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Functional viability theorems for differential inclusions with memory

by

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ABSTRACT. — This paper is a study of the relations that must exist between a multivalued dynamical system with memory and constraints depending on the past, in order to have the existence of solutions of the dynamical system satisfying the constraints.

Such solutions are called viable solutions.

RÉSUMÉ. — Ce papier est une étude des relations devant exister entre un système dynamique multivoque avec mémoire et des contraintes d'état dépendant du passé, afin d'assurer pour le système dynamique l'existence de solutions vérifiant les contraintes.

De telles solutions sont appelées des solutions viables.

INTRODUCTION

The viability problem for differential inclusions with memory is formulated as follows.

The past history up to time t will be described by the map $\mathrm{T}(t)$ from the set of continuous mappings $\mathscr{C}(]-\infty, t\,]; \mathbb{R}^n)$ into the set of continuous mappings $\mathscr{C}_0 = \mathscr{C}(]-\infty, 0\,]; \mathbb{R}^n)$ defined by

$$[\mathrm{T}(t)x](z) = x(t+z) \quad \text{for all} \quad z \leqslant 0 \quad \text{and all} \quad x \in \mathscr{C}(]-\infty, t\,]; \mathbb{R}^n)$$

A differential inclusion with memory is then defined through a set-valued map F from $\mathbb{R} \times \mathscr{C}_0$ into $\mathbb{R}^n$ which associates to the past history up to time t of a trajectory x, the subset $\mathrm{F}[t, \mathrm{T}(t)x] \subset \mathbb{R}^n$ of feasible velocities.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449/84/03/179/26/$ 4,60

We say that $x \in \mathscr{C}(]-\infty, t_0 + A]; \mathbb{R}^n)$, $A > 0$, is a solution under the initial condition $(t_0, \varphi_0) \in \mathbb{R} \times \mathscr{C}_0$ of the differential inclusion with memory defined by F if

$$* \quad \begin{cases} T(t_0)x = \varphi_0 \\ (t, T(t)x) \in \operatorname{Dom} F \qquad \text{for all} \qquad t \in [t_0, t_0 + A] \\ x \text{ is absolutely continuous on } [t_0, t_0 + A] \\ x'(t) \in F(t, T(t)x] \qquad \text{for almost all} \qquad t \in [t_0, t_0 + A] \end{cases}$$

where $$\operatorname{Dom} F = \{ (t, \varphi) \in \mathbb{R} \times \mathscr{C}_0 ; \quad F(t, \varphi) \neq \varnothing \}.$$

We say that the solution is defined on $[t_0, +\infty[$ if it verifies $(*)$ for any $A > 0$.

A simple viability problem associated to a differential inclusion with memory is formulated as follows: a nonempty subset $K \subset \mathbb{R}^n$ being given is it possible to give conditions relating F and K for the existence of solutions which verify $x(t) \in K$ for all $t \geqslant t_0$, under every initial condition (t_0, φ_0) such that $\varphi_0(0) \in K$.

A solution which satisfies such properties is called viable since the set K (called viability set) represents generally a family of constraints that the solutions should verify from initial time t_0 in order to be viable. An answer to this viability problem has been given in a previous paper [*6*].

In the present paper, we consider the more specific case where the viability condition depends at each time t on the past history of the trajectory.

We shall particularly consider the case where the solutions are asked to verify

$$x(t) \in D[t, x(t+\theta_t^{(1)}), \ldots, x(t+\theta_t^{(p)})] \quad \text{for all} \quad t \geqslant t_0,$$

with $\theta^{(1)}, \ldots, \theta^{(p)}$ given strictly negative real functions and D a given set-valued map from $\mathbb{R} \times (\mathbb{R}^n)^p$ into $\mathbb{R}^n$.

In this particular case the viability appears to be directly related to decisions taking in account at each time t the knowledge of test values considered at past times $t+\theta_t^{(1)}, \ldots, t+\theta_t^{(p)}$. The fact that $\theta^{(1)}, \ldots, \theta^{(p)}$ are time dependent, means that we take in consideration a possible variation of information delays because of technical factors.

We shall also consider the case where the solutions are asked to verify

$$x(t) = [x_1(t), \ldots, x_n(t)] \in E\left[t, \int_{-\infty}^{t+\theta_t} x_1(z)p_1(z)dz, \ldots, \int_{-\infty}^{t+\theta_t} x_n(z)p_n(z)dz\right]$$

for all $t \geqslant t_0$.
with $\theta_{(.)}$ a given strictly negative real function, $p_1, \ldots, p_n$ given real functions and E a set-valued map from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$.

In that case the viability appears to be related to decisions taking in account the cumulated values

$$\int_{-\infty}^{t+\theta_t} x_1(z)p_1(z)dz, \ldots, \int_{-\infty}^{t+\theta_t} x_n(z)p_n(z)dz$$

of the trajectory up to time $t + \theta_t < t$.

As before the fact that $\theta_{(.)}$ depends on time, means that we take in consideration a possible variation of information delays because of technical factors. These two problems appear as particular cases of the viability problem asking solutions of the differential inclusion with memory to verify:

$$\mathrm{T}(t)x \in \mathscr{K}(t) \qquad \text{for all} \qquad t \geqslant t_0 ,$$

where the viability sets $\mathscr{K}(t) \subset \mathscr{C}_0$ are defined for each time $t \in \mathbb{R}$.

We shall denote by

$$(\mathrm{M}) \quad \begin{cases} x'(t) \in \mathrm{F}[t, \mathrm{T}(t)x] \\ \mathrm{T}(t)x \in \mathscr{K}(t) \end{cases}$$

such a viability problem.

We give in this paper necessary and sufficient conditions relating the dynamical system described by F and the viability constraints described by $(\mathscr{K}(t))_{t\in\mathbb{R}}$, for the existence of viable solutions under any initial condition (t_0, φ_0) such that $\varphi_0 \in \mathscr{K}(t_0)$.

Such a viability problem (M) is a very general one. For example if the viability constraints or if the differential inclusion with memory take in account only a part of the history, it is always possible to set the problem as (M).

Indeed such a case can for example be described by

$$(\mathrm{M}_{a,b}) \quad \begin{cases} x'(t) \in \mathrm{G}[t, (\mathrm{T}(t)x)_a] \\ (\mathrm{T}(t)x)_b \in \mathscr{K}_b(t) \end{cases}$$

where for any $\varphi \in \mathscr{C}_0$, φ_a and φ_b respectively denote the restrictions of φ on $[-a, 0]$ and $[-b, 0]$ with a and b strictly positive, and where $\mathrm{G}(\cdot)$ is a set-valued map from $\mathbb{R} \times \mathscr{C}([-a, 0]; \mathbb{R}^n)$ into $\mathbb{R}^n$ and $\mathscr{K}_b(\cdot)$ a set-valued map from $\mathbb{R}$ into $\mathscr{C}([-b, 0]; \mathbb{R}^n)$.

It suffices then to define

$$\mathrm{F}(t, \varphi) = \mathrm{G}(t, \varphi_a) \qquad \text{for any} \qquad (t, \varphi) \in \mathbb{R} \times \mathscr{C}_0 .$$

and

$$\mathscr{K}(t) = \{ \varphi \in \mathscr{C}_0 ; \quad \varphi_b \in \mathscr{K}_b(t) \} .$$

Historically the viability problem has been introduced by Nagumo [*11*] in the case of ordinary differential equations and when the set of constraints is a nonempty compact of $\mathbb{R}^n$.

Viability problems have then been studied in different situations. We refer for example to Brezis [*2*], Crandall [*4*], Larrieu [*9*] and Yorke [*13*] for differential equations, to Castaing [3], Gautier [5] and Methlouthi [11] for differential inclusions and to Leela-Moauro [10], Seifert [12] for differential equations with memory.

General results on differential equations with memory can be found in [7] and on differential inclusions in [1].

I. ABSTRACT VIABILITY THEOREMS

In this part we consider the abstract viability problem

$$(\mathrm{M}) \quad \begin{cases} x'(t) \in \mathrm{F}[t, \mathrm{T}(t)x] \\ \mathrm{T}(t)x \in \mathscr{K}(t) \end{cases}$$

described at the end of the introduction.

We begin by some useful definitions for the following of the paper.

0. Definitions.

The norm on the finite dimensional vector space $\mathbb{R}^n$ will be denoted by $\|\cdot\|$.

The closed unit ball of $\mathbb{R}^n$ is defined by

$$\mathrm{B} = \{ x \in \mathbb{R}^n ;\ \| x \| \leqslant 1 \}.$$

If $\mathscr{C}$ is a nonempty subset of $\mathbb{R}^n$, then

$$d_{\mathrm{C}}(x) = \inf \{ \| x - c \| ;\ c \in \mathrm{C} \} \qquad \text{for all} \qquad x \in \mathbb{R}^n .$$

For any interval $\mathrm{I} \subset \mathbb{R}$, the topology on the set of continuous mappings $\mathscr{C}(\mathrm{I} ; \mathbb{R}^n)$ will always be the (metrizable) topology of uniform convergence on compact subsets of I.

For any compact interval $[a, b] \subset \mathbb{R}$, $a < b$, we define

$$\| x \|_{[a,b]} = \sup_{t \in [a,b]} \| x(t) \| \qquad \text{for any} \quad x \in \mathscr{C}([a, b] ; \mathbb{R}^n).$$

The graph of a set-valued map F from X into Y is defined as

$$\text{Graph } \mathrm{F} = \{ (x, y) \in \mathrm{X} \times \mathrm{Y} ;\ y \in \mathrm{F}(x) \}.$$

Let X and Y be two metric spaces, a set-valued map F from X into Y is said to be upper-semicontinuous (u. s. c.) at $x_0 \in \mathrm{X}$ if to any neighborhood Ω of $\mathrm{F}(x_0)$ in Y, we can associate a neighborhood U of x_0 in X such that $\mathrm{F}(x) \subset \Omega$ for all $x \in \mathrm{U}$.

We say that F is u. s. c. on a subset of X if F is u. s. c. at every point of this subset.

At least let F be a set-valued map from X into $\mathbb{R}^n$, we say that F is bounded on X if there exists a constant $k > 0$ such that:

$$\| y \| \leqslant k, \qquad \text{for any} \quad x \in \mathrm{X} \quad \text{and any} \quad y \in \mathrm{F}(x).$$

We call k an upper-bound of F.

1. The autonomous viability case.

In this section we consider the autonomous case

$$(\mathrm{M}_0) \quad \begin{cases} x'(t) \in \mathrm{F}\,[\mathrm{T}(t)x] \\ \mathrm{T}(t)x \in \mathscr{K} \end{cases}$$

where the differential inclusion with memory is autonomous and the viability sets $(\mathscr{K}(t))_{t\in\mathbb{R}}$ are invariant upon time.

For conveniency we only consider here initial time $t_0 = 0$ and this with no loss of generality.

DEFINITION I.1. — *For any $\varphi \in \mathscr{K}$, we define $\mathscr{D}_{\mathscr{K}}(\varphi) \subset \mathbb{R}^n$ by $v \in \mathscr{D}_{\mathscr{K}}(\varphi)$ if and only if, for any $\varepsilon > 0$, there exist $h \in\,]0, \varepsilon]$ and $x_h \in \mathscr{C}(]-\infty, h]\,; \mathbb{R}^n)$ such that*

$$(1) \quad \begin{cases} \mathrm{T}(0)x_h = \varphi \\ \mathrm{T}(h)x_h \in \mathscr{K} \\ \dfrac{x_h(h) - x_h(0)}{h} \in v + \varepsilon \mathrm{B}\,. \end{cases}$$

Then we have the following result.

THEOREM I.1. — *Let us suppose that $\mathscr{K}$ is a closed subset of $\mathscr{C}_0$ and that all element of $\mathscr{K}$ is Lipschitz with a same constant.*

Let F from $\mathscr{C}_0$ into $\mathbb{R}^n$ be u. s. c. with nonempty convex compact values on $\mathscr{K}$.

Then condition

$$(\mathrm{C}_0) \qquad \mathrm{F}(\varphi) \cap \mathscr{D}_{\mathscr{K}}(\varphi) \neq \emptyset \qquad \text{for all} \quad \varphi \in \mathscr{K},$$

is necessary and sufficient for the existence under any initial value $\varphi_0 \in \mathscr{K}$ of an associated viable solution of (M_0) *defined on* $[0, +\infty[$.

Necessity of (C_0).

Let us suppose that for the initial value $\varphi_0 \in \mathscr{K}$ there exists a solution $x \in \mathscr{C}(]-\infty, +\infty[\,; \mathbb{R}^n)$ of (M_0).

Then since x is absolutely continuous on any compact interval $[0, \mathrm{A}]$, $\mathrm{A} > 0$, we have

$$\frac{x(h) - x(0)}{h} = \frac{1}{h}\int_0^h x'(z)dz \qquad \text{for all} \quad h > 0$$

The set-valued map F being u. s. c. at φ_0, for any $\varepsilon > 0$ there exists a neighborhood V of φ_0 in $\mathscr{C}_0$ such that

$$F(\varphi) \subset F(\varphi_0) + \varepsilon B \qquad \text{for all} \quad \varphi \in V .$$

Furthermore from the topology defined on $\mathscr{C}_0$ and from the very definition of the map $T(t)$, we easily deduce the existence of $\eta > 0$ such that

$$T(t)x \in V \qquad \text{for all} \quad t \in [0, \eta] .$$

This implies that

$$F(T(t)x) \subset F(\varphi_0) + \varepsilon B \qquad \text{for all} \quad t \in [0, \eta] .$$

But since $x'(t) \in F[T(t)x]$ for almost all $t \geqslant 0$ we have

$$x'(t) \in F(\varphi_0) + \varepsilon B \qquad \text{for almost all} \quad t \in [0, \eta] .$$

We deduce that

$$\frac{x(h) - x(0)}{h} \in F(\varphi_0) + \varepsilon B \qquad \text{for all} \quad h \in]0, \eta]$$

since $F(\varphi_0) + \varepsilon B$ is convex compact, $F(\varphi_0)$ and B being convex compact.

Let us denote by ω the set of limit points of $\dfrac{x(h) - x(0)}{h}$ as $h \to 0^+$.

From the remarks made above and since we are in a finite dimensional space $\mathbb{R}^n$, we deduce that ω is nonempty and verifies

$$\omega \subset F(\varphi_0) + \varepsilon B .$$

This being satisfied for any $\varepsilon > 0$ and since $F(\varphi_0)$ is compact we have $\omega \subset F(\varphi_0)$.

We claim that $\omega \subset \mathscr{D}_{\mathscr{K}}(\varphi_0)$.

Indeed let any $v \in \omega$ be given. By the very definition of a limit point we know that for any $\varepsilon > 0$ there exists $h \in]0, \varepsilon]$ such that

$$\frac{x(h) - x(0)}{h} \in v + \varepsilon B .$$

Since $T(0)x = \varphi_0$ and $T(h)x \in \mathscr{K}$ for all $h \geqslant 0$, we easily deduce that $v \in \mathscr{D}_{\mathscr{K}}(\varphi_0)$.

Thus $\omega \subset F(\varphi_0) \cap \mathscr{D}_{\mathscr{K}}(\varphi_0)$. Q. E. D.

Sufficiency of (C_0).

Let us denote by $\lambda > 0$, the common Lipschitz constant to all elements of $\mathscr{K}$.

Let $\varphi_0 \in \mathscr{K}$ be given, we define for any $a > 0$ the set

$$\mathscr{K}_{a,\varphi_0} = \{ \varphi ; \varphi \in \mathscr{K}, \| \varphi(0) - \varphi_0(0) \| \leqslant a \}.$$

This set is compact, which is a direct consequence of Ascoli's compactness theorem.

To prove the existence of a viable solution of (M_0) associated to the initial value φ_0 we shall build a family of approximated solutions. For this we need the following lemma.

LEMMA I.1. — *For any $\varepsilon > 0$, there exists a finite sequence $h_1, \ldots, h_p$ such all $0 < h_i < \varepsilon$ for all $i = 1, \ldots, p$ and $\sum_{i=1}^{p} h_i > \frac{a}{\lambda + \varepsilon}$, to which is associated $(v_i, y_i) \in \mathbb{R}^n \times \mathscr{C}(]-\infty, h_i]; \mathbb{R}^n)$, $i = 1, \ldots, p$ such that for all i:*

$$(2) \qquad \begin{cases} T(0)y_i \in \mathscr{K}, \qquad T(h_i)y_i \in \mathscr{K}, \qquad v_i \in F[T(0)y_i] \\ \dfrac{y_i(h_i) - y_i(0)}{h_i} \in v_i + \varepsilon B \\ \| T(h_{i-1})y_{i-1} - T(0)y_i \|_{[-1/\varepsilon, 0]} < h_i \cdot \varepsilon \end{cases}$$

where $h_0 = 0$ and $y_0 = \varphi_0$.

Proof. — From (C_0), to any $\varphi \in \mathscr{K}_{a,\varphi_0}$ we can associate $v_\varphi \in F(\varphi)$, $h_\varphi \in]0, \varepsilon]$ and $y_\varphi \in \mathscr{C}(]-\infty, h_\varphi]; \mathbb{R}^n)$ such that

$$(1) \qquad \begin{cases} T(0)y_\varphi = \varphi \\ T(h_\varphi)y_\varphi \in \mathscr{K} \\ \dfrac{y_\varphi(h_\varphi) - y_\varphi(0)}{h_\varphi} \in v_\varphi + \varepsilon B \end{cases}$$

We notice that y_φ is λ-Lipschitz on $]-\infty, h_\varphi]$.

Let us define

$$V(\varphi) = \{ \psi \in \mathscr{C}_0 ; \| \psi - \varphi \|_{[-1/\varepsilon, 0]} < h_\varphi \cdot \varepsilon \}.$$

Such a set is an open neighborhood of φ in $\mathscr{C}_0$. Since $\mathscr{K}_{a,\varphi_0}$ is compact, there exists I finite such that

$$\mathscr{K}_{a,\varphi_0} \subset \bigcup_{i \in I} V(\varphi_i).$$

Then there exists $i_1 \in I$ such that $\varphi_0 \in V(\varphi_{i_1})$. We denote $h_1 = h_{\varphi_{i_1}}$, $y_1 = y_{\varphi_{i_1}}$, $v_1 = v_{\varphi_{i_1}}$. Then h_1, v_1, y_1 obviously verify (2).

Let us consider $T(h_1)y_1$, we have:

$$\begin{aligned} \| (T(h_1)y_1)(0) - \varphi_0(0) \| &\leqslant \| y_1(h_1) - y_1(0) \| + \| y_1(0) - \varphi_0(0) \| \\ &\leqslant \lambda h_1 + \varepsilon h_1 = (\lambda + \varepsilon) h_1 . \end{aligned}$$

Thus if $(\lambda + \varepsilon)h_1 > a$, we stop.

Otherwise $T(h_1)y_1 \in \mathscr{K}_{a,\varphi_0}$, then there exists φ_{i_2} such that $T(h_1)y_1 \in V(\varphi_{i_2})$. We then define $h_2 = h_{\varphi_{i_2}}$, $v_2 = v_{\varphi_{i_2}}$ and $y_2 = y_{\varphi_{i_2}}$ and easily verify that they satisfy (2).

Moreover we have:

$$\begin{aligned} &\| (T(h_2)y_2)(0) - \varphi_0(0) \| \\ &\qquad \leqslant \| (T(h_2)y_2)(0) - (T(h_1)y_1)(0) \| + \| (T(h_1)y_1)(0) - \varphi_0(0) \| \\ &\qquad \leqslant \| y_2(h_2) - y_2(0) \| + \| y_2(0) - (T(h_1)y_1)(0) \| + (\lambda + \varepsilon) h_1 \\ &\qquad \leqslant \lambda h_2 + \varepsilon h_2 + (\lambda + \varepsilon) h_1 = (\lambda + \varepsilon)(h_1 + h_2) . \end{aligned}$$

We stop if $h_1 + h_2 > \dfrac{a}{\lambda + \varepsilon}$.

Otherwise we continue. Since we have a finite number of h_{φ_i}, $i \in I$, all strictly positive, we are sure that after a finite number of operations we get a first h_p such that $\displaystyle\sum_{i=1}^{p-1} h_i \leqslant \frac{a}{\lambda + \varepsilon} < \sum_{i=1}^{p} h_i$.

Q. E. D.

Construction of the approximated solutions.

We first define the mapping y_ε from $\left]-\infty, \displaystyle\sum_{i=0}^{p} h_i\right]$ into $\mathbb{R}^n$ such that $T(0)y_\varepsilon = y_0 = \varphi_0$.

And for any $k \in \{0, \ldots, p-1\}$

$$y_\varepsilon(t) = y_{k+1}\left(t - \sum_{i=0}^{k} h_i\right) + \sum_{i=0}^{k} (y_i(h_i) - y_{i+1}(0))$$

if
$$t \in \left[\sum_{i=0}^{k} h_i, \sum_{i=0}^{k+1} h_i\right]$$

To each y_ε we associate x_ε from $\left]-\infty, \displaystyle\sum_{i=0}^{p} h_i\right]$ into $\mathbb{R}^n$ such that

$$T(0)x_\varepsilon = T(0)y_\varepsilon = \varphi_0 .$$

And for any $k \in \{0, \ldots, p-1\}$

$$x_\varepsilon(t) = \frac{t - \sum_{i=0}^{k} h_i}{h_{k+1}} \left[y_\varepsilon\left(\sum_{i=0}^{k+1} h_i\right) - y_\varepsilon\left(\sum_{i=0}^{k} h_i\right)\right] + y_\varepsilon\left(\sum_{i=0}^{k} h_i\right)$$

for any
$$t \in \left[\sum_{i=0}^{k} h_i, \sum_{i=0}^{k+1} h_i\right].$$

Then x_ε appears to be the piecewise linear mapping which interpolates $y_\varepsilon\left(\sum_{i=0}^{k} h_i\right)$ and $y_\varepsilon\left(\sum_{i=0}^{k+1} h_i\right)$ on each interval $\left[\sum_{i=0}^{k} h_i, \sum_{i=0}^{k+1} h_i\right]$.

Furthermore it is obvious that y_ε and x_ε are λ-Lipschitz since each y_i, $i = 0, \ldots, p$, is λ-Lipschitz. We have also $\| x_\varepsilon(t) - y_\varepsilon(t) \| < \lambda \cdot \varepsilon$ for all $t \in \left]-\infty, \sum_{i=0}^{p} h_i\right]$. This is obvious if $t \leqslant 0$ since $x_\varepsilon(t) = y_\varepsilon(t)$.

If $t \in \left[\sum_{i=0}^{k} h_i, \sum_{i=0}^{k+1} h_i\right]$ we easily verify that:

$$\| x_\varepsilon(t) - y_\varepsilon(t) \| \leqslant \lambda \cdot h_{k+1} < \lambda \cdot \varepsilon .$$

Moreover for any $k \in \{0, \ldots, p-1\}$ we deduce from (2) that

$$\frac{x_\varepsilon\left(\sum_{i=0}^{k+1} h_i\right) - x_\varepsilon\left(\sum_{i=0}^{k} h_i\right)}{h_{k+1}} = \frac{y_\varepsilon\left(\sum_{i=0}^{k+1} h_i\right) - y_\varepsilon\left(\sum_{i=0}^{k} h_i\right)}{h_{k+1}}$$

$$= \frac{y_{k+1}(h_{k+1}) - y_{k+1}(0)}{h_{k+1}} \in v_{k+1} + \varepsilon \mathrm{B}$$

with $v_{k+1} \in \mathrm{F}\,[\mathrm{T}(0)y_{k+1}]$.

At last we claim that for any $k \in \{0, \ldots, p-1\}$

$$\left\| \mathrm{T}\left(\sum_{i=0}^{k+1} h_i\right)x_\varepsilon - \mathrm{T}(h_{k+1})y_{k+1} \right\|_{[-1/\varepsilon, 0]} < \left(\lambda + \sum_{i=0}^{k+1} h_i\right)\varepsilon$$

Since $\| x_\varepsilon(t) - y_\varepsilon(t) \| < \lambda\varepsilon$ for any t, it suffices to prove that

$$\left\| T\left(\sum_{i=0}^{k+1} h_i\right)y_\varepsilon - T(h_{k+1})y_{k+1} \right\|_{[-1/\varepsilon,0]} < \left(\sum_{i=0}^{k+1} h_i\right)\varepsilon$$

This is done by induction on $k \in \{ 0, \ldots, p-1 \}$.

For $k = 0$, if $z \in [-h_1, 0]$ then

$$\begin{aligned}[T(h_1)y_\varepsilon](z) - [T(h_1)y_1](z) &= y_\varepsilon(z+h_1) - y_1(z+h_1) \\ = y_1(z+h_1) + y_0(0) - y_1(0) - y_1(z+h_1) & \\ &= y_0(0) - y_1(0).\end{aligned}$$

If $z \in [-1/\varepsilon, -h_1]$ then $y_\varepsilon(z+h_1) - y_1(z+h_1) = y_0(z+h_1) - y_1(z+h_1)$. But by (2) we know that $\| T(0)y_0 - T(0)y_1 \|_{[-1/\varepsilon,0]} < h_1 \cdot \varepsilon$. Thus is true that $\| T(h_1)y_\varepsilon - T(h_1)y_1 \|_{[-1/\varepsilon,0]} < h_1 \cdot \varepsilon$.

Let us suppose that for k:

$$\left\| T\left(\sum_{i=0}^{k} h_i\right)y_\varepsilon - T(h_k)y_k \right\|_{[-1/\varepsilon,0]} < \left(\sum_{i=0}^{k} h_i\right)\cdot\varepsilon.$$

Then if $z \in [-h_{k+1}, 0]$ we have

$$\left[T\left(\sum_{i=0}^{k+1} h_i\right)y_\varepsilon\right](z) - [T(h_{k+1})y_{k+1}](z)$$

$$= y_{k+1}\left(z + \sum_{i=0}^{k+1} h_i - \sum_{i=0}^{k} h_i\right) + \sum_{i=0}^{k} [y_i(h_i) - y_{i+1}(0)] - y_{k+1}(z+h_{k+1})$$

$$= \sum_{i=0}^{k} [y_i(h_i) - y_{i+1}(0)].$$

If $z \in [-1/\varepsilon, -h_{k+1}]$ then

$$\left[T\left(\sum_{i=0}^{k+1} h_i\right)y_\varepsilon\right](z) - [T(h_{k+1})y_{k+1}](z)$$

$$\begin{aligned}&= \left[T\left(\sum_{i=0}^{k} h_i\right)y_\varepsilon\right](z+h_{k+1}) - [T(h_k)y_k](z+h_{k+1}) \\ &\quad + [T(h_k)y_k](z+h_{k+1}) - y_{k+1}(z+h_{k+1}).\end{aligned}$$

Then from (2) and the hypothesis for k we easily deduce that

$$\left\| T\left(\sum_{i=0}^{k+1} h_i\right)y_\varepsilon - T(h_{k+1})y_{k+1} \right\|_{[-1/\varepsilon,0]} < \left(\sum_{i=0}^{k+1} h_i\right)\cdot\varepsilon.$$

Thus we have built a λ-Lipschitz mapping x_ε from $\left]-\infty, \sum_{i=1}^{p} h_i\right]$ into $\mathbb{R}^n$, linear on each interval $\left[\sum_{i=0}^{k} h_i, \sum_{i=0}^{k+1} h_i\right]$, such that $T(0)x_\varepsilon = \varphi_0$ and for each $k \in \{0, \ldots, p-1\}$:

$$\left\{\begin{array}{l} \dfrac{x_\varepsilon\left(\sum_{i=0}^{k+1} h_i\right) - x_\varepsilon\left(\sum_{i=0}^{k} h_i\right)}{h_{k+1}} \in v_{k+1} + \varepsilon B, \qquad v_{k+1} \in F[T(0)y_{k+1}] \\ \left\| T\left(\sum_{i=0}^{k+1}\right)x_\varepsilon - T(h_{k+1})y_{k+1} \right\|_{[-1/\varepsilon, 0]} < \left(\lambda + \sum_{i=0}^{k+1} h_i\right) \cdot \varepsilon . \end{array}\right.$$

where $\{ h_i, y_i, v_i ; i = 0, \ldots, p \}$ verifies all the properties of the preceding lemma.

Then for any ε small enough we are sure that $\dfrac{a}{2\lambda} < \sum_{i=1}^{p} h_i$. Thus using Ascoli's theorem there exists a sequence $x_{\varepsilon_n}, \varepsilon_n \to 0^+$, which converges (uniformly on compact subsets) to a mapping x defined on $\left]-\infty, \dfrac{a}{2\lambda}\right]$ which is λ-Lipschitz and verifies $T(0)x = T(0)x_{\varepsilon_n} = \varphi_0$.

We claim that x is a viable solution of (M_0). All the techniques of this proof being exactly identical to those used in [*6*].

The result being true not only for initial time zero and since $\dfrac{a}{2\lambda}$ does not depend on φ_0, it is obvious that the viable solution can be extended on $[0, +\infty[$. Q. E. D.

2. The nonautonomous viability case.

In this section we consider the nonautonomous viability case

$$(M) \quad \left\{\begin{array}{l} x'(t) \in F[t, T(t)x] \\ T(t)x \in \mathscr{K}(t) \end{array}\right.$$

described at the end of the introduction.

Here F is a set-valued map from $\mathbb{R} \times \mathscr{C}_0$ into $\mathbb{R}^n$ and $\mathscr{K}$ a set-valued map from $\mathbb{R}$ into $\mathscr{C}_0$ which graph is supposed to be nonempty.

DEFINITION I.2. — *For any* (t, φ), $\varphi \in \mathcal{K}(t)$ *we define* $\mathcal{D}_{\mathcal{K}(t)}(\varphi) \subset \mathbb{R}^n$ *by* $v \in \mathcal{D}_{\mathcal{K}(t)}(\varphi)$ *if and only if, for any* $\varepsilon > 0$, *there exist* $h \in]0, \varepsilon]$ *and* $x_h \in \mathcal{C}(]-\infty, t+h]; \mathbb{R}^n)$ *such that*

$$(3) \qquad \begin{cases} \mathrm{T}(t)x_h = \varphi \\ \mathrm{T}(t+h)x_h \in \mathcal{K}(t+h) \\ \dfrac{x_h(t+h) - x_h(t)}{h} \in v + \varepsilon \mathrm{B}\,. \end{cases}$$

Then we have the following result.

THEOREM I.2. — *Let us suppose that* $\mathcal{K}(\cdot)$ *has a closed graph and that all elements of* $\mathcal{K}(t)$ *are Lipschitz with a constant independent from* t.

Let F *be u. s. c. with nonempty convex compact values on the graph of* $\mathcal{K}(\cdot)$.

Then condition

$$(\mathrm{C}) \qquad \mathrm{F}(t, \varphi) \cap \mathcal{D}_{\mathcal{K}(t)}(\varphi) \neq \emptyset \qquad \textit{for all} \qquad (t, \varphi) \in \mathrm{Graph}\ \mathcal{K}$$

is necessary and sufficient for the existence under any initial condition $(t_0, \varphi_0) \in \mathrm{Graph}\ \mathcal{K}$ *of an associated viable solution of* (M) *defined on* $[t_0, +\infty[$.

Proof. — The proof of the necessity of (C) is similar to the one given in the autonomous case.

Let us now suppose that (C) is verified.

We denote $\mathcal{C}_* = \mathcal{C}(]-\infty, 0]; \mathbb{R} \times \mathbb{R}^n)$ and define

$$\mathcal{L} = \{ (z(\cdot), \varphi) \in \mathcal{C}_* ; \varphi \in \mathcal{K}_{z(0)}, z(\cdot) \text{ is 1-Lipschitz} \}\,.$$

as well as the set-valued map G from $\mathcal{C}_*$ into $\mathbb{R}^{n+1}$ such that

$$\mathrm{G}[z(\cdot), \varphi] = \{ 1 \} \times \mathrm{F}(z(0), \varphi) \qquad \text{for all} \qquad (z, \varphi) \in \mathcal{C}_*\,.$$

There is no difficulty to verify that $\mathcal{L}$ is nonempty closed in $\mathcal{C}_*$, that all elements of $\mathcal{L}$ are Lipschitz with a same constant and that G is u. s. c. with nonempty convex compact values on $\mathcal{L}$.

Let now $(z, \varphi) \in \mathcal{L}$ be given. Then by (C) we know the existence of $v \in \mathrm{F}[z(0), \varphi]$ such that for any $\varepsilon > 0$ there exist

$$h \in]0, \varepsilon] \qquad \text{and} \qquad x_h \in \mathcal{C}(]-\infty, z(0) + h]; \mathbb{R}^n)$$

which verify

$$(3) \qquad \begin{cases} \mathrm{T}[z(0)]x_h = \varphi \\ \mathrm{T}[z(0) + h]x_h \in \mathcal{K}(z(0) + h) \\ \dfrac{x_h(z(0) + h) - x_h(z(0))}{h} \in v + \varepsilon \mathrm{B} \end{cases}$$

We can then define z_h on $]-\infty, h]$ by

$$\begin{cases} z_h(t) = z(t) & \text{for all} \quad t \in]-\infty, 0] \\ z_h(t) = z(0) + t & \text{for all} \quad t \in [0, h]. \end{cases}$$

$z_h(\cdot)$ is obviously 1-Lipschitz since $z(\cdot)$ is 1-Lipschitz.

Let us then consider $(z_h, y_h) \in \mathscr{C}(]-\infty, h]; \mathbb{R} \times \mathbb{R}^n)$ such that

$$y_h(t) = x_h(z(0) + t) \qquad \text{for all} \qquad t \in]-\infty, h].$$

Then

$$T(0)[z_h, y_h] = [T(0)z_h, T(0)y_h] = [T(0)z_h, T(z(0))x_h] = (z, \varphi)$$

and

$$T(h)[z_h, y_h] = [T(h)z_h, T(z(0) + h)x_h] \in \mathscr{L}$$

since

$$T(z(0) + h)x_h \in \mathscr{K}(z(0) + h) \qquad \text{and} \qquad z(0) + h = (T(h)z_h)(0).$$

Moreover,

$$\frac{z_h(h) - z_h(0)}{h} = 1$$

and

$$\frac{y_h(h) - y_h(0)}{h} = \frac{x_h(z(0) + h) - x_h(z(0))}{h} \in v + \varepsilon B.$$

Thus the autonomous system defined by G and $\mathscr{L}$ verifies all the hypothesis of Theorem I.1. Then for any initial value $(z_0, \varphi_0) \in \mathscr{L}$ and any initial time t_0 there exists an associated viable solution defined on $[t_0, +\infty[$.

Let now $\varphi_0 \in \mathscr{K}(t_0)$ be given. We define z_{t_0} to be the constant function t_0 on $]-\infty, 0]$.

Then $(z_{t_0}, \varphi_0) \in \mathscr{L}$ and there exists under the initial condition $t_0, (z_{t_0}, \varphi_0)$ a viable solution (u, x) which verifies:

$$\begin{cases} T(t_0)u = z_{t_0}, \qquad T(t_0)x = \varphi_0 \\ (u, x) \text{ is Lipschitz} \\ u'(t) = 1 \quad \text{for almost all} \quad t \geqslant t_0 \\ x'(t) \in F[u(t), T(t)x] \quad \text{for almost all} \quad t \geqslant t_0 \\ T(t)(u, x) \in \mathscr{L} \qquad \text{for all} \qquad t \geqslant t_0. \end{cases}$$

It is then obvious that $u(t) = t$ for all $t \geqslant t_0$ and that $T(t)(u, x) \in \mathscr{L}$ implies

$$T(t)x \in \mathscr{K}[(T(t)u)(0)] = \mathscr{K}(u(t)) = \mathscr{K}(t) \quad \text{for all} \quad t \geqslant t_0.$$

Thus x is a viable solution of (M) under the initial condition (t_0, φ_0). Q.E.D.

II. APPLICATIONS

In this part we show that we can deduce from the preceding theorems, the specific viability problems presented in the introduction.

1. Viability depending on test values.

Let F be a given set-valued map from $\mathbb{R} \times \mathscr{C}_0$ into $\mathbb{R}^n$ and D a given set-valued map from $\mathbb{R} \times (\mathbb{R}^n)^p$ into $\mathbb{R}^n$ whose graph is supposed to be nonempty.

At last $\theta^{(1)}, \ldots, \theta^{(p)}$ be given strictly negative continuous functions from $\mathbb{R}$ into $]-\infty, 0[$, we suppose $\theta_t^{(p)} < \theta_t^{(p-1)} < \ldots < \theta_t^{(1)}$ for all $t \in \mathbb{R}$. We can now define for each $t \in \mathbb{R}$:

$$\mathscr{K}_{\mathrm{D}}(t) = \{ \varphi \in \mathscr{C}_0 ; \varphi(0) \in \mathrm{D}[t, \varphi(\theta_t^{(1)}), \ldots, \varphi(\theta_t^{(p)})] \}.$$

We easily verify that the set-valued map $\mathscr{K}_{\mathrm{D}}$ has a nonempty graph in $\mathbb{R} \times \mathscr{C}_0$ since Graph D is nonempty and since we can always build a continuous mapping φ knowing $\varphi(\theta_t^{(1)}), \ldots, \varphi(\theta_t^{(p)})$ and $\varphi(0)$. Moreover we easily verify that the viability condition $\mathrm{T}(t)x \in \mathscr{K}_{\mathrm{D}}(t)$ is equivalent to the relation $x(t) \in \mathrm{D}(t, x(t + \theta_t^{(1)}), \ldots, x(t + \theta_t^{(p)}))$.

Thus the problem of the existence of solutions for the system

$$(\mathrm{M}_{\mathrm{D}}^*) \quad \begin{cases} x'(t) \in \mathrm{F}(t, \mathrm{T}(t)x) \\ x(t) \in \mathrm{D}[t, x(t + \theta_t^{(1)}), \ldots, x(t + \theta_t^{(p)})] \end{cases}$$

is equivalent to the existence of viable solutions for

$$(\mathrm{M}_{\mathrm{D}}) \quad \begin{cases} x'(t) \in \mathrm{F}(t, \mathrm{T}(t)x) \\ \mathrm{T}(t)x \in \mathscr{K}_{\mathrm{D}}(t). \end{cases}$$

Thus we have the following result.

THEOREM II.1. — *Let us suppose that* F *is bounded, u. s. c. with nonempty convex compact values on* Graph $\mathscr{K}_{\mathrm{D}}(\cdot)$ *and that* D *has a closed graph. Then condition*

$$(\mathrm{C}_{\mathrm{D}}) \qquad \mathrm{F}(t, \varphi) \cap \mathscr{D}_{\mathscr{K}_{\mathrm{D}(t)}}(\varphi) \neq \varnothing \qquad \text{for all} \qquad (t, \varphi) \in \text{Graph } \mathscr{K}_{\mathrm{D}}$$

is equivalent to the existence under any initial condition $(t_0, \varphi_0) \in$ Graph $\mathscr{K}_{\mathrm{D}}$ *of an associated viable solution of* $(\mathrm{M}_{\mathrm{D}})$ *or* $(\mathrm{M}_{\mathrm{D}}^*)$ *defined on* $[t_0, +\infty[$.

Remark. — This result is not only a direct consequence of *Theorem I.2* since elements of $\mathscr{K}_{\mathrm{D}}(\cdot)$ are only continuous and no more Lipschitz.

The necessity of $(\mathrm{C}_{\mathrm{D}})$ is deduced identically as in *Theorem I.1.*

To prove the existence of solutions under condition $(\mathrm{C}_{\mathrm{D}})$ we shall need the following preliminary lemma.

LEMMA II.1. — *To any* $\varphi \in \mathscr{C}_0$ *and* $t_0 \in \mathbb{R}$, *we can associate a sequence of Lipschitz functions* $\varphi_{\mathrm{N}} \in \mathscr{C}_0$, $\mathrm{N} \in \mathbb{N}$, *which converges to* φ *in* $\mathscr{C}_0$ *as* $\mathrm{N} \to +\infty$ *and such that*

$$\varphi_{\mathrm{N}}(0) = \varphi(0), \qquad \varphi_{\mathrm{N}}(\theta_{t_0}^{(1)}) = \varphi(\theta_{t_0}^{(1)}), \ldots, \varphi_{\mathrm{N}}(\theta_{t_0}^{(p)}) = \varphi(\theta_{t_0}^{(p)})$$

for all N (*large enough*).

Proof. — The proof is easy considering that φ is uniformly continuous on every compact subset of $]-\infty, 0]$. Indeed let $N \in \mathbb{N}$ be large enough so that $-N < \theta_{t_0}^{(p)}$. There exists then $\eta_N > 0$ such that:

$$s, s' \in [-N, 0], \qquad |s - s'| < \eta_N \quad \text{implies} \quad \|\varphi(s) - \varphi(s')\| < \frac{1}{N}.$$

We can then define a partition of $[-N, 0]$;

$$s_m = -N < s_{m-1} < s_{m-2} < \ldots < s_1 < s_0 = 0$$

such that $|s_i - s_{i+1}| < \eta_N$ for any $i=0, \ldots, m-1$ and $\theta_{t_0}^{(j)} \in \{ s_i \}_{i=1,\ldots,m}$ for all $j = 1, \ldots, p$.

Thus we can build φ_N the piecewise linear mapping on $[-N, 0]$ which interpolates $\varphi(s_{i+1}), \varphi(s_i)$ on each interval $[s_{i+1}, s_i]$.

We then extend this mapping on $]-\infty, -N[$ by setting

$$\varphi_N(t) = \varphi_N(-N) = \varphi(-N) \quad \text{for all} \quad t < -N.$$

Then φ_N is obviously Lipschitz and we easily verify that

$$\|\varphi_N - \varphi\|_{[-N,0]} < \frac{1}{N}.$$

Furthermore by construction we have $\varphi_N(0) = \varphi(0)$ and $\varphi_N(\theta_{t_0}^{(i)}) = \varphi(\theta_{t_0}^{(i)})$ for all $i = 1, \ldots, p$. Q. E. D.

Let now $\varphi_0 \in \mathscr{K}_D(t_0)$ be given, then from the preceding lemma there exists a sequence $(\varphi_N^0)_{N\in\mathbb{N}}$, all Lipschitz such that φ_N^0 converges to φ_0 in $\mathscr{C}_0$ as $N \to +\infty$, and $\varphi_N^0(0) = \varphi_0(0)$, $\varphi_N^0(\theta_{t_0}^{(1)}) = \varphi_0(\theta_{t_0}^{(1)}), \ldots, \varphi_N^0(\theta_{t_0}^{(p)}) = \varphi_0(\theta_{t_0}^{(p)})$ for all $N \in \mathbb{N}$.

Thus $\varphi_N^0 \in \mathscr{K}_D(t_0)$ for all $N \in \mathbb{N}$, by definition of $\mathscr{K}_D(t_0)$.

We shall now prove the existence of a viable solution of (M_D) under each initial condition (t_0, φ_N^0), $N \in \mathbb{N}$. And then by now standard techniques we shall deduce the existence of a viable solution under the initial condition (t_0, φ_0).

Proof. — Let us fix $\varphi_N^0 \in \mathscr{K}_D(t_0)$ as defined above. By construction it is λ_N-Lipschitz. Since F is bounded on Graph $\mathscr{K}_D$, let us denote by $k > 0$ an upper-bound of F on Graph $\mathscr{K}_D$.

We define $\mu_N > 0$ such that for example $\mu_N > \operatorname{Max}\{\lambda_N, k + 1\}$. Let us then define the set-valued map $\mathscr{K}_N^D$ from $\mathbb{R}$ into $\mathscr{C}_0$ such that for all $t \in \mathbb{R}$:

$$\mathscr{K}_N^D(t) = \{ \psi \in \mathscr{K}_D(t);\ \psi \text{ is } \mu_N\text{-Lipschitz} \}.$$

Then $\varphi_N^0 \in \mathscr{K}_N^D(t_0)$. Furthermore since Graph D is closed and since each θ_i, $i = 1, \ldots, p$ is continuous, we easily verify using the uniform convergence on compact subsets defining the topology on $\mathscr{C}_0$, that Graph $\mathscr{K}_D$ and Graph $\mathscr{K}_N^D$ are closed in $\mathbb{R} \times \mathscr{C}_0$.

Furthermore condition (C_D) implies that for any $\psi \in \mathcal{K}_N^D(t) \subset \mathcal{K}_D(t)$ there exists $v \in F(t, \psi)$ such that for any $\varepsilon > 0$ there exist $h \in]0, \varepsilon]$ and $x_h \in \mathcal{C}(]-\infty, t+h]; \mathbb{R}^n)$ which verify

$$(3) \qquad \begin{cases} T(t)x_h = \psi \\ T(t+h)x_h \in \mathcal{K}_D(t+h) \\ \dfrac{x_h(t+h) - x_h(t)}{h} \in v + \varepsilon B \end{cases}$$

From $T(t+h)x_h \in \mathcal{K}_D(t+h)$ we have

$$x_h(t+h) \in D(t+h, x_h(t+h+\theta_{t+h}^{(1)}), \ldots, x_h(t+h+\theta_{t+h}^{(p)}))$$

But using the continuity of $\theta^{(1)}$ at point t, since $\theta_t^{(1)} < 0$, choosing α such that $\theta_t^{(1)} < \alpha < 0$ we are sure that there exists $\eta > 0$ such that $|l| < \eta$ implies $\theta_{t+l}^{(1)} < \alpha$. Thus taking $\varepsilon < \operatorname{Min}\{-\alpha, \eta\}$ we have $h + \theta_{t+h}^{(1)} < 0$ since $0 < h < \varepsilon < \eta$. Since $\theta^{(p)} < \theta^{(p-1)} < \ldots < \theta^{(1)}$ we also have $h + \theta_{t+h}^{(i)} < 0$ for all $i = 1, \ldots, p$.

Thus $x_h(t+h+\theta_{t+h}^{(i)}) = [T(t)x_h](h+\theta_{t+h}^i) = \psi(h+\theta_{t+h}^i)$ for all $i=1, \ldots, p$.

Thus $x_h(t+h) \in D(t+h, \psi(h+\theta_{t+h}^{(1)}), \ldots, \psi(h+\theta_{t+h}^{(p)}))$.

Let us now define $y_h \in \mathcal{C}(]-\infty, t+h]; \mathbb{R}^n)$ such that $T(t)y_h = \psi$, y_h is the linear mapping which interpolates $x_h(t) = \psi(0)$ and $x_h(t+h)$ on $[t, t+h]$.

Thus obviously y_h verifies $T(t+h)y_h \in \mathcal{K}_D(t+h)$ and if $\varepsilon < 1$ then $y_h(\cdot)$ is μ_N-Lipschitz on $]-\infty, t+h]$ since ψ is μ_N-Lipschitz and

$$\left\| \frac{y_h(t+h) - y_h(t)}{h} \right\| = \left\| \frac{x_h(t+h) - x_h(t)}{h} \right\| \leqslant \|v\| + \varepsilon \leqslant k+1 < \mu_N .$$

Thus $T(t+h)y_h \in \mathcal{K}_N^D(t+h)$.

So for any $\psi \in \mathcal{K}_N^D(t)$ there exists $v \in F(t, \psi)$ such that for any $\varepsilon > 0$ there exist $h \in]0, \varepsilon]$ and $y_h \in \mathcal{C}(]-\infty, t+h]; \mathbb{R}^n)$ which verify

$$(3)' \qquad \begin{cases} T(t)y_h = \psi \\ T(t+h)y_h \in \mathcal{K}_N^D(t+h) \\ \dfrac{y_h(t+h) - y_h(t)}{h} \in v + \varepsilon B . \end{cases}$$

Thus all the hypothesis of *Theorem I.2* being satisfied by the viability system

$$(M_N^D) \quad \begin{cases} x'(t) \in F(t, T(t)x) \\ T(t)x \in \mathcal{K}_N^D(t) \end{cases}$$

there exists a viable solution $x_N \in \mathcal{C}(\mathbb{R}; \mathbb{R}^n)$ of (M_N^D) under the initial condition (t_0, φ_N^0).

Such a viable solution by definition verifies

$$\begin{cases} T(t_0)x_N = \varphi_N^0 \\ T(t)x_N \in \mathscr{K}_N^D(t) \subset \mathscr{K}_D(t) \quad \text{for all} \quad t \geqslant t_0 \\ x_N \text{ is } \mu_N\text{-Lipschitz on } \mathbb{R} \\ x_N'(t) \in F(t, T(t)x_N) \quad \text{for almost all} \quad t \geqslant t_0 . \end{cases}$$

In fact since F is bounded by k on Graph $\mathscr{K}_D(\cdot)$, it is obvious that x_N is k-Lipschitz on $[t_0, +\infty[$.

Thus this being done for any φ_N^0, $N \in \mathbb{N}$, since φ_N^0 converges to φ_0 as $N \to +\infty$ and since each x_N, $N \in \mathbb{N}$, is k-Lipschitz on $[t_0, +\infty[$ with k independent from N, using Ascoli's theorem there exists a subsequence (again denoted x_N) which converges (uniformly on compact subsets) to $x \in \mathscr{C}(\mathbb{R}; \mathbb{R}^n)$.

Furthermore x is k-Lipschitz on $[t_0, +\infty[$ and verifies $T(t_0)x = \varphi_0$ as well as $T(t)x \in \mathscr{K}_D(t)$ for all $t \geqslant t_0$ since Graph $\mathscr{K}_D(\cdot)$ is closed and $T(t)x_N$ converges to $T(t)x$ in $\mathscr{C}_0$ as $N \to +\infty$.

At last $x'(t) \in F(t, T(t)x)$ for almost all $t \geqslant t_0$, which is proved by the same standard arguments given in [*6*]. The existence of a viable solution under the initial condition (t_0, φ_0) is then proved. Q. E. D.

To finish this section we give an equivalent definition of $\mathscr{D}_{\mathscr{K}_D(t)}(\varphi)$ in that example of viability problem.

PROPOSITION II.1. — *For any*

$$\varphi \in \mathscr{K}_D(t) = \{ \psi \in \mathscr{C}_0 ; \psi(0) \in D[t, \psi(\theta_t^{(1)}), \ldots, \psi(\theta_t^{(p)})] \} \quad v \in \mathscr{D}_{\mathscr{K}_D(t)}(\varphi)$$

if and only if

$$(i) \quad \liminf_{h \to 0^+} \frac{1}{h} d_{D[t+h, \varphi(h+\theta_{t+h}^{(1)}), \ldots, \varphi(h+\theta_{t+h}^{(p)})]}(\varphi(0) + hv) = 0 .$$

Remark. — We recall that t being given, since $\theta^{(1)}, \ldots, \theta^{(p)}$ are continuous and verify $\theta_t^{(p)} < \ldots < \theta_t^{(1)} < 0$, for h small enough we have

$$h + \theta_{t+h}^{(p)} < \ldots < h + \theta_{t+h}^{(1)} \leqslant 0$$

which is necessary since φ is only defined on $]-\infty, 0]$.

Proof. — If $v \in \mathscr{D}_{\mathscr{K}_D(t)}(\varphi)$, then by definition, for any $\varepsilon > 0$ there exist $h \in]0, \varepsilon]$ and $x_h \in \mathscr{C}(]-\infty, t+h]; \mathbb{R}^n)$ such that

$$(3) \qquad \begin{cases} T(t)x_h = \varphi \\ T(t+h)x_h \in \mathscr{K}_D(t+h) \\ \dfrac{x_h(t+h) - x_h(t)}{h} \in v + \varepsilon B \end{cases}$$

But we have already seen that if ε is chosen small enough, then

$$T(t+h)x_h \in \mathscr{K}_D(t+h)$$

means that $x_h(t+h) \in D(t+h, \varphi(h+\theta^{(1)}_{t+h}), \ldots, \varphi(h+\theta^{(p)}_{t+h}))$.

Thus since

$$\left\| \frac{x_h(t+h) - x_h(t)}{h} - v \right\| \leqslant \varepsilon$$

and since $x_h(t) = \varphi(0)$ we deduce that

$$\frac{1}{h} d_{D[t+h,\varphi(h+\theta^{(1)}_{t+h}),\ldots,\varphi(h+\theta^{(p)}_{t+h})]}[\varphi(0) + hv] \leqslant \varepsilon .$$

Thus (i) is verified.

Conversely if (i) is verified, then for any $\varepsilon > 0$ there exists $h \in]0, \varepsilon]$ such that

$$\frac{1}{h} d_{D[t+h,\varphi(h+\theta^{(1)}_{t+h}),\ldots,\varphi(h+\theta^{(p)}_{t+h})]}[\varphi(0) + hv] \leqslant \varepsilon$$

This implies that there exists

$$w \in D(t+h, \varphi(h+\theta^{(1)}_{t+h}), \ldots, \varphi(h+\theta^{(p)}_{t+h}))$$

$$\text{such that } \left\| \frac{w - \varphi(0)}{h} - v \right\| \leqslant \varepsilon .$$

Thus taking ε small enough, it suffices now to build $x_h \in \mathscr{C}(]-\infty, t+h]; \mathbb{R}^n)$ such that $T(t)x_h = \varphi$ and x_n is linear interpolating $\varphi(0)$ and w on $[t, t+h]$.

Indeed from this construction we have

$$\frac{w - \varphi(0)}{h} = \frac{x_h(t+h) - x_h(t)}{h} \in v + \varepsilon B$$

and $T(t+h)x_h \in \mathscr{K}_D(t+h)$ since

$$[T(t+h)x_h](0) = x_h(t+h) = w \in D[t+h, \varphi(h+\theta^{(1)}_{t+h}), \ldots, \varphi(h+\theta^{(p)}_{t+h})]$$

and since for all $i = 1, \ldots, p$ we have

$$[T(t+h)x_h](\theta^{(i)}_{t+h}) = \varphi(h+\theta^{(i)}_{t+h}) .$$

Thus $v \in \mathscr{D}_{\mathscr{K}_{D(t)}}(\varphi)$. Q. E. D.

2) Viability depending on cumulated values.

Let F be a given set-valued mapping from $\mathbb{R} \times \mathscr{C}_0$ into $\mathbb{R}^n$ and E a given set-valued mapping from $\mathbb{R}^+ \times \mathbb{R}^n$ into $\mathbb{R}^n$ whose graph is supposed to be nonempty.

Let $\theta_{(.)}$ be a continuous function from $\mathbb{R}^+$ int $]-\infty, 0[$ which we suppose is bounded from below.

Let $p_1, \ldots, p_n$ be a finite sequence of locally integrable functions from $\mathbb{R}$ into $\mathbb{R}$, all of which supposed not to be almost everywhere null on $]-\infty, a]$ where $a = \operatorname*{Inf}_{t \in \mathbb{R}^+} \theta_t$.

Moreover for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ we define $\alpha \,\square\, \beta = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n) \in \mathbb{R}^n$.

At last we denote by $p = (p_1, \ldots, p_n)$ the mapping from $\mathbb{R}$ into $\mathbb{R}^n$ such that

$$p(z) = (p_1(z), \ldots, p_n(z)) \qquad \text{for all} \quad z \in \mathbb{R}\,.$$

Thus we can define for each $t \geqslant 0$

$$\mathscr{K}_{\mathrm{E}}(t) = \left\{ \varphi \in \mathscr{C}_0 \,;\ \varphi(0) \in \mathrm{E}\left(t, \int_{-\infty}^{\theta_t} \varphi(z) \,\square\, p(z + t) dz\right) \right\}.$$

In the definition of $\mathscr{K}_{\mathrm{E}}(t)$ we implicitely assume that the mapping $z \to \varphi(z) \,\square\, p(z + t)$ belongs to $\mathrm{L}^1(]-\infty, \theta_t]\,; \mathbb{R}^n)$ and thus equivalently to $\mathrm{L}^1(]-\infty, 0]\,; \mathbb{R}^n)$ since φ is continuous on $]-\infty, 0]$ and p locally integrable on $\mathbb{R}$.

PROPOSITION II.2.1. — *The set-valued map $\mathscr{K}_{\mathrm{E}}$ has a nonempty graph in $\mathbb{R}^+ \times \mathscr{C}_0$.*

Proof. — Since E has a nonempty graph in $(\mathbb{R}^+ \times \mathbb{R}^n) \times \mathbb{R}^n$, then there exist $t \in \mathbb{R}^+$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $\mathrm{E}(t, \alpha) \neq \emptyset$.

Moreover since $t \geqslant 0$, from $\theta_t \geqslant a$ and the assumptions made on $p_1, \ldots, p_n$, we are sure that the functions

$$z \to p_1(z + t), \qquad z \to p_2(z + t), \ldots, z \to p_n(z + t)$$

cannot be almost everywhere null on $]-\infty, \theta_t]$, thus there exists $u_1, \ldots, u_n$ continuous with compact support in $]-\infty, \theta_t]$ such that

$$\int_{-\infty}^{\theta_t} u_i(z) p_i(z + t) dz = 1 \qquad \text{for all} \qquad i = 1, \ldots, n\,.$$

Thus considering the mapping $\varphi = (\alpha_1 u_1, \ldots, \alpha_n u_n)$ from $]-\infty, \theta_t]$ into $\mathbb{R}^n$ we have

$$\begin{aligned}
&\int_{-\infty}^{\theta_t} \varphi(z) \,\square\, p(z + t) dz \\
&= \left(\alpha_1 \int_{-\infty}^{\theta_t} u_1(z) p_1(z + t) dz, \ldots, \alpha_n \int_{-\infty}^{\theta_t} u_n(z) p_n(z + t) dz\right) \\
&\qquad\qquad = (\alpha_1, \ldots, \alpha_n) = \alpha\,.
\end{aligned}$$

Then choosing $w = (w_1, \dots, w_n) \in E(t,\alpha)$, it is possible to extend continuously on $]-\infty, 0]$ such that $\varphi(0) = w$.

Thus $\varphi(0) \in E\left(t, \int_{-\infty}^{\theta_t} \varphi(z) \,\square\, p(z+t)dz\right)$. Q. E. D.

At last we easily verify that for any $x = (x_1, \dots, x_n) \in \mathscr{C}(\mathbb{R}; \mathbb{R}^n)$ the viability condition $T(t)x \in \mathscr{K}_E(t)$, $t \geqslant 0$, is equivalent to

$$\begin{aligned} x(t) = [T(t)x](0) &\in E\left(t, \int_{-\infty}^{\theta_t} [T(t)x](z) \,\square\, p(z+t)dz\right) \\ &= E\left(t, \int_{-\infty}^{\theta_t} x(t+z) \,\square\, p(z+t)dz\right) \\ &= E\left(t, \int_{-\infty}^{t+\theta_t} x(z) \,\square\, p(z)dz\right) \\ &= E\left(t, \int_{-\infty}^{t+\theta_t} x_1(z)p_1(z)dz, \dots, \int_{-\infty}^{t+\theta_t} x_n(z)p_n(z)dz\right). \end{aligned}$$

Thus the problem of the existence of solutions for the system

$$(M_E^*) \quad \begin{cases} x'(t) \in F(t, T(t)x) \\ x(t) \in E\left(t, \int_{-\infty}^{t+\theta_t} x(z) \,\square\, p(z)dz\right) \end{cases}$$

is equivalent to the existence of viable solutions for

$$(M_E) \quad \begin{cases} x'(t) \in F(t, T(t)x) \\ T(t)x \in \mathscr{K}_E(t). \end{cases}$$

Then we have the following result.

THEOREM II.2.— *Let us suppose that* F *is bounded, u. s. c. with nonempty convex compact values on* Graph $\mathscr{K}_E(\cdot)$ *and that* E *has a closed graph. Then condition*

(C_E) $F(t, \varphi) \cap \mathscr{D}_{\mathscr{K}_{E(t)}}(\varphi) \neq \emptyset$ *for all* $(t, \varphi) \in$ Graph $\mathscr{K}_E$

is equivalent to the existence under any initial condition $(t_0, \varphi_0) \in$ Graph $\mathscr{K}_E$ *of an associated viable solution of* (M_E) *or* (M_E^*) *defined on* $[t_0, +\infty[$.

Remark. — Initial time t_0 is necessarily positive since E and $\mathscr{K}_E$ are only defined for $t \geqslant 0$.

The necessity of (C_E) is proved identically as in *Theorem I.1.* To prove the existence of solutions under condition (C_E) we shall need the following preliminary lemma.

LEMMA II.2. — *Let q from* $\mathbb{R}$ *into* $\mathbb{R}$ *be locally integrable, not almost* *everywhere null on* $]-\infty, \theta_t]$, $\theta_t < 0$, *and* $f \in \mathscr{C}(]-\infty, 0]; \mathbb{R})$ *be such that* $q \cdot f \in L^1(]-\infty, \theta_t]; \mathbb{R})$.

Then there exists a sequence f_N, $N \in \mathbb{N}$, *of Lipschitz functions from* $]-\infty, 0]$ *into* $\mathbb{R}$, *with compact support, converging to* f *(uniformly on compact subsets) as* $N \to +\infty$ *and verifying*

$$f_N(0) = f(0), \qquad \int_{-\infty}^{\theta_t} f_N(z)q(z)dz = \int_{-\infty}^{\theta_t} f(z)q(z)dz$$

for all N *large enough.*

Proof. — Let $N \in \mathbb{N}$ be large enough such that $-N < \theta_t$. Using the uniform continuity of f on $[-N, 0]$ we know that for any $\varepsilon > 0$ we can build as in Lemma II.1, a piecewise linear function g_N on $[-N, 0]$ such that $g_N(0) = f(0)$, $g_N(-N) = f(-N)$ and $\| g_N - f \|_{[-N,0]} < \varepsilon$ for all N.

Since q is locally integrable there exists $\eta_N > 0$ such that

$$\int_{-N-\eta_N}^{-N} | q(z) | dz < \varepsilon .$$

We then extend g_N on $[-N-\eta_N, -N]$ by the linear function which interpolates 0 and $g_N(-N)$. At last g_N is extended by zero on $]-\infty, -N-\eta_N]$. Thus g_N is Lipschitz since piecewise linear, with compact support.

Let us take for example

$$\varepsilon < \operatorname{Min} \left\{ \frac{1}{N\left[\int_{-N}^{\theta_t} | q(z) | dz + 1\right]}, \frac{1}{N[| f(-N) | + 1]} \right\} < \frac{1}{N}.$$

Then

$$\int_{-\infty}^{\theta_t} g_N(z)q(z)dz = \int_{-\eta_N-N}^{-N} g_N(z)q(z)dz + \int_{-N}^{\theta_t} g_N(z)q(z)dz$$

with

$$\left| \int_{-\eta_N-N}^{-N} g_N(z)q(z)dz \right| \leqslant \int_{-\eta_N-N}^{-N} | g_N(z) | \, | q(z) | dz$$
$$\leqslant | f(-N) | \int_{-\eta_N-N}^{-N} | q(z) | dz < \frac{1}{N}$$

Moreover

$$\left| \int_{-N}^{\theta_t} g_N(z)q(z)dz - \int_{-N}^{\theta_t} f(z)q(z)dz \right| \leqslant \varepsilon \int_{-N}^{\theta_t} | q(z) | dz < \frac{1}{N}$$

Thus obviously

$$\lim_{N\to+\infty} \int_{-\infty}^{\theta_t} g_N(z)q(z)dz = \lim_{N\to+\infty} \int_{-N}^{\theta_t} f(z)q(z)dz = \int_{-\infty}^{\theta_t} f(z)q(z)dz,$$

since $f \cdot q \in L^1(]-\infty, \theta_t]; \mathbb{R})$.

We can then write $\int_{-\infty}^{\theta_t} g_N(z)q(z)dz = \int_{-\infty}^{\theta_t} f(z)q(z)dz + \alpha_N$ with $\alpha_N \to 0$ as $N \to +\infty$.

But since q is not almost everywhere null on $]-\infty, \theta_t]$ there necessarily exists a C^∞ function u from $]-\infty, \theta_t]$ into $\mathbb{R}$, with compact support, such that

$$\int_{-\infty}^{\theta_t} u(z)q(z) = 1 .$$

We extend u on $[\theta_t, 0]$ by interpolating linearily $u(\theta_t)$ and $0 = u(0)$. Thus considering $f_N = g_N - \alpha_N u$, for any N, we see that each f_N is Lipschitz, with compact support, that $f_N(0) = g_N(0) - \alpha_N u(0) = g_N(0) = f(0)$ and that $\int_{-\infty}^{\theta_t} f_N(z)q(z)dz = \int_{-\infty}^{\theta_t} f(z)q(z)dz$.

At last since u has a compact support, since $\alpha_N \to 0$ and g_N converges to f in $\mathscr{C}(]-\infty, 0]; \mathbb{R})$ as $N \to +\infty$, it is easy to verify that f_N converges to f in $\mathscr{C}(]-\infty, 0]; \mathbb{R})$. Q. E. D.

Let now $\varphi_0 \in \mathscr{K}_E(t_0)$ be given. By the assumptions made on $p_1, \ldots, p_n$, we know that for each $i = 1, \ldots, n$ the mapping $z \to p_i(z + t_0)$ is locally integrable and not almost everywhere null on $]-\infty, \theta_{t_0}]$ since $t_0 \geqslant 0$ and $\theta_{t_0} \geqslant a$. Thus using the preceding lemma for each component of φ_0, we can build a sequence of Lipschitz with compact support mappings $\varphi_N^0 \in \mathscr{C}_0$, $N \in \mathbb{N}$, which converges to φ_0 in $\mathscr{C}_0$ as $N \to +\infty$ and verifies for all $N \in \mathbb{N}$;

$$\varphi_N^0(0) = \varphi_0(0), \int_{-\infty}^{\theta_{t_0}} \varphi_N^0(z) \ \square \, p(z + t_0)dz = \int_{-\infty}^{\theta_{t_0}} \varphi_0(z) \ \square \, p(z + t_0)dz$$

Thus obviously $\varphi_N^0 \in \mathscr{K}_E(t_0)$ for all $N \in \mathbb{N}$.

We shall, as in the preceding section, prove the existence of a viable solution of (M_E) under each initial condition (t_0, φ_N^0), $N \in \mathbb{N}$. And then deduce the existence of a viable solution for (t_0, φ_0).

Proof. — Let us fix $\varphi_N^0 \in \mathscr{K}_E(t_0)$ as defined above. We suppose that $\operatorname{Supp}(\varphi_N^0) \subset [-A_N - t_0, 0]$ with $A_N > 0$.

By construction φ_N^0 is λ_N-Lipschitz. Since F is bounded on Graph $\mathscr{K}_E(\cdot)$, we denote by $k > 0$ an upper-bound of F on Graph $\mathscr{K}_E(\cdot)$.

We define $\mu_N > 0$ such that for example $\mu_N > \max\{\lambda_N, k + 1\}$.

Let us then define the set-valued map $\mathscr{K}_N^E(\cdot)$ from $\mathbb{R}^+$ into $\mathscr{C}_0$ such that for all $t \in \mathbb{R}^+$;

$$\mathscr{K}_N^E(t) = \{\psi \in \mathscr{K}_E(t); \operatorname{Supp}(\psi) \subset [-A_N - t, 0], \psi \text{ is } \mu_N\text{-Lipschitz}\}.$$

Then $\varphi_N^0 \in \mathscr{K}_N^E(t_0) \subset \mathscr{K}_E(t_0)$.

Furthermore condition (C_E) implies that for any $\psi \in \mathscr{K}_N^E(t) \subset \mathscr{K}_E(t)$ there exists $v \in F(t, \psi)$ such that for any $\varepsilon > 0$ there exist $h \in]0, \varepsilon]$ and $x_h \in \mathscr{C}(]-\infty, t + h]; \mathbb{R}^n)$ which verify

$$(3) \qquad \begin{cases} T(t)x_h = \psi \\ T(t + h)x_h \in \mathscr{K}_E(t + h) \\ \dfrac{x_h(t + h) - x_h(t)}{h} \in v + \varepsilon B \end{cases}$$

We then define $y_h \in \mathscr{C}(]-\infty, t+h]; \mathbb{R}^n)$ such that $T(t)y_h = \psi$ and y_h linear interpolates $x_h(t)$ and $x_h(t+h)$ on $[t, t+h]$.

Since θ is continuous and $\theta_t < 0$ we have already seen that if ε is small enough then $h + \theta_{t+h} \leqslant 0$. Thus for all $z \in]-\infty, \theta_{t+h}]$ we have

$$[T(t+h)x_h](z) = x_h(t+h+z) = [T(t)x_h](z+h).$$

Then

$$[T(t+h)x_h](z) = \psi(z+h) \qquad \text{for all} \qquad z \in]-\infty, \theta_{t+h}].$$

For the same reasons

$$[T(t+h)y_h](z) = \psi(h+z) \qquad \text{for all} \qquad z \in]-\infty, \theta_{t+h}]$$

Thus $$T(t+h)x_h = T(t+h)y_h \qquad \text{on} \qquad]-\infty, \theta_{t+h}].$$

This implies that

$$\int_{-\infty}^{\theta_{t+h}} [T(t+h)x_h](z) \,\square\, p(z+t+h)dz = \int_{-\infty}^{\theta_{t+h}} [T(t+h)y_h](z) \,\square\, p(z+t+h)dz$$

But since $[T(t+h)y_h](0) = x_h(t+h) = [T(t+h)x_h](0)$ we easily deduce from the definition of $\mathscr{K}_E(t+h)$ and since

$$T(t+h)x_h \in \mathscr{K}_E(t+h) \qquad \text{that} \qquad T(t+h)y_h \in \mathscr{K}_E(t+h).$$

Furthermore since $\operatorname{Supp}(\psi) \subset [-A_N - t, 0]$ it is obvious that

$$\operatorname{Supp}[T(t+h)y_h] \subset [-A_N - t - h, 0].$$

At last y_h is μ_N-Lipschitz since

$$\left\|\frac{y_h(t+h) - y_h(t)}{h}\right\| = \left\|\frac{x_h(t+h) - x_h(t)}{h}\right\| \leqslant \|v\| + \varepsilon \leqslant k + \varepsilon < k+1,$$

when $\varepsilon < 1$. Thus $T(t+h)y_h \in \mathscr{K}_N^E(t+h)$.

To show that all the hypothesis of *Theorem I.2* are satisfied by the viability system:

$$(M_N^E) \quad \begin{cases} x'(t) \in F(t, T(t)x) \\ T(t)x \in \mathscr{K}_N^E(t) \end{cases}$$

it remains to show that $\mathscr{K}_N^E(\cdot)$ has a closed graph. For this let

$$(t_m, \psi_m) \in \operatorname{Graph} \mathscr{K}_N^E, \quad m \in \mathbb{N},$$

converges to $(t, \psi) \in \mathbb{R}^+ \times \mathscr{C}_0$ as $m \to +\infty$.

Then since $\psi_m \in \mathscr{K}_N^E(t_m)$ implies that ψ_m is μ_N-Lipschitz and

$$\operatorname{Supp}(\psi_m) \subset [-A_N - t_m, 0],$$

we easily deduce that ψ is μ_N-Lipschitz and that $\operatorname{Supp}(\psi) \subset [-A_N - t, 0]$.

It remains to show that $\psi(0) \in E\left(t, \int_{-\infty}^{\theta_t} \psi(z) \,\square\, p(z+t)dz\right)$. For this, since $\psi_m(0) \in \mathrm{E}\left(t_m, \int_{-\infty}^{\theta_{t_m}} \psi_m(z) \,\square\, p(z + t_m)dz\right)$ for all $m \in \mathbb{N}$ and since Graph E is closed, it suffices to prove that

$$\int_{-\infty}^{\theta_{t_m}} \psi_m(z) \,\square\, p(z + t_m)dz \to \int_{-\infty}^{\theta_t} \psi(z) \,\square\, p(z + t)dz \quad \text{as} \quad m \to +\infty .$$

We easily deduce from the definition of the law $\square$ that

$$\int_{-\infty}^{\theta_{t_m}} \psi_m(z) \,\square\, p(z + t_m)dz = \int_{-\infty}^{\theta_{t_m}} [\psi_m(z) - \psi(z)] \,\square\, p(z + t_m)dz$$
$$+ \int_{-\infty}^{\theta_t} \psi(z) \,\square\, p(z + t_m)dz + \int_{\theta_t}^{\theta_{t_m}} \psi(z) \,\square\, p(z + t_m)dz .$$

By the very properties of ψ and $(\psi_m)_{m\in\mathbb{N}}$ it is immediate that they are null outside a same compact interval of $]-\infty, 0]$ and that $(\psi_m)_{m\in\mathbb{N}}$ converges uniformly to ψ on this compact interval. Now since t_m and θ_{t_m}, $m \in \mathbb{N}$, are uniformly bounded and since p is locally integrable, it becomes obvious that

$$\int_{-\infty}^{\theta_{t_m}} [\psi_m(z) - \psi(z)[\,\square\, p(z + t_m)dz \to 0 \quad \text{as} \quad m \to +\infty .$$

We also verify that $\int_{\theta_t}^{\theta_{tm}} \psi(z) \,\square\, p(z + t_m)dz \to 0$ as $m \to +\infty$, since ψ is continuous, $\theta_{t_m} \to \theta_t$ as $t_m \to t$ and p is locally integrable.

At last ψ being continuous with compact support and p locally integrable it is easy (using convolution type results on each components of $\int_{-\infty}^{\theta} \psi(z) \,\square\, p(z + s)dz$) to show that the mapping $s \to \int_{-\infty}^{\theta} \psi(z) \,\square\, p(z+s)dz$ is continuous at any $s \in \mathbb{R}$.

Thus it is then true that $\int_{-\infty}^{\theta_{t_m}} \psi_m(z) \,\square\, p(z + t_m)dz \to \int_{-\infty}^{\theta_t} \psi(z) \,\square\, p(z+t)dz$ as $m \to +\infty$. Then $\psi \in \mathscr{K}_{\mathrm{N}}^{\mathrm{E}}(t)$.

Now since all the hypothesis of *Theorem I.2* are satisfied by the viability system $(\mathrm{M}_{\mathrm{N}}^{\mathrm{E}})$, there exists a viable solution $x_{\mathrm{N}} \in \mathscr{C}(\mathbb{R}; \mathbb{R}^n)$ under the initial condition $(t_0, \varphi_{\mathrm{N}}^0)$, such a solution verifies:

$$\begin{cases} \mathrm{T}(t_0)x_{\mathrm{N}} = \varphi_{\mathrm{N}}^0 & \\ \mathrm{T}(t)x_{\mathrm{N}} \in \mathscr{K}_{\mathrm{N}}^{\mathrm{E}}(t) \subset \mathscr{K}_{\mathrm{E}}(t) & \text{for all} \quad t \geqslant t_0 \\ x_{\mathrm{N}} \ \text{ is } \ \mu_{\mathrm{N}}\text{-Lipschitz} & \\ x'_{\mathrm{N}}(t) \in \mathrm{F}(t, \mathrm{T}(t)x_{\mathrm{N}}) & \text{for almost all} \quad t \geqslant t_0 \end{cases}$$

In fact since F is bounded by k on Graph $\mathscr{K}_{\mathrm{E}}(\cdot)$ it is obvious that x_{N} is k-Lipschitz on $[t_0, +\infty[$. Thus this being done for any φ_{N}^0, $\mathrm{N} \in \mathbb{N}$, we get a sequence x_N, $N \in \mathbb{N}$, which for the same reasons as in the preceding section will admits a subsequence (again denoted x_N) converging (uniformly on compact subsets) to $x \in \mathscr{C}(\mathbb{R}; \mathbb{R}^n)$.

Obviously x is k-Lipschitz on $[t_0, +\infty[$ and verifies $T(t_0)x = \varphi_0$, $x'(t) \in F(t, T(t)x)$ for almost all $t \geqslant t_0$. To end the proof of the viability of x it suffices now to verify that for all $t \geqslant t_0$, $T(t)x \in \mathscr{K}_E(t)$ which is equivalent to $x(t) \in E\left[t, \int_{-\infty}^{t+\theta_t} x(z) \,\square\, p(z)dz\right]$.

Let $t \geqslant t_0$ be given. We know that for all $N \in \mathbb{N}$

$$x_N(t) \in E\left[t, \int_{-\infty}^{t+\theta_t} x_N(z) \,\square\, p(z)dz\right].$$

As $E(\cdot)$ has a closed graph, it is then sufficient to show that

$$\int_{-\infty}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \to \int_{-\infty}^{t+\theta_t} x(z) \,\square\, p(z)dz \quad \text{when} \quad N \to +\infty.$$

Since $T(t_0)x = \varphi_0$ and since for all $N \in \mathbb{N}$

$$\int_{-\infty}^{\theta_{t_0}} \varphi_0(z) \,\square\, p(z+t_0)dz = \int_{-\infty}^{\theta_{t_0}} \varphi_N^0(z) \,\square\, p(z+t_0)dz$$

we have

$$\begin{aligned}
\int_{-\infty}^{t+\theta_t} x_N(z) \,\square\, p(z)dz &= \int_{-\infty}^{t_0+\theta_{t_0}} x_N(z) \,\square\, p(z)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \\
&= \int_{-\infty}^{\theta_{t_0}} [T(t_0)x_N](z) \,\square\, p(z+t_0)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \\
&= \int_{-\infty}^{\theta_{t_0}} \varphi_N^0(z) \,\square\, p(z+t_0)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \\
&= \int_{-\infty}^{\theta_{t_0}} \varphi_0(z) \,\square\, p(z+t_0)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \\
&= \int_{-\infty}^{\theta_{t_0}} [T(t_0)x](z) \,\square\, p(z+t_0)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \\
&= \int_{-\infty}^{t_0+\theta_{t_0}} x(z) \,\square\, p(z)dz + \int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz.
\end{aligned}$$

Then since x_N converges uniformly to x on the compact interval $[t_0 + \theta_{t_0}, t + \theta_t]$ and since p is integrable on $[t_0 + \theta_{t_0}, t + \theta_t]$ we have

$$\int_{t_0+\theta_{t_0}}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \to \int_{t_0+\theta_{t_0}}^{t+\theta_t} x(z) \,\square\, p(z)dz \qquad \text{as} \quad N \to +\infty.$$

This proves that $\int_{-\infty}^{t+\theta_t} x_N(z) \,\square\, p(z)dz \to \int_{-\infty}^{t+\theta_t} x(z) \,\square\, p(z)dz$ as $N \to +\infty$.

Thus x is actually viable. Q. E. D.

To finish we give now an equivalent definition of $\mathcal{D}_{\mathcal{K}_{E(t)}}(\varphi)$ in that example of viability problem.

PROPOSITION II.2.2. — *For any*

$$\varphi \in \mathcal{K}_E(t) = \left\{ \psi \in \mathcal{C}_0 ; \psi(0) \in E\left(t, \int_{-\infty}^{\theta t} \psi(z) \square p(z+t)dz\right)\right\}, \quad v \in \mathcal{D}_{\mathcal{K}_{E(t)}}(\varphi)$$

if and only if

$$(ii) \quad \lim_{h\to 0^+} \operatorname{Inf} \frac{1}{h} d_{E\left(t+h, \int_{-\infty}^{h+\theta_{t+h}} \varphi(z) \square p(z+t)dz\right)} [\varphi(0) + hv] = 0 .$$

Remark. — We recall that t being given, since θ is continuous and $\theta_t < 0$, for h small enough $h + \theta_{t+h} \leqslant 0$, which gives a sense to $\int_{-\infty}^{h+\theta_{t+h}} \varphi(z) \square p(z+t)dz$ since φ is only defined on $]-\infty, 0]$.

The proof presents no difficulty and is a pure adaptation of the proof of *Proposition II.1.*

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