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Existence results for first order Hamilton Jacobi equations

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ABSTRACT. — We prove existence results for the classical first-order Hamilton Jacobi Equations. These results rely on the notion of viscosity solution and they are obtained under the same assumptions as the uniqueness optimal results for bounded, uniformly continuous solutions.

Key-words: Viscosity solution, existence result, comparison result.

RÉSUMÉ. — On démontre des résultats d'existence pour les équations classiques de Hamilton Jacobi du premier ordre. Ces résultats sont basés sur la notion de solution de viscosité et sont obtenus sous les mêmes hypothèses que les résultats optimaux d'unicité pour les solutions bornées uniformément continues.

Mots-clés: Solution de viscosité, résultat d'existence, résultat de comparaison.

I. INTRODUCTION

We prove here existence results for the following problems:

(1) $$\mathrm{H}(x, u, \mathrm{D}u) = 0 \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega,$$

which we call the Dirichlet problem for Hamilton Jacobi equations, and

$$(2)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \mathrm{H}(x, t, u, \mathrm{D}u) = 0 \quad \text{in } \Omega \times \,]0, \mathrm{T}[\\ u(x, t) = \phi(x, t) \quad \text{on } \partial\Omega \times \,]0, \mathrm{T}[, \qquad u(x, 0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

which we call the Cauchy problem for Hamilton Jacobi equations. Here and below Ω is any open subset of $\mathbb{R}^{\mathrm{N}}$, z and u_0 are given functions (boundary conditions), and $\mathrm{H}(x, u, p)$ (resp $\mathrm{H}(x, t, u, p)$) is a given continuous function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}$ (resp on $\overline{\Omega} \times [0, \mathrm{T}] \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}}$) which is called the *Hamiltonian.* When we take $\Omega = \mathbb{R}^{\mathrm{N}}$, there is no boundary condition; we will only require u to be bounded.

Problems (1) and (2) are global non linear first order problems; and it is well known that, in general, they do not have classical solutions — i. e. of class C^1 — even if the Hamiltonian and boundary conditions are smooth. On the other hand, several authors proved the existence of *generalized solutions* (i. e. solutions in $\mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$ or $\mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega \times \,]0, \mathrm{T}[)$ which satisfy the equations almost everywhere) under various assumptions: e. g. A. Douglis [*5*], S. N. Kruzkov [*10*] [*11*] [*12*], W. H. Fleming [*6*] [*7*] [*8*], A. Friedman [*9*], S. H. Benton [*1*], the most general results being given by P. L. Lions [*13*]. The problem of uniqueness seems to be more difficult: one proves easily that, in general, (1) and (2) have many generalized solutions (see E. D. Conway and E. Hopf [*2*], M. G. Crandall and P. L. Lions [*4*], P. L. Lions [*13*]).

To solve this problem, M. G. Crandall and P. L. Lions [*4*] introduced the notion of viscosity solutions (called this way because the generalized solutions of (1) and (2) obtained by the vanishing viscosity method are proved to be viscosity solutions of (1) and (2)). This notion of solutions, which is defined for continuous solutions, has many interesting properties, especially uniqueness and stability results (see M. G. Crandall and P. L. Lions [*4*], M. G. Crandall, L. C. Evans and P. L. Lions [*3*] and P. L. Lions [*13*]). Existence results of viscosity solutions were then obtained by P. L. Lions [*13*] [*14*]: these results are obtained under two types of assumptions; roughly speaking, one concerns the dependence of H in p (in particular $\mathrm{H} \to +\infty$, when $|p| \to +\infty$) and the other concerns the dependence of H in x. P. E. Souganidis [*15*] has recently extended the second case.

This paper is concerned with existence results under *optimal assumptions* concerning the dependence of H in x. In fact, all the existence results are proved under the assumptions (proved to be optimal in M. G. Crandall and P. L. Lions [*4*]) which give uniqueness results in $\mathrm{BUC}(\Omega)$ (*) or

(*) BUC (0) is the space of bounded uniformly continuous functions in 0.

$\mathrm{BUC}(\Omega \times]0, T[)$ and, in the case of $\Omega \neq \mathbb{R}^N$, under the classical condition of the existence of viscosity sub- and supersolution of (1) (or (2)) in $\mathrm{BUC}(\Omega)$ (or $\mathrm{BUC}(\Omega \times]0, T[)$). Let us just mention that we explain in sections II and III when such viscosity sub- and supersolutions exist and how this type of results extends those proved in [*13*] [*14*] or [*15*].

In section II, we briefly recall the notion of viscosity solutions (for more details, the reader can refer to M. G. Crandall and P. L. Lions [*4*], M. G. Crandall, L. C. Evans and P. L. Lions [*3*] or P. L. Lions [*13*]). In section III, we prove the existence result for (1) in a general open subset Ω, then in $\Omega = \mathbb{R}^N$. The section III is devoted to the proof of the existence result for (2) in a general open subset Ω. Then we give some particular results in the case $\Omega = \mathbb{R}^N$.

II. ON THE NOTION OF VISCOSITY SOLUTION

We want in this section to recall the definition of *viscosity solution* introduced by M. G. Crandall and P. L. Lions [*4*]. We only give the most useful definition (a simple presentation can be found in M. G. Crandall, L. C. Evans and P. L. Lions [*3*]).

We will define the notion of viscosity solution of

$$\text{(3)} \qquad F(y, u(y), Du) = 0 \quad \text{in } \mathcal{O}.$$

$\mathcal{O}$ open subset of $\mathbb{R}^m$ where

$$F \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^m).$$

REMARK II.1. — (1) and (2) are special cases of (3). For (1), take $F = H$, $\mathcal{O} = \Omega$ and $y = x$; for (2), take $\mathcal{O} = \Omega \times]0, T[$ and $y = (x, t)$; then $F(y, r, P) = p_{n+1} + H(x, t, r, p)$ where $P = (p, p_{n+1})$.

DÉFINITION II.1. — Let $u \in C(\mathcal{O})$.

i) u is a viscosity subsolution of (3) if and only if, for all $\phi \in C^1(\mathcal{O})$, we have:

$$\text{(4)} \qquad \left\{ \begin{array}{l} \text{at each local maximum point } y_0 \text{ of } u - \phi \text{ in } \mathcal{O}, \text{ we have} \\ \qquad F(y_0, u(y_0), D\phi(y_0)) \leqslant 0 \end{array} \right.$$

ii) u is a viscosity supersolution of (3), if and only if, for all $\phi \in C^1(\mathcal{O})$, we have

$$\text{(5)} \qquad \left\{ \begin{array}{l} \text{at each local minimum point } y_0 \text{ of } u - \phi \text{ in } \mathcal{O}, \text{ we have} \\ \qquad F(y_0, u(y_0), D\phi(y_0)) \geqslant 0 \end{array} \right.$$

iii) u is a viscosity solution of (3) if and only if u satisfies both (4) and (5).

We do not recall here the main uniqueness results. The reader can find them in M. G. Crandall and P. L. Lions [*4*] or M. G. Crandall, L. C. Evans and P. L. Lions [*3*], P. L. Lions [*13*].

III. EXISTENCE RESULTS FOR THE DIRICHLET PROBLEM

We denote by BUC(Ω) the space of bounded uniformly continuous functions on Ω. We will use the following assumptions:

(6) H is uniformly continuous on $\overline{\Omega} \times (-\mathrm{R}, \mathrm{R}) \times \overline{\mathrm{B}}_{\mathrm{R}} \quad (\forall \mathrm{R} < \infty)$

(7) $$\begin{cases} \forall \mathrm{R} < \infty, \exists \gamma_{\mathrm{R}} > 0 \quad \mathrm{H}(x, t, p) - \mathrm{H}(x, s, p) \geqslant \gamma_{\mathrm{R}}(t - s) \\ \text{for} \quad x \in \overline{\Omega}, \quad -\mathrm{R} \leqslant s \leqslant t \leqslant \mathrm{R}, p \in \mathbb{R}^{\mathrm{N}}. \end{cases}$$

(8) $$\limsup_{\varepsilon \to 0} \{ |\mathrm{H}(x, t, p) - \mathrm{H}(y, t, p)| \, \| x - y|(1 + |p|) \leqslant \varepsilon, |t| \leqslant \mathrm{R} \} = 0 \qquad (\forall \mathrm{R} < \infty)$$

REMARK III.1. — Let us just remark that (6), (7), (8) are the optimal assumptions which give a uniqueness result for (1) in BUC(Ω).

THEOREM III.1. — Under assumptions (6), (7), (8) and if we assume that there exist $\underline{u}$ and $\overline{u} \in$ BUC(Ω) respectively viscosity sub- and supersolution of:

$$\mathrm{H}(x, u, \mathrm{D}u) = 0 \qquad \text{in } \Omega,$$

and such that $\underline{u} = \overline{u}$ on $\partial\Omega$, then there exists a unique viscosity solution u in BUC(Ω) of (1) with $\phi = \underline{u}_{/\partial\Omega} = \overline{u}_{/\partial\Omega}$.

In the case where $\Omega = \mathbb{R}^{\mathrm{N}}$, we can give the following corollary:

COROLLARY III.1. — Under assumptions (6), (7), (8) and if we assume that there exists $\mathrm{M} > 0$ such that

$$\mathrm{H}(x, \mathrm{M}, 0) \geqslant 0 \quad \text{and} \quad \mathrm{H}(x, -\mathrm{M}, 0) \leqslant 0 \qquad \forall x \in \mathbb{R}^{\mathrm{N}},$$

then there exists a unique viscosity solution u in BUC($\mathbb{R}^{\mathrm{N}}$) of

$$\mathrm{H}(x, u, \mathrm{D}u) = 0 \qquad \text{in } \mathbb{R}^{\mathrm{N}}.$$

REMARK III.2. — Let us just mention that in corollary III.1, the existence of M is insured by an assumption like:

(9) $\exists \alpha > 0, \forall x \in \mathbb{R}^{\mathrm{N}}, \forall t, s \in \mathbb{R} \quad (\mathrm{H}(x, t, 0) - \mathrm{H}(x, s, 0))(t - s) \geqslant \alpha(t - s)^2$

REMARK III.2. — The above results extends those obtained by P. L. Lions [*14*] and P. E. Souganidis [*15*]. Let us just recall that the existence result of [14] similar to theorem III.1 was proved under stronger assumptions (6) (7) and:

$$(10)\quad \begin{cases} \text{H is lipschitz in } \Omega \text{ for } (t, p) \in \mathbb{R} \times \mathbb{R}^N \\ \forall R < \infty, \exists C_0^R, C_R \text{ such that:} \\ \qquad \left|\dfrac{\partial H}{\partial x}\right| \leqslant C_0^R |p| + C_R \text{ a. e. } |t| \leqslant R, x \in \overline{\Omega}, p \in \mathbb{R}^N \end{cases}$$

and finally $\bar{u}$, $\underline{u}$ (like in theorem III.1) were supposed to be in $W^{1,\infty}(\Omega)$. Theorem III.1 (and particularly corollary III.1) generalized also an existence result of P. E. Souganidis [15] obtained essentially under assumptions (6), (7) and:

$$(11)\quad \limsup_{\varepsilon \downarrow 0} \{ |H(x,t,p) - H(y,t,p)| \backslash |x-y| \leqslant \varepsilon |x-y| \, |p| \leqslant R, |t| \leqslant R \} = 0 \quad (\forall R < \infty).$$

Let us just notice that (11) implies (8).

Finally, let us mention that a different type of existence result is given in P. L. Lions [13] [14] under the assumption: $H(x, t, p) \to +\infty$ when $|p| \to +\infty$ uniformly in $x \in \overline{\Omega}$, $|t| \leqslant R$ $(\forall R < \infty)$ (for example).

REMARK III.3. — Let us notice a particular case (interesting for applications to optimal control problems): if H is convex in p for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$, then any subsolution of

$$\underline{u} \in W^{1,\infty}(\Omega),\ H(x, \underline{u}, D\underline{u}) \leqslant 0 \text{ a. e.}$$

is a viscosity subsolution of (1) (see [4] [13] for the proof of this claim). In this case, the existence of such $\underline{u}$ is discussed in S. N. Kruskov [10] [11] [12], W. H. Fleming [7] and P. L. Lions [13]. (In [13], a necessary and sufficient condition for the existence of $\underline{u}$ is given).

Proof of theorem III.1.

STEP 1

$\Omega = \mathbb{R}^N$. Let us consider the Hamiltonian $\tilde{H}$ defined by:

$$\tilde{H}(x, t, p) = [[(-\bar{u}(x)) \vee (H(x, t, p) - t)] \wedge (-\underline{u}(x))] + t$$

where $a \wedge b = \inf(a, b)$ and $a \vee b = \sup(a, b)$.
$\tilde{H}$ is well defined because, by comparison result in $\mathbb{R}^N$ (see [3] [4] or [13]), we have

$$\underline{u} \leqslant \bar{u}.$$

We shall prove that there exists a unique viscosity solution u in $BUC(\mathbb{R}^N)$ of

$$\tilde{H}(x, u, Du) = 0 \quad \text{in } \mathbb{R}^N,$$

and that u is also the unique viscosity solution in $BUC(\mathbb{R}^N)$ of:

$$H(x, u, Du) = 0 \quad \text{in } \mathbb{R}^N.$$

a) Existence for $\tilde{H}$.

It is easy to prove that $\tilde{H}$ still satisfies the assumptions (6), (7) and (8). Let us next remark that (8) implies that the function $x \to \tilde{H}\left(\frac{x}{1+|p|}, t, p\right)$ is uniformly continuous in $\mathbb{R}^N$, uniformly with respect to $p \in \mathbb{R}^N$ and $|t| \leqslant R$ ($\forall R < \infty$).

Then, if we define H_ε by

$$H_\varepsilon(x, t, p) = \left(\tilde{H}\left(\frac{\cdot}{1+|p|}, t, p\right) * \rho_\varepsilon\right)(x) = \int_{\mathbb{R}^N} \tilde{H}\left(\frac{x-y}{1+|p|}, t, \rho\right)\rho_\varepsilon(y)dy,$$

where

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon N}\rho\left(\frac{x}{\varepsilon}\right) \quad \text{with} \quad \rho \in D^+(\mathbb{R}^N), \operatorname{supp}\rho \subset B(0, 1), \int_{\mathbb{R}^N} \rho(x)dx = 1;$$

by a well known result, we have

$$\forall R < \infty \qquad \left\| H_\varepsilon(x, t, p) - \tilde{H}\left(\frac{x}{1+|p|}, t, p\right)\right\|_{L^\infty(\mathbb{R}^N \times (-R,R) \times \mathbb{R}^N)} \to 0 \quad \text{when } \varepsilon \to 0;$$

and since $\tilde{H}$ is bounded, if $|t| \leqslant R$:

$$H_\varepsilon(\cdot, t, p) \in W^{1,\infty}(\mathbb{R}^N), \left|\frac{\partial H_\varepsilon}{\partial x}\right| \leqslant C_\varepsilon(R) \qquad \forall R < \infty.$$

Let us define $\tilde{H}_\varepsilon$ by:

$$\tilde{H}_\varepsilon(x, t, p) = H_\varepsilon((1 + |p|)\cdot x, t, p),$$

then

$$\forall R < \infty \quad \|\tilde{H}_\varepsilon - \tilde{H}\|_{L^\infty(\mathbb{R}^N \times (-R,R) \times \mathbb{R}^N)} \to 0 \qquad \text{when} \quad \varepsilon \to 0$$

$$\forall R < \infty \quad \left|\frac{\partial \tilde{H}_\varepsilon}{\partial x}\right| \leqslant C_\varepsilon(R)(1 + |p|).$$

Moreover, if $R_0 = \max(\|\bar{u}\|_{L^\infty(\mathbb{R}^N)}, \|\underline{u}\|_{L^\infty(\mathbb{R}^N)})$, we claim that $-R_0$ and R_0 are respectively viscosity sub- and supersolution of

$$\tilde{H}_\varepsilon(x, u, Du) = 0 \qquad \text{in } \mathbb{R}^N.$$

As $-R_0$ and $R_0 \in C^\infty(\mathbb{R}^N)$, it suffices to prove that $\forall x \in \mathbb{R}^N$ $\tilde{H}_\varepsilon(x, -R_0, 0) \leqslant 0$ and $\tilde{H}_\varepsilon(x, R_0, 0) \geqslant 0$.

We use the fact that

$$-\bar{u}(x) \leqslant \tilde{H}(x, t, p) - t \leqslant -\underline{u}(x)$$

and then:

$$-R_0 \leqslant \tilde{H}(x, t, p) - t \leqslant R_0.$$

Now we deduce easily that

$$-\mathrm{R}_0 \leqslant \tilde{\mathrm{H}}_\varepsilon(x, t, p) - t \leqslant \mathrm{R}_0 ;$$

and then, taking $t = \mathrm{R}_0$ or $t = -\mathrm{R}_0$ and $p = 0$, in the above inequalities, we obtain

$$\tilde{\mathrm{H}}_\varepsilon(x, \mathrm{R}_0, 0) \geqslant 0$$

and

$$\tilde{\mathrm{H}}_\varepsilon(x, -\mathrm{R}_0, 0) \leqslant 0 .$$

Then, by an existence result of P. L. Lions [*14*], we conclude that there exists a unique viscosity solution u^ε in $\mathrm{BUC}(\mathbb{R}^\mathrm{N})$ of:

$$\tilde{\mathrm{H}}_\varepsilon(x, u^\varepsilon, \mathrm{D}u^\varepsilon) = 0 \qquad \text{in } \mathbb{R}^\mathrm{N} .$$

Moreover, by comparison results, we have

$$-\mathrm{R}_0 \leqslant u^\varepsilon \leqslant \mathrm{R}_0 .$$

But, for all $\varepsilon > 0$, u^ε is a viscosity subsolution of

$$\tilde{\mathrm{H}}(x, u, \mathrm{D}u) - \| \tilde{\mathrm{H}}_\varepsilon - \tilde{\mathrm{H}} \|_{\mathrm{L}^\infty(\mathbb{R}^\mathrm{N} \times (-\mathrm{R}_0, \mathrm{R}_0) \times \mathbb{R}^\mathrm{N})} = 0 \qquad \text{in } \mathbb{R}^\mathrm{N},$$

and u^ε is a viscosity supersolution of

$$\tilde{\mathrm{H}}(x, u, \mathrm{D}u) + \| \tilde{\mathrm{H}}_\varepsilon - \tilde{\mathrm{H}} \|_{\mathrm{L}^\infty(\mathbb{R}^\mathrm{N} \times (-\mathrm{R}_0, \mathrm{R}_0) \times \mathbb{R}^\mathrm{N})} = 0 \qquad \text{in } \mathbb{R}^\mathrm{N}.$$

Then, using comparison results, we obtain, for all ε and $\varepsilon' > 0$:

$$\| u^\varepsilon - u^{\varepsilon'} \|_{\mathrm{L}^\infty(\mathbb{R}^\mathrm{N})} \leqslant \frac{1}{\gamma} (\| \tilde{\mathrm{H}}_\varepsilon - \tilde{\mathrm{H}} \|_{\mathrm{L}^\infty(\mathbb{R}^\mathrm{N} \times (-\mathrm{R}_0, \mathrm{R}_0) \times \mathbb{R}^\mathrm{N})} + \| \tilde{\mathrm{H}}_\varepsilon - \tilde{\mathrm{H}} \|_{\mathrm{L}^\infty(\mathbb{R}^\mathrm{N} \times (-\mathrm{R}_0, \mathrm{R}_0) \times \mathbb{R}^\mathrm{N})})$$

where

$$\gamma = \inf(\gamma_{\mathrm{R}_0}, 1) .$$

So, u^ε is a Cauchy sequence in $\mathrm{BUC}(\mathbb{R}^\mathrm{N})$, and thus u^ε converges to $u \in \mathrm{BUC}(\mathbb{R}^\mathrm{N})$, uniformly in $\mathbb{R}^\mathrm{N}$. Using the stability of viscosity solution, we conclude that u is a viscosity solution of

$$\tilde{\mathrm{H}}(x, u, \mathrm{D}u) = 0 \qquad \text{in } \mathbb{R}^\mathrm{N} .$$

Moreover, by comparison results, we have $\underline{u} \leqslant u \leqslant \bar{u}$, because $\underline{u}$ and $\bar{u}$ are respectively viscosity sub- and supersolution of $\tilde{\mathrm{H}}(x, u, \mathrm{D}u) = 0$ in $\mathbb{R}^\mathrm{N}$.

b) u **is a viscosity solution of** $\mathrm{H}(x, u, \mathrm{D}u) = 0$ **in** $\mathbb{R}^\mathrm{N}$**.**

We just prove that u is a viscosity subsolution of $\mathrm{H}(x, u, \mathrm{D}u) = 0$. The proof to show that u is a viscosity supersolution is exactly the same. Let $\phi \in \mathrm{C}^1(\mathbb{R}^\mathrm{N})$ and let x_0 be a local maximum point of $u - \phi$. We have:

$$[[(-\bar{u}(x_0)) \vee \mathrm{H}(x_0, u(x_0), \mathrm{D}\phi(x_0)) - u(x_0)] \wedge (-\underline{u}(x_0))] + u(x_0) \leq 0 .$$

i) If $H(x_0, u(x_0), D\phi(x_0)) - u(x_0) < -\overline{u}(x_0)$, then

$$H(x_0, u(x_0), D\phi(x_0)) < -\overline{u}(x_0) + u(x_0) \leqslant 0.$$

ii) If $-\overline{u}(x_0) \leqslant H(x_0, u(x_0), D\phi(x_0)) - u(x_0) \leqslant -\underline{u}(x_0)$, there is nothing to show.

iii) If $H(x_0, u(x_0), D\phi(x_0)) - u(x_0) > -\underline{u}(x_0)$, then $-\underline{u}(x_0) + u(x_0) \leqslant 0$, and so $\underline{u}(x_0) = u(x_0)$.

But x_0 is a local maximum point of $u - \phi$, therefore if $r > 0$ is such that we have

$$u(x_0) - \phi(x_0) \geqslant u(x) - \phi(x) \qquad \text{for} \quad x \in B(x_0, r),$$

we deduce easily

$$\underline{u}(x_0) - \phi(x_0) \geqslant \underline{u}(x) - \phi(x) \qquad \text{for} \quad x \in B(x_0, r).$$

So x_0 is a local maximum point of $\underline{u} - \phi$, and since $\underline{u}$ is a viscosity subsolution of $H(x, u, Du) = 0$ in $\mathbb{R}^N$, we have

$$H(x_0, u(x_0), D\phi(x_0)) = H(x_0, \underline{u}(x_0), D\phi(x_0)) \leqslant 0.$$

This ends to proof of the step 1.

STEP 2

We will show that solving a problem of type (1) in $\Omega \neq \mathbb{R}^N$ is equivalent to solve a problem of type (1) in $\mathbb{R}^N$.

More precisely, we consider the Hamiltonian $\tilde{H}$ defined in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ by

$$\tilde{H}(x, t, p) = \begin{cases} [[(-\overline{u}(x)) \vee (H(x, t, p) - t)] \wedge (-\underline{u}(x))] + t & \text{if} \quad x \in \overline{\Omega} \\ -\tilde{u}(x) + t & \text{if} \quad x \notin \overline{\Omega} \end{cases}$$

where $\tilde{u}$ is a convenient extension of $\underline{u}$ in $\mathbb{R}^N$ such that $\tilde{u} \in BUC(\mathbb{R}^N)$. Since $\underline{u}_{/\partial\Omega} = \overline{u}_{/\partial\Omega} = \phi$, $\tilde{H}$ satisfies the assumption (6), (7), (8). Moreover if u_1 is the function of $BUC(\mathbb{R}^N)$ defined by

$$u_1(x) = \begin{cases} \overline{u}(x) & \text{if} \quad x \in \overline{\Omega} \\ \tilde{u}(x) & \text{if} \quad x \notin \overline{\Omega}, \end{cases}$$

then obviously, $\tilde{u}$ and u_1 are respectively viscosity sub- and supersolutions of

$$\tilde{H}(x, u, Du) = 0 \qquad \text{in } \mathbb{R}^N.$$

Then, by step 1, there exists a unique viscosity solution u in $BUC(\mathbb{R}^N)$ of

$$\tilde{H}(x, u, Du) = 0 \qquad \text{in } \mathbb{R}^N.$$

and by comparison results, we have

$$\tilde{u} \leqslant u \leqslant u_1 \qquad \text{in } \mathbb{R}^N,$$

then $\tilde{u} = u$ in $\mathbb{R}^N - \overline{\Omega}$, and therefore $u = \phi$ on $\partial\Omega$.

And, by a property of viscosity solution (see M. G. Crandall and P. L. Lions [4]), since u is a viscosity of $\tilde{H}(x, u, Du) = 0$ in $\mathbb{R}^N$ and since Ω is an open subset of $\mathbb{R}^N$, then $u_{/\Omega}$ is a viscosity solution of

$$\tilde{H}(x, w, Dw) = 0 \qquad \text{in } \Omega .$$

This ends the proof of theorem III.1.

REMARK III.4. — Let us finally remark that this result is proved for any open subset Ω of $\mathbb{R}^N$: we don't need any assumption of smoothness or boundedness for Ω.

IV. EXISTENCE RESULTS FOR THE CAUCHY PROBLEM

We will use the following assumptions on H:

(9) $H(x, t, r, p)$ is bounded uniformly continuous on

$$\Omega \times [0, T] \times (-R, R) \times \overline{B}_R \qquad (\forall R < \infty),$$

$$(10) \quad \begin{cases} \forall R < \infty, \exists \gamma_R \in \mathbb{R} \quad \text{such that for} \quad x \in \Omega,\ t \in]0, T[\quad \text{and,} \\ -R \leqslant s \leqslant r \leqslant R, \quad \text{we have} \quad H(x, t, r, p) - H(x, t, s, p) \geqslant \gamma_R(r - s) . \end{cases}$$

$$(11) \quad \limsup_{\varepsilon \downarrow 0} \{ \,|\, H(x, t, r, p) - H(y, t, r, p) \,|\, \backslash \,|\, x - y \,|\, (1 + |\, p \,|) \leqslant \varepsilon, \ |\, r \,| \leqslant R,\ t \in]0, T[\,\} = 0 \qquad (\forall R < \infty) .$$

THEOREM IV.1. — Under assumptions (9), (10), (11), if there exists $\underline{u}$ and $\overline{u}$ respectively sub- and supersolution of (2) in $BUC(\Omega \times]0, T[)$, such that $\underline{u}(x, t) = \overline{u}(x, t) = \phi(x, t)$ on $\partial\Omega \times [0, T]$ and $\underline{u}(x, 0) = \overline{u}(x, 0) = u_0(x)$ in Ω, then there exists a unique viscosity solution u in $BUC(\Omega \times]0, T[)$ of

$$(2) \quad \begin{cases} \dfrac{\partial u}{\partial t} + H(x, t, u, Du) = 0 \quad \text{in } \Omega \times]0, T[\\ u(x, t) = \phi(x, t) \quad \text{in } \partial\Omega \times [0, T], \quad u(x, 0) = u_0(x) \quad \text{in } \Omega . \end{cases}$$

Let us give a more precise result in the case $\Omega = \mathbb{R}^N$.

COROLLARY IV.1. — Under assumptions (9), (10), (11), if $u_0 \in BUC(\mathbb{R}^N)$, then there exists T_0 ($0 < T_0 \leqslant T$), which depends only on H and $\| u_0 \|_{L^\infty(\mathbb{R}^N)}$, and $u \in BUC(\mathbb{R}^N \times [0, T_0])$ a viscosity solution of

$$(12) \quad \begin{cases} \dfrac{\partial u}{\partial t} + H(x, t, u, Du) = 0 \quad \text{in } \mathbb{R}^N \times]0, T_0[\\ u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N . \end{cases}$$

Moreover, if there exists $C \in \mathbb{R}$ such that $\gamma_R \geqslant C$ ($\forall R < \infty$) then $T_0 = T$.

REMARK IV.1. — The result of theorem IV.1 generalizes existence results obtained by P. L. Lions in [13] or [14] under stronger assumptions on the dependence of H in x, t, p and on the regularity of $\underline{u}$ and $\bar{u}$.

REMARK IV.2. — The result of corollary IV.1 is inspired from existence results proved by P. E. Souganidis [15] in the cases where (11) is replaced by

$$(13)\quad \begin{cases} \text{H is Lipschitz in } \mathbb{R}^N \text{ for } (t, r, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^N. \\ \forall R > 0, \exists C_R \text{ such that } \left|\dfrac{\partial H}{\partial x}\right| \leqslant C_R(1 + |p|) \text{ a. e. } x \in \mathbb{R}^N,\ t \in [0, T], \\ \qquad\qquad |r| \leqslant R,\ p \in \mathbb{R}^N \end{cases}$$

or

$$(14)\quad \limsup_{\varepsilon \downarrow 0} \{ |H(x, t, r, p) - H(y, t, r, p)| / |x - y| \leqslant \varepsilon, \\ |x - y||p| \leqslant R, |r| \leqslant R, t \in [0, T] \} = 0$$

for any $R > 0$.

Notice that (13) and (14) are more restrictive assumptions than (11).

REMARK IV.3. — The local existence result of Corollary IV.1 is quite optimal.

Take, for example

$$(15)\quad \begin{aligned} &\frac{\partial u}{\partial t} - u^2 = 0 \quad \text{in } \mathbb{R}^N \times]0, T[. \\ &u(x, 0) = \frac{1}{T_0} \quad \text{with } 0 < T_0 < T. \end{aligned}$$

The viscosity solution is given for $t < T_0$ by $\dfrac{1}{T_0 - t}$. Notice that in this example $\gamma_R = -R$.

REMARK IV.4. — We can make the some remark as in the Dirichlet problem for the existence of $\underline{u}$ and $\bar{u}$. For example, if H is convex in p for all $x \in \Omega$, $t \in [0, T]$, $r \in \mathbb{R}$, any generalized subsolution of

$$\underline{u} \in W^{1,\infty}(\Omega \times]0, T[), \frac{\partial \underline{u}}{\partial t} + H(x, t, \underline{u}, D\underline{u}) \leqslant 0 \text{ a. e.}$$

is a viscosity subsolution of (2). (See [4] [13] for the proof of this claim). Again, in this case the existence of such $\underline{u}$ is discussed in W. H. Fleming [6] [7] [8], A. Friedman [9], S. N. Kruzkov [10] [11] [12] and P. L. Lions [13] (A necessary and sufficient condition for the existence of $\underline{u}$ is given in [13]).

REMARK IV.5. — We will denote by

$$M = \max(\|\underline{u}\|_{L^\infty(\Omega \times]0,T[)}, \|\bar{u}\|_{L^\infty(\Omega \times]0,T[)})$$

REMARK IV.6. — By comparison results for the Cauchy problem (see [3], [4] or [13]), if u is viscosity solution of (2) in $\mathrm{BUC}(\Omega \times]0, T[)$, we have:

$$\underline{u} \leqslant u \leqslant \bar{u}.$$

These inequalities justify the reduction we shall make in the proof of theorem IV.1.

Proof of theorem IV.1.

STEP 1

The step 1 consists in giving the proof when $\gamma_R > 0$ ($\forall R < \infty$). We consider the Hamiltonian $\tilde{H}$ defined on $\Omega \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^N$ by:

$$\tilde{H}(x, t, r, P) = [[(-\bar{u}(x, t) \vee (p_{n+1} + H(x, t, r, p) - r)] \wedge (-\underline{u}(x, t))] + r, \quad \text{if} \quad t \in [0, T]$$

$$\tilde{H}(x, t, r, P) = \tilde{H}(x, T, r, P) \quad \text{if} \quad t \geqslant T$$

for $x \in \Omega$, $r \in \mathbb{R}$, $P \in \mathbb{R}^{N+1}$, where $P = (p, p_{n+1})$.

$\tilde{H}$ satisfies the assumptions (9), (10). Moreover, we have

$$\limsup_{\varepsilon \downarrow 0} \{ | \tilde{H}(x, t, r, P) - \tilde{H}(y, t, r, P) | / | x - y | (1 + | P |) \leqslant \varepsilon \; t \in (0, \infty), \; | r | \leqslant R \} = 0 \quad (\forall R < \infty),$$

because H satisfies (11) and

$$| x - y | (1 + | P |) \leqslant \varepsilon$$

implies

$$| x - y | (1 + | p |) \leqslant \varepsilon.$$

Now let us define the functions u_1 and u_2 of $\mathrm{BUC}(\Omega \times (0, \infty))$ by $u_1(x, t) = \underline{u}(x, t)$ if $t \leqslant T$, $u_1(x, t) = \underline{u}(x, T)$ if $t \geqslant T$ for $x \in \Omega$, and $u_2(x, t) = \bar{u}(x, t)$ if $t \leqslant T$, $u_2(x, t) = \bar{u}(x, T)$ if $t \geqslant T$ for $x \in \Omega$. Then u_1 and u_2 are respectively viscosity sub- and supersolutions in $\Omega \times (0, \infty)$ of

$$\tilde{H}\left(x, t, u, \left(Du, \frac{\partial u}{\partial t}\right)\right) = 0 \qquad \text{in } \Omega \times (0, \infty).$$

If we want to use theorem III.1, it suffices to prove that the function $t \to \tilde{H}\left(x, \dfrac{t}{1 + | P |}, r, P\right)$ is uniformly continuous, uniformly with respect to $x \in \Omega$, $| r | \leqslant R$, $P \in \mathbb{R}^{N+1}$, since this will imply that we have

$$\limsup_{\varepsilon \downarrow 0} \{ | \tilde{H}(x, t, r, P) - \tilde{H}(x, s, r, P) | \setminus | t - s | (1 + | P |) \leqslant \varepsilon, \; x \in \Omega, | r | \leqslant R \} = 0 \qquad (\forall R < \infty).$$

LEMMA IV. — Let $\eta > 0$, there exists $\varepsilon > 0$ such that

$$\sup_{\substack{x \in \Omega \\ |r| < R \\ P \in \mathbb{R}^{N+1}}} \left\{ \left| \tilde{H}\left(x, \frac{t}{1 + | P |}, r, P\right) - \tilde{H}\left(x, \frac{s}{1 + | P |}, r, P\right) \right| \, | t - s | \leqslant \varepsilon \right\} \leqslant \eta.$$

Proof of Lemma IV.1. — Let us remark that

$$\lim_{|\mathrm{P}|\to+\infty}\left(\tilde{\mathrm{H}}\left(x, \frac{t}{1+|\mathrm{P}|}, r, \mathrm{P}\right)\right) = -u_0(x) + r$$

uniformly with respect to $t \in (0, \infty)$, $|r| \leqslant \mathrm{R}$, $x \in \Omega$.
Then, there exists $\mathrm{R}_1 > 0$ such that, for $|\mathrm{P}| \geqslant \mathrm{R}_1$, $x \in \Omega$, $|r| \leqslant \mathrm{R}$ and $t, s \in (0, \infty)$, we have

$$\left|\tilde{\mathrm{H}}\left(x, \frac{t}{1+|\mathrm{P}|}, r, \mathrm{P}\right) - \tilde{\mathrm{H}}\left(x, \frac{s}{1+|\mathrm{P}|}, r, \mathrm{P}\right)\right| \leqslant \eta .$$

Now we fix R_1 with the property above. We just have to look at

$$\sup_{\substack{x\in\mathbb{R}^{\mathrm{N}}\\ |u|\leqslant \mathrm{R}\\ |\mathrm{P}|\leqslant \mathrm{R}_1}} \left\{\left|\tilde{\mathrm{H}}\left(x, \frac{t}{1+|\mathrm{P}|}, u, \mathrm{P}\right) - \tilde{\mathrm{H}}\left(x, \frac{s}{1+|\mathrm{P}|}, u, \mathrm{P}\right)\right| \Big\backslash |t-s| \leqslant \varepsilon\right\}.$$

One concludes easily using the form of $\tilde{\mathrm{H}}$ and the assumption (9). Then, there exists a viscosity solution u in $\mathrm{BUC}(\Omega \times]0, \infty[)$ of

$$\tilde{\mathrm{H}}\left(x, t, u, \left(\mathrm{D}u, \frac{\partial u}{\partial t}\right)\right) = 0 \qquad \text{in } \Omega \times]0, \infty[\,.$$

The same arguments as in the proof of theorem III.1 show that u is the unique viscosity solution of (2) in $\mathrm{BUC}(\Omega \times]0, \mathrm{T}[)$.

STEP 2

We will show in this step that we can assume $\gamma_{\mathrm{R}} > 0$ in (10). Let M defined in remark IV.1 and let H_1 the Hamiltonian defined by:

$$\begin{aligned}
&\mathrm{H}_1(x, t, r, p) = \mathrm{H}(x, t, r, p) && \text{if } |r| \leqslant \mathrm{M}\\
&\mathrm{H}_1(x, t, r, p) = \mathrm{H}(x, t, \mathrm{M}, p) + r - \mathrm{M} && \text{if } r \geqslant \mathrm{M}\\
&\mathrm{H}_1(x, t, r, p) = \mathrm{H}(x, t, -\mathrm{M}, p) + r + \mathrm{M} && \text{if } r \leqslant -\mathrm{M}.
\end{aligned}$$

By the remark IV.2, it is the same to solve (2) with H or with H_1. Moreover H_1 satisfies (9), (10), (11) and in (10) we can choose γ_{R} independent of R, i.e. $\gamma = \gamma_{\mathrm{M}} \wedge 1$. So if $\gamma_{\mathrm{M}} > 0$ we have nothing to show. If $\gamma_{\mathrm{M}} \leqslant 0$, we remark that $v = e^{(\gamma-1)t}u$ is viscosity solution of

$$(2')\quad \begin{cases} \dfrac{\partial v}{\partial t} + (1-\gamma)v + e^{(\gamma-1)t}\mathrm{H}_1(x, t, e^{(1-\gamma)t}v, e^{(1-\gamma)t}\mathrm{D}v) = 0 & \text{in } \Omega \times]0, \mathrm{T}[\,.\\ v(x, t) = e^{(\gamma-1)t}\phi(x, t) \quad \text{on} \quad \partial\Omega \times [0, \mathrm{T}],\ v(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

if and only if u is a viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + H_1(x, t, u, Du) = 0 \text{ in } \Omega \times]0, T[. \\ u(x, t) = \phi(x, t) \quad \text{on } \partial\Omega \times]0, T[, \; u(x, 0) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

Moreover, $e^{(\gamma-1)t}\underline{u}$ and $e^{(\gamma-1)t}\overline{u}$ are respectively viscosity sub- and supersolutions of (2').

Finally $H(x, t, r, p) = (1-\gamma)r + e^{(\gamma-1)t}H_1(x, t, e^{(1-\gamma)t}r, e^{(1-\gamma)t}p)$ satisfies (9), (10) and (11) with $\gamma_R = 1$ ($\forall R < \infty$). So, by step 1, there exists a viscosity solution of (2') $v \in BUC(\Omega \times]0, T[)$. Now if we consider the function u of $BUC(\Omega \times]0, T[)$ given by:

$$u(x, t) = e^{(1-\gamma)t}v(x, t)$$

one show easily that u is the unique viscosity solution of (2) in $BUC(\Omega \times]0, T[)$.

Now we prove the corollary IV.1:

Proof of Corollary IV.1. — The proof consists in building sub- and supersolution in order to use the theorem IV.1.

1st CASE

We treat the particular case when: $u_0 \in \mathscr{C}_b^\infty(\mathbb{R}^N)$.

STEP 1

$\gamma_R \geqslant 0$ ($\forall R < \infty$). Then, the function $r \to H(x, t, r, p)$ is non-decreasing.

We denote by $M = \|H(x, t, u_0(x), Du_0(x)\|_{L^\infty(\mathbb{R}^N \times [0,T])}$, $v(x, t) = Mt + u_0(x)$ and $w(x, t) = -Mt + u_0(x)$. Then v and $w \in \mathscr{C}_b^\infty(\mathbb{R}^N \times]0, T[)$. Moreover

$$\frac{\partial v}{\partial t} + H(x, t, v, Dv) = M + H(x, t, Mt + u_0(x), Du_0(x)) \geqslant M + H(x, t, u_0(x), Du_0(x)) \geqslant 0.$$

Then v is a viscosity supersolution of (12) in $\mathbb{R}^N \times [0, T]$ and in the same way w is a viscosity subsolution of (12) in $\mathbb{R}^N \times [0, T]$; so there exists a viscosity solution of (12) in $BUC(\mathbb{R}^N \times [0, T])$. Let us notice that in this case $T_0 = T$.

STEP 2

We show in this step that we can assume $\gamma_R \geqslant 0$. First we look for *a priori* bound on $\|u\|_{L^\infty(\mathbb{R}^N \times [0,t])}$ if u is viscosity solution of (12) ($t \leqslant T$).

We denote by

$K = \| H(x, t, r, 0) \|_{L^\infty(\mathbb{R}^N \times [0,T] \times (-R_0, R_0))}$ (where $R_0 = \| u_0 \|_{L^\infty(\mathbb{R}^N)}$), $u_1(x, t) = -2Kt - R_0$, $u_2(x, t) = 2Kt + R_0$, $R = 2KT + R_0$ and $\gamma = \gamma_R$.

Then

$$\begin{aligned}\frac{\partial u_1}{\partial t} + H(x, t, u_1, Du_1) &= -2K + H(x, t, -2Kt - R_0, 0)\\ &\leqslant -K(1 + 2\gamma t).\end{aligned}$$

Let T_0 be the greatest real number in $(0, T)$ such that $1 + 2\gamma t > 0$; then in $\mathbb{R}^N \times [0, T_0]$ u_1 is a viscosity subsolution of $\frac{\partial v}{\partial t} + H(x, t, v, Dv) = 0$ in $\mathbb{R}^N \times [0, T_0]$ because $u_1 \in \mathscr{C}_b^\infty(\mathbb{R}^N \times [0, T_0])$. By the same arguments, u_2 is a viscosity supersolution of $\frac{\partial v}{\partial t} + H(x, t, v, Dv) = 0$ in $\mathbb{R}^N \times [0, T_0]$. By comparison results for the Cauchy problem (see [*3*] [*4*] or [*13*]), if u is a viscosity solution of (12) in $\mathbb{R}^N \times [0, T_0]$, we have

$$u_1 \leqslant u \leqslant u_2 .$$

These inequalities justify the following reductions. Let $\tilde{H}$ defined by

$$\tilde{H}(x, t, r, p) = \begin{cases} = H(x, t, r, p) & \text{if } |r| \leqslant R. \\ = H(x, t, R, p) + r - R & \text{if } r \geqslant R. \\ = H(x, t, -R, p) + r + R & \text{if } r \leqslant -R. \end{cases}$$

By the same remarks as in the proof of theorem IV.1 $\tilde{H}$ satisfies (9), (10), (11) and, in (10), we can choose γ independant of R. Now we consider the associated problem:

$$(12^*) \quad \begin{cases} \dfrac{\partial v}{\partial t} + (1-\gamma)v + e^{(\gamma-1)t}\tilde{H}(x, t, e^{(1-\gamma)t}v, e^{(1-\gamma)t}Dv) = \text{ in } \mathbb{R}^N \times]0, T_0[\\ v(x, 0) = u_0(x). \end{cases}$$

Since the Hamiltonian $(1 - \gamma)r + e^{(\gamma-1)t}\tilde{H}(x, t, e^{(1-\gamma)t}r, e^{(1-\gamma)t}p)$ satisfies (9), (10), (11) and in (10), $\gamma_R = 1$ ($\forall R < \infty$), by step 1 there exists a unique viscosity solution v of (12*) in $BUC(\mathbb{R}^N \times [0, T_0])$. One shows easily that u given by:

$$u(x, t) = e^{(1-\gamma)t}v(x, t)$$

is the unique viscosity solution in $\mathbb{R}^N \times [0, T_0]$ of:

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x, t, u, Du) = 0 \quad \text{in } \mathbb{R}^N \times [0, T_0] \\ u(x, 0) = u_0(x) \end{cases}$$

and by *a priori* estimates on $\| u \|_{L^\infty(\mathbb{R}^N \times [0,T])}$ u is the unique viscosity solution of:

$$(16) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + H(x, t, u, Du) = 0 & \text{in } \mathbb{R}^N \times [0, T_0] \\ u(x, 0) = u_0(x). \end{cases}$$

REMARK IV.7. — Let us notice that if there exists $C \in \mathbb{R}$ such that $\gamma_R \geqslant C$ ($\forall R < \infty$), we do not need *a priori* estimates on $\| u \|_{L^\infty(\mathbb{R}^N \times [0,t])}$ ($t \leqslant T$). Considering (12*) with $T_0 = T$ and $\gamma = C$, we obtain, in the same way as Step 2 above, the existence of a viscosity solution of (12) in $\mathbb{R}^N \times [0, T]$.

REMARK IV.8. — We remark easily that T_0 built in the step 2 depends only on $\| u_0 \|_{L^\infty(\mathbb{R}^N)}$ and H.

2nd CASE

We now treat the general case when $u_0 \in BUC(\mathbb{R}^N)$. If $u_0 \in BUC(\mathbb{R}^N)$, there exists $u_0^\varepsilon \in C_b^\infty(\mathbb{R}^N)$ which converges uniformly in $\mathbb{R}^N$ to u_0 and which satisfies $\| u_0^\varepsilon \|_{L^\infty(\mathbb{R}^N)} \leqslant \| u_0 \|_{L^\infty(\mathbb{R}^N)}$. The viscosity solutions of (12) associated to u_0^ε, u^ε are defined on $\mathbb{R}^N \times [0, T_0]$ (T_0 independent of ε), and by comparison results (see [3] [4] or [13]), we have

$$\| u^\varepsilon - u^{\varepsilon'} \|_{L^\infty(\mathbb{R}^N \times [0,T_0])} \leqslant e^{-\gamma T_0} \| u_0^\varepsilon - u_0^{\varepsilon'} \|_{L^\infty(\mathbb{R}^N)}.$$

Then u^ε converges uniformly in $\mathbb{R}^N \times [0, T_0]$ to u which is the unique viscosity solution of (12) by a classical stability result (see [3] [4] or [13]).

REMARK IV.9. — If there exists $C \in \mathbb{R}$ such that $\gamma_R \geqslant C$ ($\forall R < \infty$), all the functions u^ε are viscosity solution of (12) in $\mathbb{R}^N \times [0, T]$ and so u is a viscosity solution of (12) in $\mathbb{R}^N \times [0, T]$. This ends the proof of Corollary IV.1.

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