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SOME EXACT RATES IN THE FUNCTIONAL LAW OF THE ITERATED LOGARITHM

QUELQUES VITESSES EXACTES DANS LA LOI FONCTIONNELLE DU LOGARITHME ITÉRÉ

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ABSTRACT. – We find exact convergence rate in the Strassen's functional law of the iterated logarithm for a class of elements on the boundary of the limit set. Our result applies, in particular, to the power functions $c_\alpha x^\alpha$ with $\alpha \in]1/2, 1[$, thus solving a small ball estimate problem which was open for ten years.

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RÉSUMÉ. – Nous établissons la vitesse de convergence exacte dans la loi fonctionnelle du logarithme itéré de Strassen pour une classe d'éléments de la frontière de l'ensemble limite. Notre résultat s'applique en particulier aux fonctions puissance $c_\alpha x^\alpha$ avec $\alpha \in]1/2, 1[$, résolvant ainsi un problème de probabilité de petite boule resté ouvert durant une décennie.

Mots Clés : Mouvement Brownien ; Lois fonctionnelles de Strassen et Chung ; Probabilités de petite boule

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1. Introduction

1.1. Strassen's law

Let W be a standard Brownian motion. Consider for $T > 3$ the random processes

$$W_T(x) = \frac{W(Tx)}{\sqrt{2T \log\log T}}$$

indexed by $x \in [0,1]$. Let $\mathcal{C}$ be the space of real valued continuous functions on $[0,1]$ starting from 0, equipped with the supremum norm $\|\cdot\|$. According to Strassen's functional law of the iterated logarithm (see [12]) the sequence $\{W_T, T \geqslant 3\}$ is almost surely relatively compact in $(\mathcal{C}, \|\cdot\|)$ and its almost sure limit set is

$$\mathcal{H}_1 = \left\{h\colon\ h(x) = \int_{[0,x]} h'\,d\lambda,\ J(h) \leqslant 1\right\}$$

where λ denotes the Lebesgue measure, h' any Lebesgue derivative of a λ-absolutely continuous function h and the energy of h is given by $J(h) = \int_{[0,1]} h'^2\,d\lambda$. Since $\mathcal{H}_1$ is closed, this implies

$$\liminf_{T\to\infty} \|W_T - h\| \begin{cases} = 0 & \text{if } h \in \mathcal{H}_1 \\ > 0 & \text{if } h \notin \mathcal{H}_1 \end{cases} \quad \text{a.s.} \tag{1}$$

Thus $J(h)$ quantifies at the first order the difficulty for W_T to look like h. Recall further that W satisfies the usual large deviation principle on $(\mathcal{C}, \|\cdot\|)$ with the good rate function $J(h)/2$, in the sense of Deuschel and Stroock [5]. According to De Acosta [6], $J(h)/2$ also governs the small deviations of W in the direction of enlarged h.

1.2. Functional Chung's law

Fix an accumulation point $h \in \mathcal{H}_1$. In [3] (see also [6]) Csáki proved that if $J(h) < 1$, then the exact rate in (1) depends on $J(h)$ only, namely,

$$\liminf_{T\to\infty} \log\log T\,\|W_T - h\| = \frac{\pi}{4\sqrt{1-J(h)}} \quad \text{a.s.} \tag{2}$$

This reduces to Chung's law when $h = 0$ (see [2]). Conversely, when $J(h) = 1$, the limit (2) is infinite. Moreover Goodman and Kuelbs obtained in [7]

$$c(h) = \liminf_{T\to\infty} (\log\log T)^{2/3} \|W_T - h\| < \infty \quad \text{a.s.} \tag{3}$$

In this case the increasing function, which we call the *global energy loss* of h,

$$\varphi_h(\varepsilon) = 1 - \inf_{\|g-h\|\leqslant\varepsilon} J(g)$$

is crucial with respect to the exact rate in (1). The infimum is taken among λ-absolutely continuous functions g. More precisely, if $J(h) = 1$, we distinguish between *slowest* and *intermediate* rates according to the criterion

$$d(h) = \lim_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon)}{\varepsilon} \begin{cases} \in [2, \infty[, & c(h) > 0, \\ = \infty, & c(h) = 0, \end{cases} \tag{4}$$

since $d(h)$ always exists (see [10, Section 5]) and determines $c(h)$ as recalled below.

Remark 1. – The question of exact rates and constants in (2) under various norms has been intensively investigated in the last decade (see Berthet and Shi [1] and references therein). For instance, (4) remains unchanged under L_2 metric as shown by the exact rates for the L_2 version of (1) calculated in Kuelbs, Li and Talagrand [10].

Let us picture out the situation on the border of $\mathcal{H}_1$,

$$\partial \mathcal{H}_1 = \{h \colon h \in \mathcal{H}_1,\ J(h) = 1\}.$$

1.3. Slowest functions

The behaviour of φ_h at zero is closely related to the length and smoothness of h. Let γ be a signed Borel measure on $[0, 1]$ such that $\gamma([x, 1])$ defines a version of $h'(x)$. If γ can be chosen of bounded variation, then $V(h', B)$ denotes the total variation of γ over any Borel set $B \subset [0, 1]$ and we write $V(h^{'}) = V(h', [0, 1]) < \infty$. Otherwise we set $V(h', B) = \infty$ and $V(h') = \infty$.

The *slowest* functions are

$$\partial^s \mathcal{H}_1 = \{h \colon h \in \mathcal{H}_1,\ J(h) = 1,\ V(h') < \infty\},$$

since Grill showed in [9] that $c(h) > 0$ in (3) if, and only if, $h \in \partial^s \mathcal{H}_1$ which is also equivalent to $d(h) \in [2, \infty)$ in (4) (see e.g. [7,10]). It is very difficult for the Brownian motion to follow uniformly such smooth trajectories.

The exact constant $c(h)$ is obtained by Csáki in [3,4] when h is piecewise linear or quadratic. Recently, in [8] Gorn and Lifshits extended Csáki's method to characterize $c(h)$ for any $h \in \partial^s \mathcal{H}_1$ as the unique solution of an equation and provided a procedure for its numerical calculation.

1.4. Intermediate functions

In his seminal work [9], Grill proved that any *intermediate* function $h \in \partial^i \mathcal{H}_1 = \partial \mathcal{H}_1 \backslash \partial^s \mathcal{H}_1$ satisfies

$$\liminf_{T \to \infty} \frac{\|W_T - h\|}{\varepsilon(T)} \in [1, 2] \quad \text{a.s.} \tag{5}$$

where the rate $\varepsilon(T)$ is the unique solution of

$$\sqrt{\varphi_h(\varepsilon)} = \frac{\pi}{4\varepsilon \log \log T}. \tag{6}$$

Note that $\lim_{T\to\infty}\varepsilon(T)=0$, $\lim_{T\to\infty}\varepsilon(T)\log\log T=\infty$ and $\varepsilon(T)$ is decreasing. Further, $d(h)=\infty$ and thus, by (5) and (6), $c(h)=0$ in (3). Also, in practice it suffices to find an equivalent for $\varphi_h(\varepsilon)$ and hence for $\varepsilon(T)$.

The functions of $\partial^i\mathcal{H}_1$ are more easily approached by a Brownian path because they have a larger global energy loss function. This may be due to their irregular behaviour – oscillating or just non-smooth pieces admit efficient rectification.

Remark 2. – Interestingly, the statements (5) and (6) remain true for all $h\in\mathcal{H}_1$ and the liminf in (5) is 1 whenever φ_h is slowly varying at 0. For instance, it is the case when $J(h)<1$ and (2) immediately follows. Conversely, if $h\in\partial^s\mathcal{H}_1$ then $d(h)$ in (4) is explicited in [10] and (6) then yields the right order $(\log\log T)^{-2/3}$ but comparing (5) and (3) via the result of [8] shows that $(16\pi^{-2}d(h))^{1/3}\neq c(h)$ for some $h\in\partial^s\mathcal{H}_1$. Hence the constant in (5) is not 1 in general.

Surprisingly, many years after (5) has appeared, the exact rate was not obtained even for simple power functions (cf. Example 1 below). We intend to show that the liminf in (5) is 1 for a large class of $h\in\partial^i\mathcal{H}_1$.

1.5. Typical intermediate functions

In order to illustrate our results let us introduce elementary critical functions – having Lebesgue derivative of infinite variation and unit L_2 norm. All are locally Hölder with index $\alpha\in\,]1/2,1[$ and have loss function

$$\varphi_h(\varepsilon)=\mathrm{O}\big(\varepsilon^{(2\alpha-1)/\alpha}\big)$$

with exact constants easily computed by invoking optimization arguments. Let $\Delta_\alpha=\alpha^2(1-\alpha)^{(1-\alpha)/\alpha}(2\alpha-1)^{-1}$. We assume everywhere that ε is small enough.

Example 1. – The function $h_1(x)=bx^\alpha=\pm\sqrt{2\alpha-1}x^\alpha/\alpha\in\partial^i\mathcal{H}_1$ has energy loss

$$\begin{aligned}\varphi_{h_1}(\varepsilon)&=2|b|^{1/\alpha}\Delta_\alpha\varepsilon^{(2\alpha-1)/\alpha}\\&=2(2\alpha-1)^{(1-2\alpha)/(2\alpha)}\alpha^{(2\alpha-1)/\alpha}(1-\alpha)^{(1-\alpha)/\alpha}\varepsilon^{(2\alpha-1)/\alpha}\end{aligned}$$

localized at the origin. The constant belongs to $]1,2[$.

Next, a smooth perturbation is added with almost no effect.

Example 2. – Let $0<c^2<c_\alpha^2=(2\alpha-1)(1-\alpha)^{-2}$, $a=c(-1\pm\sqrt{c^{-2}-c_\alpha^{-2}})$ and $h_2(x)=cx^\alpha+ax$. Then $h_2\in\partial^i\mathcal{H}_1$ and

$$\varphi_{h_2}(\varepsilon)=2|c|^{1/\alpha}\Delta_\alpha\varepsilon^{(2\alpha-1)/\alpha}+2a\varepsilon.$$

In Examples 3 and 4 we consider a single α-Hölder point $y\in\,]0,1[$ away from the origin. Denote $l_+\in\,]0,1-y]$ (respectively $l_-\in\,]0,y]$) the length of an interval starting (respectively ending) at y. Interestingly, y and $l_\pm$ eventually play no role.

Example 3. – Let $h_3=I_{[0,y[}h_{3-}+I_{[y,y+l_+[}h_{3\alpha}+I_{[y+l_+,1]}h_{3+}$ with $h_{3\alpha}(x)=h_{3-}(y)+b(x-y)^\alpha$, $b\neq0$ and $h'_{3\pm}$ of bounded variation. Then $h_3\in\partial^i\mathcal{H}_1$ satisfies

$$\varphi_{h_3}(\varepsilon)=2^{(3\alpha-1)/\alpha}|b|^{1/\alpha}\Delta_\alpha\varepsilon^{(2\alpha-1)/\alpha}+\mathrm{O}(\varepsilon).$$

In particular, taking $h'_{3-} = a$ and $h'_{3+} = c$ constant yields $\mathrm{O}(\varepsilon) = -\mathrm{sign}(b) \times 2a\varepsilon - y^{-1}\varepsilon^2 + \phi(\varepsilon)$ where, if $cb \in [0, \alpha b^2 l_+^{\alpha-1}]$, $\phi(\varepsilon) = 0$ and, if $cb < 0$, $\phi(\varepsilon) = |4c|\varepsilon - 4(1 - y - l_+)^{-1}\varepsilon^2$ whereas, if $cb > \alpha b^2 l_+^{\alpha-1}$, $\phi(\varepsilon) = |4c|(1 - \alpha l_+^{\alpha-1} b/c)\varepsilon + \mathrm{O}(\varepsilon^{3/2})$.

Observe that $\varphi_{h_3}(\varepsilon) \sim \varphi_{h_1}(2\varepsilon)$ when $h'_{3\pm} = 0$ because $|g(y) - h(y)| < \varepsilon$ is less restrictive than $g(0) = 0$. Compare this with another situation of the kind – a doubly Hölder point:

Example 4. – Let $y \in]0, 1[$, $a^2 < y$, $b_\pm \neq 0$, $h_{\alpha\pm}(x) = a + b_\pm |y - x|^\alpha$ and $h'_{4\pm}$ of bounded variation be such that $h_4 = I_{[0, y-l_-]} h_{4-} + I_{]y-l_-, y[} h_{\alpha-} + I_{[y, y+l_+[} h_{\alpha+} + I_{[y+l_+, 1]} h_{4+}$ belongs to $\partial\mathcal{H}_1$. If $b_- b_+ > 0$ we have

$$\varphi_{h_4}(\varepsilon) = 2^{(3\alpha-1)/\alpha} \left(|b_-|^{1/\alpha} + |b_+|^{1/\alpha}\right) \Delta_\alpha \varepsilon^{(2\alpha-1)/\alpha} + \mathrm{O}(\varepsilon)$$

whereas, if $b_- b_+ < 0$ less energy can be spared, since then

$$\varphi_{h_4}(\varepsilon) = 2^{(3\alpha-1)/\alpha} \left(|b_-|^{1/(1-\alpha)} + |b_+|^{1/(1-\alpha)}\right)^{(1-\alpha)/\alpha} \Delta_\alpha \varepsilon^{(2\alpha-1)/\alpha} + \mathrm{O}(\varepsilon).$$

We end with a natural extension of Example 3 in the spirit of Theorem 2 below.

Example 5. – For $i = 0, \ldots, n$ let $\alpha_i \in]1/2, 1[$, $b_i \neq 0$, $x_i \in]0, 1[$, $x_i < x_{i+1}$, $l_i \in]0, x_{i+1} - x_i]$, $h_{5,i}(x) = a_i + b_i (x - x_i)^{\alpha_i}$, $I_0 = [0, 1] \setminus \bigcup_{i=1}^n [x_i, x_i + l_i]$ and $h'_{5,0}$ of bounded variation be such that

$$h_5 = \sum_{i=1}^n I_{[x_i, x_i + l_i]} h_{5,i} + I_0 h_{5,0} \in \partial\mathcal{H}_1.$$

Then, for $\alpha = \min_i \alpha_i$ we have

$$\varphi_{h_5}(\varepsilon) = 2^{(3\alpha-1)/\alpha} \left(\sum_{i:\ \alpha_i = \alpha} |b_i|^{1/\alpha}\right) \Delta_\alpha \varepsilon^{(2\alpha-1)/\alpha} + \mathrm{O}(\varepsilon).$$

2. Main results

The exact constant in (5) depends on the nature of *rectified* trajectories

$$\mathcal{R}_h(\varepsilon) = \left\{h_\varepsilon\colon\ h_\varepsilon \in \mathcal{H}_1,\ \|h_\varepsilon - h\| \leqslant \varepsilon,\ J(h_\varepsilon) = 1 - \varphi_h(\varepsilon)\right\} \tag{7}$$

which are close to h with shortened paths. Unfortunately, the study of $\mathcal{R}_h$ is not an easy task – except for simple functions as h_1, h_2 or particularized h_3 – since there is no general way to evaluate the crucial function φ_h. Our answer concerns the case where the energy loss of h occurs on the neighborhood of a finite subset of $[0, 1]$ due to a few isolated critical points. This framework includes the above examples.

The main innovation which enables to solve the problem in this case, has geometric nature. Namely, for the lower estimate of probabilities $\mathbb{P}(||W_T - h|| < \varepsilon)$ we use the probabilities of the kind $\mathbb{P}(W_T \in A(h, \varepsilon))$ where $A(h, \varepsilon)$ is a subset of the ε-ball around h but it is *not a ball* itself. Instead, $A(h, \varepsilon)$ turns out to be a set of trajectories running inside of a very narrow strip at the most critical points of h and inside of a large one elsewhere.

For any Borel subset B of $[0,1]$ and h absolutely continuous we write

$$\|h\|_B = \sup_B |h|, \qquad J(h,B) = \int_B h'^2\, d\lambda$$

and consider the *local energy loss* function of h,

$$\varphi_h(\varepsilon, B) = J(h,B) - \inf_{\|g-h\|\leqslant\varepsilon} J(h,B)$$

so that $\varphi_h(\varepsilon,[0,1]) = \varphi_h(\varepsilon)$. First consider the generic situation where the energy must be spared at 0.

THEOREM 1. – *If $h \in \partial^i\mathcal{H}_1$ is such that for any $x \in]0,1[$,*

$$V\big(h',[0,x]\big) = \infty \quad \textit{and} \quad V\big(h',[x,1]\big) < \infty \tag{8}$$

then

$$\lim_{\varepsilon\to 0} \frac{\varphi_h(\varepsilon,[0,x])}{\varphi_h(\varepsilon)} = 1 \tag{9}$$

and the unique solution $\varepsilon(T)$ of Eq. (6) *satisfies*

$$\liminf_{T\to\infty} \frac{\|W_T - h\|}{\varepsilon(T)} = 1 \quad \textit{a.s.}$$

Note that (9) allows to solve (6) using any $\varphi_h(\varepsilon,[0,x])$ instead of $\varphi_h(\varepsilon)$. Hence φ_h needs to be studied locally only – optimal h_ε in (7) is not required.

Remark 3. – Concerning the relationship between (7) and (8), consider the simple situation where h follows the assumptions of Theorem 1 and h is either (i) concave on $[0,x_0]$ with $h^{'}(0)=\infty$ or (ii) convex on $[0,x_0]$ with $h^{'}(0)=-\infty$. Define δ_ε as the smallest solution of (i) $h(\delta)\geqslant \varepsilon+\delta h'(\delta)$ or (ii) $h(\delta)\leqslant -\varepsilon+\delta h'(\delta)$ so that δ_ε decreases to 0 as ε tends to 0. Then for all $\varepsilon>0$ small enough there exists $h_\varepsilon\in\mathcal{R}_h(\varepsilon)$ and $x\in(0,x_0)$ such that $\mathbf{1}_{[\delta_\varepsilon,x]}h'_\varepsilon = \mathbf{1}_{[\delta_\varepsilon,x]}h'$ a.e. Further, (9) can be refined into

$$\varphi_h(\varepsilon) = J\big(h,[0,\delta_\varepsilon]\big) - \big(|h(\delta_\varepsilon)|-\varepsilon\big)^2/\delta_\varepsilon + \mathrm{O}(\varepsilon).$$

We provide a detailed proof of Theorem 1 to help the reader in understanding what makes the following more general version work.

THEOREM 2. – *Let $h\in\partial^i\mathcal{H}_1$ be such that there exists $0\leqslant x_1<\cdots<x_n\leqslant 1$ satisfying, for any $\theta>0$, $A_{\theta,i}=[x_i-\theta,x_i+\theta]\cap[0,1]$ and $A_\theta=\bigcup_{i=1}^n A_{\theta,i}$,*

$$V(h',A_\theta) = \infty \quad \textit{and} \quad V\big(h',[0,1]\backslash A_\theta\big) < \infty. \tag{10}$$

Then the conclusion of Theorem 1 *holds true, with* (9) *replaced with*

$$\lim_{\varepsilon\to 0}\frac{\varphi_h(\varepsilon, A_\theta)}{\varphi_h(\varepsilon)} = 1.$$

Remark 4. – Comparing the quantities $\varphi_h(\varepsilon, A_{\theta,i})$ can tell us how many x_i are really essential. We call a point x_i *sub-critical* whenever for all $\theta > 0$ such that $\bigcap_{i=1}^n A_{\theta,i} = \emptyset$,

$$\lim_{\varepsilon\to 0}\frac{\varphi_h(\varepsilon, A_{\theta,i})}{\varphi_h(\varepsilon)} = 0.$$

If x_i is sub-critical, then the resulting rate $\varepsilon(T)$ is not affected by $\varphi_h(\varepsilon, A_{\theta,i})$.

Remark 5. – The actual position x_i of the most critical oscillation slightly influences φ_h and the exact constant in $\varepsilon(T)$ but not the rate. Usually, having $x_1 = 0$ leads to higher $\varepsilon(T)$ because translating the same oscillation at $x_1 > 0$ turns $\varphi_h(\varepsilon, [0,\theta])$ into $\varphi_h(2\varepsilon, [x_i, x_i + \theta])$ – compare h_1 and h_3.

We now deduce from Theorem 2 the functional Chung law for our examples.

COROLLARY 3. – *If $h \in \partial^i \mathcal{H}_1$ satisfies* (10) *and*

$$d_\rho(h) = \lim_{\varepsilon\to 0}\varepsilon^{-\rho}\varphi_h(\varepsilon) \in (0,\infty)$$

with $\rho < 1$ then

$$\liminf_{T\to\infty}(\log\log T)^{2/(\rho+2)}\|W_T - h\| = \left(\frac{\pi^2}{16 d_\rho(h)}\right)^{1/(\rho+2)} \quad a.s.$$

Corollary 3 applies to h_i for $i = 1,\dots,5$ with $\rho = 2 - 1/\alpha$ and explicit $d_\rho(h_i)$. In particular,

$$\begin{aligned}&\liminf_{T\to\infty}(\log\log T)^{2\alpha/(4\alpha-1)}\|W_T - h_1\| \\ &\quad = \frac{(\pi^2/32)^{\alpha/(4\alpha-1)}(2\alpha-1)^{(2\alpha-1)/(8\alpha-2)}}{\alpha^{(2\alpha-1)/(4\alpha-1)}(1-\alpha)^{(1-\alpha)/(4\alpha-1)}} \quad \text{a.s.}\end{aligned} \tag{11}$$

The power $2\alpha/(4\alpha - 1)$ fills the gap between $2/3$ and 1, as announced in erroneous Corollaries 1 and 4 in [9]. Our results for h_i now provide right power, exact constants and remainder terms.

Remark 6. – When $\alpha \to 1/2$, the limiting constant tends to $\pi/4$, hence (11) falls in agreement with Chung's law, that is (2) for $h = 0$. Clearly, for α very close to $1/2$ both h_1 and W_T expend most of their energy at the origin and then, roughly speaking, stay within the interval $[-\varepsilon, \varepsilon]$ while the time varies from almost zero to one. The same comment stands for h_3 when $h'_{3\pm} = 0$ but the limiting constant is smaller than Chung's one. When $\alpha \to 1$, the limiting constant tends to $(\pi^2/32)^{1/3}$, thus (11) also provides a correct interpolation towards the exact rate for $h(x) = x$ given in Csáki [3].

3. Proofs

In this section we achieve the lower bound in (5) under (8), then under (10).

3.1. Proof of Theorem 1

Our preliminary lemma justifies (9).

LEMMA 4. – *Let* $h \in \partial^i \mathcal{H}_1$ *obey* (8). *For all* $x \in]0,1[$ *we have*

$$\lim_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon,[0,x])}{\varepsilon} = \infty \quad \textit{and} \quad \limsup_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon,[x,1])}{\varepsilon} < \infty. \tag{12}$$

Further, there exists a positive function $\rho_x(\varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \rho_x(\varepsilon) = 0$ *and*

$$\varphi_h(\varepsilon) \geqslant \varphi_h\big(\varepsilon,[0,x]\big) \geqslant \big(1-\rho_x(\varepsilon)\big)\varphi_h(\varepsilon). \tag{13}$$

Proof. – Under (8), Propositions 1 and 2 in [10] respectively imply

$$\lim_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon,[x,1])}{2\varepsilon} = \big|h'(x)\big| + \big|h'(1)\big| + V\big(h',[x,1]\big) < \infty$$

and

$$\lim_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon,[0,x])}{2\varepsilon} = |h'(x)| + V\big(h',[0,x]\big) = \infty$$

whence (12). In the same way, $\lim_{\varepsilon \to 0} \varphi_h(\varepsilon)/\varepsilon = \infty$. The upper bound in (13) comes from the fact that replacing g' with h' on $[x,1]$ yields

$$\begin{aligned}
\varphi_h(\varepsilon) &= 1 - \inf_{\|g-h\| \leqslant \varepsilon} \big(J\big(g,[0,x]\big) + J\big(g,[x,1]\big)\big) \\
&\geqslant 1 - \inf_{\|g-h\| \leqslant \varepsilon} \big(J\big(g,[0,x]\big) + J\big(h,[x,1]\big)\big) = \varphi_h\big(\varepsilon,[0,x]\big).
\end{aligned}$$

Since

$$\begin{aligned}
\varphi_h\big(\varepsilon,[0,x]\big) + \varphi_h\big(\varepsilon,[x,1]\big) &= 1 - \Big(\inf_{\|g-h\| \leqslant \varepsilon} J\big(g,[0,x]\big) + \inf_{\|g-h\| \leqslant \varepsilon} J\big(g,[x,1]\big)\Big) \\
&\geqslant 1 - \inf_{\|g-h\| \leqslant \varepsilon} \big(J\big(g,[0,x]\big) + J\big(g,[x,1]\big)\big) = \varphi_h(\varepsilon),
\end{aligned}$$

we see that $\rho_x(\varepsilon) = \varphi_h(\varepsilon,[x,1])/\varphi_h(\varepsilon)$ satisfies (13) together with

$$\lim_{\varepsilon \to 0} \rho_x(\varepsilon) = \lim_{\varepsilon \to 0} \frac{\varphi_h(\varepsilon,[x,1])/\varepsilon}{\varphi_h(\varepsilon)/\varepsilon} = 0. \qquad \square$$

Fix $h \in \partial^i \mathcal{H}_1$ satisfying (8). For brevity, we write $D = (2\log\log T)^{1/2}$ and let $\varepsilon = \varepsilon(T)$ be the solution of (6). The forthcoming constants $\beta_i > 0$ are everywhere sufficiently small. The following steps aim to evaluate

$$\mathbb{P}\big(\|W_T - h\| \leqslant (1+\beta_1)\varepsilon\big) \tag{14}$$

for all sufficiently large T.

Step 1. Let us start with useful consequences of the assumptions in force. Since $d(h)=\infty$ in (4), we have $\lim_{\varepsilon\to 0}\varphi_h(\varepsilon)/\varepsilon=\infty$ whereas $\lim_{\varepsilon\to 0}\varphi_h(\varepsilon)=0$ by semi-continuity of the energy function. Moreover, (6) means

$$\frac{\varphi_h(\varepsilon)}{\varepsilon}=\frac{\pi^2}{4D^4\varepsilon^3} \tag{15}$$

thus $\lim_{T\to\infty}\varepsilon=0$, $\lim_{T\to\infty}D^{4/3}\varepsilon=0$ but $\lim_{T\to\infty}D^2\varepsilon=\infty$. Next, (8) and (12) ensure that for every fixed $x\in\,]0,1[$, $\beta_2>0$ and arbitrarily small ε,

$$\left|h'(1)\right|+V\left(h',[x,1]\right)<\beta_2\frac{\varphi_h(\varepsilon,[0,x])}{\varepsilon}. \tag{16}$$

Mixing (13) and (15) further gives, for any $\beta_3>0$ and ε small enough,

$$\frac{\varphi_h(\varepsilon,[0,x])}{\varepsilon}\geqslant(1-\beta_3)\frac{\varphi_h(\varepsilon)}{\varepsilon}=(1-\beta_3)\frac{\pi^2}{4D^4\varepsilon^3}. \tag{17}$$

For any $\varepsilon>0$ and $x\in\,]0,1[$ consider $h_\varepsilon\in\mathcal{H}_1$ such that $\|h_\varepsilon-h\|\leqslant\varepsilon$ and $J(h_\varepsilon,[0,x])=J(h,[0,x])-\varphi_h(\varepsilon,[0,x])$. We introduce the mixture $g_\varepsilon=(1-\beta_4)h_\varepsilon+\beta_4 h$. Obviously,

$$\|g_\varepsilon-h\|\leqslant(1-\beta_4)\varepsilon. \tag{18}$$

Step 2. We split the lower bound in two parts observing that the most probable way of fulfilling our small ball requirement (14) for Brownian path is to follow g_ε on $[0,x]$ very closely, then to stay in a larger tube around h on $[x,1]$. By independence and stationarity of the increments of W,

$$\begin{aligned}
&\mathbb{P}\left(\|W-Dh\|\leqslant(1+\beta_1)D\varepsilon\right)\\
&\quad\geqslant\mathbb{P}\left(\left\{\|W-Dh\|_{[0,x]}\leqslant D\varepsilon\right\}\cap\left\{\|W-Dh\|_{[x,1]}\leqslant(1+\beta_1)D\varepsilon\right\}\right)\\
&\quad\geqslant\mathbb{P}\left(\|W-Dh\|_{[0,x]}\leqslant D\varepsilon\right)\\
&\qquad\times\inf_{|a|\leqslant D\varepsilon}\mathbb{P}\left(\|a+\widetilde{W}-D\Delta_x h\|_{[0,1-x]}\leqslant(1+\beta_1)D\varepsilon\right)
\end{aligned} \tag{19}$$

where $\Delta_x h(s)=h(x+s)-h(x)$ for $s\in[0,1-x]$, $\widetilde{W}(s)=W(x+s)-W(x)$ is still a Brownian motion and $a=W(x)-Dh(x)$ is controlled by the first event.

Step 3. Using (18) and the Cameron–Martin formula, we get

$$\begin{aligned}
\mathbb{P}\left(\|W-Dh\|_{[0,x]}\leqslant D\varepsilon\right)&\geqslant\mathbb{P}\left(\|W-Dg_\varepsilon\|_{[0,x]}\leqslant\beta_4 D\varepsilon\right)\\
&\geqslant\mathbb{P}\left(\|W\|_{[0,x]}\leqslant\beta_4 D\varepsilon\right)\exp\left(-\frac{D^2}{2}J\left(g_\varepsilon,[0,x]\right)\right).
\end{aligned}$$

Now, by Chung's estimate (see [2]) and the scaling property, for every $\beta_5>0$ all $D\varepsilon$ small enough satisfy

$$\mathbb{P}\left(\|W\|_{[0,x]}\leqslant\beta_4 D\varepsilon\right)\geqslant\exp\left(-\left(\frac{\pi^2}{8}+\beta_5\right)\left(\frac{\beta_4 D\varepsilon}{\sqrt{x}}\right)^{-2}\right).$$

Recalling the definition of h_ε, assumption (8), and $J(h_\varepsilon) < J(h) = 1$ we have

$$\begin{aligned}
J\big(g_\varepsilon, [0,x]\big) &= J\big((1-\beta_4)h_\varepsilon + \beta_4 h, [0,x]\big)\\
&\leqslant (1-\beta_4)^2 J\big(h_\varepsilon, [0,x]\big) + \beta_4^2 J\big(h, [0,x]\big)\\
&\quad + 2\beta_4(1-\beta_4)\sqrt{J\big(h_\varepsilon, [0,x]\big) J\big(h, [0,x]\big)}\\
&\leqslant (1-\beta_4)^2\big(J\big(h, [0,x]\big) - \varphi_h\big(\varepsilon, [0,x]\big)\big)\\
&\quad + \beta_4^2 J\big(h, [0,x]\big) + 2\beta_4(1-\beta_4) J\big(h, [0,x]\big)\\
&= J\big(h, [0,x]\big) - (1-\beta_4)^2 \varphi_h\big(\varepsilon, [0,x]\big).
\end{aligned}$$

Therefore, ultimately in $D\varepsilon \to 0$,

$$\begin{aligned}
\mathbb{P}\big(\|W - Dh\|_{[0,x]} \leqslant D\varepsilon\big) \geqslant \exp\bigg(&-\frac{D^2}{2}\big(J\big(h,[0,x]\big) - (1-\beta_4)^2\varphi_h\big(\varepsilon,[0,x]\big)\big)\\
&- \left(\frac{\pi^2}{8} + \beta_5\right)\frac{x}{(\beta_4 D\varepsilon)^2}\bigg).
\end{aligned} \tag{20}$$

Step 4. Fix $|a| \leqslant D\varepsilon$. The Cameron–Martin formula implies

$$\begin{aligned}
&\mathbb{P}\big(\|a + \widetilde{W} - D\Delta_x h\|_{[0,1-x]} \leqslant (1+\beta_1)D\varepsilon\big)\\
&\quad = \exp\left(-\frac{D^2}{2} J\big(\Delta_x h, [0,1-x]\big)\right)\\
&\qquad \times \mathbb{E}\left(I_{\{\|a+\widetilde{W}\|_{[0,1-x]} \leqslant (1+\beta_1)D\varepsilon\}} \exp\left(-D \int\limits_{[0,1-x]} (\Delta_x h)'\, d\widetilde{W}\right)\right).
\end{aligned} \tag{21}$$

Now, remind that $h'(x+s) = \gamma([x+s,1])$ is a version of $(\Delta_x h)'$ on $s \in [0,1-x]$ and let γ_x denote the corresponding measure on $[0,1-x]$, i.e. γ translated by $-x$. Taking into account (16) and the indicator function in (21), the integration by parts then gives

$$\begin{aligned}
\int\limits_{[0,1-x]} (\Delta_x h)'\, d\widetilde{W} &= \widetilde{W}(1-x)(\Delta_x h)'(1-x) + \int\limits_{[0,1-x]} \widetilde{W}\, d\gamma_x\\
&\leqslant \sup_{[0,1-x]} |\widetilde{W}|\big(\big|h'(1)\big| + V\big(h', [x,1]\big)\big)\\
&\leqslant \big(|a| + (1+\beta_1)D\varepsilon\big)\beta_2 \frac{\varphi_h(\varepsilon, [0,x])}{\varepsilon}\\
&\leqslant (2+\beta_1)\beta_2 D\varphi_h\big(\varepsilon, [0,x]\big).
\end{aligned}$$

Next we rescale $\widetilde{W}$ to a standard Wiener process W again and apply a boundary crossing estimate (see [11] or e.g. Theorem 4.5 in [1]). Uniformly in $|a| \leqslant D\varepsilon$ we get, as $D\varepsilon \to 0$,

$$\begin{aligned}
&\mathbb{P}\big(\|a + \widetilde{W}\|_{[0,1-x]} \leqslant (1+\beta_1)D\varepsilon\big)\\
&\quad = \mathbb{P}\big(\|a + \sqrt{1-x}\,W\|_{[0,1]} \leqslant (1+\beta_1)D\varepsilon\big)
\end{aligned}$$

$$\geqslant \mathbb{P}\left(\left\{\frac{-\beta_1 D\varepsilon}{\sqrt{1-x}} \leqslant W(t) \leqslant \frac{(2+\beta_1)D\varepsilon}{\sqrt{1-x}}\colon t \in [0,1]\right\}\right)$$
$$= \exp\left(-\frac{\pi^2(1-x)+\mathrm{o}(1)}{2(2+\beta_1+\beta_1)^2(D\varepsilon)^2}\right).$$

Thus (21) is bounded below by

$$\exp\left(-\frac{D^2}{2}J\big(h,[x,1]\big) - \left(\frac{\pi^2(1-x)}{8(1+\beta_1)^2}+\beta_6\right)\frac{1}{(D\varepsilon)^2} - (2+\beta_1)\beta_2 D^2\varphi_h\big(\varepsilon,[0,x]\big)\right) \quad (22)$$

for every $\beta_6 > 0$ provided ε and $D\varepsilon$ are small enough.

Step 5. Combining (19), (20) and (22), all small ε satisfy

$$\mathbb{P}\big(\|W-Dh\| \leqslant (1+\beta_1)D\varepsilon\big)$$
$$\geqslant \exp\left(-\frac{D^2}{2} - \left(\frac{1-x}{(1+\beta_1)^2} + \left(1+\frac{8\beta_5}{\pi^2}\right)\frac{x}{\beta_4^2} + \frac{8\beta_6}{\pi^2}\right)\frac{\pi^2}{8(D\varepsilon)^2}\right.$$
$$\left. + \big((1-\beta_4)^2 - 2(2+\beta_1)\beta_2\big)\frac{D^2}{2}\varphi_h\big(\varepsilon,[0,x]\big)\right)$$

which, in view of (17), yields

$$\mathbb{P}\big(\|W-Dh\| \leqslant (1+\beta_1)D\varepsilon\big)$$
$$\geqslant \exp\left(-\frac{D^2}{2} + \frac{\pi^2}{8(D\varepsilon)^2}\left(\big((1-\beta_4)^2 - 2(2+\beta_1)\beta_2\big)(1-\beta_3)\right.\right.$$
$$\left.\left. - \frac{1}{(1+\beta_1)^2} - \left(1+\frac{8\beta_5}{\pi^2} - \frac{\beta_4^2}{(1+\beta_1)^2}\right)\frac{x}{\beta_4^2} - \frac{8\beta_6}{\pi^2}\right)\right)$$
$$\geqslant \exp\left(-\frac{D^2}{2} + \frac{\beta_7}{(D\varepsilon)^2}\right)$$

where $\beta_7 > 0$ provided $\beta_4 < (1+\beta_1)^{-1}\beta_1$ and $\beta_2, \beta_3, \beta_6, x$ are chosen sufficiently small with respect to β_1 and β_4. Hence, if T is so large that $D^{4/3}\varepsilon \leqslant \sqrt{\beta_7}$, and (15), (20), and (22) simultaneously hold, we obtain the precise estimate

$$\mathbb{P}\big(\|W-Dh\| \leqslant (1+\beta_1)D\varepsilon\big) \geqslant \exp\left(-\frac{D^2}{2} + D^{2/3}\right) = \frac{1}{\log T}\exp\big((2\log\log T)^{1/3}\big).$$

Step 6. The lower bound of Step 5 allows to conclude the proof by the following standard argument, as in [3]. Applying divergent part of Borel–Cantelli lemma along the sequence $T_n = n^n$ with slightly modified W_T and h to ensure independence, we easily deduce that for every $\beta_1 > 0$

$$\liminf_{T\to\infty} \frac{\|W_T - h\|}{\varepsilon(T)} \leqslant 1+\beta_1 \quad \text{a.s.}$$

3.2. Proof of Theorem 2

Fix $\theta > 0$ so small that $\bigcap_{i\leqslant n} A_{\theta,i} = \emptyset$ and hence $B_\theta = [0,1]\backslash A_\theta = \bigcup_{j\leqslant m_h} B_{\theta,j}$ is a union of m_h disjoint intervals. Clearly, $m_h = n$ exept when $(x_1, x_n) = (0,1)$ $(m_h = n-1)$ or $0 < x_1 < x_n < 1$ $(m_h = n+1)$.

Under (10), Lemma 4 holds with $[0,x]$ changed into A_θ and $[x,1]$ into B_θ, by the same arguments. Also, $V(h, A_{\theta,i}) < \infty$ implies $\lim_{\varepsilon\to 0} \varphi_h(\varepsilon, A_{\theta,i})/\varphi_h(\varepsilon, A_\theta) = 0$ thus we can assume with no loss of generality that

$$\inf_{i\leqslant n} V(h, A_{\theta,i}) = \infty. \tag{23}$$

In step 1, (16) becomes

$$\begin{aligned}\sum_{j=1}^{m_h} \psi(B_{\theta,j}) &= \sum_{j=1}^{m_h} \bigl(\bigl|h'(\inf B_{\theta,j})\bigr| + \bigl|h'(\sup B_{\theta,j})\bigr| + V(h', B_{\theta,j})\bigr) \\ &< \beta_2 \frac{\varphi_h(\varepsilon, A_\theta)}{\varepsilon}.\end{aligned} \tag{24}$$

In step 2, we progressively enlarge the size of the main strip around h by using constants $\beta_0^1 = 0 < \beta_i^1 < \beta_{i+1}^1 < \beta_{m_h+n-1}^1 = \beta_1$. Let $(\tau,\delta) = (\theta, 1)$ whenever $x_1 = 0$ and $(\tau,\delta) = (x_1 - \theta, 0)$ otherwise. Writing $\alpha_i = \lambda(A_{\theta,i})$ and $\eta_j = \lambda(B_{\theta,j}))$ the basic decomposition (19) now reads

$$\begin{aligned}&\mathbb{P}\bigl(\|W - Dh\| \leqslant (1+\beta_1)D\varepsilon\bigr) \\ &\quad\geqslant \mathbb{P}\bigl(\|W - Dh\|_{[0,\tau]} \leqslant D\varepsilon\bigr) \\ &\quad\prod_{i=1+\delta}^{n} \inf_{|b|\leqslant(1+\beta^1_{2i-2-\delta})D\varepsilon} \mathbb{P}\bigl(\|b + W_i - D\Delta_i h\|_{[0,\alpha_i]} \leqslant \bigl(1+\beta^1_{2i-1-\delta}\bigr)D\varepsilon\bigr) \\ &\quad\prod_{j=2-\delta}^{m_h} \inf_{|a|\leqslant(1+\beta^1_{2j-3+\delta})D\varepsilon} \mathbb{P}\bigl(\bigl\{\|a + \widetilde{W}_j - D\widetilde{\Delta}_j h\|_{[0,\eta_j]} \leqslant \bigl(1+\beta^1_{2j-2+\delta}\bigr)D\varepsilon\bigr\}\bigr)\end{aligned}$$

where an empty product is 1, W_i and $\widetilde{W}_j$ are mutually independent standard Brownian paths,

$$\Delta_i h(s) = h(x_i - \theta + s) - h(x_i - \theta)$$

and

$$\widetilde{\Delta}_j h(s) = h(\inf B_{\theta,j} + s) - h(\inf B_{\theta,j}).$$

The following estimates do not depend on the exit levels a from $A_{\theta,i}$ and b from $B_{\theta,j}$ controlling the chain of conditioning events since θ is fixed and crucial rectifications occur very close to the x_i's, as $\varepsilon \to 0$.

In step 3, a new argument is required when $b \neq 0$. For fixed $i \geqslant 1+\delta$ we consider $h_{\varepsilon,i} \in \mathcal{H}_1$ such that $\|h_{\varepsilon,i} - h\|_{A_{\theta,i}} \leqslant \varepsilon$ and $J(h_{\varepsilon,i}, A_{\theta,i}) = J(h, A_{\theta,i}) - \varphi_h(\varepsilon, A_{\theta,i})$. Under (23) we have, for $A^-_{\theta,i} = [x_i - \theta, x_i - \theta/2]$,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\varphi_h(\varepsilon, A_{\theta,i})} = \lim_{\varepsilon\to 0} \frac{\varphi_h(\varepsilon, A^-_{\theta,i})}{\varphi_h(\varepsilon, A_{\theta,i})} = 0.$$

Therefore, $h_{\varepsilon,i}$ can be modified on $A^-_{\theta,i}$ at almost no energy cost.

LEMMA 5. – *If ε is small enough, then, for any $i \geqslant 1+\delta$ and $|c| \leqslant \varepsilon$ one can find $h_{c,\varepsilon,i} \in \mathcal{H}_1$ such that $h_{c,\varepsilon,i}(x_i-\theta) = h(x_i-\theta)+c$, $\|h_{c,\varepsilon,i}-h\|_{A_{\theta,i}} \leqslant \varepsilon$ and $J(h_{c,\varepsilon,i}, A_{\theta,i}) \leqslant J(h, A_{\theta,i}) - (1-\beta_9)\varphi_h(\varepsilon, A_{\theta,i})$.*

A solution is given by letting $h'_{c,\varepsilon,i} = h' + 2(h_{\varepsilon,i}(x_i-\theta/2) - h(x_i-\theta/2) - c)/\theta$ on $A^-_{\theta,i}$ and $h'_{c,\varepsilon,i} = h'_{\varepsilon,i}$ on $A_{\theta,i}\backslash A^-_{\theta,i}$ since then

$$\begin{aligned} J(h_{c,\varepsilon,i}, A_{\theta,i}) &= J(h, A^-_{\theta,i}) + \mathrm{O}(\varepsilon) + J(h_{\varepsilon,i}, A_{\theta,i}\backslash A^-_{\theta,i}) \\ &= J(h, A_{\theta,i}) - \varphi_h(\varepsilon, A_{\theta,i}) + \mathrm{O}(\varepsilon) + \bigl(J(h, A^-_{\theta,i}) - J(h_{\varepsilon,i}, A^-_{\theta,i})\bigr). \end{aligned}$$

Let $0 < \beta_4 < \inf_{0\leqslant k\leqslant m_h+n-2}(\beta^1_{k+1} - \beta^1_k)$. Combining Lemma 5 with the arguments of step 3 we obtain, uniformly in $|b| \leqslant (1+\beta^1_k)D\varepsilon < (1+\beta^1_{k+1})D\varepsilon$,

$$\begin{aligned} &\mathbb{P}\bigl(\|b + W_i - D\Delta_i h\|_{[0,\alpha_i]} \leqslant (1+\beta^1_{k+1})D\varepsilon\bigr) \\ &\quad\geqslant \mathbb{P}\bigl(\|W_i - D\Delta_i h_{b/D,(1+\beta^1_{k+1})\varepsilon,i}\|_{[0,\alpha_i]} \leqslant \beta_4 D\varepsilon\bigr) \\ &\quad\geqslant \exp\biggl(-\frac{D^2}{2}\bigl(J(h, A_{\theta,i}) - (1-\beta_9)\varphi_h(\varepsilon, A_{\theta,i})\bigr) - \frac{(1+\beta_5)\pi^2\alpha_i}{8(\beta_4 D\varepsilon)^2}\biggr). \end{aligned}$$

Next, along the lines of step 4, we get, for $|a| \leqslant (1+\beta^1_k)D\varepsilon$ and $D\varepsilon$ small enough,

$$\begin{aligned} &\mathbb{P}\bigl(\bigl\{\|a + \widetilde{W}_j - D\widetilde{\Delta}_j h\|_{[0,\eta_j]} \leqslant (1+\beta^1_{k+1})D\varepsilon\bigr\}\bigr) \\ &\quad\geqslant \exp\biggl(-\frac{D^2}{2}J(h, B_{\theta,j}) - \frac{\pi^2(1+\beta_6)\eta_j}{8(1+\beta^1_{k+1})^2(D\varepsilon)^2} \\ &\qquad - (2+\beta^1_k+\beta^1_{k+1})\psi(B_{\theta,j})D^2\varepsilon\biggr). \end{aligned}$$

Taking care of $\mathbb{P}(\|W - Dh\|_{[0,\tau]} \leqslant D\varepsilon)$ in one or the other way – according to δ and recalling that $a = b = 0$ in this case – it follows that

$$\begin{aligned} &\mathbb{P}\bigl(\|W - Dh\| \leqslant (1+\beta_1)D\varepsilon\bigr) \\ &\quad\geqslant \exp\biggl(-\frac{\pi^2}{8(D\varepsilon)^2}\biggl(\frac{1+\beta_6}{(1+\beta^1_1)^2}\sum_{j=1}^{m_h}\eta_j + \biggl(\frac{1+\beta_5}{\beta_4^2}\biggr)\sum_{i=1}^{n}\alpha_i\biggr) \\ &\qquad - \frac{D^2}{2}\biggl(1 - (1-\beta_9)\varphi_h(\varepsilon, A_\theta) + 4(1+\beta_1)\sum_{j=1}^{m_h}\psi(B_{\theta,j})\varepsilon\biggr)\biggr) \end{aligned}$$

where we used $\beta^1_1 \leqslant \beta^1_k \leqslant \beta_1$ for $k \geqslant 1$ and $\varphi_h(\varepsilon, A_\theta) \leqslant \sum_{i\leqslant n}\varphi_h(\varepsilon, A_{\theta,i})$. Since

$$1 - \sum_{j=1}^{m_h}\eta_j = \sum_{i=1}^{n}\alpha_i \leqslant 2n\theta$$

is arbitrarily small, we conclude as for Theorem 1, by (24) and $\varphi_h(\varepsilon, A_\theta) \geqslant (1-\beta_3)\pi^2/4D^4\varepsilon^2$. $\square$

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