ANNALES DE L'I. H. P., SECTION C

L. C. EVANS
H. ISHII

A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities

Annales de l'I. H. P., section C, tome 2, nº 1 (1985), p. 1-20
<http://www.numdam.org/item?id=AIHPC_1985__2_1_1_0>

Ann. Inst. Henri Poincaré,
Vol. 2, n° 1, 1985, p. 1-20. *Analyse non linéaire*

A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities

by

L. C. EVANS (*)
Department of Mathematics,
University of Maryland,
College Park, MD 20742 USA

and

H. ISHII
Department of Mathematics,
Chuo University,
Tokyo 112 Japan

ABSTRACT. — We illustrate the effectiveness of viscosity solution methods for Hamilton-Jacobi PDE by demonstrating a new approach to a method of W. Fleming ([*10*], [*11*]) for proving WKB-type representations. We present new proofs of three examples, due originally to Ventcel-Freidlin, Varadhan, and Fleming.

RÉSUMÉ. — On étudie, par une méthode issue de la théorie des équations aux dérivées partielles, certains problèmes asymptotiques concernant les équations différentielles stochastiques, quand l'intensité du bruit tend vers zéro. La méthode employée est celle des solutions de viscosité pour les équations de Hamilton-Jacobi. Ceci permet une approche nouvelle de la méthode de Fleming ([*10*], [*11*]) pour établir des représentations du type WKB. On donne des démonstrations nouvelles de trois exemples, dus à Ventcel-Freidlin, Varadhan et Fleming.

0. INTRODUCTION

The study of nonlinear, first-order PDE of Hamilton-Jacobi type has recently been facilitated with the introduction, by M. G. Crandall and

(*) Supported in part by NSF grant MCS-8301265 and the Alfred P. Sloan Foundation.

P. L. Lions [*6*], of the new notion of weak or so-called « viscosity » solutions of such PDE. This concept, reformulated in part by Crandall-Evans-Lions [*5*], introduces a body of new, simple, and fairly flexible techniques for investigating various questions, principally the problems of uniqueness of weak solutions. These ideas have been applied to control theory in [*2*] [*4*] [*16*], etc. and to game theory in [*3*] [*8*] [*17*].

Our purpose in this paper is to expound another class of applications by illustrating in some detail the usefulness of these new methods in studying various asymptotic problems concerning random differential equations with small noise intensities. We present here new proofs of three related convergence theorems of WKB-type; these various results are due originally to Ventcel-Freidlin [*19*], Varadhan [*18*], Fleming [*9*] [*10*], and Friedman [*12*], among others. More precisely, given a differential equation subject to random disturbances of « size » ε, we consider some sample path functional u^ε, which should be very small as $\varepsilon \to 0$, and try to estimate how fast it goes to zero. We look for a representation of the form

$$u^\varepsilon = e^{\frac{-\mathrm{I}+o(1)}{\varepsilon^2}} \quad \text{as} \quad \varepsilon \to 0, \tag{0.1}$$

where I is to be determined. Our technique for proving (0.1) and calculating I consists in giving new justification—by viscosity solution methods—to an approach of W. Fleming [*10*] (and Holland [*13*]). The basic plan is this:

STEP 1: Make the logarithmic change of unknown

$$v^\varepsilon \equiv -\varepsilon^2 \log u^\varepsilon \tag{0.2}$$

and find a PDE that v^ε solves.

STEP 2: Obtain estimates, independent of ε, on v^ε and its first derivatives.

STEP 3: Show that for some subsequence $\varepsilon_k \to 0$, v^{ε_k} converges to a function v, which solves a certain first-order PDE.

STEP 4: Interpret this PDE as the dynamic programming equation of a deterministic control theory problem, and then show that $v = \mathrm{I}$ is the associated value function.

In carrying out Step 3 and especially Step 4 we make essential use of the properties of viscosity solutions and, most importantly, of the representation formulas for viscosity solutions of certain PDE (see the appendix, § 4).

There are several advantages of our procedures over the original proofs in [*10*] [*12*] [*18*] [*19*]. First and principally, we present a unified approach to questions therefore attached by rather different methods. We have taken some pains in the exposition (§ 1-3) to emphasize the similarities among our solutions of the three problems studied. Secondly our techniques are almost totally analytic and are much simpler than the probabilistic proofs in [*12*] [*19*]. Roughly speaking we replace the difficult Ventcel-Freidlin estimates by the not-so-difficult PDE estimates required for Step 2 above. Fleming in [*10*] similarly circumvents the need for the Ventcel-Freidlin bounds, but his proofs require an interpretation of v^ε as the value of a certain stochastic control theory problem. This complication we avoid by carrying out Step 3 above using viscosity solution methods; we then need only deterministic control theory to discover a formula for $v = \mathrm{I}$. It should be noted that we do encounter some difficulties in implementing Step 4 above, because of the (infinite) boundary layers which occur in examples 2, 3, but these can be handled by fairly obvious approximation procedures.

The three applications mentioned above concern the asymptotic behavior of exit times (§ 1), the probability of exiting through a given portion of the boundary (§ 2), and the probability of remaining in a given set for a given time (§ 3). The appendix (§ 4) collects facts about representation formulas, from control theory, of viscosity solutions of various PDE. We will not recall here the definitions and basic properties of viscosity solutions, referring the reader instead to [*5*].

Below are certain standing hypothesis which are assumed to hold throughout the paper. We are given mappings $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathrm{M}^{n\times n}$, and then define $a \equiv c^{\mathrm{T}}c : \mathbb{R}^n \to \mathrm{S}^{n\times n}$, where $\mathrm{M}^{n\times n}$ (resp. $\mathrm{S}^{n\times n}$) is the space of all (resp. all symmetric) $n \times n$ matrices. We suppose

$$(\mathrm{A})_1 \quad \begin{cases} b \text{ is } \mathrm{C}^2, \quad a \text{ is } \mathrm{C}^2, \\ \text{and there exists a constant C such that} \\ \sup\limits_{\mathbb{R}^n} |\, b \,|, |\, \mathrm{D}b \,|, |\, a \,|, |\, \mathrm{D}a \,| \leq \mathrm{C}\,. \end{cases}$$

We assume further

$$(\mathrm{A})_2 \begin{cases} \text{there exists a real number } \theta > 0 \text{ such that} \\ a_{ij}(x)\xi_i\xi_j \geq \theta \,|\, \xi \,|^2 \qquad \text{for all} \qquad x, \xi \in \mathbb{R}^n\,. \end{cases}$$

Examples 1-3 concern diffusions within a region Ω. We require that

$$(\mathrm{A})_3 \ \Omega \subset \mathbb{R}^n \text{ is open and bounded, with smooth boundary } \partial\Omega\,.$$

Hypotheses $(\mathrm{A})_1$-$(\mathrm{A})_3$ taken together form condition (A). Let us note here that we have made no attempt to discover minimal assumptions under which our methods still hold; in particular it is clear using obvious approximations that we really need only assume b to the Lipschitz.

Finally, it is convenient to introduce certain notational conventions used throughout. Each curve $x(\cdot)$ mentioned below is assumed to belong to the Sobolev space $H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$ and in particular to be absolutely continuous on compact subsets of $[0,\infty)$ and so differentiable a. e. Given such an $x(\cdot)$, we define for a. e. $s>0$

$$\| \dot{x}(s) - b(x(s)) \|^2 \equiv a^{kl}(x(s))(\dot{x}(s) - b(x(s)))_k(\dot{x}(s) - b(x(s)))_l ,$$

where $((a^{kl})) = a^{-1}$ is the inverse matrix of a. For any $x(\cdot)$ as above with $x(t) = x \in \Omega$, we get

$$\tau_x = \tau_x(x(\cdot)) = \inf \{ s > t \mid x(s) \in \mathbb{R}^n \backslash \Omega \}$$

and

$$\bar{\tau}_x = \bar{\tau}_x(x(\cdot)) = \inf \{ s > t \mid x(s) \in \mathbb{R}^n \backslash \bar{\Omega} \} ,$$

where the infimum of the empty set is $+\infty$. These are the first exit times from Ω and Ω, respectively, after time t. Lastly, we let $W_s (s \geq 0)$ denote a standard n-dimensional Wiener process.

We also note here that our techniques extend without excessive difficulty to cover various other WKB-type estimates. For example we have in unpublished work given new proofs of the asymptotics for fundamental solutions due to Varadhan [*18*, § 4] and Friedman [*12*, Chap. 14, § 5]; however, our techniques are not self-contained and, like theirs, require some hard estimates of Aronson [*1*].

Finally let us explicitly remark that our presentation throughout is fairly terse, both to moderate the length of the paper and to avoid excessive repetition of routine facts from PDE, control theory, and probability. In particular a good understanding of [*5*] [*10*] and parts of [*14*] is more-or-less a prerequisite.

1. EXAMPLE 1: ASYMPTOTIC BEHAVIOR OF $E(e^{-\lambda\tau_x^\varepsilon})$

We begin with a fairly simple example, inspired by Varadhan [*18*, § 3]. Fix $\varepsilon > 0$, $x \in \Omega$, and consider the Itô stochastic differential equation

$$(1.1)_\varepsilon \qquad \begin{cases} dX_s^\varepsilon = \varepsilon c(X_s^\varepsilon) dW_s & (s > 0) \\ X_0^\varepsilon = x \quad \text{a.s.} \end{cases}$$

Let τ_x^ε be the exit time from Ω and set for fixed $\lambda > 0$

$$(1.2) \qquad u^\varepsilon(x) \equiv E(e^{-\lambda\tau_x^\varepsilon}) .$$

According to standard theory u^ε is smooth in $\bar{\Omega}$ and solves the PDE

$$(1.3)_\varepsilon \qquad \begin{cases} \lambda u^\varepsilon - \dfrac{\varepsilon^2}{2} a_{ij} u_{ij}^\varepsilon = 0 \quad \text{in} \quad \Omega . \\ u^\varepsilon = 1 \quad \text{on} \quad \partial\Omega . \end{cases}$$

Denote by dist $(x, \partial\Omega)$ the distance from x to $\partial\Omega$ in the Riemannian metric determined by $a^{-1}(\cdot)$; that is,

$$\text{dist}\,(x, \partial\Omega) \equiv \inf\left\{ \int_0^{\tau_x} \| \dot{x}(s) \| \, ds \mid x(0) = x \right\}.$$

Set

$$\mathrm{I}(x) \equiv \sqrt{2\lambda}\ \text{dist}\,(x, \partial\Omega). \tag{1.4}$$

THEOREM 1.1. — (Varadhan [*18*]). We have

$$\lim_{\varepsilon\to 0} -\varepsilon \log u^{\varepsilon} = \mathrm{I}, \tag{1.5}$$

uniformly on $\overline{\Omega}$.

We present below a new proof of this result, following the general scheme outlined in § 0. To this end, we set

$$v^{\varepsilon} \equiv -\varepsilon \log u^{\varepsilon} \geq 0; \tag{1.6}$$

An easy calculation then reveals

$$(1.7)_{\varepsilon} \qquad \begin{cases} -\dfrac{\varepsilon}{2} a_{ij} v^{\varepsilon}_{ij} + \dfrac{1}{2} a_{ij} v^{\varepsilon}_i v^{\varepsilon}_j = \lambda & \text{in} \quad \Omega \\ \qquad\qquad\qquad v^{\varepsilon} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

LEMMA 1.2. — There exists a constant C, independent of ε, such that

$$\max_{\Omega} | v^{\varepsilon} |, | \mathrm{D}v^{\varepsilon} | \leq \mathrm{C}.$$

Proof. — To simplify notation we omit the superscript « ε ».

An easy argument using barriers gives

$$\max_{\Omega} | v |, \max_{\partial\Omega} | \mathrm{D}v | \leq \mathrm{C}. \tag{1.8}$$

We propose next to estimate $| \mathrm{D}v |$ in Ω by studying the auxiliary function

$$z \equiv | \mathrm{D}v |^2.$$

Should z attain its maximum at some point $x_0 \in \Omega$, then at x_0

$$z_i = 2v_k v_{ki} = 0 \qquad (i = 1, \ldots, n) \tag{1.9}$$

and

$$0 \leq -\frac{\varepsilon}{2} a_{ij} z_{ij} = -\varepsilon a_{ij} v_{ki} v_{kj} + 2v_k\left(-\frac{\varepsilon}{2} a_{ij} v_{ijk} \right).$$

Thus

$$\begin{aligned}\varepsilon\theta \mid D^2v \mid^2 &\leq 2v_k\left(-\frac{\varepsilon}{2}a_{ij}v_{ij}\right)_k + 2v_k\left(\frac{\varepsilon}{2}a_{ij,k}v_{ij}\right)\\ &= 2v_k\left(\lambda - \frac{1}{2}a_{ij}v_iv_j\right)_k + \varepsilon a_{ij,k}v_{ij}v_k\\ &\leq C \mid Dv \mid^3 + C \mid Dv \mid^2 + \frac{\varepsilon}{2}\theta \mid D^2v \mid^2 ;\end{aligned}$$

here we used (1.9).

Therefore

$$\varepsilon \mid D^2v \mid^2 \leq C(\mid Dv \mid^3 + 1) \quad \text{at} \quad x_0 . \tag{1.10}$$

However

$$\left|\frac{1}{2}a_{ij}v_iv_j - \lambda\right|^2 = \left|\frac{\varepsilon}{2}a_{ij}v_{ij}\right|^2 \leq \varepsilon^2 C \mid D^2v \mid^2 .$$

Hence

$$\begin{aligned}z^2 = \mid Dv \mid^4 &\leq C + C\left|\frac{1}{2}a_{ij}v_iv_j - \lambda\right|^2\\ &\leq C + \varepsilon C(\mid Dv \mid^3 + 1) \quad \text{at} \quad x_0 . \qquad \square\end{aligned}$$

LEMMA 1.3. — The function $I \in C^{0,1}(\overline{\Omega})$ and is the viscosity solution of

$$\begin{cases} \frac{1}{2}a_{ij}I_iI_j = \lambda & \text{in} \quad \Omega \\ I = 0 & \text{on} \quad \partial\Omega . \end{cases} \tag{1.11}$$

Proof. — A calculation shows

$$I(x) = \inf\left\{ \int_0^{\tau_x} \frac{1}{2} \| \dot{x}(s) \|^2 + \lambda ds \mid x(0) = x \right\}. \tag{1.12}$$

Consider now the control problem of minimizing

$$J(x, \alpha(\cdot)) \equiv \int_0^{\tau_x} \frac{1}{2} a_{ij}(x(s))\alpha_i(s)\alpha_j(s) + \lambda ds , \tag{1.13}$$

where

$$\begin{cases} \dot{x}(s) = a(x(s))\alpha(s) & (s > 0) \\ x(0) = x \end{cases} \tag{1.14}$$

for each control $\alpha(\cdot) \in L^1(0, \infty ; \mathbb{R}^n)$. The value function

$$J(x) \equiv \inf_{\alpha(.)} J(x, \alpha(\cdot))$$

is then a viscosity solution of the dynamic programming PDE

$$\max_{\alpha\in\mathbb{R}^n}\left\{-a_{ij}\alpha_i \mathrm{J}_j - \frac{1}{2}a_{ij}\alpha_i\alpha_j\right\} = \lambda \quad \text{in} \quad \Omega$$
$$\mathrm{J} = 0 \quad \text{on} \quad \partial\Omega :$$

see [14] for details. However

$$\max_{\alpha\in\mathbb{R}^n}\left\{-a_{ij}\alpha_i\beta_j - \frac{1}{2}a_{ij}\alpha_i\alpha_j\right\} = \frac{1}{2}a_{ij}\beta_i\beta_j$$

and so J is a viscosity solution of (1.11). Finally, note from (1.14) that $\mathrm{I}(x) \equiv \mathrm{J}(x)$. □

Proof of Theorem 1.1. — In view of Lemma 1.2 there is a sequence $\varepsilon_k \to 0$ and a function $v \in \mathrm{C}^{0,1}(\overline{\Omega})$ such that $v^{\varepsilon_k} \to v$ uniformly on $\overline{\Omega}$. From $(1.7)_\varepsilon$ it follows that v is a viscosity solution of

$$(1.15) \qquad \begin{cases} \frac{1}{2}a_{ij}v_iv_j = \lambda & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega . \end{cases}$$

However, a viscosity solution of (1.15) is unique: cf. Crandall-Lions [6, Proposition III.2]. Consequently, $v = \mathrm{I} = \sqrt{2\lambda}\,\mathrm{dist}\,(x, \partial\Omega)$, by Lemma 1.3. □

2. EXAMPLE 2: ASYMPTOTIC PROBABILITY OF EXITING THROUGH A GIVEN SUBSET OF $\partial\Omega$

This next application of our methods is based on Fleming [10]. Fix $\varepsilon > 0$, $x \in \Omega$. We turn our attention to the Itô random differential equation

$$(2.1)_\varepsilon \qquad \begin{cases} d\mathrm{X}_s^\varepsilon = b(\mathrm{X}_s^\varepsilon)ds + \varepsilon c(\mathrm{X}_s^\varepsilon)d\mathrm{W}_s \quad (s > 0) \\ \mathrm{X}_0^\varepsilon = x \quad \text{a.s.} \end{cases}$$

Suppose Ω is connected and assume also $\Gamma \subset \partial\Omega$ is a given smooth, nonempty, relatively open subset of the boundary.

Now set

$$(2.2) \qquad u^\varepsilon(x) \equiv \mathrm{P}(\mathrm{X}_{\tau_x^\varepsilon}^\varepsilon \in \Gamma),$$

the probability that a sample path of the solution of $(2.1)_\varepsilon$ will first exit Ω through the « window » Γ. Then u^ε is smooth on $\overline{\Omega}\backslash\partial\Gamma$ and

$$(2.3)_\varepsilon \qquad \begin{cases} -\frac{\varepsilon^2}{2}a_{ij}u_{ij}^\varepsilon - b_iu_i^\varepsilon = 0 \quad \text{in} \quad \Omega \\ u^\varepsilon = 1 \quad \text{on} \quad \overline{\Gamma} \\ u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega\backslash\overline{\Gamma}. \end{cases}$$

Finally we define for $x \in \Omega$

$$\text{(2.4)} \quad \mathrm{I}(x) \equiv \inf \left\{ \frac{1}{2} \int_0^{\tau_x} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(0)=x, x(\tau_x) \in \Gamma \text{ if } \tau_x < \infty \right\}.$$

Following Fleming [10], we introduce this strong hypothesis on the vector field $b(\cdot)$:

$$\text{(B)} \quad \begin{cases} \text{If} \quad x(\cdot) \in \mathrm{H}^1_{\mathrm{loc}}([0, \infty); \mathbb{R}^n) \quad \text{and} \quad x(s) \in \overline{\Omega} \quad \text{for all} \quad s \geq 0, \\ \text{then} \quad \displaystyle\int_0^\infty | \dot{x}(s) - b(x(s)) |^2 ds = \infty . \end{cases}$$

Condition (B) says that it requires an infinite amount of energy to resist the flow determined by $b(\cdot)$ and stay always within $\overline{\Omega}$. In the appendix (§ 4) we derive certain consequences of (B) which are employed below.

THEOREM 2.1 (Fleming [10]). — Assume in addition to hypotheses listed before that $b(\cdot)$ satisfies condition (B). Then

$$\text{(2.5)} \qquad \lim_{\varepsilon \to 0} - \varepsilon^2 \log u^\varepsilon = \mathrm{I},$$

uniformly on compact subsets of $\Omega \cup \Gamma$.

Applications may be found in [10] [12] and [19].

We begin the proof of (2.5) by setting

$$\text{(2.6)} \qquad v^\varepsilon \equiv - \varepsilon^2 \log u^\varepsilon \geq 0 ;$$

a straightforward calculation using $(2.3)_\varepsilon$ gives

$$(2.7)_\varepsilon \qquad \begin{cases} - \dfrac{\varepsilon^2}{2} a_{ij} v^\varepsilon_{ij} + \dfrac{1}{2} a_{ij} v^\varepsilon_i v^\varepsilon_j - b_i v^\varepsilon_i = 0 \quad \text{in} \quad \Omega \\ v^\varepsilon = 0 \quad \text{on} \quad \overline{\Gamma} \\ v^\varepsilon(x) \to + \infty \quad \text{as} \quad x \to \partial\Omega \backslash \overline{\Gamma} . \end{cases}$$

LEMMA 2.2. — For each $\Omega' \subset \subset \Omega \cup \Gamma$ there exists a constant $\mathrm{C}(\Omega')$, independent of ε, such that

$$\sup_{\Omega'} | v^\varepsilon |, | \mathrm{D}v^\varepsilon | \leq \mathrm{C}(\Omega') .$$

Proof. — To simplify notation we drop the superscript « ε ». Choose any $\Omega'' \subset \subset \Omega \cup \Gamma$ and set $\Gamma'' = \partial\Omega'' \cap \Gamma$. A barrier argument shows

$$\text{(2.8)} \qquad \sup_{\Gamma''} | \mathrm{D}v | \leq \mathrm{C} .$$

Consider now the auxiliary function

$$z \equiv \zeta^4 | \mathrm{D}v |^2 ,$$

where ζ is a smooth cutoff function with support in Ω''. Should z attain its maximum at some point $x_0 \in \Omega$, then at x_0

$$z_i = 2\zeta^4 v_k v_{ki} + 4\zeta^3 \,|\, Dv \,|^2 \zeta_i = 0 \qquad (i = 1, \ldots, n) \tag{2.9}$$

and

$$\begin{aligned} 0 \leq -\frac{\varepsilon^2}{2} a_{ij} z_{ij} &\leq -\frac{\varepsilon^2}{2}\theta\zeta^4 \,|\, D^2 v \,|^2 + 2\zeta^4 v_k \left(-\frac{\varepsilon^2}{2} a_{ij} v_{ij} \right)_k \\ &\qquad + C\zeta^2 \,|\, Dv \,|^2 \\ &= -\frac{\varepsilon^2}{2}\theta\zeta^4 \,|\, D^2 v \,|^2 + 2\zeta^4 v_k \left(-\frac{1}{2} a_{ij} v_i v_j + b_i v_i \right)_k \\ &\qquad + C\zeta^2 \,|\, Dv \,|^2 \\ &\leq -\frac{\varepsilon^2}{2}\theta\zeta^4 \,|\, D^2 v \,|^2 + C\zeta^3 \,|\, Dv \,|^3 + C, \quad \text{by} \quad (2.9). \end{aligned} \tag{2.10}$$

Furthermore

$$\left| \frac{1}{2} a_{ij} v_i v_j - b_i v_i \right|^2 = \left| \frac{\varepsilon^2}{2} a_{ij} v_{ij} \right|^2 \leq C\varepsilon^4 \,|\, D^2 v \,|^2 ,$$

and so (2.10) gives

$$\zeta^4 \,|\, Dv \,|^4 \leq C + \varepsilon^2 C(\zeta^3 \,|\, Dv \,|^3 + 1) \quad \text{at} \quad x_0 . \qquad \square$$

LEMMA 2.3. — The function $I \in C^{0,1}(\overline{\Omega})$ and is a viscosity solution of

$$\begin{cases} \dfrac{1}{2} a_{ij} I_i I_j - b_i I_i = 0 & \text{in} \quad \Omega \\ I = 0 & \text{on} \quad \overline{\Gamma} \end{cases} \tag{2.11}$$

Proof. — It is routine to establish that I is uniformly Lipschitz on Ω, $I = 0$ on $\overline{\Gamma}$.

To see I solves (2.11) in the viscosity sense consider the control problem of minimizing

$$J(x, \alpha(\cdot)) \equiv \frac{1}{2} \int_0^{\tau_x} a_{ij}(x(s))\alpha_i(s)\alpha_j(s) ds , \tag{2.12}$$

where

$$\begin{cases} \dot{x}(s) = a(x(s))\alpha(s) + b(x(s)) & (s > 0) \\ x(0) = x , \end{cases} \tag{2.13}$$

for all controls $\alpha(\cdot)$ such that $x(\tau_x) \in \Gamma$ if $\tau_x < \infty$. Then

$$J(x) \equiv \inf_{\alpha(.)} \{ J(x, \alpha(\cdot)) \,|\, x(\tau_x) \in \Gamma \quad \text{if} \quad \tau_x < \infty \}$$

is a viscosity solution of

$$\max_{\alpha \in \mathbb{R}^n} \left\{ - a_{ij}\alpha_i J_j - b_j J_j - \frac{1}{2} a_{ij}\alpha_i\alpha_j \right\} = 0. \tag{2.14}$$

But notice $I \equiv J$; so that (2.14) becomes (2.11). $\square$

LEMMA 2.4. — For each $x \in \Omega \cup \Gamma$

$$I(x) = \inf \left\{ \frac{1}{2} \int_0^{\bar{\tau}_x} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(0) = x, \right. \\ \left. x(\bar{\tau}_x) \in \bar{\Gamma} \quad \text{if} \quad \bar{\tau}_x < \infty \right\}. \tag{2.15}$$

Proof. — Denote the right hand side of (2.15) by $\bar{I}(x)$. Clearly $\bar{I}(x) \leq I(x)$, $I = \bar{I} = 0$ on Γ.

Fix $x \in \Omega$, $\sigma > 0$, and then choose an absolutely continuous function $x(\cdot)$ so that

$$\left\{ \begin{array}{l} x(0) = x, \qquad \bar{\tau}_x \leq T, \qquad x(\bar{\tau}_x) \in \bar{\Gamma} \\ \displaystyle \frac{1}{2} \int_0^{\bar{\tau}_x} \| \dot{x}(s) - b(x(s)) \|^2 ds \leq \bar{I}(x) + \sigma, \end{array} \right. \tag{2.16}$$

for some constant T independent of σ; such an $x(\cdot)$ exists owing to Lemma 4.2 in the appendix.

Since $\partial\Omega$ is smooth there exists a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\left\{ \begin{array}{ll} \partial\Omega = \{ \phi = 0 \}, & \Omega = \{ \phi > 0 \}, \\ | D\phi | = 1 \quad \text{on} \quad \partial\Omega, & D\phi(x) = 0. \end{array} \right. \tag{2.17}$$

Note then that for $\sigma > 0$ sufficiently small, the curve

$$x^\sigma(s) \equiv x(s) + \sigma D\phi(x(s)) \qquad (0 \leq s \leq \bar{\tau}_x) \tag{2.18}$$

lies in Ω, with

$$\| x(\cdot) - x^\sigma(\cdot) \|_{H^1([0,\bar{\tau}_x];\mathbb{R}^n)} = 0(\sigma), \qquad \text{as} \quad \sigma \to 0. \tag{2.19}$$

Extend $x^\sigma(\cdot)$ for $t > \bar{\tau}_x$ so that $x^\sigma(\tau_x^\sigma) \in \Gamma$ and

$$\begin{aligned} I(x) &\leq \frac{1}{2} \int_0^{\tau_x^\sigma} \| \dot{x}^\sigma(s) - b(x^\sigma(s)) \|^2 ds \\ &\leq \bar{I}(x) + 0(\sigma), \qquad \text{by} \quad (2.16), (2.18), (2.19). \end{aligned}$$ $\square$

Proof of Theorem 2.1. — According to Lemma 2.2 there exist a sequence $\varepsilon_k \to 0$ and a function $v \in C^{0,1}_{\text{loc}}(\Omega \cup \Gamma)$ such that $v^{\varepsilon_k} \to v$ uniformly on compact subsets of $\Omega \cup \Gamma$. From $(2.7)_\varepsilon$ we see v to be a viscosity solution of

$$\begin{aligned} &\frac{1}{2} a_{ij} v_i v_j - b_i v_i = 0 \quad \text{in} \quad \Omega \\ &v = 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.20}$$

This PDE implies $|Dv| \leq C$ in all of Ω (cf. Crandall-Lions [*6*, § I.4]) and thus $v \in C^{0,1}(\overline{\Omega})$; that is, v has a unique, Lipschitz extension to $\overline{\Omega}$.

Next, because $b(\cdot)$ satisfies (B) we know from Theorem 4.1 in the appendix that

$$v(x) = \inf\left\{ \frac{1}{2}\int_0^{\tau_x} \| \dot{x}(t) - b(x(t)) \|^2 dt + v(x(\tau_x)) \mid x(0) = x \right\}. \tag{2.21}$$

But then, since $v \doteq 0$ on $\overline{\Gamma}$,

$$v \leq I \quad \text{in} \quad \overline{\Omega}. \tag{2.22}$$

We must prove the opposite inequality, and for this will exploit the second boundary condition in $(2.7)_\varepsilon$. Let us therefore choose some smooth, open $\Omega' \supset \supset \Omega$, with a smooth, relatively open subset $\Gamma' \subset \partial\Omega'$, such that

$$0 < \operatorname{dist}(\partial\Omega', \partial\Omega) \leq \beta \quad \text{and} \quad \operatorname{dist}(x, \Gamma') \leq \beta, \quad \operatorname{dist}(x', \Gamma) \leq \beta \quad \text{for each} \quad x \in \Gamma, \quad x' \in \Gamma', \quad \text{and some number } \beta > 0. \tag{2.23}$$

Choose $\lambda > 0$ and define

$$I'_\lambda(x) \equiv \inf\left\{ \frac{1}{2}\int_0^{\tau'_x} e^{-\lambda s} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(0) = x, \quad x(\tau'_x) \in \Gamma' \ \text{if} \ \tau'_x < \infty \right\}, \tag{2.24}$$

where τ'_x is the exit time from Ω'. Then I'_λ is Lipschitz and

$$\begin{cases} \lambda I'_\lambda + \frac{1}{2} a_{ij} I'_{\lambda,i} I'_{\lambda,j} - b_i I'_{\lambda,i} = 0 & \text{in} \quad \Omega' \\ I'_\lambda = 0 & \text{on} \quad \Gamma' \end{cases} \tag{2.25}$$

in the viscosity sense.

Now for sufficiently small $\gamma > 0$ the usual convolution $I^\gamma_\lambda \equiv I'_\lambda * \rho_\gamma$ is defined in $\overline{\Omega}$. Furthermore according to (2.23)

$$I^\gamma_\lambda \leq C\beta \quad \text{on} \quad \Gamma; \tag{2.26}$$

whereas Jensen's inequality implies

$$\lambda I^\gamma_\lambda - \frac{\varepsilon^2}{2} a_{ij} I^\gamma_{\lambda,ij} + \frac{1}{2} a_{ij} I^\gamma_{\lambda,i} I^\gamma_{\lambda,j} - b_i I^\gamma_{\lambda,i} \leq C\left(\gamma + \frac{\varepsilon^2}{\gamma^2}\right) \quad \text{in} \quad \Omega. \tag{2.27}$$

Fix a constant M such that

$$I^\gamma_\lambda \leq M \quad \text{on} \quad \partial\Omega/\Gamma \qquad (\text{for all small } \lambda, \gamma > 0) \tag{2.28}$$

Now consider the solution v^ε_λ of the PDE

$$\begin{cases} \lambda v^\varepsilon_\lambda - \frac{\varepsilon^2}{2} a_{ij} v^\varepsilon_{\lambda,ij} + \frac{1}{2} a_{ij} v^\varepsilon_{\lambda,i} v^\varepsilon_{\lambda,j} - b_i v^\varepsilon_{\lambda,i} = 0 & \text{in} \quad \Omega \\ v^\varepsilon_\lambda = C\beta & \text{on} \quad \overline{\Gamma} \\ v^\varepsilon_\lambda = M & \text{on} \quad \partial\Omega \backslash \overline{\Gamma}. \end{cases}$$

As $v_\lambda^\varepsilon > 0$, the maximum principle and $(2.7)_\varepsilon$ imply

$$v_\lambda^\varepsilon \leq v^\varepsilon + C\beta \quad \text{in} \quad \overline{\Omega}.$$

But then the maximum principle and (2.26)-(2.28) give

$$I_\lambda^\gamma \leq v^\varepsilon + C\beta + \frac{C}{\lambda}\left(\gamma + \frac{\varepsilon^2}{\gamma^2}\right) \quad \text{in} \quad \overline{\Omega}$$

Let $\varepsilon = \varepsilon_k \to 0$ and $\gamma \to 0$, in that order, to find

$$(2.29) \qquad I'_\lambda \leq v + C\beta \quad \text{in} \quad \overline{\Omega},$$

Thus

$$\overline{I}_\lambda \leq v \quad \text{in} \quad \overline{\Omega},$$

for

$$\overline{I}^\lambda(x) \equiv \inf\left\{ \frac{1}{2}\int_0^{\overline{\tau}_x} \overline{e}^{\lambda s} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(0) = x, x(\overline{\tau}_x) \in \overline{\Gamma} \text{ if } \overline{\tau}_x < \infty \right\}.$$

Hence Lemma 4.2 from the appendix (§ 4) implies

$$\overline{I} \leq v \quad \text{in} \quad \overline{\Omega},$$

where

$$\overline{I}(x) \equiv \inf\left\{ \frac{1}{2}\int_0^{\overline{\tau}_x} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(0) = x, x(\overline{\tau}_x) \in \overline{\Gamma} \quad \text{if} \quad \overline{\tau}_x < \infty \right\},$$

But $I \equiv \overline{I}$ in Ω, according to Lemma 2.4. □

Remark. — For this proof we borrowed some ideas from P. L. Lions [*14*, § 6.1]. □

3. EXAMPLE 3: ASYMPTOTIC PROBABILITY OF REMAINING IN Ω FOR A GIVEN AMOUNT OF TIME

The following application gives a new approach to certain results of Ventcel and Freidlin [*19*].

Fix $\varepsilon > 0$, $x \in \Omega$, $T > 0$. For each $0 \leq t < T$ we consider the stochastic differential equation

$$(3.1)_\varepsilon \qquad \begin{cases} dX_s^\varepsilon = b(X_s^\varepsilon)ds + \varepsilon c(X_s^\varepsilon)dW_s & (s \geq t) \\ X_t^\varepsilon = x \quad \text{a.s.} \end{cases}$$

Define $Q \equiv \Omega \times (0, T)$.

Set

$$(3.2) \qquad u^\varepsilon(x, t) \equiv P\left(X_s^\varepsilon \in \Omega \mid t \leq s \leq T\right),$$

the probability that a sample path of the solution of $(3.1)_\varepsilon$ will remain within Ω at least until time T. Then u^ε is smooth in $\overline{Q}\backslash\partial\Omega \cap \{ t = T \}$ and solves the PDE

$$(3.3)_\varepsilon \qquad \begin{cases} u_t^\varepsilon + \dfrac{\varepsilon^2}{2} a_{ij}u_{ij}^\varepsilon + b_i u_i^\varepsilon = 0 & \text{in} \quad Q \\ u^\varepsilon = 1 \quad \text{on} & \Omega \times \{ t = T \} \\ u^\varepsilon = 0 \quad \text{on} & \partial\Omega \times (0, T). \end{cases}$$

Next define

$$(3.4) \quad I(x, t) \equiv \inf \left\{ \frac{1}{2} \int_t^T \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x, \right. \\ \left. x(s) \in \Omega \quad \text{for} \quad t \leq s \leq T \right\}.$$

THEOREM 3.1. (cf. Ventcel-Freidlin [*19*]). — We have

$$(3.5) \qquad \lim_{\varepsilon \to 0} - \varepsilon^2 \log u^\varepsilon = I,$$

uniformly for $0 \leq t \leq T$ and x in compact subsets of Ω.

We begin the proof by setting

$$(3.6) \qquad v^\varepsilon \equiv - \varepsilon^2 \log u^\varepsilon \geq 0$$

and calculating

$$(3.7)_\varepsilon \qquad \begin{cases} v_t^\varepsilon + \dfrac{\varepsilon^2}{2} a_{ij}v_{ij}^\varepsilon - \dfrac{1}{2} a_{ij}v_i^\varepsilon v_j^\varepsilon + b_i v_i^\varepsilon = 0 & \text{in} \quad Q \\ v^\varepsilon = 0 \quad \text{on} \quad \Omega \times \{ t = T \} \\ v^\varepsilon \to \infty \quad \text{as} \quad (x, t) \to \partial\Omega \times [0, T). \end{cases}$$

LEMMA 3.2. — *i*) For each $\Omega' \subset \subset \Omega$ there exists a constant $C(\Omega')$, independent of ε, such that

$$\sup_{Q'} | v^\varepsilon |, \; | Dv^\varepsilon | \leq C(\Omega'). \qquad (Q' \equiv \Omega' \times (0, T)).$$

ii) Furthermore, for each $\Omega' \subset \subset \Omega$ and each $0 < \alpha < 1$ there exists a constant $C(\alpha, \Omega')$, independent of ε, such that

$$[v^\varepsilon]_{C^\alpha(\overline{Q'})} \leq C(\alpha, \Omega').$$

Proof. — First we set $w(x, t) \equiv v^\varepsilon(x, T - t)$. Then

$$(3.8)_\varepsilon \qquad \begin{cases} w_t - \dfrac{\varepsilon^2}{2} a_{ij}w_{ij} + \dfrac{1}{2} a_{ij}w_i w_j - b_i w_i = 0 & \text{in} \quad Q \\ w = 0 \quad \text{on} \quad \Omega \times \{ t = 0 \} \\ w \to \infty \quad \text{as} \quad (x, t) \to \partial\Omega \times (0, T]. \end{cases}$$

a) Estimate of $|v^\varepsilon|$.

We may assume $B(0, R) \subset \Omega$. Define

$$z(x, t) \equiv \frac{1}{R^2 - |x|^2} + \mu t \qquad (x \in B(0, R), \qquad 0 \le t \le T) \tag{3.9}$$

and calculate

$$\begin{aligned} z_t - \frac{\varepsilon^2}{2} a_{ij} z_{ij} + \frac{1}{2} a_{ij} z_i z_j - b_i z_i &= \mu - \frac{\varepsilon^2}{2}\left(\frac{2a_{ij}\delta_{ij}}{(R^2 - |x|^2)^2} + \frac{8a_{ij}x_i x_j}{(R^2 - |x|^2)^3}\right) \\ &+ \frac{2a_{ij}x_i x_j}{(R^2 - |x|^2)^4} - \frac{2b_i x_i}{(R^2 - |x|^2)^2} \\ &\ge \mu - C\varepsilon^2\left(\frac{1}{(R^2 - |x|^2)^2} + \frac{|x|^2}{(R^2 - |x|^2)^3}\right) + \frac{\theta |x|^2}{(R^2 - |x|^2)^4} - C \ge 0, \end{aligned} \tag{3.10}$$

provided $\mu = \mu(R)$ is chosen sufficiently large. As $z \ge 0$, and $z = +\infty$ on $\partial B(0, R)$, we have

$$w \le z \qquad \text{in} \qquad B(0, R) \times (0, T)$$

and consequently

$$0 \le w \le C(R) \qquad \text{in} \qquad B\left(0, \frac{R}{2}\right) \times (0, T). \tag{3.11}$$

Finally, cover any given $\Omega' \subset \subset \Omega$ with finitely many balls $B\left(x_i, \frac{R_i}{2}\right)$, such that $B(x_i, R_i) \subset \Omega$, and apply (3.11).

b) Estimate of $|Dv^\varepsilon|$.

It is clear from (3.2) that $t \mapsto u^\varepsilon(x, t)$ is nondecreasing. Hence $u_t^\varepsilon \ge 0$, $v_t^\varepsilon \le 0$, and so

$$w_t \ge 0. \tag{3.12}$$

Consequently

$$\frac{1}{2} a_{ij} w_i w_j - b_i w_i = -w_t + \frac{\varepsilon^2}{2} a_{ij} w_{ij} \le \varepsilon^2 C |D^2 w|,$$

which estimate in turn implies

$$|Dw|^4 \le C + C\varepsilon^4 |D^2 w|^2.$$

With this in hand we may use the method explained in the proof of Lemma 2.2 to derive the requisite bound on $|Dw|$.

c) Estimate of Hölder norm.

In view of the preceding w is Lipschitz in x on Q', and so we need only estimate the Hölder constant in t.

Let us assume $0 \in \Omega' \subset\subset \Omega'' \subset\subset \Omega$. Define

$$\begin{cases} \tilde{w}(x,t) \equiv w(\varepsilon x, t), \qquad \tilde{a}_{ij}(x) \equiv a_{ij}(\varepsilon x), \\ \tilde{f}(x,t) \equiv -\dfrac{1}{2}\tilde{a}_{ij}w_i(\varepsilon x,t)w_j(\varepsilon x,t) + b_i(\varepsilon x)w_i(\varepsilon x,t), \\ \qquad \tilde{Q} \equiv \{(x,t) \mid (\varepsilon^2 x, t) \in Q\}; \end{cases}$$

then

$$\begin{cases} \tilde{w}_t - \dfrac{1}{2}\tilde{a}_{ij}\tilde{w}_{ij} = \tilde{f} \\ \tilde{w} = 0 \quad \text{on} \quad \{t = 0\} \end{cases} \tag{3.13}$$

Since $\tilde{w}, \tilde{f} \in L^\infty$, standard parabolic estimates give

$$|\tilde{w}(0,t) - \tilde{w}(0,\hat{t})| \leq C(\alpha, \Omega')|t - \hat{t}|^\alpha;$$

and consequently

$$|w(0,t) - w(0,\hat{t})| \leq C(\alpha,\Omega')|t - \hat{t}|^\alpha \quad \text{for} \quad 0 < \alpha < 1,\ 0 \leq \hat{t},\ t \leq T. \quad \square$$

LEMMA 3.3. — The function $I \in C^{0,1}(\bar{Q})$ and is a viscosity solution of

$$\begin{cases} I_t - \dfrac{1}{2}a_{ij}I_iI_j + b_iI_i = 0 \quad \text{in} \quad Q \\ I = 0 \quad \text{on} \quad \Omega \times \{t = T\} \end{cases} \tag{3.14}$$

Proof. — The proof is similar to that for Lemma 2.3. □

LEMMA 3.4. — For each $(x,t) \in Q$

$$I(x,t) = \inf\left\{\frac{1}{2}\int_t^T \|\dot{x}(s) - b(x(s))\|^2 ds \mid x(t) = x,\ x(s) \in \bar{\Omega} \ \text{for} \ t \leq s \leq T\right\}. \tag{3.15}$$

Proof. — It suffices to prove $H^1((0,T);\Omega)$ is dense in $H^1((0,T);\bar{\Omega})$, and this in turn is a consequence of methods used in the proof of Lemma 2.4. □

Proof of Theorem 3.1. — According to Lemma 3.2 there exist $\varepsilon_k \to 0$ and a function v such that $v^{\varepsilon_k} \to v$ uniformly for $0 \leq t \leq T$ and x in compact subsets of Ω. Furthermore $(3.7)_\varepsilon$ implies v is a viscosity solution of

$$\begin{cases} v_t - \dfrac{1}{2}a_{ij}v_iv_j + b_iv_i = 0 \quad \text{in} \quad Q \\ v = 0 \quad \text{on} \quad \Omega \times \{t = T\}. \end{cases} \tag{3.16}$$

Since Lemma 3.2 gives the bound

$$\operatorname*{ess\,sup}_{Q'} | Dv | \leq C(\Omega')$$

for each $\Omega' \subset\subset \Omega$, we have also (cf. Crandall-Lions [*6*, Thm. I.15])

$$\operatorname*{ess\,sup}_{Q'} | v_t | \leq C(\Omega')$$

and thus $v \in C^{0,1}(Q')$ for each $Q' \equiv \Omega' \times (0,T)$, $\Omega' \subset\subset \Omega$. Now in view of the representation formulas for viscosity solutions of (3.16) (cf. Theorem 4.4) we have for $0 \leq t \leq T$, $x \in \Omega' \subset\subset \Omega$,

$$v(x,t) = \inf\left\{ \frac{1}{2}\int_t^{T \wedge \tau'_x} \| \dot{x}(s) - b(x(s)) \|^2 ds + v(x(T \wedge \tau'_x), T \wedge \tau'_x) \mid x(t) = x \right\},$$

where $\tau'_x = \tau'_x(x(\cdot))$ is the first exit time from Ω' after time t. But then

$$v(x,t) \leq \inf\left\{ \frac{1}{2}\int_t^{T} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x,\ x(s) \in \Omega' \quad \text{for} \quad t \leq s \leq T \right\}.$$

This holds for all $\Omega' \subset\subset \Omega$ and thus

$$(3.17)\quad v(x,t) \leq \inf\left\{ \frac{1}{2}\int_t^{T} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x, \quad x(s) \in \Omega \ \text{for}\ t \leq s \leq T \right\} = I(x,t).$$

We now prove the opposite inequality by utilizing the second boundary condition in $(3.7)_\varepsilon$. Let us therefore choose some smooth, open $\Omega' \supset\supset \Omega$, fix $\lambda > 0$, and set

$$I'_\lambda(x,t) \equiv \inf\left\{ \frac{1}{2}\int_t^{T+\lambda} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x, \quad x(s) \in \Omega' \ \text{for}\ t \leq s \leq T + \lambda \right\}.$$

According to Lemma 3.3, $I'_\lambda \in C^{0,1}(\overline{Q'_\lambda})$ and solves

$$(3.18)\qquad \begin{cases} I'_{\lambda,t} - \dfrac{1}{2} a_{ij} I'_{\lambda,i} I'_{\lambda,j} + b_i I'_{\lambda,i} = 0 & \text{in} \quad Q'_\lambda \\ I'_\lambda = 0 \quad \text{on} & \Omega' \times \{ t = T + \lambda \} \end{cases}$$

in the viscosity sense; here $Q'_\lambda \equiv \Omega' \times (-\lambda, T + \lambda)$.

For sufficiently small $\gamma > 0$ the usual convolution $I^\gamma_\lambda \equiv I'_\lambda * \rho_\gamma$ is defined in $\overline{Q}$, with

$$I^\gamma_\lambda \leq C\lambda \qquad \text{on} \qquad \Omega \times \{ t = T \}$$

Furthermore Jensen's inequality implies

$$I^\gamma_{\lambda,t} + \frac{\varepsilon^2}{2} a_{ij} I^\gamma_{\lambda,ij} - \frac{1}{2} a_{ij} I^\gamma_{\lambda,i} I^\gamma_{\lambda,j} + b_i I^\gamma_{\lambda,i} \geq -C\left(\gamma + \frac{\varepsilon^2}{\gamma^2}\right).$$

But then $(3.7)_\varepsilon$ and the maximum principle give

$$(3.19)\qquad \mathrm{I}_\lambda^\gamma \leq v^\varepsilon + \mathrm{C}\left(\lambda + \gamma + \frac{\varepsilon^2}{\gamma^2}\right) \quad \text{in } \overline{\mathrm{Q}}.$$

Let $\varepsilon = \varepsilon_k \to 0$, $\gamma \to 0$, $\lambda \to 0$, in that order, to find

$$(3.20)\qquad \mathrm{I}' \leq v \quad \text{in } \overline{\mathrm{Q}},$$

where

$$\mathrm{I}'(x, t) = \inf\left\{ \frac{1}{2}\int_t^{\mathrm{T}} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x, \quad x(s) \in \Omega' \text{ for } t \leq s \leq \mathrm{T} \right\}.$$

Therefore

$$(3.21)\qquad \overline{\mathrm{I}} \leq v \quad \text{in } \overline{\mathrm{Q}},$$

for

$$\overline{\mathrm{I}}(x, t) \equiv \inf\left\{ \frac{1}{2}\int_t^{\mathrm{T}} \| \dot{x}(s) - b(x(s)) \|^2 ds \mid x(t) = x, \quad x(s) \in \overline{\Omega} \text{ for } t \leq s \leq \mathrm{T} \right\}.$$

But $\mathrm{I} \equiv \overline{\mathrm{I}}$ in Q, according to Lemma 3.4. □

4. APPENDIX: REPRESENTATION FORMULAS FOR VISCOSITY SOLUTIONS OF CERTAIN FIRST-ORDER PDE

A. The equation $\frac{1}{2}(\mathrm{D}v)^{\mathrm{T}} a \mathrm{D}v - b.\mathrm{D}v = 0$.

For this subsection we assume in addition to the standing hypothesis (A) (see § 0) that the vector field $b(\cdot)$ satisfies condition (B) (see § 2).

THEOREM 4.1. — Assume (B) holds and that $v \in \mathrm{C}^{0,1}(\overline{\Omega})$ is a viscosity solution of

$$(4.1)\qquad \begin{cases} \frac{1}{2} a_{ij} v_i v_j - b_i v_i = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega. \end{cases}$$

Then for each $x \in \Omega$

$$(4.2)\qquad v(x) = \inf\left\{ \frac{1}{2}\int_0^{\tau_x} \| \dot{x}(s) - b(x(s)) \|^2 ds + g(x(\tau_x)) \mid x(0) = x \right\}.$$

The proof of (4.2) follows a preliminary lemma:

LEMMA 4.2. — There exist $\alpha > 0$, $T > 0$ such that

$$\int_0^S |\dot{x}(s) - b(x(s))|^2 ds \geq \alpha S$$

if $S \geq T$ and $x(\cdot) \in H^1([0, S]; \overline{\Omega})$.

Proof. — If not, there would exist $S_m \geq m$ and $x_m(\cdot) \in H^1([0, S_m]); \overline{\Omega})$ such that

$$\int_0^{S_m} |\dot{x}_m(s) - b(x_m(s))|^2 ds < \frac{S_m}{m^2} \qquad (m = 1, 2, \ldots).$$

Consequently

$$\int_{mk_m}^{m(k_m+1)} |\dot{x}_m(s) - b(x_m(s))|^2 ds \leq \frac{2}{m} \quad \text{for some} \quad k_m \in \{0, 1, \ldots, [S_m/m] - 1\}.$$

Set

$$z_m(s) \equiv x_m(s + mk_m) \qquad (0 \leq s \leq m);$$

then $z_m(\cdot) \in H^1([0, m]; \overline{\Omega})$ and

$$\int_0^m |\dot{z}_m(s) - b(z_m(s))|^2 ds \leq \frac{2}{m}.$$

Hence there exist $m_k \to \infty$ and $z(\cdot) \in H^1_{loc}([0, \infty); \overline{\Omega})$ such that

$$z_{m_k} \to z$$

uniformly on compact subsets of $[0, \infty)$ and

$$\dot{z} = b(z) \qquad \text{a. e.}$$

This is a contradiction to (B). □

Remark. — Cf. Fleming [*10*, Lemma 3.1]. □

REMARK 4.3. — A simple consequence of this lemma is that given any $A > 0$, there exist $T_0, \lambda_0 > 0$ such that

$$\frac{1}{2}\int_0^T e^{-\lambda s} \| \dot{x}(s) - b(x(s)) \|^2 ds \geq A \quad \text{if} \quad T \geq T_0,\ 0 \leq \lambda \leq \lambda_0,\ x(\cdot) \in H^1([0, T); \overline{\Omega}).$$

□

Proof of Theorem 4.1. — We can rewrite the first equation in (4.1) to read

$$\lambda v + \max_{\alpha \in \mathbb{R}^n} \left\{ - a_{ij}\alpha_i v_j - b_i v_i - \frac{1}{2} a_{ij}\alpha_i\alpha_j \right\} = \lambda v \qquad \text{in} \quad \Omega$$

for any $\lambda > 0$. According to the results of P. L. Lions [*16*] and Evans-Ishii [7] we therefore have

$$v(x) = \inf\left\{ \int_0^{\tau_x} e^{-\lambda s}\left(\frac{\| \dot{x}(s) - b(x(s)) \|^2}{2} + \lambda v(x(s))\right)ds + e^{-\lambda\tau_x} g(x(\tau_x)) \mid x(0) = x \right\}.$$

Using Remark 4.3 we see that therefore if $\lambda > 0$ is sufficiently small

$$v(x) = \inf\left\{ \int_0^{\tau_x} e^{-\lambda s}\left(\frac{\| \dot{x}(s) - b(x(s)) \|^2}{2} + \lambda v(x(s))\right)ds + e^{-\lambda\tau_x} g(x(\tau_x)) \mid x(0) = x, \quad \tau_x \leq \mathrm{T}_0 \right\}$$

for some constant T_0, independent of λ. From this follows (4.2). □

B. The equation $v_t - \frac{1}{2}(\mathrm{D}v)^{\mathrm{T}} a \mathrm{D}v + b.\mathrm{D}v = 0$.

The following proposition does not require condition (B).

THEOREM 4.4. — Assume $v \in \mathrm{C}^{0,1}(\overline{\mathrm{Q}})$ is a viscosity solution of

$$(4.3) \qquad \begin{cases} v_t - \frac{1}{2} a_{ij} v_i v_j + b_i v_i = 0 & \text{in } \mathrm{Q} \\ v = g & \text{on } \partial\mathrm{Q}\backslash\{ t = 0 \}. \end{cases}$$

Then for each $(x, t) \in \mathrm{Q}$

$$(4.4) \quad v(x, t) = \inf\left\{ \frac{1}{2}\int_t^{\mathrm{T}\wedge\tau_x} \| \dot{x}(s) - b(x(s)) \|^2 ds + g(x(\mathrm{T}\wedge\tau_x), \mathrm{T}\wedge\tau_x) \mid x(t) = x \right\},$$

where $\tau_x = \tau_x(x(\cdot))$ is the first exit time from Ω after time t.
This is a consequence of results of P. L. Lions [*15*] [*16*].

REFERENCES

[*1*] D. G. ARONSON, The fundamental solution of a linear parabolic equation containing a small parameter, *Ill. J. Math.*, t. 3, 1959, p. 580-619.

[*2*] G. BARLES, *Thèse de 3e cycle*, Univ. Paris IX, Dauphine, Paris, 1983.

[*3*] E. N. BARRON, L. C. EVANS, and R. JENSEN, Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls, to appear in *J. Diff. Eq.*

[*4*] I. CAPUZZO DOLCETTA and L. C. EVANS, Optimal switching for ordinary differential equations, to appear in *SIAM J. Control and Op.*, t. **22**, 1984, p. 143-161.

[*5*] M. G. GRANDALL, L. C. EVANS, and P. L. LIONS, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. AMS.*, t. **282**, 1984, p. 487-502.

[6] M. G. CRANDALL and P. L. LIONS, Viscosity solutions of Hamilton-Jacobi equations, *Trans. AMS*, t. **277**, 1983, p. 1-42.

[7] L. C. EVANS, and H. ISHII, Differential games and nonlinear first-order PDE on bounded domains, to appear in *Manuscripta Math.*

[8] L. C. EVANS and P. E. SOUGANIDIS, Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations, to appear in *Indiana U. Math. J.*

[9] W. H. FLEMING, *Inclusion probability and optimal stochastic control*, IRIA Seminars Review, 1977.

[10] W. H. FLEMING, Exit probabilities and optimal stochastic control, *Appl. Math. Op.*, t. **4**, 1978, p. 329-346.

[11] W. H. FLEMING, Logarithmic transformations and stochastic control, in *Advances in Filtering and Stochastic Control* (ed. by Fleming and Gorostiza), Springer, New York, 1983.

[12] A. FRIEDMAN, *Stochastic Differential Equations and Applications*, Vol. II, Academic Press, New York, 1976.

[13] C. J. HOLLAND, A minimum principle for the principal eigenvalue of second order linear elliptic equations with natural boundary conditions, *Comm. Pure Appl. Math.*, t. *31*, 1978, p. 509-519.

[14] P. L. LIONS, *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, Boston, 1982.

[15] P. L. LIONS, *Existence results for first-order Hamilton-Jacobi equations*, to appear.

[16] P. L. LIONS, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations I-III, to appear in *Comm. PDE.*

[17] P. E. SOUGANIDIS, *Thesis*, U. of Wisconsin, 1983.

[18] S. R. S. VARADHAN, On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math.*, t. **20**, 1967, p. 431-455.

[19] A. D. VENTCEL and M. I. FREIDLIN, On small random perturbations of dynamical systems, *Russian Math. Surveys*, t. **25**, 1970, p. 1-56.

(Manuscrit reçu le 11 janvier 1984)
(révisé le 20 août 1984)