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Solutions in the large for certain nonlinear parabolic systems

by

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ABSTRACT. — We prove the global existence of smooth solutions for certain systems of the form $u_t + f(u)_x = \mathrm{D}u_{xx}$. Here u and f are vectors and D is a constant, positive matrix. We assume that the Cauchy data u_0 satisfies $\| u_0 - \bar{u} \|_{L^\infty(\mathbb{R})} < r$, where $\bar{u}$ is a fixed vector and f is defined in an r-ball about $\bar{u}$, and that $\| u_0 - \bar{u} \|_{L^2(\mathbb{R})}$ is sufficiently small. We show how our results apply to the equations of (nonisentropic) gas dynamics, and we include a result which shows that for the Navier-Stokes equations of compressible flow, smoothing of initial discontinuities must occur for the velocity and energy, but cannot occur for the density.

RÉSUMÉ. — Nous démontrons l'existence globale de solutions régulières pour certains systèmes de la forme $u_t + f(u)_x = \mathrm{D}u_{xx}$, où u et f sont des vecteurs et D une matrice constante définie positive. Nous supposons que la donnée initiale u_0 vérifie $\| u_0 - \bar{u} \|_\infty < r$ où $\bar{u}$ est fixé, f défini dans la boule de centre $\bar{u}$ et de rayon r, et $\| u_0 - \bar{u} \|_2$ est suffisamment petit. Nous montrons ensuite comment nos résultats s'appliquent aux équations de la dynamique des gaz (cas non isentropique), et nous prouvons, en particulier, que pour les équations de Navier-Stokes pour les fluides compressibles, la régularisation des discontinuités initiales doit apparaître pour la vitesse et l'énergie, mais ne peut pas se produire pour la densité.

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§ 1. INTRODUCTION

In this paper we prove the global existence of solutions of certain parabolic systems

$$u_t + f(u)_x = \mathrm{D}u_{xx}, \qquad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x). \tag{1.2}$$

Here $u = (u_1, \ldots, u_n)$, f is a smooth vector function, and D is a constant, diagonalizable matrix with positive eigenvalues. We assume that f is defined in a ball of radius r centered at a fixed vector $\bar{u}$, and we first obtain the existence of a local solution when $u_0 - \bar{u} \in \mathrm{L}^\infty(\mathbb{R})$ with $\| u_0 - \bar{u} \|_{\mathrm{L}^\infty(\mathbb{R})} < r$. These local solutions are then extended globally under the assumption that there is a suitable entropy-entropy flux pair for (1.1) (to be defined below), and that $u_0 - \bar{u} \in \mathrm{L}^2(\mathbb{R})$ with $\| u_0 - \bar{u} \|_{\mathrm{L}^2(\mathbb{R})}$ sufficiently small. These existence theorems are presented in section 2.

In section 3 we apply our results to the equations of gas dynamics, that is, the laws of conservation of mass, momentum, and energy, with diffusion terms $\mathrm{D}u_{xx}$ included as in (1.1). We construct explicity the required entropy-entropy flux pair for the simplest formulation of these equations, and we prove a general result which shows how the pair then carries over to equivalent formulations. In this way we establish the global existence of smooth solutions of the gas dynamics equations (with diffusion terms added as above) in the energy as well as entropy formulation, and in either Lagrangean or Eulerian coordinates. A broad class of diffusion matrices D is allowed, and no assumptions are made about the smoothness of u_0.

Finally, in section 4, we present a result concerning the smoothing of initial discontinuities for the same equations of gas dynamics in which more realistic diffusion terms are included. Specifically, diffusion is added to the momentum and energy equations, but not to the mass equation; (the compressible Navier-Stokes equations are included here). We show that, for suitable weak solutions of the resulting systems, smoothing of initial discontinuities *must* occur for the velocity and energy, but *cannot* occur for the density. Thus initial discontinuities in the density must persist for all time. This result stands in marked contrast to expectations based upon physical reasoning, which suggests that the effects of viscosity and heat conduction, together with the coupling in the equations, would serve to smooth out all initial discontinuities. See for example the remarks in [*3*], p. 135.

Local solutions of (1.1) can easily be obtained by applying the contraction mapping principle to an integral representation for solutions of (1.1).

In order that a local solution may be extended globally, it is necessary that the values of u remain in a set in which the flux $f(u)$ is defined. Now, in some cases of interest, there is an invariant region for (1.1) which serves to control the term $f(u)$. Global existence of solutions can then be obtained as in [5] or [14]. (Such results usually require that D be a multiple of the identity, however.) In the present paper, however, we control the sup-norm of u by obtaining *a priori* bounds for $\| u(.,t) \|_{L^2(\mathbb{R})}$ and $\| u_x(.,t) \|_{L^2(\mathbb{R})}$ which are independent of time, and then applying a standard Sobolev inequality. To derive these *a priori* bounds, we first show that when $u_0 - \bar{u} \in L^2$, there is enough smoothing so that $u_x(.,t) \in L^2(\mathbb{R})$ for small $t > 0$. Then, by exploiting the existence of an entropy-entropy flux pair, we are able to avoid a Gronwall-type inequality in the standard energy estimates, thereby obtaining bounds for $\| u(.,t) - \bar{u} \|_{L^2(\mathbb{R})}$ and $\| u_x(.,t) \|_{L^2(\mathbb{R})}$ which are independent of t for $t \geq t_0 > 0$. This technique seems to have been applied first by Kanel, [7].

We shall now give a brief survey of the literature. The isentropic gas dynamics equations with $D = kI$ have been studied by Kanel' [7], in the case that $u_0 \in C^1$ and u_0 has small H^1 norm. In [5] and [14], these equations have been studied again with $D = kI$ but the data was only of class b. v., while in [16], $D = kI$, the data was smooth, but not necessarily small. In all of these papers, the restriction $D = kI$ was made in order that invariant regions could be found. The isentropic gas dynamics equations, with the « physical » viscosity was studied in [8]; here the data was smooth, with small H^1 norm. In the works [9] [10] and [13], the full gas dynamics equations are studied, with the restriction that the data is smooth and has small H^s norm, $s > 1$. In [6], these equations are considered for a particular equation of state, and the data is not required to be in L_2.

§ 2. GLOBAL EXISTENCE OF SOLUTIONS

In this section we prove our main result, which is the global existence of solutions for the problem (1.1)-(1.2) under suitable restrictions on u_0, f, and D. First we derive a local existence result for the case in which $D > 0$ is diagonal, $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i > 0$, $1 \leq i \leq n$:

$$u_t + f(u)_x = Du_{xx} \tag{2.1}$$

$$u(x, 0) = u_0(x). \tag{2.2}$$

Let $K(x, t)$ be the fundamental solution associated with the operator $\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}$. That is, $K(x, t)$ is an n-vector whose jth component is

$$K_j(x, t) = \frac{1}{\sqrt{4\pi d_j t}} e^{-x^2/4d_j t}.$$

The solution of (2.1)-(2.2) then satisfies the representation

$$u(t) = K(t) * u_0 - \int_0^t K_x(t-s) * f(u(s))ds,$$

where * denotes convolution in space, taken componentwise. We shall make repeated use of the following bounds for K and its derivatives:

$$\left\| \frac{\partial^k}{\partial x^k} K(.,t) \right\|_1 \leq \frac{C(k)}{t^{k/2}}, \qquad k = 0, 1, 2, \ldots \tag{2.3}$$

(Of course, $C(0) = 1$).

We assume that f is defined and is of class C^3 in a closed ball $\overline{B}_r(\overline{u})$ of radius r about a point $\overline{u}$, and that $f(\overline{u}) = 0$. Finally, we denote by C a generic positive constant which may depend on K and on the properties of f in $\overline{B}_r(\overline{u})$.

To begin, define the set of functions $\mathcal{G}_T$ by

$$\mathcal{G}_T = \{ u \in L^\infty([0,T] \times \mathbb{R}) : \| u(t) - \overline{u} \|_\infty \leq r \},$$

and the operator $\mathcal{L}$ on $\mathcal{G}_T$ by

$$\mathcal{L}(u)(t) = K(t) * u_0 - \int_0^t K_x(t-s) * f(u(s))ds, \qquad u \in \mathcal{G}_T. \tag{2.4}$$

Our local existence result will follow from the properties of $\mathcal{L}$ given in the following lemma:

LEMMA 2.1. — Assume that $u_0 - \overline{u} \in L^\infty \cap L^2$ and that $\| u_0 - \overline{u} \|_\infty = s < r$. Then if $T > 0$ is sufficiently small (depending on s), the following hold:

a) $\mathcal{L}$ maps $\mathcal{G}_T$ into itself.

b) $\mathcal{L}$ is a contraction in the L^∞ topology on $\mathcal{G}_T$.

c) There is a constant C_0 depending only on K and f such that, whenever $u \in \mathcal{G}_T$ satisfies

$$\| u(t) \|_2 \leq C_0 \| u_0 \|_2, \qquad t \in [0,T], \tag{2.5}$$

then $\mathcal{L}(u)$ also satisfies (2.5).

d) There is a constant C_1 depending only on K and f such that, whenever $u \in \mathcal{G}_T$ satisfies

$$\| u_x(t) \|_p \leq \frac{C_1 \| u_0 \|_p}{\sqrt{t}}, \qquad 0 < t \leq T, \tag{2.6}$$

for $p = 2$ or $p = \infty$, then $\mathcal{L}(u)$ also satisfies (2.6).

e) Given $t_0 \in (0, T)$, there is a constant C_2 depending only on K, *f*, and t_0, such that, whenever $u \in \mathcal{G}_T$ satisfies (2.6) and

$$\| u_{xx}(t) \|_2 \leq \frac{C_2(\| u_0 \|_2 + \| u_0 \|_2^2)}{\sqrt{t - t_0}}, \qquad t_0 < t \leq T. \tag{2.7}$$

then $\mathcal{L}(u)$ also satisfies (2.7).

f) Given $t_1 \in (t_0, T)$, there is a constant C_3 depending only on K, F, t_0, and t_1, such that, whenever $u \in \mathcal{G}_T$ satisfies (2.6), (2.7) and

$$\| u_{xxx}(t) \|_2 \leq \frac{C_3(\| u_0 \|_2 + \| u_0 \|_2^2 + \| u_0 \|_2^3)}{\sqrt{t - t_1}}, \qquad t_1 < t \leq T \tag{2.8}$$

then $\mathcal{L}(u)$ also satisfies (2.8).

Proof. — Without loss of generality, we take $\bar{u} = 0$. If $u \in \mathcal{G}_T$, (2.3) and (2.4) show that

$$\begin{aligned} \| \mathcal{L}(u)(t) \|_\infty &\leq \| K(t) \|_1 \| u_0 \|_\infty + \int_0^t \| K_x(t - s) \|_1 \| f(u(s)) \|_\infty ds \\ &\leq s + Cr \int_0^t \frac{ds}{\sqrt{t - s}} \\ &\leq r\left(\frac{s}{r} + C\sqrt{T}\right) \\ &\leq r \end{aligned}$$

provided that $C\sqrt{T} \leq \dfrac{r - s}{r}$. This proves (*a*). To prove (*b*) we let $u, v \in \mathcal{G}_T$ and compute

$$\begin{aligned} \| \mathcal{L}(u)(t) - \mathcal{L}(v)(t) \|_\infty &\leq \int_0^t \| K_x(t - s) \|_1 \| u - v \|_\infty ds \\ &\leq \int_0^t \frac{C ds}{\sqrt{t - s}} \| u - v \|_\infty \\ &\leq C\sqrt{T} \| u - v \|_\infty . \end{aligned}$$

c) is proved much like (*a*): if $u \in \mathcal{G}_T$ satisfies (2.5), then from (2.3) and (2.4),

$$\begin{aligned} \| \mathcal{L}(u)(t) \|_2 &\leq \| u_0 \|_2 + \int_0^t \frac{CC_0 \| u_0 \|_2 ds}{\sqrt{t - s}} \\ &\leq \| u_0 \|_2 (1 + CC_0\sqrt{T}) \\ &\leq C_0 \| u_0 \|_2 \end{aligned}$$

if $C_0 > 1$ and T is small.

The prove (d) we differentiate in (2.4) and apply (2.3) to obtain

$$\begin{aligned}\| \mathscr{L}(u)_x(t) \|_p &\leq \| K_x(t) \|_1 \| u_0 \|_p + \int_0^t \| K_x(t-s) \|_1 \| f(u(s))_x \|_p ds \\ &\leq \frac{C \| u_0 \|_p}{\sqrt{t}} + \int_0^t \frac{CC_1 \| u_0 \|_p ds}{\sqrt{t-s}\sqrt{s}} \\ &\leq \frac{C \| u_0 \|_p}{\sqrt{t}} (1 + C_1\sqrt{T}) \\ &\leq \frac{C_1 \| u_0 \|_p}{\sqrt{t}}\end{aligned}$$

if C_1 is large and T is small.

To prove (e) we let $v = \mathscr{L}(u)$. Then the semigroup property of K implies that, for $t > t_0$

$$v(t) = K(t - t_0) * v(t_0) - \int_{t_0}^t K_x(t-s) * f(u(s)) ds \,.$$

Thus

$$(2.9) \quad \| v_{xx}(t) \|_2 \leq \| K_x(t-t_0) \|_1 \;\; \| v_x(t_0) \|_2 + \int_{t_0}^t \| K_x(t-s) \|_1 \;\; \| f(u(s))_{xx} \|_2 ds \,.$$

However, for $s \geq t_0$,

$$\begin{aligned}\| f(u(s))_{xx} \|_2 &\leq C(\| u_x(s) \|_\infty \;\; \| u_x(s) \|_2 + \| u_{xx}(s) \|_2) \\ &\leq CN\left(1 + \frac{C_2}{\sqrt{s-t_0}}\right)\end{aligned}$$

by our hypotheses (2.6) and (2.7). Here $N = \| u_0 \|_2 + \| u_0 \|_2^2$, and C may depend on t_0. Applying (d) to the term $\| v_x(t_0) \|_2$. we thus obtain from (2.9) that

$$\begin{aligned}\| v_{xx}(t) \|_2 &\leq \frac{CN}{\sqrt{t-t_0}} + \int_{t_0}^t CN\left(\frac{1}{\sqrt{t-s}} + \frac{C_2}{\sqrt{t-s}\sqrt{s-t_0}}\right) ds \\ &= \frac{CN}{\sqrt{t-t_0}} + CN\sqrt{t-t_0} + CC_2N \\ &\leq \frac{N}{\sqrt{t-t_0}} (C + CC_2\sqrt{T}) \\ &\leq \frac{C_2N}{\sqrt{t-t_0}}\end{aligned}$$

if C_2 is large and T is small.

The proof of (f) is nearly identical to that of (e) and so will be omitted. ∎

We can now obtain the local existence of solutions of (2.1)-(2.2).

THEOREM 2.2. — Assume that $u_0 - \bar{u} \in L^\infty \cap L^2$ and $\| u_0 - \bar{u} \|_\infty = s < r$. Then there is a unique solution u of (2.1)-(2.2) defined in a strip $[0, T] \times \mathbb{R}$. where T depends only on K, f, and s. Moreover, u_t, u_x, and u_{xx} are Hölder continuous in $t \geq t_0 > 0$; $u_t(t)$, $u_x(t)$, $u_{xx}(t)$, $u_{tx}(t)$, and $u_{xxx}(t)$ are in $L^2(\mathbb{R})$ for $t > 0$; and the following bounds hold:

$$\| u(t) - \bar{u} \|_2 \leq C_0 \| u_0 - \bar{u} \|_2, \tag{2.10}$$

and

$$\| u_x(t) \|_2 \leq \frac{C_1 \| u_0 - \bar{u} \|_2}{\sqrt{t}} \tag{2.11}$$

Here C_0 and C_1 are as defined in Lemma 2.1.

Proof. — Without loss of generality, we take $\bar{u} = 0$. Let $u^0 \equiv 0$ and $u^n = \mathscr{L}(u^{n-1})$. Then by induction the estimates (2.5)-(2.8) hold for each u^n. Thus by Lemma 2.1 (a)-(b), u^n converges to a function u in $\mathscr{L}^\infty([0, T] \times \mathbb{R})$, for some small time T. We shall apply Lemma 2.1 (c)-(f) to deduce the regularity properties of u in a strip $(t_2, T) \times \mathbb{R}$, where $0 < t_0 < t_1 < t_2 < T$, and t_0 and t_1 are as in Lemma 2.1. Throughout this proof, C will denote a positive constant which depends on K, f, t_0, and t_1.

First, Lemma 2.1 (e) shows that $\| u^n_{xx}(t) \|^2 \leq C$ for $t \geq t_2$, so that the functions $u^n_x(., t)$ are uniformly Lipschitz continuous in x. Next if $t_2 \leq t' \leq t'' \leq T$ then

$$\begin{aligned} u^n_x(x, t'') - u^n_x(x, t') &= \frac{1}{\varepsilon} \int_x^{x+\varepsilon} [u^n_x(y, t'') - u^n_x(y, t')] dy + O(\varepsilon) \\ &= \frac{1}{\varepsilon} \int_{t'}^{t''} \int_x^{x+\varepsilon} u^n_{xt}(y, t) dy dt + O(\varepsilon) \\ &\leq \frac{1}{\varepsilon} (t'' - t') \sqrt{\varepsilon} \sup_{t_2 \leq t \leq T} \| u^n_{xt}(t) \|_2 + O(\varepsilon) \\ &\leq C(t'' - t')^{2/3} \end{aligned}$$

if $\varepsilon = O(t'' - t')^{2/3}$. Here we have used the equation

$$u^n_t - D u^n_{xx} = - f(u^{n-1})_x \tag{2.12}$$

together with Lemma 2.1 (e)-(f) to bound $u^n_{xt} = D u^n_{xxx} - f(u^{n-1})_{xx}$ *in* L^2. We have thus shown that the functions $\{ u^n_x \}$ are uniformly Hölder continuous in $[t_2, T] \times \mathbb{R}$. It then follows from standard results (see [*11*], p. 320) applied to (2.12) that the functions u^n and u^n_{xx} are uniformly Hölder continuous in $[t_2, T] \times \mathbb{R}$. Since $t_2 > 0$ is arbitrary, we then have by the Ascoli-

Arzela theorem that u_t^n and u_{xx}^n converge uniformly on compact sets in $(0, T] \times \mathbb{R}$ to u_t and u_{xx}, which are therefore also Hölder continuous. The bounds (2.10) and (2.11) then follow directly from Lemma 2.1. Finally, since the $u_{xxx}^n(t)$ are uniformly bounded in $L^2(\mathbb{R})$ for fixed $t > 0$, they converge weakly in $L^2(\mathbb{R})$: $u_{xxx}^n(t) \to v(t) \in L^2(\mathbb{R})$. But $v(t)$ must coincide with the distribution derivative $u_{xxx}(t)$, which is therefore in $L^2(\mathbb{R})$. It then follows from the equation $u_t = DU_{xx} - f(u)_x$ that $u_{xt}(t) \in L^2(\mathbb{R})$. ∎

In order to extend these solutions globally, that is, to all of $t > 0$, we shall make use of so-called entropy-entropy flux pairs; these are defined as follows.

DÉFINITION 2.3. — The functions α, β: $\overline{B_r(\bar{u})} \to \mathbb{R}$ are said to be an entropy-entropy flux pair for f in $\overline{B_r(\bar{u})}$ if the relation

$$\nabla\alpha(u)^t f'(u) = \nabla\beta(u)^t \tag{2.13}$$

holds in $\overline{B_r(\bar{u})}$. The entropy α will always be assumed to satisfy

$$\delta \,|\, u - \bar{u} \,|^2 \leq \alpha(u) \leq \frac{1}{\delta} |\, u - \bar{u} \,|^2, \qquad u \in \overline{B_r(\bar{u})}, \tag{2.14}$$

for some positive constant δ. Finally, α is said to be consistent with the diagonal matrix D if there is an $\varepsilon > 0$ such that

$$w^t D\alpha''(u) w \geq \varepsilon \,|\, w \,|^2 \tag{2.15}$$

for all $u \in \overline{B_r(\bar{u})}$ and $w \in \mathbb{R}^n$.

The existence of such a pair (α, β) will enable us to derive certain *a priori bounds* for solutions of (2.1). These bounds will be crucial for extending our local solutions to global ones.

LEMMA 2.4. — Assume that there is an entropy-entropy flux pair as described in Def. 2.3, satisfying (2.13), (2.14), and (2.15). Then there are positive constants C_4 and C_5 such that, whenever u is a smooth solution of (2.1) in $(t_0, t_1) \times \mathbb{R}$ (in the sense that $u(t) - \bar{u}$, $u_t(t)$, and $u_{xx}(t)$ are continuous and in $L^2(\mathbb{R})$ for $t > 0$, and $u(t) - u(t_0) \to 0$ in $L^2(\mathbb{R})$ as $t \downarrow t_0$), then

$$\| u(t) - \bar{u} \|_2 \leq C_4 \| u(t_0) - \bar{u} \|_2, \qquad t_0 \leq t \leq t_1. \tag{2.16}$$

If in addition, $u_{xxx}(t)$ and $u_{xt}(t)$ are in $L^2(\mathbb{R})$ for $t > t_0$, then

$$\| u_x(t) \|_2 \leq C_5(\| u_x(t_0) \|_2 + \| u(t_0) \|_2), \qquad t_0 \leq t \leq t_1, \tag{2.17}$$

provided that $u_x(t_0) \in L^2(\mathbb{R})$.

Proof. — Since (2.1) holds for $t \in (t_0, t_1)$, we may multiply 2.1 on the left by $\nabla\alpha^t$ and use (2.13) and (2.15) to obtain

$$\begin{aligned} \alpha(u)_t + \beta(u)_x &= \nabla\alpha(u)^t \mathrm{D} u_{xx} \\ &= (\nabla\alpha(u)^t \mathrm{D} u_x)_x - (\alpha''(u)u_x)^t \mathrm{D} u_x \\ &\leq (\nabla\alpha(u)^t \mathrm{D} u_x)_x - \varepsilon \mid u_x \mid^2 . \end{aligned}$$

Integrating over $[t_0, t] \times \mathbb{R}$, we find

$$\int \alpha(u(x, .)) \big|_{t_0}^{t} dx \leq - \varepsilon \int_{t_0}^{t} \int \mid u_x(x, t) \mid^2 dxdt ,$$

so that

$$\int \alpha(u(x, t))dx + \varepsilon \int_{t_0}^{t} \int \mid u_x \mid^2 dxdt \leq \int \alpha(u(x, t_0))dx . \tag{2.18}$$

This together with (2.14) gives (2.16).

To prove (2.17), we differentiate (2.1), multiply by ${}^t u_x$, and integrate over $[t_0, t] \times \mathbb{R}$. The result is

$$\begin{aligned} \frac{1}{2} \int \mid u_x(x, .) \mid^2 \big|_{t_0}^{t} dx &= \int_{t_0}^{t} \int \left[f(u)_x{}^t u_{xx} - (\mathrm{D} u_{xx})^t u_{xx} \right] dxdt \\ &\leq \iint (\mathrm{C} \mid u_x \mid \, \mid u_{xx} \mid - \underline{d} \mid u_{xx} \mid^2) dxdt, \end{aligned}$$

where $\underline{d} = \min d_i$. However

$$\mathrm{C} \mid u_x \mid \, \mid u_{xx} \mid \leq \mathrm{C}\left(\frac{\mathrm{C}}{4\underline{d}} \mid u_x \mid^2 + \frac{d}{\mathrm{C}} \mid u_{xx} \mid^2 \right).$$

so that

$$\| u_x(t) \|_2^2 \leq \| u_x(t_0) \|_2^2 + \mathrm{const} \int_{t_0}^{t} \int \mid u_x \mid^2 dxdt .$$

This together with (2.19) yields (2.17). ■

We can now state our global existence result.

THEOREM 2.5. — Assume that there is an entropy-entropy flux pair as described in Def. 2.3, satisfying (2.13), (2.14), and (2.15). Let $u_0 - \bar{u} \in \mathrm{L}^\infty \cap \mathrm{L}^2$ with $\| u_0 - \bar{u} \|_\infty = s < r$, and let $\mathrm{C}_0 - \mathrm{C}_5$ and T be as defined in Lemmas 2.1 and 2.4. Then the problem (2.1)-(2.2) has a global solution provided that

$$\left[2\mathrm{C}_4 \overline{\mathrm{C}}_5 \left(\frac{\mathrm{C}_1}{\sqrt{\mathrm{T}}} + \mathrm{C}_4 \right) \right]^{1/2} \| u_0 - \bar{u} \|_2 \leq s .$$

Here $\overline{\mathrm{C}}_5 = \max(\mathrm{C}_5, 1)$.

Proof. — Again we may take $\bar{u} = 0$. Let

$$a = C_4 \| u_0 \|_2 \quad \text{and} \quad b = \bar{C}_5\left(\frac{C_1}{\sqrt{T}} + C_4\right) \| u_0 \|_2 .$$

Our hypothesis is then that

$$\sqrt{2ab} \leq s .$$

By Theorem 2.2 there is a solution u defined up to time T, and from Lemma 2.1 (*a*) and (*d*), together with (2.16), we see that u satisfies

$$\| u(t) \|_\infty \leq r, \qquad 0 \leq t \leq T,$$
$$\| u(T) \|_2 \leq C_4 \| u_0 \|_2 = a ,$$

and

$$\| u_x(T) \|_2 \leq \frac{C_1 \| u_0 \|_2}{T^{1/2}} \leq b . \tag{2.19}$$

Now suppose that u has been defined up to time kT for some $k \in \mathbb{Z}_+$, and that

$$\| u(t) \|_\infty \leq r, \qquad 0 \leq t \leq kT , \tag{2.20}$$
$$\| u(kT) \|_2 \leq a , \tag{2.21}$$
$$\| u_x(kT) \|_2 \leq b . \tag{2.22}$$

Then

$$\| u(kT) \|_\infty \leq (2 \| u(kT) \|_2 \; \| u_x(kT) \|_2)^{1/2}$$
$$\leq \sqrt{2ab} \leq s ,$$

so that, by Theorem 2.1, u can be extended up to time $(k + 1)$T with $\| u(t) \|_\infty \leq r$ and $u(t) \in L^2(\mathbb{R})$ for $t \leq (k + 1)T$. But then Lemma 2.4 applies to show that

$$\| u((k + 1)T) \|_2 \leq C_4 \| u_0 \|_2 = a$$

and

$$\| u_x((k + 1)T) \|_2 \leq C_5(\| u_x(T) \|_2 + \| u(T) \|_2) .$$
$$\leq \bar{C}_5\left(\frac{C_1}{\sqrt{T}} + C_4\right) \| u_0 \|_2 = b$$

by (2.19). Thus (2.20), (2.21), and (2.22) hold up to time $(k + 1)$T. Proceeding inductively, we thus establish the existence of the solution u in all of $t \geq 0$. ■

Finally, we can dispense with the requirement that D be a diagonal matrix by making a simple change of variable.

COROLLARY 2.6. — Assume that there is an entropy-entropy flux pair (α, β) for f in $\overline{B_r}(\bar{u})$ satisfying (2.13) and (2.14). Let D be a diagonalizable matrix with positive eigenvalues, say

$$P^{-1}DP = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} > 0 .$$

Suppose that for each $u \in \overline{B_r}(\bar{u})$,

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} P^t\alpha''(u)P > 0 . \tag{2.23}$$

Then the problem (1.1)-(1.2) has a global solution provided that the data $u_0 - \bar{u}$ has suitably restricted L^2 and L^∞ norms; i. e., $P^{-1}(u_0 - \bar{u})$ satisfies the hypotheses of the last theorem.

Remark. — When D is symmetric, we may take P to be an orthogonal matrix, and the condition (2.23) simplifies to the requirement that $D\alpha''$ be positive throughout $\overline{B_r}(\bar{u})$.

Proof of Corollary 2.6. — Let $v = P^{-1}u$. Then v satisfies

$$v_t + g(v)_x = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} v_{xx}, \tag{2.24}$$

where $g(v) = P^{-1}f(Pv)$. An easy computation shows that $A(v) = \alpha(Pv)$ and $B(v) = \beta(Pv)$ satisfy (2.13) and (2.14) for g, and

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} A'' = P^{-1}DPP^t\alpha''P$$

is positive by assumption. The corollary now follows from Theorem 2.4. ∎

§ 3. APPLICATIONS TO THE EQUATIONS OF GAS DYNAMICS

In this section we apply the global existence result, Corollary 2.6, to the equations of one-dimensional gas dynamics. We refer the reader to [3] for a complete description and derivation of these equations.

First consider the equations of isentropic gas dynamics in Lagrangean coordinates:

$$\begin{bmatrix} v \\ u \end{bmatrix}_t + \begin{bmatrix} -u \\ p(v) \end{bmatrix}_x = D\begin{bmatrix} v \\ u \end{bmatrix}_{xx} . \tag{3.1}$$

Here v, u, and p are scalars which represent, respectively, the specific volume (= 1/density), velocity, and pressure. We assume that p is defined in $\{ v > 0 \}$ and that $p'(v) < 0$. Now, in the simplest case that the diffusion matrix D is a multiple of the identity, there is an invariant region in $v - u$ space (see [2] or [4]), which serves to control the nonlinear function $p(v)$. Global existence of solutions then follows from the results of Nishida-Smoller [14] and Hoff-Smoller [5]. For more general D we shall construct an entropy-entropy flux pair and appeal to Corollary 2.6. To this end, define

$$\alpha(v, \bar{u}) = \frac{(u - \bar{u})^2}{2} + \int_{\bar{v}}^{v} [p(\bar{v}) - p(s)]ds$$

and

$$\beta(v, u) = (u - \bar{u})[p(v) - p(\bar{v})],$$

where $\bar{v} > 0$. We now check that the requirements of Def. 2.3 are satisfied. First

$$\begin{aligned} \nabla\alpha^t f' &= [p(\bar{v}) - p(v), u - \bar{u}]\begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \\ &= [p'(v)(u - \bar{u}), p(v) - p(\bar{v})] \\ &= \nabla\beta^t. \end{aligned}$$

Next, $\dfrac{\partial\alpha}{\partial v}(\bar{v}, u) = 0$ and $\dfrac{\partial^2\alpha}{\partial v^2}(v, u) = -p'(v)$, so that

$$\alpha(v, u) = \frac{(u - \bar{u})^2}{2} - p'(\xi)(v - \bar{v})^2,$$

and (2.14) is thus satisfied in compact sets in $\{ v > 0 \}$. Finally, we work out the compatibility condition (2.15) for the special case that $D = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is symmetric. By the remark after Corollary 2.6, the requirement is that the matrix

$$D\alpha'' = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -p' & 0 \\ 0 & 1 \end{bmatrix}$$

be positive throughout $\overline{B_r}(\bar{v}, \bar{u})$. One easily checks that this condition is satisfied if and only if

$$(3.2) \qquad \begin{cases} a, c > 0, \quad \text{and} \\ \dfrac{b^2}{4ac} < \min\limits_{|v-\bar{v}| \le r} \dfrac{-p'(v)}{[1 - p'(v)]^2}. \end{cases}$$

The application of Corollary 2.6 to this problem may then be formulated as follows: *Assume that $\bar{v} > \bar{v} - r > 0$ and that the symmetric matrix* D *satisfies (3.2). Then the system (3.1) with initial data (v_0, u_0) has a global smooth solution provided that $(v_0(x), u_0(x)) \in \overline{B}_s(\bar{v}, \bar{u})$ for some $s < r$, and* *that the* L^2*-norm of* $(v_0 - \bar{v}, u_0 - \bar{u})$ *is suitably restricted (in the sense of Corollary 2.6).*

Observe that (3.2) is satisfied when D is any positive diagonal matrix. In addition, since (2.15) is an open condition in D, the above global existence result remains valid when the matrix D is sufficiently close to a symmetric matrix satisfying (3.2). Finally, we note that in the important case $p(v) = v^{-\gamma} (\gamma > 1)$, condition (3.2) forces D to be nearly diagonal when $\overline{B}_r(\bar{v}, \bar{u})$ includes states of either very high or very low density.

Next we turn to the equations of nonisentropic gas dynamics. It is well-known that these equations can be formulated in a number of ways, all of which are essentially equivalent for the application of Corollary 2.6. We shall therefore construct the entropy-entropy flux pair for the formulation in which the computations are simplest, and then prove a general result which shows how the entropy-entropy flux pair carries over for the equivalent systems.

Thus consider first the entropy formulation of these equations in Lagrangean coordinates, and assume for simplicity that D is a diagonal matrix:

$$\begin{bmatrix} v \\ u \\ S \end{bmatrix}_t + \begin{bmatrix} -u \\ p(v, S) \\ 0 \end{bmatrix}_x = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ S \end{bmatrix}_{xx}. \tag{3.3}$$

Here v, u, and p are the same as in (3.1), and S is the specific entropy. We assume that p is defined for $v > 0$ and all S, and that $p_v < 0$. Now define

$$\alpha(v, u, S) = \frac{(u - \bar{u})^2}{2} + \int_{\bar{v}}^{v} [p(\bar{v}, \bar{S}) - p(\tau, S)]d\tau + \frac{K(S - \bar{S})^2}{2}$$

and

$$\beta(v, u, S) = (u - \bar{u})[p(v, S) - p(\bar{v}, \bar{S})].$$

We shall show that, when the constant K is sufficiently large, the requirements (2.14), (2.14), and (2.15) of Def. 2.3 are satisfied in a ball $\overline{B}_r(\bar{v}, \bar{u}, \bar{S})$ (where, as before, $\bar{v} > \bar{v} - r > 0$). First, if $\bar{p} = p(\bar{v}, \bar{S})$,

$$\begin{aligned} \nabla\alpha^t f' &= \left[\bar{p} - p, u - \bar{u}, \int_{\bar{v}}^{v} - p_S(\tau, S)ds + K(S - \bar{S}) \right] \begin{bmatrix} 0 & -1 & 0 \\ p_v & 0 & p_S \\ 0 & 0 & 0 \end{bmatrix} \\ &= [p_v(u - \bar{u}), p - \bar{p}, p_S(u - \bar{u})] \\ &= \nabla\beta^t, \end{aligned}$$

so that (2.13) is satisfied. To establish (2.14), we first expand the term

$$g(v, S) = \int_{\bar{v}}^{v} [\bar{p} - p(\tau, S)]d\tau$$

about $(\overline{v}, \overline{S})$. Since g, g_v, and g_S vanish at $(\overline{v}, \overline{S})$, we have that

$$g(v, S) = \frac{1}{2}\begin{bmatrix} v - \overline{v} \\ S - \overline{S} \end{bmatrix}^t g''(\tilde{v}, \tilde{S})\begin{bmatrix} v - \overline{v} \\ S - \overline{S} \end{bmatrix}$$

$$\geq \frac{-p'(\tilde{v})}{2}(v - \overline{v})^2 - \text{const. } [(v - \overline{v})(S - \overline{S}) + (S - \overline{S})^2]$$

$$\geq \frac{-p(\tilde{v})}{4}(v - \overline{v})^2 - \text{const. } (S - \overline{S})^2 .$$

Thus for K sufficiently large,

$$\alpha(v, u, S) \geq \frac{(u - \overline{u})^2}{2} - \frac{p'(\tilde{v})}{4}(v - \overline{v})^2 + \left(\frac{K}{2} - \text{const.}\right)(S - \overline{S})^2$$

$$\geq \text{const.} \left\| \begin{bmatrix} v \\ u \\ S \end{bmatrix} - \begin{bmatrix} \overline{v} \\ \overline{u} \\ \overline{S} \end{bmatrix} \right\|^2$$

as required. (The other inequality in (2.14) is trivial.) Last, we check that D is compatible with α. We have that

$$D\alpha'' = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} -p_v & 0 & -p_S \\ 0 & 1 & 0 \\ -p_S & 0 & K - \int_{\overline{v}}^{v} p_{SS}(\tau, S)d\tau \end{bmatrix},$$

which is positive definite if $\begin{bmatrix} d_1 & 0 \\ 0 & d_3 \end{bmatrix}$ A is positive definite, where

$$A = \begin{bmatrix} -p_v & -p_S \\ -p_S & K - \int_{\overline{v}}^{v} p_{SS}d\tau \end{bmatrix}.$$

However, since $-p_v > 0$ and p_S and p_{SS} are bounded in $\overline{B}_r(\overline{v}, \overline{u}, \overline{S})$, it is easy to see that $\begin{bmatrix} d_1 & 0 \\ 0 & d_3 \end{bmatrix}$ A is positive definite when K is sufficiently large. Therefore every positive diagonal matrix is compatible with α. We have now checked that the requirements of Def. 2.3 are satisfied, so that Corollary 2.6 applies. We state our conclusion formally as follows: *Assume that $\overline{v} > \overline{v} - r > 0$ and that* D *is a positive diagonal matrix. Then the system (3.3) with initial data (v_0, u_0, S_0) has a global smooth solution provided that $(v_0(x), u_0(x), S_0(x)) \in \overline{B}_s(\overline{v}, \overline{u}, \overline{S})$ a. e. for some $s < r$, and that the L^2-norm of $(v_0 - \overline{v}, u_0 - \overline{u}, S_0 - \overline{S})$ is suitably restricted (in the sense of Corollary 2.6).*

Again, since (2.15) is an open condition in D, the same global existence result holds when D is sufficiently close to a positive diagonal matrix. (We emphasize, however, that we took D to be diagonal, or nearly diagonal, only as a matter of convenience. Certainly the compatibility condition (2.15) is satisfied by a much broader class of diffusion matrices.)

Next, we consider the alternative formulations of the laws of conservation of mass, momentum, and energy. These systems, without diffusion, are the following:

$$(3.4) \qquad \begin{cases} v_t - u_x = 0 \\ u_t + p_x = 0 \\ S_t \qquad = 0, \qquad p = p(v, S). \end{cases}$$

$$(3.5) \qquad \begin{cases} v_t - u_x \quad = 0 \\ u_t + p_x \quad = 0 \\ E_t + (up)_x = 0 \end{cases}$$

$$(3.6) \qquad \begin{cases} \rho_\tau + (\rho u)_\xi \qquad = 0 \\ (pu)_\tau + (\rho u^2 + p)_\xi = 0 \\ (\rho S)_\tau + (\rho u S)_\xi \quad = 0, \qquad p = p(\rho, S). \end{cases}$$

$$(3.7) \qquad \begin{cases} \rho_\tau \quad + (\rho u)_\xi \qquad = 0 \\ (\rho u)_\tau + (\rho u^2 + p)_\xi \;= 0, \\ (\rho E)_\tau + (\rho u E + up)_\xi = 0, \qquad p = p(\rho, u, E). \end{cases}$$

In these systems, v, u, p, and S are the same as in (3.3), $\rho = 1/v$ is the density, and $E = e + u^2/2$ is the total energy, where e is the internal energy. These systems become closed when taken together with a fundamental relation, which gives e in terms of v and S, or S in terms of v and e. The pressure p is then defined by

$$p = -\frac{\partial e}{\partial v}(v, S).$$

We can then formally derive the third equation (3.5) from (3.4) and the definition of E as follows:

$$\begin{aligned} E &= e_t + uu_t = e_v v_t + e_S S_t + uu_t \\ &= -pu_x - up_x = -(up)_x. \end{aligned}$$

Systems (3.6) and (3.7) are derived formally from (3.4) and (3.5) by making the change of dependent variable $\rho = 1/v$ and a particular change of independent variables $(x, t) \to (\xi, t)$. Here the Eulerian coordinates ξ and τ denote real space and time, and are related to the Lagrangean coordinates x and t by

$$t = \tau \qquad \text{and} \qquad x = \int_{-\infty}^{\xi(x,t)} \rho(s, \tau)ds.$$

Thus

$$\frac{\partial(\xi, \tau)}{\partial(x, t)} = \begin{bmatrix} 1/\rho & u \\ 0 & 1 \end{bmatrix}. \tag{3.8}$$

For example, the equation $v_t - u_x = 0$ may be rewritten in Eulerian coordinates as

$$\begin{aligned} 0 &= uv_\xi + v_\tau - \frac{1}{\rho} u_\xi \\ &= -\frac{\rho_\tau}{\rho^2} - \frac{u\rho_\xi}{\rho^2} - \frac{u_\xi}{\rho}. \end{aligned}$$

Multiplying by $-\rho^2$, we thus obtain the first equation in either (3.6) or (3.7). The other equations in (3.6) and (3.7) are derived in a similar manner.

Now the purpose of the above discussion is not to suggest that solutions of (3.4) carry over to solutions of (3.5)-(3.7), (indeed, this is known to be false; see, e. g. [*15*]), but rather to indicate the relation among the various fluxes appearing in these systems, since it is the flux which determines the entropy, entropy-flux pair (α, β) required for global existence of solutions of the corresponding viscous systems.

It is apparent that the above transformations fit into the following abstract framework: Suppose that u satisfies the system of conservation laws

$$u_t + f(u)_x = 0 \tag{3.9}$$

(u, v, f, etc., will again denote vectors). We then formally make both a change of dependent variable $u = h(v)$ and a change of independent variables $(x, t) \to (\xi, \tau)$, where

$$\frac{\partial(\xi, \tau)}{\partial(x, t)} = \begin{bmatrix} q_1(u) & q_2(u) \\ 0 & 1 \end{bmatrix}.$$

Then (3.9) transforms to

$$u_\tau + (q_2 + q_1 f')u_\xi = 0.$$

Multiplying on the left by $(h^{-1})'$, we find that

$$v_\tau + [q_2 + q_1(h^{-1})'f'h']v_\xi = 0.$$

However, in all cases of interest, (3.5)-(3.7), we found that v formally satisfied the system of conservation laws

$$v_\tau + g(v)_\xi = 0. \tag{3.10}$$

It must therefore be that

$$g' = q_2 + q_1(h^{-1})'f'h'. \tag{3.11}$$

The following proposition indicates how an entropy-entropy flux pair for a system (3.9) transforms under these changes of variables.

PROPOSITION 3.1. — Suppose that (α, β) is an entropy-entropy flux pair for f in $\overline{B_r}(\overline{u})$, satisfying both (2.13) and (2.14) of Def. 2.3, and that $\alpha'' > 0$ in $\overline{B_r}(\overline{u})$. Let q_1, q_2, h, and g be as above, and assume that

$$q_1 > 0 \tag{3.12}$$

and

$$\nabla q_1{}^t f' + \nabla q_2 = 0 \tag{3.13}$$

in $\overline{B_r}(\overline{u})$. Then provided that r is sufficiently small, the functions

$$A(v) \equiv \alpha(h(v))/q_1(h(v)) \qquad \text{and} \qquad B(v) \equiv \left(\beta + \frac{q_2\alpha}{q_1}\right)(h(v))$$

satisfy

$$\nabla_v A^t g' = \nabla_v B^t ; \tag{3.14}$$

$$\delta \,|\, v - h^{-1}(\overline{u}) \,|^2 \leq A(v) \leq \frac{1}{\delta} \,|\, v - h^{-1}(\overline{u}) \,|^2, \qquad \delta > 0 ; \tag{3.15}$$

and

$$A'' > 0 \tag{3.16}$$

in

$$h^{-1}(\overline{B}_r(\overline{u})) .$$

Proof. — We have from (2.13), (3.11), (3.13), and the definition of A that

$$\begin{aligned} \nabla_v A^t g' &= \left(\frac{\nabla\alpha}{q_1} - \frac{\alpha\nabla q_1}{q_1^2}\right)^t h' \,[q_2 + q_1 (h')^{-1} f' h'] \\ &= \left(\frac{q_2 \nabla\alpha^t}{q_1} - \frac{\alpha q_2}{q_1^2} \nabla q_1^t - \frac{\alpha \nabla q_2^t}{q_1} + \nabla\alpha^t f'\right) h' \\ &= \nabla\left(\frac{q_2\alpha}{q_1} + \beta\right)^t h' \\ &= \nabla_v B^t , \end{aligned}$$

as required. (3.15) is obvious, and to prove (3.16) we let $\gamma(u) = \alpha(u)/q_1(u)$, so that $A(v) = \gamma(h(v))$. Then

$$\frac{\partial^2 A}{\partial v_i \partial v_j} = \sum_{k,l} \frac{\partial^2 \gamma}{\partial u_k \partial u_l} \frac{\partial h^k}{\partial v_i} \frac{\partial h^l}{\partial v_j} + \sum_k \frac{\partial \gamma}{\partial u_k} \frac{\partial^2 h^k}{\partial v_i \partial v_j} . \tag{3.17}$$

However, the condition (2.14) shows that $\nabla\alpha(\overline{u}) = 0$, so that $\nabla\alpha(u) = 0(r)$ in $\overline{B}(\overline{u})$. Thus

$$\frac{\partial \gamma}{\partial u_k} = \frac{1}{q_1} \frac{\partial \alpha}{\partial u_k} - \frac{\alpha}{q_1^2} \frac{\partial q_1}{\partial u_k} = 0(r) ,$$

and similarly

$$\frac{\partial^2 \gamma}{\partial u_k \partial u_l} = \frac{1}{q_1} \frac{\partial^2 \alpha}{\partial u_k \partial u_l} + 0(r) .$$

We therefore obtain from (3.17) that for $w \in \mathbb{R}^n$,

$$\begin{aligned} w^t A'' w &= (h'w)^t \frac{\alpha''}{q_1}(h'w) + 0(r) \mid w \mid^2 \\ &\geq C_1 \mid h'w \mid^2 - C_2 r \mid w \mid^2 \\ &\geq C \mid w \mid^2 , \end{aligned}$$

for some $C > 0$, when r is sufficiently small. ■

Recall that we have already constructed an entropy-entropy flux pair (α, β) for the system (3.4), and that $\alpha'' > 0$. Now, the transformation of (3.4) to (3.5) involves only a change of the dependent variables. Therefore $q_1 = 1$ and $q_2 = 0$ in this case, so that the hypotheses (3.12) and (3.13) of Proposition 3.1 are satisfied. For the transformation to the Eulerian-coordinates systems (3.6) and (3.7), the functions q_1 and q_2 are given by (3.8); namely, $q_1 = 1/\rho = v$ and $q_2 = u$. Thus $q_1 > 0$ as long as the density is positive, and from (3.5),

$$\begin{aligned} \nabla q_1{}^t f' + \nabla q_2 &= [1, 0, 0] \begin{bmatrix} 0 & -1 & 0 \\ & * & \end{bmatrix} + [0, 1, 0] \\ &= 0 , \end{aligned}$$

so that (3.13) is satisfied. Thus Proposition 3.1 applies to show that there exists an entropy-entropy flux pair (A, B) for each of the systems (3.5)-(3.7), satisfying (3.13), (3.14), and (3.15). Corollary 2.6 then implies the following existence result for these systems:

Assume that $\overline{B}_r(\overline{u})$ is a sufficiently small closed ball in phase space in which the density is positive, and assume that the (vector) initial data $u_0(x) - \overline{u}$ is in $B_s(\overline{u})$ a. e., for some $s < r$, and that $\| u_0 - \overline{u} \|_2$ is suitably restricted (again as in Corollary 2.6). Then each of the systems (3.5)-(3.7) modified by the addition of terms DU_{xx} as in (1.1), where D is sufficiently close to a positive multiple of the identity matrix, has a global smooth solution.

§ 4. NON-SMOOTHNESS IN GAS DYNAMICS

In this last section, we consider the gas dynamics equations with the usual dissipative mechanisms (viscosity and thermal conductivity) taken into account. We shall show that if the solution is of bounded variation, and if the specific volume (= reciprocal of the density) is initially discontinuous, then it remains discontinuous for positive time. In particular, our result applies to « Riemann problem » data; i. e., data consisting of two constant states separated by a jump discontinuity. This result is somewhat surprising since it contradicts statements found in certain well-known texts; e. g. see [3, p. 135].

We can write the gas dynamics equations in Lagrangean coordinates in a single space variable as follows:

$$(4.1)\qquad \begin{aligned} &v_t - u_x = 0 \\ &u_t + p_x = (k(v)u_x)_x \\ &E_t + (up)_x = \left[\left\{ \varepsilon \frac{u^2}{2} + \frac{\lambda}{c_v}(E - u^2/2) \right\}_x /v \right]_x . \end{aligned}$$

Here v, u, $p(v, e)$ and E are the same variables as in (3.5). We assume that p and k are C^1 functions of their arguments in the region $v > 0$. The dissipative mechanisms ε and λ are called the viscosity and thermal conductivity coefficients, respectively, and c_v is the specific heat at constant volume. We assume that $\varepsilon > 0$ and $\lambda \geq 0$. The function $k(v)$ is assumed to be positive in $v > 0$; in gas dynamics, one usually takes $k(v) = v^{-1}$.

We consider the initial value problem for (4.1), where the initial data is given by

$$(4.2)\qquad (v, u, E)(x, 0) = (v_0, u_0, E_0)(x), \qquad x \in \mathbb{R}.$$

It is required to solve (4.1) in the upper half-plane, $x \in \mathbb{R}$, $t > 0$, subject to the initial data (4.2). Since we wish to consider discontinuous data, it is necessary to define precisely our notion of solution.

DÉFINITION 4.1. — By a (weak) solution of (4.1), (4.2), we mean a triple of functions $(v, u, E)(x, t)$, defined in $t > 0$, such that the following hold:

a) v, u, and E are in $L^1_{loc}(\mathbb{R})$ for each $t > 0$. v is such that $p(v, e)$ is in $L^1_{loc}(t \geq 0)$ and $k(v)$ is in $L^\infty_{loc}(t \geq 0)$ (*Note*: in gas dynamics $k(v) = v^{-1}$, $p(v, e) = (\gamma - 1)ev^{-1}$, and it suffices to assume that v is locally bounded away from zero).

b) The distribution derivatives, v_t, u_x, and $\left[\varepsilon \frac{u^2}{2} + \frac{\lambda}{c_v}\left(E - \frac{u^2}{2}\right)\right]_x$ are in $L^1_{loc}(t > 0)$, and $v_t = u_x$ a. e. in $t > 0$.

c) u, v, and E are in $L^1_{loc}(t > 0)$, and for every $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $v(x, t) > 0$.

d) For every $\phi \in C^1$, where ϕ has its support contained in a set of the form $\{ x_1 \leq x \leq x_2 \} \times \{ 0 \leq t_1 \leq t \leq t_2 \}$, we have

$$(4.2)\qquad \int_{-\infty}^{\infty} u\phi \,|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_{-\infty}^{\infty} u\phi_t + p\phi_x = - \int_{t_1}^{t_2} \int_{-\infty}^{\infty} k(v)u_x\phi_x$$

$$(4.3)\qquad \int_{-\infty}^{\infty} E\phi \,|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_{-\infty}^{\infty} E\phi_t + up\phi_x = - \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \frac{\left\{ \varepsilon u^2/2 + \frac{\lambda}{c_v}\left(E - \frac{u^2}{2}\right)\right\}_x}{v} \phi_x$$

e) $[v(.,t), u(.,t), E(.,t)] \to [v_0(.), u_0(.), E_0(.)]$ in L^1_{loc}, as $t \searrow 0$.

We remark that the assumptions *b)* and *c)* seem to be the minimal assumptions one can make in order that the integrals in (4.2) and (4.3) are defined. Since $u_x \in L^1_{loc}(t > 0)$ is needed in order for (4.2) to be defined we see that the first equation in (4.1) demands that $v_t \in L^1_{loc}(t > 0)$, and that $v_t = u_x$ a. e. in $t > 0$.

THEOREM 4.2. — Let (v, u, E) be a solution of (4.1) in the sense of definition (4.1). Suppose also that $v(.,t)$ is locally bounded away from 0 and that, for every interval $[a, b]$, the variation on $[a, b]$ of each of the functions $u(.,t)$, $v(.,t)$, and $E(.,t)$ is bounded independently of t for $t \in [0, T]$. Then if $v(.,t)$ is continuous for $0 < t \leq T$, the initial function $v(.,0)$ must also be continuous.

Proof. — Fix an interval $[a, b]$ and let V be a bound for the variation of each of $u(.,t)$, $v(.,t)$, and $E(.,t)$ on $[a,b]$ for $0 \leq t \leq T$. Let t_1 and t_2 be times in $(0, T]$ and let $\phi(x)$ be a test function with support on $[a, b]$. We then have from (4.2) that

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} k(v) u_x \phi_x = - \int_{-\infty}^{\infty} u\phi \,|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_{-\infty}^{\infty} p\phi_x$$
$$\leq \| \phi \|_\infty \, \| u(t_1) - u(t_2) \|_{L^1[a,b]} + C \mid t_1 - t_2 \mid \; \| \phi \|_\infty V,$$

for some constant C. Hence

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} k u_x \phi_x = \| \phi \|_\infty \omega(\mid t_2 - t_1 \mid), \tag{4.4}$$

where $\omega(\delta) \to 0$ as $\delta \to 0$. But from def. 4.1, $k(v)v_+ = k(v)u_x$ a. e. Thus (4.4) gives

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} k v_t \phi_x = \| \phi \|_\infty \omega(\mid t_2 - t_1 \mid). \tag{4.5}$$

Now let K be a primitive of k, i. e. $K' = k$. We claim that

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} k v_t \phi_x = \int_{-\infty}^{\infty} K(v(x,t)) \,|_{t_1}^{t_2} \phi_x dx. \tag{4.6}$$

To see this, let j_n be the usual mollifying kernel, and set $v_n = j_n * v$, $K_n = K(v_n)$. Then

$$\int_{-\infty}^{\infty} K \,|_{t_1}^{t_2} \phi_x = \lim_{n\to\infty} \int_{-\infty}^{\infty} K_n \,|_{t_1}^{t_2} \phi_x = \lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t} K_n \right) \phi_x$$
$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v_n) \frac{\partial v_n}{\partial t} \phi_x$$
$$= \lim_{n\to\infty} \left\{ \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v_n) \left[\frac{\partial v}{\partial t} \phi_x \right] + \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v_n) \left[\frac{\partial (v_n - v)}{\partial t} \phi_x \right] \right\}.$$

Now $\partial(v_n - v)/\partial_t \to 0$ in $L^1_{loc}(t > 0)$ (since $\partial v/\partial t$ is in $L^1_{loc}(t > 0)$), and since $k(v_n)$ is bounded, we see that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v_n)\left[\frac{\partial(v_n - v)}{\partial t} \phi_x\right] = 0 .$$

Also,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v_n)\left[\frac{\partial v}{\partial t} \phi_x\right] = \int_{-\infty}^{\infty} \int_{t_1}^{t_2} k(v) \frac{\partial v}{\partial t} \phi_x$$

since k' is bounded and $v_n \to v$ in $L^1_{loc}([t_1, t_2] \times \mathbb{R})$. It follows from this that (4.6) holds.

Using (4.6) in (4.5), we get

$$\int_{-\infty}^{\infty} K(v) \Big|_{t_1}^{t_2} \phi_x dx = \| \phi \|_\infty \omega(| t_2 - t_1 |),$$

for all smooth ϕ with support in $[a, b]$. Thus

$$\operatorname*{Var}_{[a,b]} [\, | K(v(., t_2) - K(v(., t_1) | \,] \leq \omega(| t_2 - t_1 |).$$

This implies that

$$\lim_{t_1, t_2 \to 0} \sup_{x \in [a,b]} | K(v(x, t_2)) - K(v(x, t_1)) | = 0 ,$$

and thus since v is assumed to be continuous in $t > 0$, we see that $K(v(x, t))$ converges, uniformly in x, to a continuous function. But from e) in definition (4.1),

$$\lim_{t\to 0} K(v(., t)) = K(v_0(.))$$

in L^1_{loc}. Thus $K(v_0(x))$ is a continuous function, and since $K' = k > 0$, we see that v_0 is a continuous function. This completes the proof.

Next, it is of some interest to investigate the smoothness of the functions u and E. For this we rely on a theorem of Aronson and Serrin, [*1*], concerning solutions of parabolic equations with discontinuous coefficients. We give here only a corollary of their result, which we need. Thus, consider the linear equation $u_t = (A(x, t)u_x + B(x, t))_x$, where A is bounded and measurable on a bounded domain $Q = (0, T) \times \Omega \subset \mathbb{R} \times \mathbb{R}_+$, $A \geq \delta > 0$ on Q, and B lies in $L^{p,q}(Q)$, i. e. $\left\{ \int_0^T \left(\int_\Omega | B |^p dx \right)^{q/p} dt \right\}^{1/q} < \infty$, where $q \geq 1$ and $p > 2$ and $\frac{1}{2p} + \frac{1}{q} < \frac{1}{2}$. Under these hypotheses, it is proved in [*1*] that in Q, if u is a weak solution (consistent with our definition), then u is Hölder continuous in x, and uniformly Hölder continuous in t in $Q \cap \{ t \geq \varepsilon > 0 \}$ for $\varepsilon > 0$.

We shall apply this result first to the second equation in (4.1) which we write in the form

$$u_t = [k(v)u_x - p(v, e)]_x.$$

Since the functions $k(v)$ and $p(v, e)$ are bounded in Q, and $k(v) \geq \delta > 0$ in Q, for some $\delta = \delta_Q > 0$, we see that the Aronson-Serrin theorem shows that u is Hölder continuous in Q. In order to show that E is continuous, we find it necessary to assume that $(u^2)_x \in L^{p,q}$. With this hypothesis, we can apply the Aronson-Serrin result to the third equation in (4.1) to conclude that E is Hölder continuous in Q. Therefore, if the classical Riemann problem (see [5]), can be solved in these function classes, it follows that u and E must be continuous in $t > 0$, and initial discontinuities in v must persist in $t > 0$. We thus have the following corollary to our theorem.

COROLLARY 4.3. — Suppose that the hypotheses of the last theorem is valid, and that $(u^2)_x \in L^{p,q}$, where $q \geq 1$ $p > 2$ and $\frac{1}{2^p} + \frac{1}{q} < \frac{1}{2}$. Then in $t > 0$, u and E are uniformly Hölder continuous in x, and are locally continuous in t.

Note that if we consider the isentropic gas dynamics equations:

$$v_t - u_x = 0, \qquad u_t + p(v)_x = (k(v)u_x)_x,$$

where k satisfies the same hypothesis as before (c. f. theorem 4.2), then if (v, u) is a solution in the sense of definition 4.1, and both $v(., t)$ and $u(., t)$ are of class b. v., and $v(., t)$ is continuous for $0 < t \leq T$, then $v(., 0)$ is continuous and $u(x, t)$ is continuous in $t > 0$. These conclusions follow from our previous methods, together with the Aronson-Serrin theorem.

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