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GIOVANNA CERAMI
DONATO FORTUNATO
MICHAËL STRUWE

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Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents

by

Giovanna CERAMI (1), **Donato FORTUNATO** (2), **Michaël STRUWE** (3)
Istituto Matematico, Universita', via Archirafi 34,
90123 Palermo, (Italy) and Mathematics Research Center,
610 Walnut Street, Madison, WI 53705 (U. S. A.)

Dipartimento di Matematica, Universita', via Nicolai 2,
70100 Bari (Italy)

Mathematishes Institut der Universität, Beringstrasse 4-6,
D5300, Bonn 1, West Germany

ABSTRACT. — In this paper we study the existence of nontrivial solutions for the boundary value problem

$$\begin{cases} -\Delta u - \lambda u - u\,|\,u\,|^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when $\Omega \subset \mathrm{R}^n$ is a bounded domain, $n \geqslant 3$, $2^* = \dfrac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $\mathrm{H}_0^1(\Omega) \subset \mathrm{L}^p(\Omega)$, λ is a real parameter. We prove that there is bifurcation from any eigenvalue λ_j of $-\Delta$ and we give an estimate of the left neighbourhoods $]\lambda_j^*, \lambda_j]$ of λ_j, $j \in \mathrm{N}$, in which the bifurcation branch can be extended. Moreover we prove that, if $\lambda \in]\lambda_j^*, \lambda_j[$, the number of nontrivial solutions is at least twice the multiplicity of λ_j.

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The same kind of results holds also when Ω is a compact Riemannian manifold of dimension $n \geqslant 3$, without boundary and Δ is the relative Laplace-Beltrami operator.

Key-words: Boundary value problem, critical Sobolev exponent, bifurcation, critical points, eigenvalue, variational problem, Riemannian manifold.

RÉSUMÉ. — Dans cet article, nous étudions l'existence de solutions non triviales pour le problème aux limites

$$\begin{cases} -\Delta u - \lambda u - u\,|\,u\,|^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

où $\Omega \subset \mathrm{R}^n$ est un domaine borné, $n \geqq 3$, $2^* = \dfrac{2n}{n-2}$ est l'exposant critique pour le plongement de Sobolev $\mathrm{H}_0^1(\Omega) \subset \mathrm{L}^p(\Omega)$, λ est un paramètre réel.

Nous démontrons que toute valeur propre λ_j de $-\Delta$ est une valeur de bifurcation, et nous donnons une estimation des voisinages $[\lambda_j^*, \lambda_j]$ de λ_j où existent des solutions non triviales. Nous montrons en outre que le nombre de celles-ci est au moins le double de la multiplicité de λ_j.

On a les mêmes résultats quand Ω est une variété riemannienne compacte de dimension $n \geqq 3$, et Δ l'opérateur de Laplace-Beltrami.

AMS (MOS) Subject Classifications: 35 A 15, 35 J 20, 58 E 99.

INTRODUCTION

Let Ω be a bounded domain in R^n, $n \geqslant 3$, $2^* = \dfrac{2n}{n-2}$ the critical exponent for the Sobolev embedding $\mathrm{H}_0^1(\Omega) \to \mathrm{L}^q(\Omega)$. For a real parameter $\lambda \in \mathbb{R}$ consider the boundary value problem

$$(0.1) \qquad \begin{cases} -\Delta u - \lambda u - u\,|\,u\,|^{2^*-2} = 0 & \text{in } \Omega \\ u\,|_{\partial\Omega} = 0 & \end{cases}$$

corresponding to the functional $f_\lambda : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ given by

$$(0.2) \qquad f_\lambda(u) = 1/2\int_\Omega (\,|\,\nabla u\,|^2 - \lambda\,|\,u\,|^2)dx - 1/2^*\int_\Omega |\,u\,|^{2^*}dx\,.$$

Since the embedding $\mathrm{H}_0^1(\Omega) \to \mathrm{L}^{2^*}(\Omega)$ is not compact the functional f_λ in general will not satisfy the Palais-Smale condition.

However, recently Brezis and Nirenberg [5] were able to establish the existence of positive solutions of (0.1) for any λ in a certain range $]\lambda^*, \lambda_1[$, where λ_j, $j \in \mathbb{N}$ ($\lambda_1 < \lambda_2 < \ldots$), denote the eigenvalues of the operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega) = (H_0^1(\Omega))^*$, and $\lambda^* \geqslant 0$ is some constant depending on n and Ω.

In this paper we study the existence of nontrivial solutions for (0.1) also for $\lambda > \lambda_j$ to obtain bifurcation from any eigenvalue λ_j. We give an estimate of the left neighbourhoods $]\lambda_j^*, \lambda_j]$ of λ_j, in which the bifurcation branch « can be extended »; moreover we prove that, if $\lambda \in]\lambda_j^*, \lambda_j[$, the number of nontrivial solutions of (0.1) is at least twice the multiplicity of λ_j (cp. Theorem 1.1).

Our results are based on the observation that although the Palais-Smale condition does not hold globally for f_λ (cp. Remark 2.3) it is satisfied locally in a certain energy range (cp. Lemma 2.1 or [5, Remark 2.2]).

We observe that the tools used in proving the above results do not depend on the shape of Ω and on the dimension n.

With suitable modifications the existence and bifurcation results also apply to problem (0.1) posed on a compact Riemannian manifold without boundary of dimension $n \geqslant 3$ (cp. Theorem 1.3).

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1. RESULTS

Let $\| u \| = \left(\int_\Omega | \nabla u |^2 dx \right)^{1/2}$, $| u |_p = \left(\int_\Omega | u |^p dx \right)^{1/p}$ denote the norms in $H_0^1(\Omega)$, $L^p(\Omega)$, respectively, and let

$$S = \inf \{ \| u \|^2 / | u |_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{ 0 \} \}$$

denote the best constant for the embedding $H_0^1(\Omega) \to L^{2^*}(\Omega)$.

THEOREM 1.1. — *For $\lambda > 0$ let $\lambda_+ = \min \{ \lambda_j \mid \lambda < \lambda_j \}$, and suppose*

$$\lambda_+ - \lambda < S\,[\text{meas}\,(\Omega)]^{-2/n}.$$

Let m be the multiplicity of λ_+. Then problem (0.1) *admits at least m pairs of nontrivial solutions*

$$\{ u_k(\lambda), - u_k(\lambda) \} \qquad k = 1, \ldots, m$$

such that

$$\| u_k(\lambda) \| \to 0 \quad \text{as} \quad \lambda \to \lambda_+ .$$

■

REMARK 1.2. — If Ω is starshaped, it is well known that (0.1) admits only the trivial solution for $\lambda \leqslant 0$ (cp. [5] [8]).

A result analogous to Theorem 1.1 holds for the problem

$$- \Delta_M u - \lambda u - u \,| u |^{2^*-2} = 0 \tag{1.1}$$

on a compact Riemannian manifold M of dimension $\geqslant 3$ and without boundary. Here Δ_M is the Laplace-Beltrami operator on M, $\lambda \geqslant 0$ a parameter and $2^* = \dfrac{2n}{n-2}$ as before. Denote by $H^1(M)$ the closure of $C^\infty(M)$ with respect to the norm

$$\| u \|_M = \left(\int_M (| \nabla u |^2 + | u |^2) dM \right)^{1/2}$$

which in local coordinates on a covering $\{ T_h \}$ of M is given by

$$\| u \|_M = \left(\sum_h \int_{T_h} \left(\sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + | u |^2 \right) \sqrt{g} dx \right)^{1/2}$$

g^{ij} denoting the metric tensor, and $g = \det(g^{ij})$. Note that the quadratic form $\int_M | \nabla u |^2 dM$ is only positive semidefinite in $H^1(M)$, then the operator

$$- \Delta_M : H^1(M) \to H^{-1}(M) := (H^1(M))'$$

possesses eigenvalues $\mu_1 < \mu_2 < \dots \mu_k < \dots$ which are $\geqslant 0$ (cp. Appendix 1 of [*4*]).

THEOREM 1.3. — *For $\lambda > 0$ let $\mu_+ = \min \{ \mu_j | \lambda < \mu_j \}$ and suppose*

$$\mu_+ - \lambda < S \left(\int_M dM \right)^{-2/n}.$$

Let m be the multiplicity of μ_+. Then problem (1.1) *admits at least m pairs of nonconstant solutions*

$$\{ u_k(\lambda), - u_k(\lambda) \} \qquad k = 1, \dots, m$$

such that

$$\| u_k(\lambda) \|_M \to 0 \quad as \quad \lambda \to \mu_+ .$$

■

2. PROOF OF THEOREMS 1.1, 1.3

The proof of Theorem 1.1 requires some lemmata.

LEMMA 2.1. — *For any $\lambda \in \mathbb{R}$ the functional f_λ (see* (0.2)) *satisfies the Palais-Smale condition in $\left] -\infty, \frac{1}{n} S^{n/2} \right[$ in the following sense:*

(P. S.) *If* $c < \frac{1}{n} S^{n/2}$ *and* $\{u_m\}$ *is a sequence in* $H_0^1(\Omega)$ *such that as* $m \to \infty$ $f_\lambda(u_m) \to c$, $df_\lambda(u_m) \to 0$ *strongly in* $H^{-1}(\Omega)$, *then* $\{u_m\}$ *contains a subsequence converging strongly in* $H_0^1(\Omega)$.

REMARK 2.2. — An analogous result has been proved in [*5*]. Nevertheless for completeness we give here a proof of lemma 2.1 which is slightly different from that contained in [*5*].

Proof. — Let $\lambda \in \mathbb{R}$, and suppose $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ such that as $m \to \infty$

$$f_\lambda(u_m) \to c_1 < \frac{1}{n} S^{n/2} \tag{2.1}$$

$$df_\lambda(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega)\,. \tag{2.2}$$

As in [*5*, estimates (2.18)] from (2.1), (2.2) we obtain that

$$\{ \| u_m \| \} \text{ is bounded}\,. \tag{2.3}$$

Hence we may extract a subsequence $\{u_m\}$ (relabeled) such that

$$u_m \to u \text{ weakly in } H_0^1(\Omega) \tag{2.4}$$

$$u_m \to u \text{ strongly in } L^p(\Omega) \text{ for any } p \in [1, 2^*[. \tag{2.5}$$

Moreover u is a solution of (0.1). Indeed, letting $\phi \in C_0^\infty(\Omega)$, by (2.4), (2.5) and (2.2) we deduce that

$$\langle df_\lambda(u), \phi \rangle = \langle df_\lambda(u_m), \phi \rangle + o(1) = o(1)\,.$$

Hence u weakly solves (0.1). But by regularity results (cp. [*5*] [*6*] [*7*] and [*10*]) it follows that

$$u \in L^\infty(\Omega) \tag{2.6}$$

and hence that u is regular and is a solution of (0.1) in the classical sense. To show that $u_m \to u$ strongly in $H_0^1(\Omega)$ as $m \to \infty$, let $v_m = u_m - u$. Testing (2.2) with v_m we obtain

$$\begin{aligned} o(1) &= \langle df_\lambda(u_m), v_m \rangle \\ &= \int_\Omega (\nabla u \nabla v_m + |\nabla v_m|^2 - \lambda(u+v_m)v_m - |u+v_m|^{2^*-2}(u+v_m)v_m)dx\,. \end{aligned} \tag{2.7}$$

By (2.4) and (2.5) we have

$$\int_\Omega (\nabla u \nabla v_m - \lambda(u + v_m)v_m)dx = o(1)\,. \tag{2.8}$$

Whence from (2.7), (2.8) we deduce that

$$\| v_m \|^2 = \int_\Omega | u + v_m |^{2^*-2}(u + v_m)v_m dx + o(1)\,. \tag{2.9}$$

Now we claim that

$$\| v_m \|^2 = | v_m |_{2^*}^{2^*} + o(1). \tag{2.10}$$

In fact, by using (2.5) and (2.6), we have

$$\begin{aligned}(2.11)\quad & \left| \int_\Omega (u + v_m) | u + v_m |^{2^*-2} v_m dx - \int_\Omega | v_m |^{2^*} dx \right| \\ & \qquad = \left| \int_\Omega \int_0^{u(x)} \frac{\partial}{\partial \xi} [(v_m + \xi) | v_m + \xi |^{2^*-2}] v_m d\xi dx \right| \\ & = \left| (2^* - 1) \int_\Omega \int_0^1 | v_m + tu |^{2^*-2} v_m u dt dx \right| \\ & \qquad \leqslant \text{const.} \left[\int_\Omega (| u | | v_m |^{2^*-1} + | v_m | | u |^{2^*-1}) dx \right] = o(1)\end{aligned}$$

and (2.10) easily follows from (2.9) and (2.11).
Since

$$\langle df_\lambda(u_m), u_m \rangle = o(1)$$

we have

$$| u_m |_{2^*}^{2^*} = \int_\Omega (| \nabla u_m |^2 - \lambda | u_m |^2) dx + o(1).$$

Inserting into the expression for $f_\lambda(u_m)$ we obtain

$$\begin{aligned}(2.12)\quad f_\lambda(u_m) &= \frac{1}{n} \int_\Omega (| \nabla u_m |^2 - \lambda | u_m |^2) dx + o(1) \\ &= \frac{1}{n} \int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx + \frac{1}{n} \int_\Omega | \nabla v_m |^2 dx + o(1).\end{aligned}$$

Moreover, since u is a solution of (0.1)

$$\int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx - \int_\Omega | u |^{2^*} dx = \langle df_\lambda(u), u \rangle = 0.$$

Whence in particular

$$\int_\Omega (| \nabla u |^2 - \lambda | u |^2) dx \geqslant 0. \tag{2.13}$$

From (2.12) and (2.13) we now infer

$$\| v_m \|^2 \leqslant n f_\lambda(u_m) + o(1).$$

Then, by (2.1), for m sufficiently large we obtain

$$\| v_m \|^2 \leqslant c_2 < S^{n/2}. \tag{2.14}$$

Now, by (2.10)

$$\| v_m \|^2 \leqslant S^{-2^*/2} \| v_m \|^{2^*} + o(1).$$

Or equivalently

$$\| v_m \|^2(S^{2^*/2} - \| v_m \|^{2^*-2}) \leqslant o(1).$$

Taking account of (2.14) this implies that $v_m \to 0$ strongly in $H_0^1(\Omega)$. concluding the proof. ■

REMARK 2.3. — Complementing the preceding lemma we have a non-compactness result for energies $\geqslant \frac{1}{n} S^{n/2}$. In fact we now show that for any $\lambda \in \mathbb{R}$ there exists a sequence $\{ u_m \} \subset H_0^1(\Omega)$ satisfying the P-S assumptions in $c = \frac{1}{n} S^{n/2}$, which is not relatively compact in $H_0^1(\Omega)$.

Let $x_0 \in \Omega$ and choose a function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ in a neighbourhood $\mathcal{N}$ of x_0. The functions $u_\mu : \mathbb{R}^n \to \mathbb{R}$

$$u_\mu(x) = \frac{[n(n-2)\mu^2]^{\frac{n-2}{4}}}{[\mu^2 + |x - x_0|^2]^{\frac{n-2}{2}}}$$

solve the equation

$$(2.15) \qquad - \Delta u_\mu = u_\mu |u_\mu|^{2^*-2} \quad \text{in } R^n.$$

Let

$$u_m = \phi u_{\mu_m} \qquad \mu_m = \frac{1}{m}.$$

Note that $u_m \in H_0^1(\Omega)$ and moreover

$$(2.16) \qquad \{ u_m \} \text{ is uniformly bounded in } H_0^1(\Omega).$$

Also we easily derive that as $m \to +\infty$

$$(2.17) \qquad \nabla u_{\mu_m} \to 0 \quad \text{in } L^2(\mathbb{R}^n \backslash \mathcal{N})$$

$$(2.18) \qquad u_m \to 0 \quad \text{in } L^\infty_{\text{loc}}(\Omega \backslash \{ x_0 \}).$$

Hence also

$$(2.19) \qquad u_m \to 0 \text{ weakly in } H_0^1(\Omega) \quad (m \to \infty).$$

Using (2.17) and (2.18) we deduce that

$$(2.20) \quad f_\lambda(u_m) = 1/2 \int_{R^n} |\nabla u_{\mu_m}|^2 dx - 1/2^* \int_{R^n} |u_{\mu_m}|^{2^*} dx + o(1)$$

$$= \frac{1}{n} S^{n/2} + o(1) \qquad \text{(cp. [1] [9]).}$$

Also using (2.15)-(2.18) we obtain

$$\| df_\lambda(u_m) \|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1} = 1}} \int_{R^n} (\nabla u_{\mu_m} \nabla v - u_{\mu_m} | u_{\mu_m} |^{2^*-2} v) dx + o(1) = o(1)$$

Hence $\{u_m\}$ satisfies the (P-S) assumptions with $c = \frac{1}{n} S^{n/2}$, however, by (2.19) and (2.20), $\{u_m\}$ cannot be relatively compact in $H_0^1(\Omega)$.

LEMMA 2.4. — *For* $\lambda > 0$ *let* $\lambda_+ = \inf\{\lambda_j \mid \lambda < \lambda_j\}$ *and set*

$$M_+ = \overline{\bigoplus_{\lambda_j \geq \lambda_+} M(\lambda_j)} \qquad \text{(the closure is taken in } H_0^1(\Omega))$$

$$M_- = \bigoplus_{\lambda_j \leq \lambda_+} M(\lambda_j)$$

where $M(\lambda_j)$ *denotes the eigenspace of* $-\Delta$ *corresponding to* λ_j. *Then*

$$\beta_\lambda := \sup_{u \in M_-} f_\lambda(u) \leq (\lambda_+ - \lambda)^{n/2} \frac{\text{meas}(\Omega)}{n},$$

moreover, there exist constants $\rho_\lambda > 0$, $\delta_\lambda \in]0, \beta_\lambda[$ *such that*

$$f_\lambda(u) \geq \delta_\lambda \qquad \text{for any } u \in M_+, \ \|u\| = \rho_\lambda .$$

Proof. — For any $u \in M_-$ we have

$$\begin{aligned} f_\lambda(u) &= 1/2 \int_\Omega (|\nabla u|^2 - \lambda |u|^2)dx - 1/2^* \int_\Omega |u|^{2^*}dx \\ &\leq 1/2(\lambda_+ - \lambda) \int_\Omega |u|^2 dx - 1/2^* \int_\Omega |u|^{2^*}dx \\ &\leq 1/2(\lambda_+ - \lambda)\,\text{meas}(\Omega)^{2/n} \left\{ \int_\Omega |u|^{2^*}dx \right\}^{2/2^*} - 1/2^* \int_\Omega |u|^{2^*}dx . \end{aligned}$$

Let

$$g(\rho) = 1/2(\lambda_+ - \lambda)\,\text{meas}(\Omega)^{2/n}\rho^2 - 1/2^* \rho^{2^*} .$$

Then

$$\sup_{u \in M_-} f_\lambda(u) \leq \sup_{\rho \geq 0} g(\rho) = \frac{1}{n}(\lambda_+ - \lambda)^{n/2}\,\text{meas}(\Omega)$$

proving the first part of the lemma.
Since for $u \in M_+$ we obtain

$$\int_\Omega (|\nabla u|^2 - \lambda |u|^2)dx \geq \left(1 - \frac{\lambda}{\lambda_+}\right) \|u\|^2$$

while

$$|u|_{2^*}^{2^*} \leq \text{const}\, \|u\|^{2^*} .$$

The second part of the claim is immediate. ■

By lemmata 2.1, 2.4, Theorem 1.1 can be deduced by the following result of Bartolo, Benci, Fortunato (cp. Theorem 2.4 of [*3*]), which is a variant of some results contained in [0].

THEOREM 2.5. — *Let* H *be a real Hilbert space with norm* $\|\cdot\|$ *and suppose* $I \in C^1(H, \mathbb{R})$ *is a functional on* H *satisfying the following conditions:*

I_1) $I(u) = I(-u)$, $I(0) = 0$;

I_2) *There exists a constant* $\beta > 0$ *such that the Palais-Smale condition* (P-S) *holds in* $]0, \beta[$;

I_3) *There exist two closed subspaces* $V, W \subset H$ *and positive constants* ρ, δ, β', *with* $\delta < \beta' < \beta$ *such that*

i) $I(u) \leqslant \beta'$ *for any* $u \in W$
ii) $I(u) \geqslant \delta$ *for any* $u \in V$, $\| u \| = \rho$
iii) $\operatorname{codim} V < +\infty$ and $\dim W \geqslant \operatorname{codim} V$.

Then there exists at least

$$\dim W - \operatorname{codim} V$$

pairs of critical points of I *with critical values belonging to the interval* $[\delta, \beta']$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. — Let $H = H_0^1(\Omega)$, $I = f_\lambda$, $V = M_+$, $W = M_-$, $\beta = \frac{1}{n} S^{n/2}$, $\beta' = \beta_\lambda$, $\delta = \delta_\lambda$, $\rho = \rho_\lambda$ and apply Theorem 2.5 together with lemmata 2.1, 2.4. ■

For the proof of Theorem 1.3 the following result from [*2*] is needed.

LEMMA 2.6. — *If* $\{ v_m \}$ *is a sequence in* $H^1(M)$ *such that* $v_m \rightharpoonup 0$ *weakly in* $H^1(M)$ *as* $m \to \infty$, *then*

$$\left(\int_M | v_m |^{2^*} dM \right)^{2/2^*} \leqslant S^{-1} \| v_m \|_M^2 + o(1).$$

Proof. — By [*2*, Theorem 2.21] for all $\phi \in H^1(M)$, $\varepsilon > 0$

$$\left(\int_M | \phi |^{2^*} dM \right)^{2/2^*} \leqslant (S^{-1} + \varepsilon) \int_M | \nabla \phi |^2 dM + A(\varepsilon) \int_M | \phi |^2 dM$$

with a constant $A(\varepsilon)$ independent of ϕ. Applying this inequality with $\phi = v_m$, and noting that by weak convergence $v_m \to 0$ $(m \to +\infty)$ we have

$$\int_M | v_m |^2 dM \to 0 \qquad m \to +\infty$$

we deduce that for any $\varepsilon > 0$

$$\left(\int_M | v_m |^{2^*} dM \right)^{2/2^*} \leqslant (S^{-1} + \varepsilon) \| v_m \|_M^2 + o(1).$$

The lemma follows on letting $\varepsilon \to 0$. ■

Proof of Theorem 1.3. — Going through the proof of Lemma 2.1 — keeping in mind Lemma 2.6 and the fact that, for any sequence $\{ v_m \}$

in $H^1(M)$ tending to 0 weakly in this space, $\| v_m \|_2 = o(1)$ — it is now immediate that also for the functional on $H^1(M)$

$$f_\lambda(u) = 1/2\int_M (|\nabla u|^2 - \lambda |u|^2)dM - 1/2^*\int |u|^{2^*}dM$$

corresponding to problem (1.1) the Palais-Smale condition is satisfied in the interval $\left]-\infty, \frac{1}{n} S^{n/2}\right[$.

Moreover it is easy to see that the same estimates of lemma 2.4 continue to hold (obviously λ_j, λ_+, $H_0^1(\Omega)$, meas Ω are replaced respectively by $\mu_j \cdot \mu_+$, $H^1(M)$, $\left(\int_M dM\right)$. Then Theorem 1.3 can be proved by using again the abstract critical point Theorem 2.5. ■

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