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SPECTRAL DENSITIES DESCRIBING OFF-WHITE NOISES

DENSITÉ SPECTRALE DE "OFF-WHITE NOISE"

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ABSTRACT. – For the white noise, the spectral density is constant, and the past (restriction to $(-\infty, 0)$) is independent from the future (restriction to $(0, +\infty)$). If the spectral density is not too far from being constant, then dependence between the past and the future can be eliminated by an equivalent measure change. A necessary and sufficient condition for a spectral density to have such a property (in other words, to describe an off-white noise) is derived here from well-known results.

RÉSUMÉ. – Pour le bruit blanc, la densité spectrale est constante et le passé (restriction à $(-\infty, 0)$) est indépendant du futur (restriction à $(0, +\infty)$). Si la densité spectrale n'est pas trop éloignée d'une constante, la dépendance entre passé et futur peut être éliminée en remplaçant la mesure par une mesure équivalente. Un tel processus gaussien stationnaire sera appelé « off-white noise ». Nous donnons une condition nécessaire et suffisante sur la densité spectrale pour que cette propriété soit vérifiée.

0. Introduction

'Past and future' is a well-known topic in the theory of stationary Gaussian random processes. The restriction of a process $X(t)$ to $t \in (-\infty, 0)$ is the past; the future is its restriction to $(0, +\infty)$ or, more generally, $(s, +\infty)$. Typically one shows that the past and the future are nearly independent if the separation s is large enough, under appropriate conditions on the spectral density of the process. In contrast, the present work deals with the case $s = 0$ (no separation). For a continuous process, of course, $X(0)$ belongs both to the past and to the future, making them heavily dependent. However, for the white

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noise they are independent anyway. An off-white noise [2] introduced in [6] (motivated by the theory of continuous tensor products of Hilbert spaces) generalizes the white noise. It is defined as a stationary Gaussian generalized random process such that the joint distribution of the past and the future is equivalent (that is, mutually absolutely continuous) to the product of their marginal distributions. The present work derives from well-known results about 'past and future' a necessary and sufficient condition for a spectral density to describe an off-white noise. I feel that it is basically a folklore worth to be written.

In the Hilbert space H of all linear measurable functionals over a Gaussian random process, the past $\mathcal{P}$ and the future $\mathcal{F}$ are linear subspaces. For the white noise, $H = \mathcal{P} \oplus \mathcal{F}$, the subspaces being orthogonal. For an off-white noise the corresponding relation is

$$H = \mathcal{P} \oplus \mathcal{F} \quad \text{in the FHS sense,}$$

as defined in [6]; it means that the orthogonal projection from $\mathcal{F}$ to $\mathcal{P}$ is a Hilbert–Schmidt operator, and $H = \mathcal{P} \oplus \mathcal{F}$ in the topological sense. The latter means that $\mathcal{P} + \mathcal{F}$ is dense in H and $\mathcal{P}, \mathcal{F}$ are at positive angle (that is, the projection is of norm less than 1).

1. Analytic functions inside and outside the circle

DEFINITION 1.1. – (a) *A* past-and-future structure (*or 'PaF structure'*) *consists of*:

- *a separable Hilbert space* H;
- *a two-sided sequence* $(\mathcal{P}_n)_{n\in\mathbb{Z}}$ *of* (*closed linear*) *subspaces* $\mathcal{P}_n \subset H$, *increasing* (*in the sense that* $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ *for all* n) *and such that the union of all* $\mathcal{P}_n$ *is dense in* H;
- *a two-sided sequence* $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ *of subspaces* $\mathcal{F}_n \subset H$, *decreasing and such that the union of all* $\mathcal{F}_n$ *is dense in* H;
- *a unitary operator* $T : H \to H$ *such that* $T\mathcal{P}_n = \mathcal{P}_{n+1}$ *and* $T\mathcal{F}_n = \mathcal{F}_{n+1}$ *for all* n.

(b) *Two PaF structures* $(H, (\mathcal{P}_n), (\mathcal{F}_n), T)$ *and* $(H', (\mathcal{P}'_n), (\mathcal{F}'_n), T')$ *are* isomorphic, *if there exists an invertible linear isometry* $U : H \to H'$ *such that* $U\mathcal{P}_n = \mathcal{P}'_n$ *and* $U\mathcal{F}_n = \mathcal{F}'_n$ *for all* n, *and* $UT = T'U$.

(c) *A* PaF geometry *is a PaF structure treated up to isomorphism.*

(d) *For any PaF geometry* $\mathcal{G} = (H, (\mathcal{P}_n), (\mathcal{F}_n), T)$ *and any* $k \in \mathbb{Z}$ *define* $\mathcal{G} + k$ (*the shifted PaF geometry*) *as* $(H, (\mathcal{P}_{n+k}), (\mathcal{F}_n), T)$ (*or equivalently* $(H, (\mathcal{P}_n), (\mathcal{F}_{n-k}), T)$). *Also define the* time-reversed PaF geometry *as* $\overline{\mathcal{G}} = (H, (\mathcal{F}_{-n}), (\mathcal{P}_{-n}), T^{-1})$.

Let μ be a (positive) σ-finite Borel measure on the unit circle $\{z \in \mathbb{C}: |z| = 1\}$. The set of all polynomials P such that $\int |P|^2 \, d\mu < \infty$ is an ideal in the commutative ring of polynomials. If the ideal contains not only 0, then it is generated by a single polynomial P_μ (not identically 0), since every ideal in that ring is principal. It is easy to see that (up to a coefficient that may be ignored), $P_\mu(z) = (z - z_1) \cdots (z - z_m)$ for some $z_1, \ldots, z_m$ on the circle. If μ is finite then $m = 0$ and $P_\mu(z) = 1$.

[2] The term is suggested by William Arveson; I called it a 'slightly coloured noise'.

DEFINITION 1.2. – (a) *A nonatomic σ-finite Borel measure μ on the circle will be called* moderate, *if the ideal of polynomials P satisfying $\int |P|^2 \, d\mu < \infty$ is different from* $\{0\}$.

(b) Poles *of a moderate measure μ are roots of the polynomial P_μ that generates the ideal.* Multiplicity *of a pole of μ is its multiplicity as a root of P_μ.*

The set of all moderate measures is a linear space, closed under multiplication by functions of the form $f/|P|^2$ where f is a bounded nonnegative Borel function on the circle and P is a polynomial (not identically 0).

Conjugation $z \mapsto \bar{z}$ maps the circle onto itself, and sends each measure μ on the circle to another measure, denote it $\tilde{\mu}$. Clearly, μ is moderate if and only if $\tilde{\mu}$ is moderate, and if they are, then $P_\mu(z) = \overline{P_{\tilde{\mu}}(\bar{z})}$ for all z. Also, each function $f \in L_2(\mu)$ corresponds to another function $\tilde{f} \in L_2(\tilde{\mu})$ such that $f(z) = \tilde{f}(\bar{z})$. However, if f is a polynomial P restricted to the circle then $\tilde{f}$ is rather the rational function $z \mapsto P(1/z)$ restricted to the circle. In particular, if $P(z) = z - z_1$ where $|z_1| = 1$ then $P(1/z) = -z_1(z - \bar{z}_1)/z = -z_1\overline{P(\bar{z})}/z$. Accordingly, if $P(z) = (z - z_1)\cdots(z - z_m)$ for some $z_1, \ldots, z_m$ on the circle then $P(1/z) = \text{const} \cdot z^{-m}\overline{P(\bar{z})}$. Therefore

$$P_{\tilde{\mu}}\left(\frac{1}{z}\right) = \text{const} \cdot z^{-m} P_\mu(z) \quad \text{for all } z \in \mathbb{C} \setminus \{0\}; \text{ here } m = \deg P_\mu;$$

and $\tilde{P}_{\tilde{\mu}}(z) = \text{const} \cdot z^{-m} P_\mu(z)$ for $|z| = 1$. Functions P_μ and $\tilde{P}_{\tilde{\mu}}$ have the same zeros (on the circle); however, P_μ has a pole (of multiplicity m) at ∞, while $\tilde{P}_{\tilde{\mu}}$, or rather its analytic continuation $\text{const} \cdot z^{-m} P_\mu(z)$, has a pole (of multiplicity m) at 0.

Every moderate measure μ determines a PaF structure $(H, (\mathcal{P}_n), (\mathcal{F}_n), T)$ as follows:

- $H = L_2(\mu)$;
- $\mathcal{F}_n$ is spanned by functions $z \mapsto z^k P_\mu(z)$ for $k \in \mathbb{Z}$, $k \geqslant n$;[3]
- $\mathcal{P}_n$ is spanned by functions $z \mapsto z^k \tilde{P}_{\tilde{\mu}}(z)$ for $k \in \mathbb{Z}$, $k \leqslant n$;
- $(Tf)(z) = zf(z)$ for $f \in L_2(\mu)$.

In other words, $\mathcal{P}_n$ is spanned by functions $z \mapsto z^k P_\mu(z)$ for $k \in \mathbb{Z}$, $k \leqslant n - \deg P_\mu$.

Treating the PaF structure up to isomorphism, we get a PaF geometry; denote it by $\mathcal{G}_\mu$. The time-reversed PaF geometry (as defined by 1.1(d)) corresponds to $\tilde{\mu}$:

$$\overline{\mathcal{G}_\mu} = \mathcal{G}_{\tilde{\mu}};$$

an isomorphism is $L_2(\mu) \ni f \mapsto \tilde{f} \in L_2(\tilde{\mu})$. If μ is symmetric (that is, $\mu = \tilde{\mu}$) then $\mathcal{G}_\mu$ is time-symmetric (that is, $\mathcal{G}_\mu = \overline{\mathcal{G}_\mu}$).

PROPOSITION 1.3. – *Let μ, μ' be moderate measures, z_0 a point on the circle, and $\mu'(dz) = |z - z_0|^2 \mu(dz)$. Then*[4]

$$\mathcal{G}_{\mu'} = \mathcal{G}_\mu + 1.$$

[3] Here P_μ may be replaced with any polynomial P (not identically 0) satisfying $\int |P|^2 \, d\mu < \infty$; redundant roots of P do not influence the *closed* subspace.

[4] $\mathcal{G}_\mu + 1$ is the shifted PaF geometry, recall 1.1(d).

Proof. – We have $\mathcal{G}_\mu = (H, (\mathcal{P}_n), (\mathcal{F}_n), T)$, $\mathcal{G}_{\mu'} = (H', (\mathcal{P}'_n), (\mathcal{F}'_n), T')$. Multiplication by $1/(z - z_0)$ is an invertible linear isometry $L_2(\mu) \to L_2(\mu')$, that is, $H \to H'$; it intertwines T with T'. We'll prove that it sends $\mathcal{F}_0$ to $\mathcal{F}'_0$ and $\mathcal{P}_1$ to $\mathcal{P}'_0$.

First, $\frac{1}{z-z_0}\mathcal{F}_0 \supset \mathcal{F}'_0$ for a trivial reason: for all $k \geqslant 0$ the function $z \mapsto (z - z_0)z^k P_{\mu'}(z)$ belongs to $\mathcal{F}_0$, since $\int |(z - z_0)P_{\mu'}(z)|^2\, \mu(dz) = \int |P_{\mu'}(z)|^2\, \mu'(dz) < \infty$.[5]

In order to prove that $\frac{1}{z-z_0}\mathcal{F}_0 \subset \mathcal{F}'_0$ take polynomials P_n such that $P_n(z) \to \frac{1}{z-z_0}$ for $n \to \infty$, and $|P_n(z)| \leqslant 2\left|\frac{1}{z-z_0}\right|$, whenever $|z| = 1$; say, we may take

$$P_n(z) = -(1 - \varepsilon_n)\bar{z}_0 \frac{1 - (1 - \varepsilon_n)^n \bar{z}_0^n z^n}{1 - (1 - \varepsilon_n)\bar{z}_0 z}$$

choosing $\varepsilon_n \to 0+$ such that $n\varepsilon_n \to \infty$. We have (for every $k \geqslant 0$) $P_n(z)z^k P_\mu(z) \to \frac{1}{z-z_0} z^k P_\mu(z)$ pointwise, and $|P_n(z)z^k P_\mu(z)| \leqslant \frac{2}{|z-z_0|}|P_\mu(z)|$. The majorant belongs to $L_2(\mu')$; polynomials $z \mapsto P_n(z)z^k P_\mu(z)$ belong to $L_2(\mu)$, therefore to $L_2(\mu')$, and to $\mathcal{F}'_0$. So, $\frac{1}{z-z_0}\mathcal{F}_0 = \mathcal{F}'_0$.

Now we apply the equality $\frac{1}{z-z_0}\mathcal{F}_0 = \mathcal{F}'_0$ to measures $\tilde{\mu}$, $\tilde{\mu}'$ (symmetric to μ, μ'); these are related by $\tilde{\mu}'(dz) = |z - \bar{z}_0|^2\, \tilde{\mu}(dz)$; thus, $\frac{1}{z-\bar{z}_0}\mathcal{F}_0(\tilde{\mu}) = \mathcal{F}_0(\tilde{\mu}')$. The isomorphism $f \mapsto \tilde{f}$ between $\mathcal{G}_\mu$ and $\mathcal{G}_{\tilde{\mu}}$ (as well as $\mathcal{G}_{\mu'}$ and $\mathcal{G}_{\tilde{\mu}'}$) transforms $\mathcal{F}_0(\tilde{\mu})$ to $\mathcal{P}_0(\mu)$, $\mathcal{F}_0(\tilde{\mu}')$ to $\mathcal{P}_0(\mu')$, and the function $z \mapsto \frac{1}{z-\bar{z}_0}$ into the function $z \mapsto \frac{1}{\bar{z}-\bar{z}_0}$. So,

$$\frac{1}{\bar{z} - \bar{z}_0}\mathcal{P}_0(\mu) = \mathcal{P}_0(\mu').$$

However, $\frac{1}{\bar{z}-\bar{z}_0} = \text{const} \cdot z \cdot \frac{1}{z-z_0}$ for $|z| = 1$ (namely, $\text{const} = -\bar{z}_0$); therefore

$$\frac{1}{\bar{z} - \bar{z}_0}\mathcal{P}_0(\mu) = \frac{1}{z - z_0} z\mathcal{P}_0(\mu) = \frac{1}{z - z_0}\mathcal{P}_1(\mu).$$

So, $\frac{1}{z-z_0}\mathcal{P}_1(\mu) = \mathcal{P}_0(\mu')$, that is, $\frac{1}{z-z_0}\mathcal{P}_1 = \mathcal{P}'_0$. □

Given a PaF geometry $\mathcal{G} = (H, (\mathcal{P}_n), (\mathcal{F}_n), T)$, we may ask, whether or not two spaces $\mathcal{P}_n, \mathcal{F}_{n+k+1}$ are at positive angle.[6] It depends on k, not n. If it holds for k then it surely holds for $k + 1$. We define the index, $\mathrm{Ind}(\mathcal{G})$, as the least $k \in \mathbb{Z}$ possessing the property.[7] Evidently,

$$\mathrm{Ind}(\mathcal{G} + k) = \mathrm{Ind}(\mathcal{G}) + k. \tag{1.4}$$

Combined with Proposition 1.3 it means that $\mathrm{Ind}(G_{\mu'}) = \mathrm{Ind}(G_\mu) + 1$ whenever $\mu'(dz) = |z - z_0|^2\, \mu(dz)$.

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[5] It may happen that z_0 is a pole of μ, then $P_\mu(z) = (z - z_0)P_{\mu'}(z)$; otherwise $P_\mu = P_{\mu'}$. In any case $z \mapsto (z - z_0)P_{\mu'}(z)$ belongs to the ideal generated by P_μ.

[6] Alternatively we could ask whether or not they are orthogonal, have trivial intersection, etc. Every such property leads to its 'index' satisfying (1.4).

[7] If all k possess the property then $\mathrm{Ind}(\mathcal{G}) = -\infty$; if no one does then $\mathrm{Ind}(\mathcal{G}) = +\infty$.

Assume for a while that μ is finite. We have $\mathrm{Ind}(\mathcal{G}_\mu) \geqslant 0$, since constant functions belong both to $\mathcal{P}_0$ and to $\mathcal{F}_0$. It is well-known (see [2, Sect. 9] or [3, Th. 4 in Sect. V.2]) that $\mathrm{Ind}(\mathcal{G}_\mu) = N$ if and only if $d\mu = |P|^2 d\nu$ for some polynomial P of degree N with all roots on the circle, and some *finite* measure ν such that $\mathrm{Ind}(\mathcal{G}_\nu) = 0$. Finiteness of ν ensures that $\mathcal{P}_0^{(\nu)} + \mathcal{F}_1^{(\nu)}$ is dense in $H^{(\nu)}$. Thus $H^{(\nu)} = \mathcal{P}_0^{(\nu)} \oplus \mathcal{F}_1^{(\nu)}$ in the topological sense. Taking into account that $\mathcal{G}_\mu = \mathcal{G}_\nu + N$ we see that the two following conditions are equivalent for every finite measure μ:

$$\mathrm{Ind}(\mathcal{G}_\mu) = N; \tag{1.5a}$$

$$H = \mathcal{P}_0 \oplus \mathcal{F}_{N+1} \text{ in the topological sense.} \tag{1.5b}$$

Therefore (due to 1.3) these conditions are equivalent for every *moderate* measure μ.

In order to get $H = \mathcal{P}_0 \oplus \mathcal{F}_{N+1}$ in the FHS sense, (one of) the following two equivalent conditions must be added:

1.6a. The orthogonal projection from $\mathcal{F}_{N+1}$ to $\mathcal{P}_0$ is a Hilbert–Schmidt operator.

1.6b. The product $P_0 F_{N+1} P_0$ is a trace-class operator; here P_0 and F_{N+1} are orthogonal projections (from H) to $\mathcal{P}_0$ and $\mathcal{F}_1$ respectively.

Recall that a real-valued function φ on the circle belongs to Sobolev space $W_2^{1/2}$ if and only if it satisfies the following two equivalent conditions:

$$\iint \frac{|\varphi(z_1) - \varphi(z_2)|^2}{|z_1 - z_2|^2} \,\mathrm{L}(dz_1)\,\mathrm{L}(dz_2) < \infty, \tag{1.7a}$$

where L stands for Lebesgue measure;

$$\sum_{-\infty}^{+\infty} |n||\hat{\varphi}_n|^2 < \infty, \tag{1.7b}$$

where $\hat{\varphi}_n$ are Fourier coefficients of φ.

A well-known deep result of Ibragimov and Solev (see [3, Sect. IV.4], see also [5, Sect. 7]) states that a finite measure μ satisfies *both* (1.5a) *and* (1.6b) if and only if μ has a density w (w.r.t. Lebesgue measure) of the form $w = |P|^2 \exp\varphi$ where $\varphi \in W_2^{1/2}$ and P is a polynomial of degree N with all zeros on the circle.

Combining the deep result with Proposition 1.3 we generalize the former from finite to moderate measures as follows.

PROPOSITION 1.8. – *For every moderate measure μ on the circle and integer N, the following two conditions are equivalent.*

(a) *$H = \mathcal{P}_0 \oplus \mathcal{F}_{N+1}$ in the FHS sense.*

(b) *μ has a density of the form*

$$\frac{d\mu}{d\mathrm{L}}(z) = \left| \frac{(z - z_1)\cdots(z - z_l)}{(z - z_1')\cdots(z - z_m')} \right|^2 \exp\varphi(z)$$

for some $l, m \in \{0, 1, 2, \ldots\}$ such that $l - m = N$, some points $z_1, \ldots, z_l, z_1', \ldots, z_m'$ on the circle, and some function $\varphi \in W_2^{1/2}$.

Proof. – A moderate measure μ is related to a finite measure ν by $d\nu = |P_\mu|^2 d\mu$; thus $\mathcal{G}_\nu = \mathcal{G}_\mu + m$ where $m = \deg P_\mu$. Condition (a) for μ is equivalent to the condition $H^{(\nu)} = \mathcal{P}_0^{(\nu)} \oplus \mathcal{F}_{N+m+1}^{(\nu)}$ for ν. The latter holds if and only if ν has a density w of the form $w = |P|^2 \exp\varphi$, where $\varphi \in W_2^{1/2}$ and $\deg P = N + m$. It means that μ has the density $\frac{|P|^2}{|P_\mu|^2} \exp\varphi$; note that $\deg P - \deg P_\mu = N$. □

The following remarks will not be used.

If μ satisfies condition 1.8(b) then l, m and $z_1, \dots, z_l$, $z'_1, \dots, z'_m$ are uniquely determined by μ (provided that $z_i \neq z'_j$ for all i, j, of course).

A proposition similar to 1.8 holds for "$H = \mathcal{P}_0 \oplus \mathcal{F}_{N+1}$ in the topological sense"; here the condition "$\varphi \in W_2^{1/2}$" is replaced with the Helson–Szegö condition: $\varphi = \tilde{\psi} + \chi$ with $\|\psi\|_\infty < \frac{\pi}{2}$ and $\|\chi\|_\infty < \infty$, where ψ, χ belong to L_∞ on the circle, and $\tilde{\psi}$ is the conjugate function to ψ. (Or alternatively, Muckenhoupt's condition (A_2) may be used.)

2. Generalized random processes in continuous time

Consider a Gaussian measure γ in the space of (tempered, Schwartz; real-valued) distributions (generalized functions) over $\mathbb{R}$; assume that γ is invariant under shifts of $\mathbb{R}$. Such measures are probability distributions [8] of stationary Gaussian generalized random processes [4]. The space of tempered distributions is dual to the space of rapidly decreasing infinitely differentiable functions φ on $\mathbb{R}$. Such φ gives a linear functional on the space of distributions; w.r.t. γ it gives a normally distributed random variable, whose variance is a quadratic form of φ and may be written as $\int |\hat{\varphi}|^2 d\nu$ where $\hat{\varphi}$ is Fourier transform of φ, and ν is so-called spectral measure (of γ). It is a positive σ-finite Borel measure on $\mathbb{R}$, symmetric (that is, invariant under the map $\lambda \to -\lambda$) and such that

$$\int_{-\infty}^{+\infty} \frac{1}{(1+\lambda^2)^m} \nu(d\lambda) < \infty \tag{2.1}$$

for m large enough, see [4, Th. 3.3, 3.4]. Let $m \in \{0, 1, 2, \dots\}$ be the least number satisfying (2.1). If ν is finite then $m = 0$.

Consider the Sobolev space $W_1^m(\mathbb{R})$ of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi, \varphi', \dots, \varphi^{(m)} \in L_1(\mathbb{R})$. If $\varphi \in W_1^m(\mathbb{R})$ then functions $\lambda \mapsto \hat{\varphi}(\lambda)$, $\lambda \mapsto \lambda\hat{\varphi}(\lambda)$, $\dots$, $\lambda \mapsto \lambda^m \hat{\varphi}(\lambda)$ belong to the space $C_0(\mathbb{R})$ of all (bounded) continuous functions on $\mathbb{R}$ vanishing at ∞, which means that the function $\lambda \mapsto (1+\lambda^2)^{m/2}\hat{\varphi}(\lambda)$ belongs to $C_0(\mathbb{R})$. Taking into account (2.1) we have

$$\forall \varphi \in W_1^m(\mathbb{R}), \quad \hat{\varphi} \in L_2(\nu).$$

[8] Sorry, 'a distribution in the space of distributions' may be confusing. A 'probability distribution' is just a probability measure (intended to describe a random element of the corresponding space). In contrast, a generalized function, called also 'distribution', is a more singular (than a measure) object over $\mathbb{R}$, generally not positive; for example, a derivative $\delta^{(n)}$ of Dirac's delta-function.

Thus, the quadratic form $\varphi \mapsto \int |\hat{\varphi}|^2 \, d\nu$ extends naturally from the space of rapidly increasing infinitely differentiable functions to $W_1^m(\mathbb{R})$. Of course, the former space is dense in the latter. [9]

Introduce two subspaces $\mathcal{P}_0(\nu), \mathcal{F}_0(\nu) \subset L_2(\nu)$; namely, $\mathcal{P}_0(\nu)$ is spanned by functions $\hat{\varphi}$ where $\varphi \in W_1^m(\mathbb{R})$, $\varphi(t) = 0$ for $t \in [0, \infty)$; the same for $\mathcal{F}_0(\nu)$, but $\varphi(t) = 0$ for $t \in (-\infty, 0]$. The map $\lambda \mapsto -\lambda$ sends ν to itself, and $\mathcal{P}_0(\nu)$ to $\mathcal{F}_0(\nu)$. That is, $f \in \mathcal{F}_0(\nu)$ if and only if $\tilde{f} \in \mathcal{P}_0(\nu)$; here $\tilde{f}(-\lambda) = f(\lambda)$.

We use the well-known conformal map $z = \frac{z'-i}{z'+i}$, $z' = -i\frac{z+1}{z-1}$ of the real line $\operatorname{Im} z' = 0$ to the unit circle $|z| = 1$; it also maps half-planes $\operatorname{Im} z' > 0$, $\operatorname{Im} z' < 0$ onto the disk $|z| < 1$ and the region $|z| > 1$ respectively. Denote by μ the image of ν under the map $\lambda \mapsto \frac{\lambda-i}{\lambda+i}$; the σ-finite measure μ on the circle is symmetric (that is, $\mu = \tilde{\mu}$) and satisfies

$$\int |1 - z|^{2m} \, \mu(dz) < \infty,$$

which is the same as (2.1), since $\left|1 - \frac{\lambda-i}{\lambda+i}\right|^2 = \frac{4}{\lambda^2+1}$. In terms of Definition 1.2, μ is a moderate measure; it has a pole of multiplicity m at 1 (or it is a finite measure, and $m = 0$); $P_\mu(z) = (z-1)^m$; $\tilde{P}_{\tilde{\mu}}(z) = (\frac{1}{z} - 1)^m = (-1)^m z^{-m}(z-1)^m$.

Denote by $\mathcal{P}_0(\mu), \mathcal{F}_0(\mu)$ subspaces $\mathcal{P}_0, \mathcal{F}_0$ appearing in the PaF structure $\mathcal{G}_\mu = (H, (\mathcal{P}_n), (\mathcal{F}_n), T)$. That is, $\mathcal{F}_0 \subset L_2(\mu)$ is spanned by functions $z \mapsto z^k(1-z)^n$ for $k \geqslant 0$, and $\mathcal{P}_0(\mu)$ is spanned by functions $z \mapsto z^k(\frac{1}{z} - 1)^m$ or $z \mapsto z^{k-m}(z-1)^m$ for $k \leqslant 0$.

The next lemma is well-known for finite measures (see [1, Sect. XII.5, before Theorem 5.1]); here is a generalization to moderate measures.

LEMMA 2.2. – *Let two functions, f on the circle and g on $\mathbb{R}$, be related by*

$$f\left(\frac{\lambda - i}{\lambda + i}\right) = g(\lambda) \quad \textit{for all } \lambda \in \mathbb{R}. \tag{2.3}$$

Then $f \in \mathcal{P}_0(\mu)$ if and only if $g \in \mathcal{P}_0(\nu)$. Also, $f \in \mathcal{F}_0(\mu)$ if and only if $g \in \mathcal{F}_0(\nu)$.

Proof. – It suffices to prove the latter, $f \in \mathcal{F}_0(\mu) \Leftrightarrow g \in \mathcal{F}_0(\nu)$, since $f \in \mathcal{P}_0(\mu) \Leftrightarrow \tilde{f} \in \mathcal{F}_0(\mu)$, and $g \in \mathcal{P}_0(\nu) \Leftrightarrow \tilde{g} \in \mathcal{F}_0(\nu)$, and $\tilde{f}(\frac{\lambda-i}{\lambda+i}) = f(\frac{\lambda+i}{\lambda-i}) = g(-\lambda) = \tilde{g}(\lambda)$.

In order to prove that $f \in \mathcal{F}_0(\mu)$ implies $g \in \mathcal{F}_0(\nu)$, consider $f(z) = z^k(1-z)^n$ for some $k \geqslant 0$; we have to prove that $g \in \mathcal{F}_0(\nu)$, where $g(\lambda) = (\frac{\lambda-i}{\lambda+i})^k(\frac{2i}{\lambda+i})^n = (2i)^n \frac{1}{(\lambda+i)^n}(1 - 2i\frac{1}{\lambda+i})^k$ is a linear combination of functions $\lambda \mapsto \frac{1}{(\lambda+i)^{n+l}}$, $l = 0, 1, \ldots, k$. Such g is Fourier transform of a linear combination of functions $h_{n+l}(t) = t^{n+l-1}\mathrm{e}^{-t}$ for $t > 0$ (otherwise 0), except for the case $n = l = 0$; in that case g is constant, and we need Fourier transform of a measure (concentrated at the origin) rather than a function of L_1. The same difficulty appears for $n > 0$, when $l = 0$; in that case $h_{n+l} = h_n$ does not belong to $W_1^n(\mathbb{R})$, since $h_n^{(n-1)}$ jumps at the origin, and $h_n^{(n)}$ is a finite measure rather

[9] Note also that $C_0(\mathbb{R})$ could be replaced with $C(\mathbb{R})$ (all bounded continuous functions on $\mathbb{R}$); accordingly, $\varphi^{(m)}$ could be a finite measure rather than a function of $L_1(\mathbb{R})$, which will be used in the proof of Lemma 2.2.

than a function of L_1. However, a smoothing, say, $t \mapsto \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} h_n(t+u)\,du$, does the job for $l = 0$. For $l > 0$ the function h_{n+l} belongs to $W_1^n(\mathbb{R})$. So, $g \in \mathcal{F}_0(\nu)$.

In order to prove that $g \in \mathcal{F}_0(\nu)$ implies $f \in \mathcal{F}_0(\mu)$, consider $g = \hat{\varphi}$ where $\varphi \in W_1^n(\mathbb{R})$, $\varphi(t) = 0$ for $t \in (-\infty, 0]$. The function $\lambda \mapsto (\lambda + i)^n g(\lambda)$ on the closed half-plane $\operatorname{Im}\lambda \geqslant 0$ is continuous, and tends to 0 for $|\lambda| \to \infty$. Therefore the function $z \mapsto (1-z)^{-n} f(z)$ on the closed disk $|z| \leqslant 1$ is continuous (and vanishes at 1). Take polynomials P_n such that $P_n(z) \to (1-z)^{-n} f(z)$ uniformly on the disk; then functions $z \mapsto (1-z)^n P_n(z)$ belong to $\mathcal{F}_0(\mu)$ and converge to f in $L_2(\mu)$. So, $f \in \mathcal{F}_0(\mu)$. □

3. Off-white noises

Return to a Gaussian measure γ in the space of distributions, its spectral measure ν, and the corresponding stationary Gaussian generalized random process. The spaces $\mathcal{P}_0(\nu), \mathcal{F}_0(\nu)$ of $L_2(\nu)$, defined in Section 2, correspond unitarily (via Fourier transform) to subspaces of the Hilbert space of all γ-measurable linear functionals. Namely, $\mathcal{P}_0(\nu)$ corresponds to functionals localized (on the time axis) on $(-\infty, 0)$ ("the past"), and $\mathcal{F}_0(\nu)$ corresponds to functionals localized on $(0, \infty)$ ("the future"). Thus, orthogonality of $\mathcal{P}_0(\nu), \mathcal{F}_0(\nu)$ means independence of the past and the future (which is the case for the white noise, whose spectral measure is Lebesgue measure on $\mathbb{R}$). The property

$$L_2(\nu) = \mathcal{P}_0(\nu) \oplus \mathcal{F}_0(\nu) \quad \text{in the FHS sense} \tag{3.1}$$

means that dependence between the past and the future boils down to a density. That is, γ is equivalent (mutually absolutely continuous) to another measure that makes the past and the future independent. [10] Such a process will be called an *off-white noise.*

THEOREM 3.2. – *The following two conditions are equivalent.*
(a) ν *is the spectral measure of an off-white noise*;
(b) μ *has a density of the form*

$$\frac{d\mu}{d\mathrm{L}}(z) = \left|\frac{(z-z_1)\cdots(z-z_{m-1})}{(z-1)^m}\right|^2 \exp\varphi(z)$$

for some $m \in \{1, 2, \ldots\}$*, some points* $z_1, \ldots, z_{m-1}$ *on the circle, different from* 1*, and some function* $\varphi \in W_2^{1/2}$.

Proof. – Condition (a) is equivalent to (3.1). By Lemma 2.2, (3.1) is equivalent to $L_2(\mu) = \mathcal{P}_0(\mu) \oplus \mathcal{F}_0(\mu)$ in the FHS sense. The latter is 1.8(a) for $N = -1$. By Proposition 1.8 it is equivalent to 1.8(b) for $N = -1$. It remains to note that μ has no poles except for 1. □

[10] I mean $(-\infty, 0)$ and $(0, \infty)$, not $(-\infty, t)$ and (t, ∞) for all t simultaneously.

Clearly, m in 3.2(b) is the same as m in (2.1). We are mostly interested in the case $m=1$; in that case (2.1) becomes

$$\int_{-\infty}^{+\infty} \frac{1}{1+\lambda^2}\,\nu(d\lambda) < \infty, \tag{3.3}$$

and 3.2(b) becomes

$$\frac{d\mu}{d\mathrm{L}}(z) = \frac{1}{|1-z|^2} w(z).$$

However, [11]

$$\frac{d\nu}{d\mathrm{L}}(\lambda) = \frac{1}{2}|1-z|^2 \frac{d\mu}{d\mathrm{L}}(z)$$

where $z = \frac{\lambda - i}{\lambda + i}$; thus

$$w\left(\frac{\lambda - i}{\lambda + i}\right) = 2W(\lambda), \tag{3.4}$$

where $W(\lambda) = \frac{d\nu}{d\mathrm{L}}(\lambda)$.

The condition $\ln w \in W_2^{1/2}$ can be rewritten in terms of W,

$$\iint \frac{|\ln W(\lambda_1) - \ln W(\lambda_2)|^2}{|\lambda_1 - \lambda_2|^2}\, d\lambda_1\, d\lambda_2 < \infty, \tag{3.5}$$

which is (1.7a) combined with the fact that $\frac{dz_1 dz_2}{(z_1 - z_2)^2}$ is invariant under linear-fractional transformations. [12] Recall that $W(-\lambda) = W(\lambda)$.

PROPOSITION 3.6. – (a) *If W satisfies* (3.5) *then*

$$\int_0^\infty \left| \ln W(2\lambda) - \ln W(\lambda) \right|^2 \frac{d\lambda}{\lambda} < \infty. \tag{3.7}$$

(b) *Let W be strictly positive, have a continuous derivative, and*

$$\int_0^\infty \left| \frac{d}{d\lambda} \ln W(\lambda) \right|^2 \lambda\, d\lambda < \infty. \tag{3.8}$$

Then W satisfies (3.5).

---

[11] I denote by L both Lebesgue measure on the circle and Lebesgue measure on $\mathbb{R}$. We have $dz = \frac{2i}{(\lambda+i)^2}\, d\lambda$, thus $\mathrm{L}(dz) = \frac{2}{\lambda^2+1}\mathrm{L}(d\lambda)$; also, $|1-z|^2 = \frac{4}{\lambda^2+1}$.

[12] That is, if $f(z) = \frac{az+b}{cz+d}$ then $\frac{1}{(z_1-z_2)^2} = \frac{f'(z_1) f'(z_2)}{(f(z_1)-f(z_2))^2}$.

Proof. – First, integration in (3.5) may be restricted from $\mathbb{R}\times\mathbb{R}$ to $(0,\infty)\times(0,\infty)$. Indeed, using the property $W(\lambda)=W(-\lambda)$ we get the kernel $\frac{2}{|\lambda_1-\lambda_2|^2}+\frac{2}{|\lambda_1+\lambda_2|^2}$ equivalent to $\frac{1}{|\lambda_1-\lambda_2|^2}$.

Second,

$$\int_0^\infty\int_0^\infty \frac{|\ln W(\lambda_1)-\ln W(\lambda_2)|^2}{|\lambda_1-\lambda_2|^2}\,d\lambda_1\,d\lambda_2 = \int_0^\infty \frac{du}{(u-1)^2}\int_0^\infty \bigl|\ln W(u\lambda)-\ln W(\lambda)\bigr|^2\frac{d\lambda}{\lambda}, \tag{3.9}$$

which is just a change of variable, $\lambda_1=\lambda_2 u$.

Let W satisfy (3.5); we have to check (3.7). Consider

$$f(u)=\left(\int_0^\infty |\ln W(u\lambda)-\ln W(\lambda)|^2\,\frac{d\lambda}{\lambda}\right)^{1/2}.$$

The triangle inequality gives $f(uv)\leqslant f(u)+f(v)$, since

$$\int_0^\infty \bigl|\ln W(uv\lambda)-\ln W(v\lambda)\bigr|^2\frac{d\lambda}{\lambda}=\int_0^\infty \bigl|\ln W(u\lambda)-\ln W(\lambda)\bigr|^2\,\frac{d\lambda}{\lambda}.$$

Also, $f(u)<\infty$ for almost all u due to (3.5) and (3.9). Taking u such that $f(u)<\infty$ and $f(\frac{2}{u})<\infty$ we get $f(2)<\infty$, which is (3.7).

Let W satisfy (3.8); we need to check (3.5), or equivalently, $\int_0^\infty f^2(u)\,\frac{du}{(u-1)^2}<\infty$. We have

$$\left(\int_0^\infty\left|\int_1^u \frac{W'(\lambda x)}{W(\lambda x)}dx\right|^2\lambda\,d\lambda\right)^{1/2}\leqslant\int_1^u\left(\int_0^\infty\left|\frac{W'(\lambda x)}{W(\lambda x)}\right|^2\lambda\,d\lambda\right)^{1/2}dx;$$

$$\left(\int_0^\infty \bigl|\ln W(\lambda u)-\ln W(\lambda)\bigr|^2\frac{d\lambda}{\lambda}\right)^{1/2}\leqslant\left(\int_1^u\frac{dx}{x}\right)\left(\int_0^\infty\left|\frac{W'(\lambda)}{W(\lambda)}\right|^2\lambda\,d\lambda\right)^{1/2};$$

$f(u)\leqslant \mathrm{const}\cdot\ln u$ for all $u\in[1,\infty)$; similarly, $f(u)\leqslant\mathrm{const}\cdot|\ln u|$ for all $u\in(0,1]$. So,

$$\int_0^\infty f^2(u)\frac{du}{(u-1)^2}\leqslant \mathrm{const}\cdot\int_0^\infty\left(\frac{\ln u}{u-1}\right)^2du<\infty. \qquad\square$$

Example 3.10. – Assume that $W(\lambda)=|\lambda|^\alpha$ for $|\lambda|$ large enough, and W is strictly positive and smooth everywhere. Then condition (3.5) is satisfied if and only if $\alpha=0$ (just the white noise).

Example 3.11. – Assume that $W(\lambda)=(\ln|\lambda|)^\alpha$ for $|\lambda|$ large enough, and W is strictly positive and smooth everywhere. Then condition (3.5) is satisfied for all α. Condition (3.3) is also satisfied. Thus, every such W describes an off-white noise.

Example 3.12. – Assume that $W(\lambda) = \exp(-\ln^{\alpha}|\lambda|)$ for $|\lambda|$ large enough (here $\alpha > 0$), and W is strictly positive and smooth everywhere. Then condition (3.5) is satisfied if and only if $\alpha < 1/2$. Condition (3.3) is also satisfied. So, for $\alpha \in (0, 1/2)$ every such W describes an off-white noise.

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REFERENCES

[1] J.L. Doob, Stochastic Processes, Wiley, 1953.
[2] H. Helson, D. Sarason, Past and future, Math. Scand. 21 (1967) 5–16.
[3] I.A. Ibragimov, Y.A. Rozanov, Gaussian Random Processes, Springer, 1978 (Transl. from Russian 1970).
[4] K. Itô, Stationary random distributions, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. 28 (3) (1954) 209–223.
[5] V.V. Peller, Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximations, Gaussian processes, the problem of majorizing operators), Math. USSR Sbornik 41 (4) (1982) 443–479 (Transl. from Russian 1980).
[6] B. Tsirelson, From slightly coloured noises to unitless product systems, preprint, math.FA/0006165.