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Conservation laws for the nonlinear Schrödinger equation

by

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ABSTRACT. — We propose a method of calculating the operator densities $\hat{h}_n$ $n = 0, 1, \ldots$ of the conservation laws for the quantum nonlinear Schrödinger equation. It follows from the method that $\hat{h}_n$ are polynomials in fields and their derivatives and in the coupling constant. The densities $\hat{h}_n$ $n \leq 4$ are explicitly calculated. Comparison with the integral densities b_n $n = 0, 1, \ldots$ for the classical nonlinear Schrödinger equation shows that the correspondence between $\hat{h}_n$ and b_n breaks down after $n = 3$.

RÉSUMÉ. — On propose une méthode pour calculer les densités opératoires $\hat{h}_n$ $n = 0, 1, \ldots$ pour les intégrales de l'équation de Schrödinger non linéaire quantique. Il s'ensuit que les $\hat{h}_n$ sont des fonctions polynomiales des champs, de leurs dérivées et de la constante de couplage. Les densités $\hat{h}_n$, $n \leqslant 4$, sont calculées explicitement. En les comparant avec les densités intégrales b_n $n = 0, 1, \ldots$ pour l'équation de Schrödinger non linéaire classique, on voit que la correspondance entre b_n et $\hat{h}_n$ n'est plus valable pour $n > 3$.

1. INTRODUCTION

We consider the quantum nonlinear Schrödinger equation (NLSE) in 1 + 1 space-time dimensions

$$i\Psi_t = -\Psi_{xx} + 2c\Psi^\dagger\Psi^2 . \tag{1.1}$$

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Its Hamiltonian

$$\hat{\mathrm{H}}_2 = - \int dx(\Psi^\dagger \Psi_{xx} - c\Psi^{\dagger 2}\Psi^2) \tag{1.2}$$

is the second quantized form of the many body Hamiltonian

$$\mathrm{H}_2^{(\mathrm{N})} = - \sum_{i=1}^{\mathrm{N}} \frac{\partial^2}{\partial x_i^2} + c \sum_{i \neq j} \delta(x_i - x_j) \tag{1.3}$$

Hamiltonian (1.3) describes the interaction of N identical particles on the line via elastic collisions and c is the strength of interaction. The famous « Bethe Ansatz » [1] [2] exhibits the system of generalized eigenstates $| \Psi_\mathrm{N}(k_1, \ldots, k_\mathrm{N}) \rangle = | \Psi_\mathrm{N}(k) \rangle$ of $\mathrm{H}_2^{(\mathrm{N})}$ which is complete if $c \geq 0$. We have

$$\mathrm{H}_2^{(\mathrm{N})} | \Psi_\mathrm{N}(\vec{k}) > = \left(\sum_{i=1}^{\mathrm{N}} k_i^2 \right) | \Psi_\mathrm{N}(\vec{k}) > . \tag{1.4}$$

Since Bethe Ansatz eigenstates depend on N quantum numbers $k_1, \ldots, k_\mathrm{N}$ the Hamiltonian (1.3) must be completely integrable. This means that there are N independent operators $\mathrm{H}_n^{(\mathrm{N})}$ $n = 1, \ldots, \mathrm{N}$ such that

$$\mathrm{H}_n^{(\mathrm{N})} | \Psi_\mathrm{N}(\vec{k}) > = \left(\sum_{i=1}^{\mathrm{N}} k_i^n \right) | \Psi_\mathrm{N}(\vec{k}) > \tag{1.5}$$

$\mathrm{H}_1^{(\mathrm{N})} = (-i) \sum_{i=1}^{\mathrm{N}} \partial/\partial x_i$ is of course the total momentum and $\mathrm{H}_2^{(\mathrm{N})}$ is the Hamiltonian (1.3). Existence of $\mathrm{H}_n^{(\mathrm{N})}$ should imply the infinite sequence of independent conservation laws $\hat{\mathrm{H}}_n$ $n = 1, 2, \ldots$ for the NLSE given by their operator densities $\hat{h}_n$

$$\hat{\mathrm{H}}_n = \int dx\, \hat{h}_n(x) . \tag{1.6}$$

We have

$$\hat{h}_1 = (-i)\Psi^\dagger \Psi_x \tag{1.7}$$

$$\hat{h}_2 = (-i)^2(\Psi^\dagger \Psi_{xx} - c\Psi^{\dagger 2}\Psi^2) . \tag{1.8}$$

Operators $\hat{\mathrm{H}}_n$ are completely characterized by the property that for any N

$$\hat{\mathrm{H}}_n | \Psi_\mathrm{N}(\vec{k}) > = \left(\sum_{i=1}^{\mathrm{N}} k_i^n \right) | \Psi_\mathrm{N}(\vec{k}) > . \tag{1.9}$$

It is desirable to have explicit expressions for the operator densities $\hat{h}_n$. In this paper I suggest a method for calculating $\hat{h}_n$ for any n. Using this method I calculate $\hat{h}_3$ and $\hat{h}_4$. In section 4 I compare $\hat{h}_n$ with the functional densities b_n of the integrals of motion for the classical NLSE

$$i\varphi_t = -\varphi_{xx} + 2c|\varphi|^2\varphi \tag{1.10}$$

Thacker [3] has obtained $\hat{h}_3$ using a completely different approach. Kulish and Sklyanin [4] and Thacker [4] have integrated (1.1) using the quantum inverse scattering method. Their method however does not yield explicit formulas for $\hat{h}_n$ in terms of the fields (*).

2. N-PARTICLE SECTOR

In this section we fix N and omit the superscript N in formulas. The Hamiltonian H_2 is equal to the Laplacean $-\sum_{i=1}^{N} \partial^2/\partial x_i^2$ with the boundary conditions

$$(\partial/\partial x_j - \partial/\partial x_i)F = cF \tag{2.1}$$

on hyperplanes $\{ x_i - x_j = 0 \}$ $i, j = 1, \ldots, N$.

Because of the symmetry of function F it suffices to restrict it to $R_\dagger^N = \{ x_1 \leq x_2 \leq \ldots \leq x_N \}$ and to impose boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = cF \tag{2.2}$$

on hyperplanes $x_k = x_{k+1}$ $k = 1, \ldots, N-1$.

I will use the following fact. There is an operator P on symmetric functions in R^N that intertwines Laplacean with the Neumann boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = 0 \tag{2.3}$$

and Laplacean with boundary conditions (2.2) for $c \geq 0$. The operator P constructed as follows. For any $i \neq j$ let P_{ij} be given by

$$(P_{ij}f)(x_1, \ldots, x_N) = \int_0^\infty dt e^{-ct} f(x_1, \ldots, x_i - t, \ldots, x_j + t, \ldots, x_N). \tag{2.4}$$

Denote by S the operator from all functions f on $\mathbb{R}^N$ into symmetric functions on R^N obtained by restricting f to $R_\dagger^N$ and then extending it to $\mathbb{R}^N$ by symmetry. Then [5]

$$P = S \prod_{i<j} (1 - cP_{ij}). \tag{2.5}$$

(*) *Added in proofs:* in a forthcoming paper I show that the formulas for integrals of the NLSE obtained in [4] via the quantum scattering method are false.

Denoting by Δ_2 the Laplacean with boundary conditions (2.3) we express the intertwining property of P by

$$H_2P = P\Delta_2 . \quad (2.6)$$

Let Δ_n be given by $(-i)^n \sum_{i=1}^{N} \partial^n/\partial x_i^n$ with « higher » Neumann boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i+1} f = 0 \quad (2.7)$$

for $i = 0, 1, \ldots, [n/2] - 1$ on hyperplanes $\{ x_k = x_{k+1} \}$ $k = 1, \ldots, N-1$. Let H_n be defined from

$$P\Delta_n = H_nP \quad (2.8)$$

for $n = 1, \ldots$. Since operators Δ_n commute, H_n also commute. It follows from (2.5) that P takes boundary conditions (2.7) into boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i+1} f = c(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i} f \quad (2.9)$$

So H_n is equal to $(-i)^n \sum_{i=1}^{N} \partial^n/\partial x_i^n$ with boundary conditions (2.9) for $i = 0, \ldots, [n/2] - 1$. It remains to obtain formulas for H_n similar to the formula (1.3) for H_2.

Let $g(x_1, \ldots, x_N)$ be an infinitely differentiable function and let f satisfy the boundary conditions (2.9). Then

$$\langle g \,|\, H_3 \,|\, f \rangle = (-i)^3 \int d^N x \bar{g} \left(\sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} f \right). \quad (2.10)$$

Integrating by parts and taking (2.9) into account we get

$$\langle g \,|\, H_3 \,|\, f \rangle = -(-i)^3 \int d^N x \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^2}{\partial x_i^2} f$$

$$- c(-i)^3 \sum_{i \neq j} \int d^N x \delta(x_i - x_j) \bar{g} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f. \quad (2.11)$$

Integrating by parts again

$$\langle g \mid H_3 \mid f \rangle = (-i)^3 \int d^N x \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \bar{g} \frac{\partial}{\partial x_i} f + (-i)^3 c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \frac{\partial}{\partial x_i} \bar{g} f + (-i)^3 c \int d^N x \sum_{i \neq j} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \delta(x_i - x_j) \bar{g} f. \quad (2.12)$$

After one more integration by parts and obvious transformations (2.12) becomes

$$\langle g \mid H_3 \mid f \rangle = i^3 \int d^N x \sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} \bar{g} f + (-i)^3 \frac{3}{2} c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} f \quad (2.13)$$

which yields

$$H_3 = (-i)^3 \left(\sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} - \frac{3}{2} c \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \right). \quad (2.14)$$

For H_4 we have

$$\langle g \mid H_4 \mid f \rangle = \int d^N x \bar{g} \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} f. \quad (2.15)$$

Integrating by parts the right hand side

$$- \int d^N x \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^3}{\partial x_i^3} f - c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \bar{g} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) f. \quad (2.16)$$

Integrating by parts the first term in (2.16) we get

$$\int d^N x \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \bar{g} \frac{\partial^2}{\partial x_i^2} f + \frac{1}{2} c \int d^N x \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f. \quad (2.17)$$

Integrating (2.17) by parts again and remembering the second term in (2.16) we have

$$\begin{aligned}\langle g \mid \mathrm{H}_4 \mid f\rangle = &\int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial^4}{\partial x_i^4} \bar{g} f \\ &- \frac{c}{2}\int d^{\mathrm{N}}x \sum_{i\neq j} \delta(x_i - x_j)\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right)\bar{g} f \\ &- \frac{c}{2}\int d^{\mathrm{N}}x \sum_{i\neq j} \delta(x_i - x_j)\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right)^2 \bar{g} f \\ &- c\int d^{\mathrm{N}}x \sum_{i\neq j} \delta(x_i - x_j)\bar{g}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right) f. \end{aligned} \quad (2.18)$$

The last term in (2.18) can again be integrated by parts yielding

$$- c\int d^{\mathrm{N}}x \sum_{i\neq j} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i^2 \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right)\delta(x_i - x_j)\bar{g} f + 3c^2 \int d^{\mathrm{N}}x \sum_{i\neq j\neq k} \delta(x_i - x_j)\delta(x_j - x_k)\bar{g} f. \quad (2.19)$$

From (2.18) and (2.19) we have

$$\begin{aligned}\mathrm{H}_4 = &\sum_{i=1}^{\mathrm{N}} \frac{\partial^4}{\partial x_i^4} - c\sum_{i\neq j} \delta(x_i - x_j)\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right) \\ &- c\sum_{i\neq j}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right)\delta(x_i - x_j) + 3c^2 \sum_{i\neq j\neq k} \delta(x_i - x_j)\delta(x_j - x_k). \end{aligned} \quad (2.20)$$

3. SECOND QUANTIZED FORM OF H_n

A standard calculation gives

$$\begin{aligned}\hat{\mathrm{H}}_3 &= (-i)^3 \int dx \Psi^\dagger \Psi_{xxx} \\ &\quad - \frac{3}{2} c(-i)^3 \int dx \int dy \Psi^\dagger(x)\Psi^\dagger(y)\delta(x-y)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\Psi(x)\Psi(y) \\ &= (-i)^3 \int dx [\Psi^\dagger \Psi_{xxx} - 3c\Psi^{\dagger 2}\Psi\Psi_x]. \end{aligned} \quad (3.1)$$

Thus

$$\hat{h}_3(x) = (-i)^3(\Psi^\dagger\Psi_{xxx} - 3c\Psi^{\dagger 2}\Psi\Psi_x). \tag{3.2}$$

Also

$$\begin{aligned}\hat{\mathrm{H}}_4 = &\int dx \Psi^\dagger\Psi_{xxxx}\\ &- c\int dx\int dy \Psi^\dagger(x)\Psi^\dagger(y)\delta(x-y)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\right)\Psi(x)\Psi(y)\\ &- c\int dx\int dy \Psi^\dagger(x)\Psi^\dagger(y)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\right)\delta(x-y)\Psi(x)\Psi(y)\\ &+ 3c^2\int dx\int dy\int dz \Psi^\dagger(x)\Psi^\dagger(y)\Psi^\dagger(z)\delta(x-y)\delta(y-z)\Psi(x)\Psi(y)\Psi(z).\end{aligned} \tag{3.3}$$

After obvious integrations by parts we have

$$\begin{aligned}\hat{\mathrm{H}}_4 = \int dx[&\Psi^\dagger\Psi_{xxx} - 2c\Psi^{\dagger 2}\Psi\Psi_{xx} - c\Psi^{\dagger 2}\Psi_x^2 - 2c\Psi^\dagger\Psi^\dagger_{xx}\Psi^2\\ &- c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3].\end{aligned} \tag{3.4}$$

Thus

$$\begin{aligned}\hat{h}_4(x) = &\Psi^\dagger\Psi_{xxxx} - 2c\Psi^{\dagger 2}\Psi\Psi_{xx} - c\Psi^{\dagger 2}\Psi_x^2 - 2c\Psi^\dagger\Psi_{xx}\Psi^2\\ &- c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3.\end{aligned} \tag{3.5}$$

4. COMPARISON WITH CLASSICAL INTEGRALS

The classical NLSE (or Zakharov-Shabat equation [6])

$$i\varphi_t = -\varphi_{xx} + 2c|\varphi|^2\varphi \tag{4.1}$$

is a completely integrable Hamiltonian system with infinitely many degrees of freedom [7]. In particular (4.1) has an infinite number of integrals of motion $\mathrm{B}_n(\overline{\varphi}, \varphi)$. The functionals B_n are determined by the local densities b_n

$$\mathrm{B}_n(\overline{\varphi}, \varphi) = \int_{-\infty}^{\infty} dx b_n(\overline{\varphi}(x), \varphi(x)). \tag{4.2}$$

The densities b_n are found from the recurrence relation

$$b_{n+1} = \overline{\varphi}\frac{d}{dx}\left(\frac{b_n}{\overline{\varphi}}\right) - c\sum_{i+j=n-1} b_i b_j \tag{4.3}$$

and

$$b_0 = \overline{\varphi}\varphi. \tag{4.4}$$

From (4.3) and (4.4) we get

$$b_1 = \overline{\varphi}\varphi_x \tag{4.5}$$

$$b_2 = \overline{\varphi}\varphi_{xx} - c|\varphi|^4 \tag{4.6}$$

$$b_3 = \overline{\varphi}\varphi_{xxx} - 2c\overline{\varphi}^2(\varphi^2)_x - c\overline{\varphi}\overline{\varphi}_x\varphi^2 \qquad (4.7)$$

$$b_4 = \overline{\varphi}\varphi_{xxxx} - 2c\overline{\varphi}^2(\varphi^2)_{xx} - 2c\overline{\varphi}^2\varphi\varphi_{xx} - c\overline{\varphi}^2\varphi_x^2 - 3c\overline{\varphi}\overline{\varphi}_x(\varphi^2)_x - c\overline{\varphi}\overline{\varphi}_x\varphi^2 + 2c^2 \mid \varphi \mid^6 . \qquad (4.8)$$

The local densities h and g that differ by a total derivative are equivalent $h \simeq g$ because they define the same functional $\int_{-\infty}^{\infty} dxh(x) = \int_{-\infty}^{\infty} dxg(x)$. We have

$$b_3 \simeq \overline{\varphi}\varphi_{xxx} - \frac{3}{2} c\overline{\varphi}^2(\varphi^2)_x = b'_3 \qquad (4.9)$$

$$b_4 \simeq \overline{\varphi}_{xx}\varphi_{xx} + 2c(\overline{\varphi}^2)_x(\varphi^2)_x + c\overline{\varphi}^2\varphi_x^2 + c\overline{\varphi}_x^2\varphi^2 + 2c^2 \mid \varphi \mid^6 = b'_4 . \qquad (4.10)$$

We see from (3.2) that $\hat{h}_3$ differs from b'_3 by the factor $(-i)^3$ only. On the other hand the difference between $\hat{h}_4$ (3.5) and b_4 is essential. Replacing $\hat{h}_4$ by an equivalent operator density h'_4 the closest that we can get to b'_4 is

$$\hat{h}'_4 = \Psi^\dagger_{xx}\Psi_{xx} + 2c(\Psi^{\dagger 2})_x(\Psi^2)_x + c\Psi^{\dagger 2}\Psi_x^2 + c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3 . \qquad (4.11)$$

The difference corresponds to $c^2 \mid \varphi \mid^6$ which is a nontrivial density.

5. CONCLUSION

The nonlinear Schrödinger equation (1.1) has an infinite sequence of conservation laws $\hat{\mathrm{H}}_n$ given by the operator densities $\hat{h}_n(\Psi^\dagger(x), \Psi(x))$. The densities $\hat{h}_n$ can be found using the method of sections 2 and 3. It is clear from the method that $\hat{h}_n$ are polynomials in the fields and their derivatives. Besides $\hat{h}_n$ are polynomials in the coupling constant c. The degree of $\hat{h}_n$ in c is $[n/2]$.

Correspondence between $\hat{h}_n$ and the integral densities b_n of the classical NLSE (4.1) breaks down at $n = 4$.

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