}\,(R), \quad f'_u(x, u)^* Gx \neq 0 . \end{cases}$$

Then the heavy viable trajectories on the sphere are the solutions to the system of differential equations

$$(5.12) \quad \begin{cases} i)\ x' = f(x, u) \\ ii)\ u' = -\dfrac{f'_u(x, u)^* Gx}{\| f'_u(x, u)^* Gx \|^2} (\langle Gf(x, u), f(x, u) \rangle \\ \qquad\qquad + \langle Gx, f'_u(x, u) f(x, u) \rangle). \end{cases} \qquad \blacksquare$$

Remark. — Consider the case when

$$(5.13) \qquad K = \{ x \in X \mid g(x) = 0 \} \cap A^{-1}(M)$$

where g is a C^2-map from X to Y and A is a C^1-map from X to Z, Y and Z being finite dimensional spaces. We assume that $y \to T_M(y)$ is lower semicontinuous, that

$$(5.14) \qquad \forall x \in K, \qquad A'(x) \operatorname{Ker} g'(x) - T_M(Ax) = X ,$$

and that $\forall y, z \in Z \times Z$, $\exists v$, a solution to the inclusion:

$$(5.15) \quad \begin{cases} z \in DT_M(Ax, A'f)(A'v - y) \cap [A'f'_x v + A''(f, v) \\ \qquad\qquad - A'g'^{-1}(g''(f, v) + g'f'_x v)] \end{cases}$$

where we have set $f = f(x, u)$, $g' = g'(x)$, $A' = A'(x)$, $f'_x = f'_x(x, u)$, $f'_u = f'_u(x, u)$, $A''(u, v) = A''(x)(u, v)$, $g''(u, v) = g''(x)(u, v)$.

Assumption (5.14) and Aubin-Ekeland [1984] p. 440 imply that

$$(5.16) \quad R(x) = \{ u \in U \mid g'(x) f(x, u) = 0 \quad \text{and} \quad A'(x) f(x, u) \in T_M(Ax) \}$$

and assumption (5.15) implies that

$$(5.16)\quad \begin{cases} \mathrm{DR}(x, u)(v) := -(g'f'_u)^{-1}(g''(f, v) + g'f'_x v) \cap \\ \qquad \cap (\mathrm{A}'f'_u)^{-1}(\mathrm{DT_M}(\mathrm{A}(x), \mathrm{A}'f)(\mathrm{A}'v) - \mathrm{A}''(f, v) - \mathrm{A}'f'_x v). \end{cases}$$

6. HEAVY TRAJECTORIES OF DECENTRALIZED DYNAMICAL ALLOCATION MECHANISMS

We consider the decentralized dynamical allocation mechanism proposed in Aubin [1981] *b*) (see Aubin-Cellina [1984], Section 5.5, p. 245-256 and Stacchetti [1983]).

We interpret $\mathrm{Y} := \mathbb{R}^l$ as a *commodity space*, its dual Y^* as the *price space*.

The description of the exchange economy begins with

$$(6.1)\qquad \text{the subset} \quad \mathrm{M} \subset \mathrm{Y} \quad \text{of available commodities}.$$

The problem is to devise decentralized mechanisms allowing to allocate an available commodity $y \in \mathrm{M}$ among n consumers $i = 1, \ldots, n$.

Each consumer chooses a commodity in a consumption set $\mathrm{L}_i \subset \mathrm{Y}$. By setting

$$\mathrm{X} := \mathrm{Y}^n, \quad \mathrm{L} := \prod_{i=1}^{n} \mathrm{L}_i, \quad x = (x_1, \ldots, x_n) \in \mathrm{X}$$

the set K of *allocations* is defined by

$$(6.2)\qquad \mathrm{K} := \left\{ x \in \mathrm{L} \,\middle|\, \sum_{i=1}^{n} x_i \in \mathrm{M} \right\}.$$

Let $\mathrm{P} \subset \mathrm{Y}^*$ denote the subset of prices. We describe the behavior of each agent by *change functions* c_i satisfying

$$(6.3)\quad \begin{cases} i)\ \forall i = 1, \ldots, n, \quad c_i : \mathrm{L}_i \times \mathrm{P} \to \mathrm{Y} \text{ is } \mathrm{C}^1 \quad \text{around} \quad \mathrm{L}_i \times \mathrm{P} \\ ii)\ \forall x \in \mathrm{L}_i, \quad \forall p \in \mathrm{P}, \quad c_i(x, p) \in \mathrm{T}_{\mathrm{L}_i}(x). \end{cases}$$

The decentralized dynamical allocation mechanism we study is described by a system of differential equations: Given $x_0 = (x_{01}, \ldots, x_{0n}) \in \mathrm{K}$, find solutions $x_i(t)$ to the system of differential equations

$$(6.4)\qquad \forall i = 1, \ldots, n, \qquad x'_i(t) = c_i(x_i(t), p(t)), \quad x_i(0) = x_{i0}$$

controlled by prices $p(t) \in \mathrm{P}$.

The viability condition requires that at each instant, $x(t)=(x_1(t), \ldots, x_n(t))$ is an allocation:

$$(6.5) \qquad \forall t \geqslant 0, \quad \forall i = 1, \ldots, n, \qquad x_i(t) \in \mathrm{L}_i \quad \text{and} \quad \sum_{i=1}^{n} x_i(t) \in \mathrm{M}.$$

In order to have an explicit description of the feedback map R, we posit the following assumptions:

$$(6.6) \qquad \begin{cases} i) \text{ the subsets } \mathrm{L}_i, \mathrm{M} \text{ and } \mathrm{P} \text{ are closed convex} \\ ii) \ 0 \in \operatorname{Int}\left(\sum_{i=1}^{n} \mathrm{L}_i - \mathrm{M}\right) \end{cases}$$

which imply that

$$\mathrm{T_K}(x) = \left\{ v \in \prod_{i=1}^{n} \mathrm{T}_{\mathrm{L}_i}(x_i) \,\middle|\, \sum_{i=1}^{n} v_i \in \mathrm{T_M}\left(\sum_{i=1}^{n} x_i\right)\right\}$$

(see Aubin-Cellina [1984] p. 241).

The feedback map R is defined by

$$(6.7) \qquad \mathrm{R}(x) = \left\{ p \in \mathrm{P} \,\middle|\, \sum_{i=1}^{n} c_i(x_i, p) \in \mathrm{T_M}\left(\sum_{i=1}^{n} x_i\right)\right\}$$

thanks to assumption (6.3) *ii*).

The first order condition requires that

$$(6.8) \qquad \forall x \in \mathrm{K}, \qquad \mathrm{R}(x) \neq \varnothing.$$

Remark. — We recall that when $\mathrm{P} = \mathrm{S}^l$, $\mathrm{M} = \mathrm{M} - \mathbb{R}^l_+$ and the change functions c_i are affine with respect to p, then a sufficient condition for (6.8) is that the budgetary constraints

$$\forall p \in \mathrm{P}, \quad \forall x \in \mathrm{L}, \qquad \sum_{i=1}^{n} \langle p, c_i(x, p) \rangle \leqslant 0.$$

are satisfied. ■

In order to express the second-order condition, we have to couch the derivative of the feedback map in terms of the derivatives of the change functions c_i and the set-valued maps $\mathrm{T}_{\mathrm{L}_i}(\cdot)$, $\mathrm{T_P}(\cdot)$ and $\mathrm{T_M}(\cdot)$.

A proof analogous to the proof of Lemma 4.1 implies

LEMMA 6.1. — We posit assumptions (6.3) and (6.6). We assume that

(6.9) the set-valued map $\mathrm{T_M}$ is soft and that for all $(x, p) \in \mathrm{Graph}(\mathrm{R})$, the following transversality property holds true:

$$(6.10)\quad \begin{cases} \forall y, z \in \mathrm{Y} \times \mathrm{Y}, \quad \exists \pi \in \mathrm{T_P}(p), \quad \exists v_i \in \mathrm{T}_{\mathrm{L}_i}(x_i) \quad \text{such that} \\ z \in \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} c_i(x_i, p) v_i + \sum_{i=1}^{n} \frac{\partial}{\partial p} c_i(x_i, p) \pi \\ \qquad\qquad - \mathrm{DT_M}\left(\displaystyle\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} c_i(x_i, p)\right)\left(\sum_{i=1}^{n} v_i - y\right). \end{cases}$$

Then the derivative $\mathrm{DR}(x, p)(v)$ of the feedback map R is empty when v_i does not belong to $\mathrm{T}_{\mathrm{L}_i}(x_i)$ for some i and is equal to

$$(6.11)\quad \begin{cases} \mathrm{DR}(x, p)(v) = \\ \mathrm{T_P}(p) \cap \left(\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial p} c_i(x_i, p)\right)^{-1} \left(\mathrm{DT_M}\left(\displaystyle\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} c_i(x_i, p)\right)\left(\sum_{i=1}^{n} v_i\right)\right. \\ \qquad\qquad \left. - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} c_i(x_i, p) v_i\right) \end{cases}$$

when $v_i \in \mathrm{T}_{\mathrm{L}_i}(x_i)$ for all $i = 1, \ldots, n$. ▲

We then obtain the following theorem:

THEOREM 6.2. — We posit assumptions (6.3), (6.6), (6.10),

$$(6.12)\quad \{\ \mathrm{L}_i \text{ and P are subspaces}$$

and

(6.13) the feedback map R is lower semicontinuously differentiable.

Then, for all initial allocation $x_0 \in \mathrm{K}$ and all initial price p_0 satisfying

$\displaystyle\sum_{i=1}^{n} c_i(x_{0i}, p_0) \in \mathrm{T_M}\left(\sum_{i=1}^{n} x_{0i}\right)$ if the graph of $\mathrm{T_M}(\cdot)$ is locally compact or

$\displaystyle\sum_{i=1}^{n} c_i(x_{0i}, p_0) \in \mathrm{D_M}\left(\sum_{i=1}^{n} x_{0i}\right)$ if not,

there exist $\mathrm{T} > 0$ and a heavy viable trajectory of the decentralized dynamical allocation mechanism (6.4), a solution to the system of differential equations

$$(6.14)\qquad \begin{cases} i)\ x_i'(t) = c_i(x_i(t), p(t))), \quad x_i(0) = x_{0i}, \quad i = 1, \ldots, n \\ ii)\ p'(t) = m(\mathrm{DR}(x(t), p(t))(c(x(t), p(t)))), \ p(0) = p_0. \end{cases}$$

▲

We cannot have an explicit analytical expression of the element of minimal norm of $\mathrm{DR}(x, p)(c(x, p))$ except in special situations. Assume for instance that we do not take into account the constraints on prices and individual consumptions. We take $\mathrm{P} = \mathrm{Y}^*$ and $\mathrm{L}_i = \mathrm{Y}$ for all i. Then, for all $x \in \mathrm{X}$,

$$(6.15)\qquad \mathrm{R}(x) = \left\{ p \in \mathrm{Y}^* \mid \sum_{i=1}^{n} c_i(x_i, p) \in \mathrm{T_M}\left(\sum_{i=1}^{n} x_i\right)\right\}.$$

The surjectivity property

$$(6.16)\qquad \forall (x, p) \in \mathrm{Graph}\,(\mathrm{R}), \qquad \sum_{i=1}^{n} \frac{\partial}{\partial p} c_i(x_i, p) \quad \text{is surjective}$$

implies that the transversality property (6.10) is satisfied. Then, if we assume that

$$(6.17)\qquad \text{the set-valued map } \mathrm{T_M} \text{ is soft},$$

we obtain

$$(6.18)\qquad \begin{cases} \mathrm{DR}(x, p)(v) = \\ \left(\sum_{i=1}^{n} \frac{\partial}{\partial p} c_i(x_i, p)\right)^{-1} \left(\mathrm{DT_M}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} c_i(x_i, p)\right)\left(\sum_{i=1}^{n} v_i\right)\right. \\ \qquad\qquad\qquad\qquad \left. - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} c_i(x_i, p) \cdot v_i\right). \end{cases}$$

Let us set now

$$(6.19)\qquad \varphi(x, p) := \left(\sum_{i=1}^{n} \frac{\partial}{\partial p}\, c_i(x_i, p)\right)^{-1} \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} c_i(x_i, p) c_i(x_i, p)\right)$$

and

$$(6.20)\qquad \mathrm{E}(x, p) := \left(\sum_{i=1}^{n} \frac{\partial}{\partial p} c_i(x_i, p)\right)^{-1} \mathrm{DT_M}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} c_i(x_i, p)\right)\left(\sum_{i=1}^{n} c_i(x_i, p)\right).$$

Then the element with minimal norm can be written

$$(6.21)\qquad m(\mathrm{DR}(x, p)(c(x, p))) = (1 - \pi_{\mathrm{E}(x,p)})\varphi(x, p)\,.$$

EXAMPLE. — We consider the case when the set M of available commodities is the cone of commodities smaller than or equal to a given commodity $w \in \mathbb{R}^l$:

$$(6.22)\qquad \mathrm{M} := w - \mathbb{R}^l_+\,.$$

We consider specific change functions of the form

$$(6.23)\qquad c_i(x_i, p) := \gamma_i(x_i) - p$$

where γ_i are continuous functions defined on consumptions sets $\mathrm{L}_i := \mathbb{R}^l$.

Remark. — This example of change function is motivated by the case when the behavior of a consumer is described by a concave differentiable utility function u_i. Hence change functions of the form $\nabla u_i(x) - p$ are selections of the super-differential of the restriction of u_i to a budget set $\{ x \mid \langle p, x \rangle = r \}$ and differential equations of the form

$$x_i'(t) = \nabla u_i(x_i(t)) - p(t)$$

translate a behavior of consumer i whose increases his consumption « proportionally » to a generalized gradient of the restriction of its utility to his budget set. ■

Finally, we introduce a C^1-function φ from the set K of allocations to the price set $\mathbb{R}^l$, which associates to each allocation $x \in \mathrm{K}$ the minimum price set by producers. It defines a set-valued map feedback map P by

$$(6.24)\qquad \mathrm{P}(x) := \{ p \in \mathbb{R}^l \mid p \geqslant \varphi(x) \}\,.$$

We set:

$$(6.25)\qquad \mathrm{H}_0(x) := \left\{ h = 1, \ldots, l \,\middle|\, \sum_{i=1}^{n} x_{i_h} = w_h \right\}.$$

We shall assume that φ satisfies

$$(6.26)\qquad \forall x \in \mathrm{K}, \quad \forall h \in \mathrm{H}_0(x), \qquad \varphi(x)_h \geqslant \frac{1}{n}\sum_{i=1}^{n} \gamma_i(x_i)_h.$$

We also set, for any $x \in \mathrm{K}$ and $p \in \mathrm{P}(x)$,

$$(6.27)\qquad \mathrm{H}(x, p) := \{ h = 1, \ldots, l \mid \varphi(x)_h = p_h \}\,.$$

Hence, the differential equations governing the evolution of heavy viable trajectories are defined by

$$(6.28)\qquad \begin{cases} i)\ x_i'(t) = \gamma_i(x_i(t)) - p(t) & (i = 1, \ldots, n) \\ ii)\ x_i(0) = x_{i_0} \end{cases}$$

and for all $h = 1, \ldots, l$

$$(6.29) \qquad \begin{cases} p'_h(t) = \begin{cases} 0 & \text{if} \quad h \notin \mathrm{H}(x(t), p(t)) \\ \displaystyle\sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial x_i}(x(t))(\gamma_i(x_i(t)) - p(t)) \right)_h \end{cases} \\ p(0) \in \mathrm{P}(x_0) \end{cases}$$

Indeed, assumption (6.26) implies that $\mathrm{R}(x) = \mathrm{P}(x)$ because

$$\mathrm{R}(x) := \left\{ p \in \mathrm{P}(x) \mid \forall h \in \mathrm{H}_0(x),\ p_h \geqslant \frac{1}{n} \sum_{i=1}^{n} \gamma_i(x_i)_h \right\}.$$

Hence the graph of R is closed and its derivative is given by

$$(6.30) \qquad \mathrm{DR}(x, p)(v) = \left\{ \pi \mid \forall h \in \mathrm{H}(x, p),\ \pi_h \geqslant \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial x_i}(x) \cdot v_i \right)_h \right\}.$$

Therefore:

$$(6.31) \qquad m(\mathrm{DR}(x, p)(v))_h = \begin{cases} 0 & \text{if} \quad h \notin \mathrm{H}(x, p) \\ \displaystyle\sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial x_i}(x) v_i \right)_h & \text{if} \quad h \in \mathrm{H}(x, p). \end{cases}$$

■

ANNEX

Let $L \subset X$ and $M \subset Y$ be two closed convex subsets and $A \in \mathscr{L}(X, Y)$ satisfying

$$(*) \qquad 0 \in \operatorname{Int}(A(L) - M).$$

Then $K := L \cap A^{-1}(M)$ is a nonempty closed convex subset. Let π_L and π_M denote the projectors of best approximation onto L and M. Then we can write

$$m(K) = \pi_L(-A^*q)$$

where q is a solution to the equation

$$(**) \qquad A\pi_L(-A^*q) = \pi_M(q + A\pi_L(-A^*q)).$$

Furthermore, $Am(K) = \pi_M(q + A\pi_L(-A^*q))$. ■

NUMERICAL EXAMPLE

We provide now a numerical example due to Klaus Schilling dealing with 3 consumers and 2 goods. We assume here that for $i = 1, 2 3$

$$\gamma_i(x_i) = \begin{pmatrix} \alpha_i x_{i1}^{\alpha_i - 1} \cdot x_{i2}^{1-\alpha_i} \\ (1 - \alpha_i) x_{i1}^{\alpha_i} x_{i2}^{-\alpha_i} \end{pmatrix}$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.5, 0.7).$$

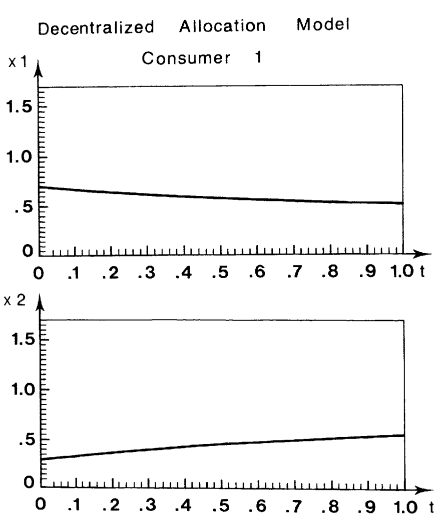

FIG. 1

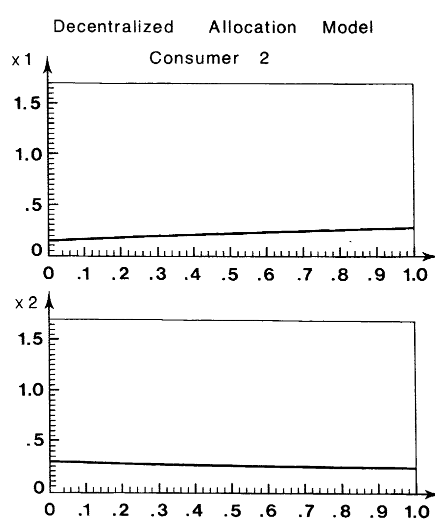

FIG. 2

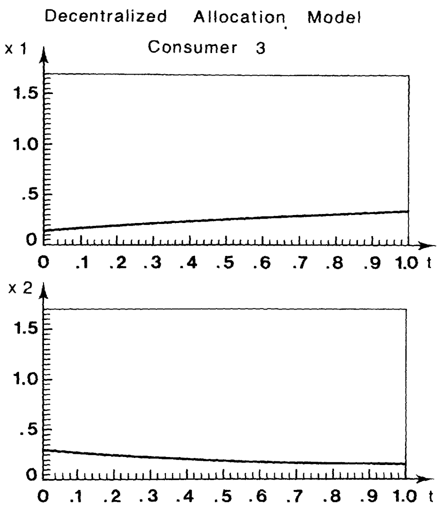

FIG. 3

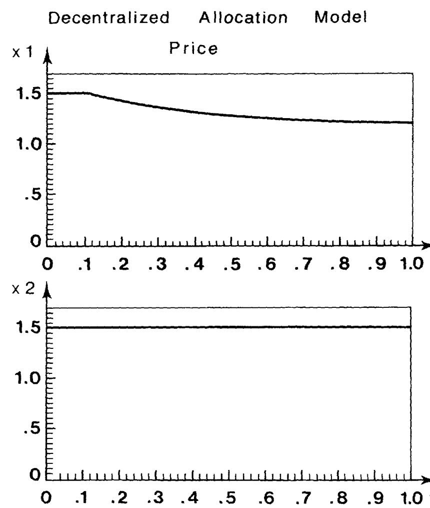

FIG. 4

The consumption level w is fixed by

$$w = (1, 1).$$

The initial conditions are

$$p(0) = (1.51, 1.51), \quad x_1(0) = (0.7, 0.3), \quad x_2(0) = x_3(0) = (0.14, 0.3).$$

One can see on Fig. 4 that in this model the price of the first good is evolving in a heavy way. The initial conditions for the second good are such that the price can be kept constant on the time interval $[0, 1]$.

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