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SPECTRAL GAP AND LOGARITHMIC SOBOLEV INEQUALITY FOR UNBOUNDED CONSERVATIVE SPIN SYSTEMS

TROU SPECTRAL ET INÉGALITÉS DE SOBOLEV LOGARITHMIQUES POUR DES SYSTÉMES DE SPINS CONSERVATIFS ET NON BORNÉS

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ABSTRACT. – We consider reversible, conservative Ginzburg–Landau processes, whose potential are bounded perturbations of the Gaussian potential, evolving on a d-dimensional cube of length L. Following the martingale approach introduced in (S.L. Lu, H.T. Yau, Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics, Comm. Math. Phys. 156 (1993) 433–499), we prove in all dimensions that the spectral gap of the generator and the logarithmic Sobolev constant are of order L^{-2}. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous considérons des processus de Ginzburg–Landau réversibles, dont le potentiel est une perturbation bornée du potential Gaussien, évoluent sur un cube d-dimensionel de largeur L. Suivant la méthode martingale introduite dans (S.L. Lu, H.T. Yau, Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics, Comm. Math. Phys. 156 (1993) 433–499), nous démontrons qu'en toute dimension le trou spectral et la constante de Sobolev logarithmique sont d'ordre L^{-2}. © 2002 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

In recent years some progress has been made in the investigation of convergence to equilibrium of reversible conservative interacting particle systems [1,2,9,8,11,4,5].

In finite volume the techniques used to obtain the rate of convergence to equilibrium rely mostly on the estimation of the spectral gap of the generator. In general, one shows that the generator of the particle system restricted to a cube of length N has a gap of order N^{-2} in any dimension. This estimate together with standard spectral arguments permits to prove that the particle system restricted to a cube of size N decays to equilibrium in the variance sense at the exponential rate $\exp\{-ct/N^2\}$: for any function f in L^2,

$$\|P_t f - E_\pi[f]\|_2^2 \leqslant \exp\{-ct/N^2\}\|f - E_\pi[f]\|_2^2,$$

where $\{P_t,\ t \geqslant 0\}$ stands for the semi-group of the process, π for the invariant measure, $E_\pi[f]$ for the expectation of f with respect to π and $\|\cdot\|_2$ for the L^2 norm with respect to π.

In infinite volume, since the spectrum of the generator of a conservative system has no gap at the origin, instead of exponential convergence to equilibrium, one expects a polynomial convergence. In this context, the main difficulty is to use the local information on the gap of the spectrum of the generator restricted to a finite cube to deduce the global behavior of the system in infinite volume.

On the other hand, the relation between the logarithmic Sobolev inequality and the hypercontractivity has long been established. The hypercontractivity in turn permits to prove upper and lower Gaussian estimates of the transition probability of a reversible Markov process (cf. [6,11]).

In this article we present a sharp estimate of the spectral gap and of the logarithmic Sobolev constant for the Ginzburg–Landau process whose potential is a bounded perturbation of the Gaussian potential. The precise assumptions are given in Section 2. We follow here the martingale approach introduced in [14]. The main ideas are essentially the same but there are several technical difficulties coming from the unboundedness of the spins. The main ingredients are a local central limit theorem, uniform over the parameter and from which follows the equivalence of ensembles, and some sharp large deviations estimates.

The article is divided as follows. In Section 2 we state the main results and introduce the notation. In Section 3 we prove the spectral gap and in Section 4 the logarithmic Sobolev inequality. In Section 5 we prove a uniform local central limit theorem and deduce some results regarding the equivalence of ensembles. In Section 6 we obtain some large deviations estimates which play a central role in the proof of the logarithmic Sobolev inequality.

2. Notation and results

For $L \geqslant 1$, denote by Λ_L the cube $\{1, \dots, L\}^d$. Configurations of the state space $\mathbb{R}^{\Lambda_L}$ are denoted by the Greek letters η, ξ, so that η_x indicates the value of the spin at $x \in \Lambda_L$ for the configuration η. The configuration η undergoes a diffusion on $\mathbb{R}^{\Lambda_L}$

whose infinitesimal generator $\mathcal{L}_{\Lambda_L}$ is given by

$$\mathcal{L}_{\Lambda_L} = \frac{1}{2} \sum_{\substack{x,y\in\Lambda_L \\ |x-y|=1}} (\partial_{\eta_x} - \partial_{\eta_y})^2 - \frac{1}{2} \sum_{\substack{x,y\in\Lambda_L \\ |x-y|=1}} \big(V'(\eta_y) - V'(\eta_x)\big)(\partial_{\eta_y} - \partial_{\eta_x}).$$

$V:\mathbb{R}\to\mathbb{R}$ represents the potential $V(a) = (1/2)a^2 + F(a)$, where $F:\mathbb{R}\to\mathbb{R}$ is a smooth bounded function such that $\|F'\|_\infty < \infty$,

$$\int \mathrm{e}^{-V(x)}\,dx = 1.$$

We assumed the convex part of the potential to be Gaussian for simplicity. All proofs of the results presented in Section 5 on the uniform local central limit theorems rely strongly on this hypothesis. We believe, however, that the approach presented here extend to the case where we have a bounded perturbation of a convex potential. In this respect, it was recently observed by Caputo [3] that when the potential is a purely convex function, the L^2 behavior of the inverse of the spectral gap and of the logarithmic Sobolev constant can be easily obtained by techniques introduced for models with convex interactions (see [13] and references therein).

Denote by $Z:\mathbb{R}\to\mathbb{R}$ the partition function

$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{\lambda a - V(a)}\,da, \tag{2.1}$$

by $R:\mathbb{R}\to\mathbb{R}$ the density function $\partial_\lambda \log Z(\lambda)$, which is smooth and strictly increasing, and by Φ the inverse of R so that

$$\alpha = \frac{1}{Z(\Phi(\alpha))} \int_{-\infty}^{\infty} a\,\mathrm{e}^{\Phi(\alpha)a - V(a)}\,da$$

for each α in $\mathbb{R}$.

For λ in $\mathbb{R}$, denote by $\bar\nu_\lambda^{\Lambda_L}$ the product measure on $\mathbb{R}^{\Lambda_L}$ defined by

$$\bar\nu_\lambda^{\Lambda_L}(d\eta) = \prod_{x\in\Lambda_L} \frac{1}{Z(\lambda)} \mathrm{e}^{\lambda\eta_x - V(\eta_x)}\,d\eta_x$$

and let $\nu_\alpha^{\Lambda_L} = \bar\nu_{\Phi(\alpha)}^{\Lambda_L}$. Notice that $E_{\nu_\alpha}[\eta_x] = \alpha$ for all α in $\mathbb{R}$, x in Λ_L. Most of the times omit the superscript Λ_L. For each M in $\mathbb{R}$, denote by $\nu_{\Lambda_L,M}$ the canonical measure on Λ_L with total spin equal to M:

$$\nu_{\Lambda_L,M}(\,\cdot\,) = \nu_\alpha^{\Lambda_L}\bigg(\cdot\,\bigg|\, \sum_{x\in\Lambda_L} \eta_x = M\bigg).$$

Expectation with respect to $\nu_{\Lambda_L,M}$ is denoted by $E_{\Lambda_L,M}$.

An elementary computation shows that the product measures $\{\bar{\nu}_\lambda,\ \lambda\in\mathbb{R}\}$ are reversible for the Markov process with generator $\mathcal{L}_{\Lambda_L}$. The Dirichlet form D_{Λ_L} associated to $\mathcal{L}_{\Lambda_L}$ is given by

$$D_{\Lambda_L}(\mu, f)=\frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\|x-y|=1}}\big\langle (T^{x,y}f)^2\big\rangle_\mu.$$

In this formula and below, for a probability measure μ, $\langle\cdot\rangle_\mu$ stands for expectation with respect to μ. Furthermore, for x, y in $\mathbb{Z}^d$, $T^{x,y}$ represents the operator that acts on smooth functions f as

$$T^{x,y}f=\frac{\partial f}{\partial\eta_x}-\frac{\partial f}{\partial\eta_y}$$

and μ stands for the invariant measures ν_α, $\nu_{\Lambda_L,M}$.

For a positive integer L and M in $\mathbb{R}$, denote by $W(L,M)$ the inverse of the spectral gap of the generator $\mathcal{L}_{\Lambda_L}$ with respect to the measure $\nu_{\Lambda_L,M}$:

$$W(L,M)=\sup_f\frac{\langle f;f\rangle_{\nu_{\Lambda_L,M}}}{D_{\Lambda_L}(\nu_{\Lambda_L,M}, f)}.$$

In this formula the supremum is carried over all smooth functions f in $L^2(\nu_{\Lambda_L,M})$ and $\langle f;f\rangle_\mu$ stands for the variance of f with respect to μ. We also denote this variance by the symbol $\mathbf{Var}(\mu,f)$. Let

$$W(L)=\sup_{M\in\mathbb{R}}W(L,M).$$

THEOREM 2.1. – *There exists a finite constant C_0 depending only on F such that*

$$W(L)\leqslant C_0L^2$$

for all $L\geqslant 2$.

A lower bound of the same order is easy to derive. Fix a smooth function $H:[0,1]^d\to\mathbb{R}$ such that $\int H(u)\,du=0$ and let $f_H(\eta)=\sum_{x\in\Lambda_L}H(x/L)\eta_x$. An elementary computation shows that

$$\begin{aligned}\langle f_H;f_H\rangle_{\nu_{\Lambda_L,M}}&=\Big(\sum_x H(x/L)\Big)^2\langle\eta_{2e_1};\eta_{e_1}\rangle_{\nu_{\Lambda_L,M}}\\&\quad+\sum_x H(x/L)^2\big\{\langle\eta_{e_1};\eta_{e_1}\rangle_{\nu_{\Lambda_L,M}}-\langle\eta_{2e_1};\eta_{e_1}\rangle_{\nu_{\Lambda_L,M}}\big\},\end{aligned}$$

$$D_{\Lambda_L}(\nu_{\Lambda_L,M}, f_H)=(1/2)\sum_{|x-y|=1}\big[H(y/L)-H(x/L)\big]^2.$$

In this formula $\{e_j,\ 1\leqslant j\leqslant d\}$ stands for the canonical basis of $\mathbb{R}^d$. By Corollary 5.3, as $L\uparrow\infty$, $M/L^d\to\alpha$, $\langle f_H;f_H\rangle_{\nu_{\Lambda_L,M}}/L^2D_{\Lambda_L}(\nu_{\Lambda_L,M},f_H)$ converges to

$\langle \eta_{e_1}; \eta_{e_1} \rangle_{\nu_\alpha} \int H(u)^2\, du / \int \|(\nabla H)(u)\|^2\, du$. This proves that

$$\liminf_{L\to\infty} L^{-2} W(L) > 0.$$

For $L \geqslant 2$, a probability measure ν on $\mathbb{R}^{\Lambda_L}$ and a function f such that $\langle f^2 \rangle_\nu = 1$, denote by $S_{\Lambda_L}(\nu, f)$ the entropy of $f^2\, d\nu$ with respect to ν:

$$S_{\Lambda_L}(\nu, f) = \int f^2 \log f^2\, d\nu$$

and by $\theta(L, M)$ the inverse of the logarithmic Sobolev constant of the Ginzburg–Landau process on the cube Λ_L with respect to the measure $\nu_{\Lambda_L, M}$:

$$\theta(L, M) = \sup_f \frac{S_{\Lambda_L}(\nu_{\Lambda_L, M}, f)}{D_{\Lambda_L}(\nu_{\Lambda_L, M}, f)}.$$

In this formula, the supremum is carried over all smooth functions f in $L^2(\nu_{\Lambda_L, M})$ such that $\langle f^2 \rangle_{\nu_{\Lambda_L, M}} = 1$. Let

$$\theta(L) = \sup_{M\in\mathbb{R}} \theta(L, M).$$

THEOREM 2.2. – *Assume that $\|F''\|_\infty < \infty$. There exists a finite constant C depending only on F such that $\theta(L) \leqslant C L^2$ for all $L \geqslant 2$.*

We follow here the martingale method developed by Lu and Yau [14] to prove the spectral gap and a bound on the logarithmic Sobolev constant for a conservative interacting particle system. This approach relies on two a-priori estimates. First, a local central limit theorem for i.i.d. random variables with marginals equal to the marginals of the product measure $\bar{\nu}_\lambda$, *uniform* over the parameter λ in $\mathbb{R}$. Second, a spectral gap or a logarithmic Sobolev inequality, *uniform* over the density, for a Glauber dynamics on one site which is reversible with respect to the one-site marginal of the canonical invariant measure.

3. Spectral gap

To fix ideas, we prove Theorem 2.1 in dimension 1. The reader can find in Section A.3.3 of [10] the arguments needed to extend the proof to higher dimensions. To detach the main ideas, we divide the proof in four steps. The proof goes by induction on L. We start with $L = 2$.

In this section all constants are allowed to depend on $\|F\|_\infty$, $\|F'\|_\infty$. In the case they depend on some other parameter, the dependence is stated explicitly.

Step 1. One-site spectral gap. Consider a smooth function $f : \mathbb{R}^{\Lambda_2} \to \mathbb{R}$. We want to estimate $\langle f; f \rangle_{\nu_{\Lambda_2, M}}$ in terms of the Dirichlet form of f. Since for the measure $\nu_{\Lambda_2, M}$ the total spin is fixed to be equal to M, let $g(a) = f(M - a, a)$ and notice that $\langle f; f \rangle_{\nu_{\Lambda_2, M}}$ is equal to $\langle g; g \rangle_{\nu_{\Lambda_2, M}}$.

The following result will be of much help. Fix $L \geqslant 2$ and M in $\mathbb{R}$. Denote by $\nu^1_{\Lambda_L,M}$ the marginal distribution of η_L with respect to $\nu_{\Lambda_L,M}$. The Glauber dynamics has a positive spectral gap which is uniform with respect to M:

LEMMA 3.1. – *There is a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\mathbf{Var}(\nu^1_{\Lambda_L,M}, f) \leqslant C_0 E_{\nu^1_{\Lambda_L,M}}\left[\left(\frac{\partial f}{\partial \eta_L}\right)^2\right]$$

for every $L \geqslant 2$, every M in $\mathbb{R}$ and every smooth function $f:\mathbb{R} \to \mathbb{R}$ in $L^2(\nu^1_{\Lambda_L,M})$.

Remark 3.2. – In the case of grand canonical measures, this result is true under the more general hypothesis of strict convexity at infinity of the potential (cf. [13] and references therein). In case of canonical measures the main problem is to obtain a good approximation of the one-site marginal in terms of the one-site marginal of grand canonical measures.

Before proving this result, we conclude the first step. Applying this result to the function g defined above, we obtain that its variance is bounded by $C_0 E_{\nu^1_{\Lambda_2,M}}[(\partial g/\partial \eta_2)^2]$. Since $\partial g/\partial \eta_2 = (\partial f/\partial \eta_2 - \partial f/\partial \eta_1)$, we have that

$$\begin{aligned}\langle f; f\rangle_{\nu_{\Lambda_2,M}} &= \langle g; g\rangle_{\nu_{\Lambda_2,M}} = \langle g; g\rangle_{\nu^1_{\Lambda_2,M}} \\ &\leqslant C_0 E_{\nu^1_{\Lambda_2,M}}\left[\left(\frac{\partial g}{\partial \eta_2}\right)^2\right] = C_0\, E_{\nu_{\Lambda_2,M}}\left[\left(\frac{\partial g}{\partial \eta_2}\right)^2\right] \\ &= C_0\, E_{\nu_{\Lambda_2,M}}\left[\left(\frac{\partial f}{\partial \eta_2} - \frac{\partial f}{\partial \eta_1}\right)^2\right].\end{aligned}$$

This shows that $W(2) \leqslant C_0$, proving Theorem 2.1 in the case $L = 2$. We conclude this step with the

Proof of Lemma 3.1. – We first prove the lemma for the grand canonical measure. Fix λ in $\mathbb{R}$ and denote by $\bar{\nu}^1_\lambda$ the one-site marginal of the product measure $\bar{\nu}^{\Lambda_L}_\lambda$. Fix x_λ in $\mathbb{R}$, that will be specified later, and f in $L^2(\bar{\nu}^1_\lambda)$. The variance of f is bounded above by

$$\int_{\mathbb{R}} \big(f(x) - f(x_\lambda)\big)^2 \mathrm{e}^{-V_\lambda(x)}\, dx,$$

where $V_\lambda(x) = -\lambda x + \log Z(\lambda) + V(x)$. By Schwarz inequality, the previous expression is less than or equal to

$$\int_{x_\lambda}^{\infty} dx\,[f'(x)]^2 \mathrm{e}^{-V_\lambda(x)} \left\{\mathrm{e}^{V_\lambda(x)} \int_x^{\infty} dy\,(y - x_\lambda)\mathrm{e}^{-V_\lambda(y)}\right\} + \int_{-\infty}^{x_\lambda} dx[f'(x)]^2 \mathrm{e}^{-V_\lambda(x)} \left\{\mathrm{e}^{V_\lambda(x)} \int_{-\infty}^{x} dy\,(x_\lambda - y)\mathrm{e}^{-V_\lambda(y)}\right\}.$$

It remains to show that the expressions inside braces are uniformly bounded in x and λ for an appropriate choice of x_λ. Both expressions are handled in the same way and we consider, to fix ideas, the first one where we need to estimate

$$\sup_{x\geqslant x_\lambda}\left\{e^{(1/2)(x-\lambda)^2+F(x)}\int_x^\infty dy\,(y-x_\lambda)e^{-(1/2)(y-\lambda)^2-F(y)}\right\}.$$

Choose $x_\lambda=\lambda$ and change variables to reduce the previous expression to

$$\sup_{x\geqslant 0}\left\{e^{(x^2/2)+F_\lambda(x)}\int_x^\infty dy\,y e^{-(y^2/2)-F_\lambda(y)}\right\},$$

where $F_\lambda(a)=F(a+\lambda)$. In the case where $F=0$, this expression is bounded above by some universal constant C_0. Since F is bounded, this expression is less than or equal to $C_0\exp\{2\|F\|_\infty\}$ uniformly over λ. This concludes the proof of the lemma in the case of grand canonical measures.

We turn now to the case of canonical measures. We need to introduce some notation. For λ in $\mathbb{R}$, let $\{X_j^\lambda,\ j\geqslant 1\}$ be a sequence of i.i.d. random variables with density $Z(\lambda)^{-1}\exp\{\lambda x-V(x)\}$. For a positive integer L, denote by $f_{\lambda,L}$ the density of $(\sigma(\lambda)^2L)^{-1/2}\sum_{1\leqslant j\leqslant L}\{X_j^\lambda-\gamma_1(\lambda)\}$, where $\gamma_k(\lambda)$ is the kth truncated moment of X_1^λ and $\sigma(\lambda)^2$ is its variance: $\gamma_1(\lambda)=E[X_1^\lambda]$, $\gamma_k(\lambda)=E[(X_1^\lambda-\gamma_1(\lambda))^k]$. We prove in Section 5 an Edgeworth expansion for $f_{\lambda,L}$ uniform over the parameter λ.

We may write the measure $\nu_{L,M}^1(dx)$ in terms of the density $f_{\lambda,L}$. Choose λ so that $\gamma_1(\lambda)=M/L$: $\lambda=\Phi(M/L)$. Then, $\nu_{L,M}^1(dx)=\sqrt{L/(L-1)}g_\lambda(x)f_{\lambda,L-1}([\gamma_1-x]/\sigma\sqrt{L-1})\ f_{\lambda,L}(0)^{-1}\,dx$, where g_λ stands for the density $Z(\lambda)^{-1}\exp\{\lambda x-V(x)\}$. Hereafter, we will omit the dependence of γ_j and σ on λ.

Denote the Radon–Nikodym derivative of $\nu_{L,M}^1(dx)$ with respect to the Lebesgue measure by $R(x)=R_{L,M}(x)$. Fix a function f in $L^2(\nu^1_{\Lambda_L,M})$ and x_λ in $\mathbb{R}$ to be specified later. Following the proof for the grand canonical measure, we bound the variance of f by

$$\int_{\mathbb{R}}\big(f(x)-f(x_\lambda)\big)^2R(x)\,dx.$$

We now repeat the arguments presented in the case of the grand canonical measures. After few steps, we reduce the proof of the lemma to the proof that

$$\sup_{x\geqslant x_\lambda}\left\{R(x)^{-1}\int_x^\infty dy(y-x_\lambda)R(y)\right\}$$

is bounded, uniformly in M. Choose $x_\lambda = \lambda$, change variables and recall the notation introduced above to rewrite the previous expression as

$$\sup_{x\geqslant 0}\left\{\int_x^\infty dy\, y\,\frac{g_\lambda(y+\lambda)}{g_\lambda(x+\lambda)}\,\frac{f_{\lambda,L-1}([\gamma_1 - y - \lambda]/\sigma\sqrt{L-1})}{f_{\lambda,L-1}([\gamma_1 - x - \lambda]/\sigma\sqrt{L-1})}\right\}.$$

By the explicit formula for the density g_λ and since F is bounded, this expression is less than or equal to

$$e^{2\|F\|_\infty}\sup_{x\geqslant 0}\left\{e^{x^2/2}\int_x^\infty dy\, y e^{-y^2/2}\,\frac{f_{\lambda,L-1}([\gamma_1 - y - \lambda]/\sigma\sqrt{L-1})}{f_{\lambda,L-1}([\gamma_1 - x - \lambda]/\sigma\sqrt{L-1})}\right\}.$$

We need now to estimate the ratio of the densities inside the integral. For a positive integer L, denote by $g_{\lambda,L}(x)$ the density of $\sum_{1\leqslant j\leqslant L} X_j^\lambda$. An elementary induction argument shows that $g_{\lambda,L}(x) = Z(\lambda)^{-L}\exp\{\lambda x\}g_{0,L}(x)$ so that $g_{\lambda,L}(x)/g_{\mu,L}(x) = (Z(\mu)/Z(\lambda))^L\exp\{(\lambda-\mu)x\}$ for any parameter μ. Choose μ so that $\gamma_1(\lambda) - \gamma_1(\mu) = x/(L-1)$ and notice that $\mu\leqslant\lambda$ because $x\geqslant 0$ and γ_1 is an increasing function. The previous identity gives that

$$\frac{f_{\lambda,L-1}([\gamma_1(\lambda) - y - \lambda]/\sigma(\lambda)\sqrt{L-1})}{f_{\lambda,L-1}([\gamma_1(\lambda) - x - \lambda]/\sigma(\lambda)\sqrt{L-1})} = \frac{f_{\mu,L-1}([\gamma_1(\lambda) - \lambda + x - y]/\sigma(\mu)\sqrt{L-1})}{f_{\mu,L-1}([\gamma_1(\lambda) - \lambda]/\sigma(\mu)\sqrt{L-1})}\,e^{(\lambda-\mu)(x-y)}.$$

The exponential is bounded by 1 because $\mu\leqslant\lambda$ and $x\leqslant y$. To conclude the proof of the lemma it is therefore enough to show that the previous ration is bounded.

In the proof of Lemma 5.1 we show that $|\gamma_1(\lambda)-\lambda|$ is bounded, uniformly in λ, by a constant C_1 which depends only on $\|F\|_\infty$ and that $\sigma(\mu)$ is bounded above and below by a finite positive constant for all λ in $\mathbb{R}$ and x in $\mathbb{R}_+$. In particular, by Theorem 5.2, there exists L_0 such that for $L\geqslant L_0$, the ratio on the right hand side of the previous formula is bounded by a constant that depends only on $\|F\|_\infty$. On the other hand, for $2\leqslant L\leqslant L_0$, by Lemma 5.6 and explicit computations to express $f_{\mu,L}$ in terms of $\tilde f_{\mu,L}$, this ratio is bounded by $\exp\{CL\}$ for some constant C depending only on $\|F\|_\infty$. This concludes the proof of the lemma. □

Step 2. Decomposition of the variance. We will obtain now a recursive equation for $W(L)$. Assume that we already estimated $W(K)$ for $2\leqslant K\leqslant L-1$. Let us write the identity

$$f - E_{\Lambda_L,M}[f] = \{f - E_{\Lambda_L,M}[f\mid\eta_L]\} + \{E_{\Lambda_L,M}[f\mid\eta_L] - E_{\Lambda_L,M}[f]\}.$$

Through this decomposition we may express the variance of f as

$$E_{\Lambda_L,M}\big[(f - E_{\Lambda_L,M}[f])^2\big] = E_{\Lambda_L,M}\big[(f - E_{\Lambda_L,M}[f\mid\eta_L])^2\big] + E_{\Lambda_L,M}\big[(E_{\Lambda_L,M}[f\mid\eta_L] - E_{\Lambda_L,M}[f])^2\big]. \quad (3.1)$$

The first term on the right-hand side is easily analyzed through the induction assumption and a simple computation on the Dirichlet form. We write

$$E_{\Lambda_L,M}\big[(f - E_{\Lambda_L,M}[f \mid \eta_L])^2\big] = E_{\Lambda_L,M}\big[E_{\Lambda_L,M}\big[(f - E_{\Lambda_L,M}[f \mid \eta_L])^2 \mid \eta_L\big]\big]$$
$$= E_{\Lambda_L,M}\big[E_{\Lambda_{L-1},M-\eta_L}\big[(f_{\eta_L} - E_{\Lambda_{L-1},M-\eta_L}[f_{\eta_L}])^2\big]\big].$$

Here we used the fact that $E_{\Lambda_L,M}[\cdot \mid \eta_L] = E_{\Lambda_{L-1},M-\eta_L}[\cdot]$. In this formula and below f_{η_L} stands for the real function defined on $\mathbb{R}^{\Lambda_{L-1}}$ whose value at $(\xi_1,\dots,\xi_{L-1})$ is given by $f_{\eta_L}(\xi_1,\dots,\xi_{L-1}) = f(\xi_1,\dots,\xi_{L-1},\eta_L)$. By the induction assumption this last expectation is bounded above by

$$W(L-1)E_{\Lambda_L,M}\big[D_{\Lambda_{L-1}}(\nu_{\Lambda_{L-1},M-\eta_L}, f_{\eta_L})\big] \leqslant W(L-1)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

In conclusion, we proved that

$$E_{\Lambda_L,M}\big[(f - E_{\Lambda_L,M}[f \mid \eta_L])^2\big] \leqslant W(L-1)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f). \tag{3.2}$$

The second term in (3.1) is nothing more than the variance of $E_{\Lambda_L,M}[f \mid \eta_L]$, a function of one variable. Lemma 3.1 provides an estimate for this expression:

$$E_{\Lambda_L,M}\big[(E_{\Lambda_L,M}[f \mid \eta_L] - E_{\Lambda_L,M}[f])^2\big] \leqslant C_0 E_{\Lambda_L,M}\left[\left(\frac{\partial}{\partial \eta_L} E_{\Lambda_L,M}[f \mid \eta_L]\right)^2\right] \tag{3.3}$$

for some constant C_0 depending only on $\|F\|_\infty$.

Step 3. Bounds on Glauber dynamics, small values of L. We now estimate the right hand side of (3.3), which is the Glauber Dirichlet form of $E_{\Lambda_L,M}[f \mid \eta_L]$, in terms of the Kawasaki Dirichlet form of f. A straightforward computation gives that

$$\frac{\partial}{\partial \eta_L} E_{\Lambda_L,M}[f \mid \eta_L] = \frac{1}{L-1}\sum_{x=1}^{L-1} E_{\Lambda_L,M}\left[\frac{\partial f}{\partial \eta_L} - \frac{\partial f}{\partial \eta_x} \,\middle|\, \eta_L\right]$$
$$+ E_{\Lambda_L,M}\left[f; \frac{1}{L-1}\sum_{x=1}^{L-1} V'(\eta_x) \,\middle|\, \eta_L\right]. \tag{3.4}$$

In this formula $E[g;h \mid \mathcal{F}]$ stands for the conditional covariance of g and h: $E[g;h \mid \mathcal{F}] = E[gh \mid \mathcal{F}] - E[g \mid \mathcal{F}]E[h \mid \mathcal{F}]$. We examine these two terms separately.

The first expression on the right hand side of (3.4) is easily estimated. Recall the definition of the operator $T^{x,y}f$. Since $T^{L,x}f = \sum_{x\leqslant y\leqslant L-1} T^{y+1,y}f$, by Schwarz inequality, we have that

$$E_{\Lambda_L,M}\left[\left(E_{\Lambda_L,M}\left[\frac{1}{L-1}\sum_{x=1}^{L-1} T^{L,x}f \,\middle|\, \eta_L\right]\right)^2\right]$$
$$\leqslant \frac{1}{L-1}\sum_{x=1}^{L-1}(L-x)\sum_{y=x}^{L-1} E_{\Lambda_L,M}[(T^{y,y+1}f)^2]$$
$$\leqslant L\sum_{x=1}^{L-1} E_{\Lambda_L,M}[(T^{x,x+1}f)^2] = LD_{\Lambda_L}(\nu_{\Lambda_L,M}, f). \tag{3.5}$$

The second term in (3.4) is also easy to handle for small values of L. Since $V(\varphi) = (1/2)\varphi^2 + F(\varphi)$ and since $\sum_{1\leqslant x\leqslant L-1}\eta_x$ is fixed for the measure $E_{\Lambda_L,M}[\cdot \mid \eta_L]$, the square of the second term on the right hand side of (3.4) is equal to

$$E_{\Lambda_L,M}\left[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\,\bigg|\,\eta_L\right]^2$$
$$= E_{\Lambda_{L-1},M-\eta_L}\left[f_{\eta_L};\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\right]^2$$
$$\leqslant E_{\Lambda_{L-1},M-\eta_L}[f_{\eta_L};f_{\eta_L}]\,E_{\Lambda_{L-1},M-\eta_L}\left[\left(\frac{1}{L-1}\sum_{x=1}^{L-1}\widetilde{F}(\eta_x)\right)^2\right].$$

In this formula, $\widetilde{F}$ stands for $F' - \langle F'\rangle_{\nu_{\Lambda_{L-1},M-\eta_L}}$. The second term is bounded by $4\|F'\|_\infty^2$. On the other hand, by the induction assumption, the first term is bounded by $W(L-1)D_{\Lambda_{L-1}}(\nu_{\Lambda_{L-1},M-\eta_L}, f_{\eta_L})$. Hence, taking expectation with respect to $\nu_{\Lambda_L,M}$, we obtain that

$$E_{\Lambda_L,M}\left[\left(E_{\Lambda_L,M}\left[f;\frac{1}{L-1}\sum_{x=1}^{L-1}V'(\eta_x)\,\bigg|\,\eta_L\right]\right)^2\right] \leqslant C_0 W(L-1)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f)$$

for some constant C_0 depending on F only. Without much effort and using the local central limit theorem, one can obtain an estimate of type $C_0W(L-1)L^{-1}D_{\Lambda_L}(\nu_{\Lambda_L,M}, f)$ for the left hand side of the last expression. However, for small values of L this improvement is irrelevant.

From this estimate and (3.5) we get that the left hand side of (3.3), which is the second term of (3.1), is bounded above by

$$C_0\{L + W(L-1)\}D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

Putting together this estimate with (3.2), we obtain that

$$\langle f; f\rangle_{\nu_{\Lambda_L,M}} \leqslant \big\{[1+C_0]W(L-1) + C_0L\big\}D_{\Lambda_L}(\nu_{\Lambda_L,M}, f)$$

or, taking a supremum over smooth functions f, that

$$W(L) \leqslant C_1W(L-1) + C_0L. \tag{3.6}$$

This inequality permits to iterate the estimate $W(2) \leqslant C$ obtained in Step 1 to derive estimates of $W(L)$ for small values of L. We now consider large values of L.

Step 4. Bounds on Glauber dynamics, large values of L. Here again we want to estimate the second term of (3.1). Applying Lemma 3.1, we bound this expression by the right hand side of (3.3). The first term of (3.4) is handled as before, giving (3.5). The second one requires a deeper analysis. Its square is equal to

$$E_{\Lambda_L,M}\left[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\,\middle|\,\eta_L\right]^2=E_{\Lambda_{L-1},M-\eta_L}\left[f;\frac{1}{L-1}\sum_{x=1}^{L-1}F'(\eta_x)\right]^2. \tag{3.7}$$

Here and below we omit the subscript η_L of f. Fix $1\leqslant K\leqslant\sqrt{L}$ and divide the interval $\{1,\dots,L-1\}$ into $\ell=\lfloor(L-1)/K\rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a\rfloor$ represents the integer part of a. Denote by I_j the jth interval, by M_j the total spin on I_j: $M_j=\sum_{x\in I_j}\eta_x$ and by E_{I_j,M_j} the expectation with respect to the canonical measure ν_{I_j,M_j}. The right hand side of the previous formula is bounded above by

$$\begin{aligned}&2E_{\Lambda_{L-1},M-\eta_L}\left[f;\frac{1}{L-1}\sum_{j=1}^{\ell}\sum_{x\in I_j}\{F'(\eta_x)-E_{I_j,M_j}[F']\}\right]^2\\&\quad+2E_{\Lambda_{L-1},M-\eta_L}\left[f;\frac{1}{L-1}\sum_{j=1}^{\ell}|I_j|E_{I_j,M_j}[F']\right]^2.\end{aligned} \tag{3.8}$$

Taking conditional expectation with respect to M_j, we rewrite the first term as

$$\begin{aligned}&2\left(\frac{1}{L-1}\sum_{j=1}^{\ell}E_{\Lambda_{L-1},M-\eta_L}\left[E_{I_j,M_j}\left[f;\sum_{x\in I_j}F'(\eta_x)\right]\right]\right)^2\\&\quad\leqslant\frac{2\ell}{(L-1)^2}\sum_{j=1}^{\ell}E_{\Lambda_{L-1},M-\eta_L}\left[\mathbf{Var}(\nu_{I_j,M_j},f)\mathbf{Var}\left(\nu_{I_j,M_j},\sum_{x\in I_j}F'(\eta_x)\right)\right].\end{aligned}$$

By the induction assumption, $\mathbf{Var}(\nu_{I_j,M_j},f)$ is bounded above by $W(|I_j|)D_{I_j}(\nu_{I_j,M_j},f)$. On the other hand, by Corollary 5.4, the variance of $|I_j|^{-1}\sum_{x\in I_j}F'(\eta_x)$ with respect to ν_{I_j,M_j} is bounded above by $C_0|I_j|^{-1}\|F'\|_\infty^2$ uniformly over M_j, where C_0 is a finite constant depending only on $\|F\|_\infty$. The previous expression is thus less than or equal to

$$\begin{aligned}&\frac{C_1\ell}{L^2}\sum_{j=1}^{\ell}W(|I_j|)|I_j|E_{\Lambda_{L-1},M-\eta_L}[D_{I_j}(\nu_{I_j,M_j},f)]\\&\quad\leqslant\frac{C_2}{L}\sum_{j=1}^{\ell}W(|I_j|)E_{\Lambda_{L-1},M-\eta_L}[D_{I_j}(\nu_{I_j,M_j},f)].\end{aligned}$$

Since $W(K+1)\leqslant CW(K)$, which follows from (3.6) and from the bound $W(K)\geqslant CK^2$, and since the previous sum is bounded by the global Dirichlet form $D_{\Lambda_{L-1}}(\nu_{\Lambda_{L-1},M-\eta_L},f)$, we proved that the first term of (3.8) is bounded above by

$$\frac{C_3W(K)}{L}D_{\Lambda_{L-1}}(\nu_{\Lambda_{L-1},M-\eta_L},f). \tag{3.9}$$

We turn now to the second term of (3.8). It is equal to

$$2\left(E_{\Lambda_{L-1},M-\eta_L}\left[f;\frac{1}{L-1}\sum_{j=1}^{\ell}|I_j|(E_{I_j,M_j}[F']-a-b[m_j-m])\right]\right)^2,$$

where $m_j = M_j/|I_j|$, $m = (M-\eta_L)/(L-1)$ and a, b are constants to be chosen later. We were allowed to add the terms a, $b[m_j-m]$ in the covariance because a, $b\sum_{j=1}^{\ell}|I_j|[m_j-m]$ are constants. Let $G(m_j) = E_{I_j,M_j}[F'] - a - b[m_j-m]$. By Schwarz inequality, the previous expression is bounded above by

$$2E_{\Lambda_{L-1},M-\eta_L}[f;f]\,E_{\Lambda_{L-1},M-\eta_L}\left[\left(\frac{1}{L-1}\sum_{j=1}^{\ell}|I_j|G(m_j)\right)^2\right].$$

We claim that

$$E_{\Lambda_{L-1},M-\eta_L}\left[\left(\frac{1}{L-1}\sum_{j=1}^{\ell}|I_j|G(m_j)\right)^2\right] \leqslant \frac{C_0}{KL} \tag{3.10}$$

for some finite constant C_0. Indeed, developing the square, we write this expectation as

$$\frac{1}{(L-1)^2}\sum_{j=1}^{\ell}|I_j|^2E_{\Lambda_{L-1},M-\eta_L}[G(m_j)^2] + \frac{1}{(L-1)^2}\sum_{i\neq j}|I_j||I_i|E_{\Lambda_{L-1},M-\eta_L}[G(m_i)G(m_j)]. \tag{3.11}$$

Recall that

$$m = (M-\eta_L)/(L-1), \quad m_j = M_j/|I_j|.$$

By Corollary 5.3, $E_{\Lambda_{L-1},M-\eta_L}[G(m_j)^2]$ is bounded above by

$$E_{\nu_m}[G(m_j)^2] + \frac{C_0|I_j|}{L}\sqrt{E_{\nu_m}[G(m_j)^4]}. \tag{3.12}$$

Let $A(\alpha) = E_{\nu_\alpha}[F'(\eta_1)]$ and set $a = A(m)$, $b = A'(m)$. With this choice, $G(m_j) = E_{I_j,M_j}[F'(\eta_x)] - A(m_j) + A(m_j) - A(m) - A'(m)[m_j-m]$. By Corollary 5.3, $|E_{I_j,M_j}[F'(\eta_x)] - A(m_j)|$ is less than or equal to $C\|F'\|_\infty/|I_j|$. On the other hand, $A(m_j)-A(m)-A'(m)[m_j-m]$ is bounded in absolute value by $(1/2)\|A''\|_\infty[m_j-m]^2$. In particular,

$$\frac{1}{(L-1)^2}\sum_{j=1}^{\ell}|I_j|^2E_{\nu_m}[G(m_j)^2] \leqslant \frac{C_0\ell}{L^2} + \frac{\|A''\|_\infty^2}{2L^2}\sum_{j=1}^{\ell}|I_j|^2E_{\nu_m}[(m_j-m)^4] \tag{3.13}$$

for some constant C_0 depending only on F. By Lemma 5.1, since ν_m is a product measure, the expectation on the right hand side of the previous inequality is bounded above by $C|I_j|^{-2}$. In Lemma 3.3 below we prove that $\|A''\|_\infty$ is bounded by a constant. The last expression is thus less than or equal to $C_0\ell/L^2 \leqslant C/KL$.

The same arguments give that

$$\frac{C_0}{(L-1)^3}\sum_{j=1}^{\ell}|I_j|^3\sqrt{E_{\nu_m}[G(m_j)^4]} \leqslant \frac{C_0}{L^2}.$$

Therefore, the first line of (3.11) is bounded above by C_0/KL.

We proceed in the same way to bound the second term of (3.11). Fix $i \neq j$. By Corollary 5.3, $E_{\Lambda_{L-1},M-\eta_L}[G(m_i)G(m_j)]$ is bounded above by

$$E_{\nu_m}[G(m_i)G(m_j)] + \frac{C_0K}{L}\sqrt{E_{\nu_m}[G(m_i)^2G(m_j)^2]}.$$

Notice that the first term vanishes because ν_m is a product measure and

$$E_{\nu_m}\left[E_{I_j,M_j}[F'(\eta_x)]\right] = E_{\nu_m}\left[E_{\nu_m}\left[F'(\eta_x)\,\middle|\,\sum_{y\in I_j}\eta_y = M_j\right]\right] = E_{\nu_m}[F'(\eta_x)] = A(m),$$

$E_{\nu_m}[m_j] = m$. On the other hand, since ν_m is a product measure, $E_{\nu_m}[G(m_i)^2G(m_j)^2] = E_{\nu_m}[G(m_i)^2]E_{\nu_m}[G(m_j)^2]$. Hence,

$$\frac{1}{(L-1)^2}\sum_{i\neq j}|I_j||I_i|E_{\Lambda_{L-1},M-\eta_L}[G(m_i)G(m_j)] \leqslant \frac{1}{(L-1)}\sum_{i=1}^{\ell}|I_i|\frac{C_0K}{L}E_{\nu_m}[G(m_i)^2]$$

because $\sum_j |I_j| = L-1$. The right hand side of the previous formula is exactly the first term in (3.13) that we showed to be bounded by $\widetilde{C}_1/KL$. This estimate together with the bounds obtained in (3.13) and in the paragraph that follows (3.13) prove (3.10). Therefore, the second term of (3.8) is bounded above by $C_0(KL)^{-1}E_{\Lambda_{L-1},M-\eta_L}[f;f]$. This bound together with (3.9) gives that (3.8), and therefore (3.7), is less than or equal to

$$\frac{C_3W(K)}{L}D_{\Lambda_{L-1}}(\nu_{\Lambda_{L-1},M-\eta_L}, f) + \frac{C}{KL}E_{\Lambda_{L-1},M-\eta_L}[f;f].$$

Since (3.7) is just the square of the second term of (3.4), taking expectation with respect to $\nu_{\Lambda_L,M}$ in (3.7) and recalling (3.5), we have that (3.3) is bounded above by

$$C\left(L + \frac{W(K)}{L}\right)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f) + \frac{C}{KL}E_{\Lambda_L,M}[f;f].$$

Choose K large enough for $\varepsilon = C/K$ to be strictly smaller than 2. Adding this term to (3.2), in view of the decomposition (3.1), we deduce that

$$E_{\Lambda_L,M}\left[(f - E_{\Lambda_L,M}[f])^2\right] \leqslant \left(1 - \frac{\varepsilon}{L}\right)^{-1}\left(W(L-1) + C_3L + \frac{C_3}{L}\right)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

Taking supremum over smooth functions $f:\mathbb{R}^{\Lambda_L}\to\mathbb{R}$ in $L^2(\nu_{\Lambda_L,M})$, we obtain that

$$W(L)\leqslant\left(1-\frac{\varepsilon}{L}\right)^{-1}\left(W(L-1)+C_3L+\frac{C_3}{L}\right).$$

It is not difficult to deduce from this recursive relation the existence of a constant C_4 such that $W(L)\leqslant C_4L^2$ for all $L\geqslant 2$. This concludes the proof of Theorem 2.1. □

We conclude this section proving a result needed earlier in the proof.

LEMMA 3.3. – *There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\sup_{\alpha\in\mathbb{R}}|A''(\alpha)|\leqslant C_0.$$

Proof. – We claim that $A(\alpha)=\Phi(\alpha)-\alpha$. Indeed, since $A(\alpha)=E_{\nu_\alpha}[F'(\eta_1)]$, we have that

$$A(\alpha)=\Phi(\alpha)-\alpha+\frac{1}{Z(\Phi(\alpha))}\int\{-\Phi(\alpha)+a+F'(a)\}\mathrm{e}^{\Phi(\alpha)a-V(a)}\,da.$$

An integration by parts shows that the integral vanishes proving that $A(\alpha)=\Phi(\alpha)-\alpha$.

It follows from this identity that $A''(\alpha)=\Phi''(\alpha)$. On the other hand, since $\Phi=R^{-1}$,

$$\Phi''(\alpha)=-\frac{R''(\Phi(\alpha))}{[R'(\Phi(\alpha))]^3}.$$

Recall that $\{\gamma_k,\ k\geqslant 2\}$ stands for the truncated moments of the variables X_1^λ. We obtain from the definition of R that $R'(\Phi(\alpha))=\gamma_2(\Phi(\alpha))$, $R''(\Phi(\alpha))=\gamma_3(\Phi(\alpha))$. Therefore, $A''(\alpha)=-\gamma_3(\Phi(\alpha))/\gamma_2(\Phi(\alpha))^3$ and the statement follows from Lemma 5.1. □

4. Logarithmic Sobolev inequality

We prove in this section Theorem 2.2. The approach is similar to the one presented in last section for the spectral gap. We will derive a recursive formula for $\theta(L)$ in terms of $\theta(L-1)$ and L in four steps. As before, all constants are allowed to depend on $\|F\|_\infty$, $\|F'\|_\infty$ and $\|F''\|_\infty$.

Step 1. One-site logarithmic Sobolev inequality. We start our proof with the case $L=2$. Let $f:\mathbb{R}^{\Lambda_2}\to\mathbb{R}$ be a smooth function such that $\langle f^2\rangle_{\nu_{\Lambda_2,M}}=1$. Let $g(\eta_2)=f(M-\eta_2,\eta_2)$. Since the total spin is fixed to be M, we have that $\langle g^2\rangle_{\nu_{\Lambda_2,M}}=\langle f^2\rangle_{\nu_{\Lambda_2,M}}=1$ and that $S_{\Lambda_2}(\nu_{\Lambda_2,M},g)=S_{\Lambda_2}(\nu_{\Lambda_2,M},f)$. The next lemma permits to estimate the entropy of $S_{\Lambda_2}(\nu_{\Lambda_2,M},g)$ in terms of the Glauber Dirichlet form of g. This result is in fact a logarithmic Sobolev inequality for the Glauber dynamics obtained when restricting the Kawasaki exchange dynamics to one site. Recall that $\nu^1_{\Lambda_L,M}$ represents the one-site marginal of $\nu_{\Lambda_L,M}$.

LEMMA 4.1. – *There exists a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\int H(\eta_L)^2 \log H(\eta_L)^2 \, d\nu^1_{\Lambda_L,M}(\eta_L) \leqslant C_0 E_{\nu^1_{\Lambda_L,M}}\left[\left(\frac{\partial H}{\partial \eta_L}\right)^2\right] \tag{4.1}$$

for every $L \geqslant 2$, every M in $\mathbb{R}$ and every smooth function $H:\mathbb{R}\to\mathbb{R}$ in $L^2(\nu^1_{\Lambda_L,M})$ such that $\langle H^2\rangle_{\nu^1_{\Lambda_L,M}} = 1$.

Same comments presented in Remark 3.2 apply here.

We conclude the first step before proving the lemma. From the previous statement applied to $L=2$ and $H=g$ we have that

$$\begin{aligned} S_{\Lambda_2}(\nu_{\Lambda_2,M}, f) = S_{\Lambda_2}(\nu_{\Lambda_2,M}, g) &\leqslant C_0 E_{\nu^1_{\Lambda_2,M}}\left[\left(\frac{\partial g}{\partial \eta_2}\right)^2\right] \\ &= C_0 E_{\nu_{\Lambda_2,M}}\left[\left(\frac{\partial g}{\partial \eta_2}\right)^2\right] = C_0 E_{\nu_{\Lambda_2,M}}\left[\left(\frac{\partial f}{\partial \eta_2} - \frac{\partial f}{\partial \eta_1}\right)^2\right] \end{aligned}$$

because $\partial g/\partial\eta_2 = \partial f/\partial\eta_2 - \partial f/\partial\eta_1$. This proves that $\theta(2) \leqslant C_0$.

Proof of Lemma 4.1. – We first prove the lemma in the case of grand canonical measures. Recall that we denote by $\bar\nu^1_\lambda$ the one-site marginal of the measure $\bar\nu^{\Lambda_L}_\lambda$. We want to show that there exists a constant C_0, independent of λ, such that

$$\int H(a)^2 \log H(a)^2 \, \bar\nu^1_\lambda(da) \leqslant C_0 \int [H'(a)]^2 \, \bar\nu^1_\lambda(da) \tag{4.2}$$

for all smooth functions $H:\mathbb{R}\to\mathbb{R}$ such that $\langle H^2\rangle_{\bar\nu^1_\lambda} = 1$. Since the potential V is a bounded perturbation of the Gaussian potential, by Corollary 6.2.45 in [7], the previous inequality holds with a constant C_0 that might depend on λ. All the matter here is to show that we may find a finite constant independent of λ.

Recall the definition of the potential V_λ introduced in the proof of Lemma 3.1. A change of variable permits to rewrite the left hand side of (4.2) as

$$\int H_\lambda(a)^2 \log H_\lambda(a)^2 \, e^{-\widetilde{F}_\lambda(a)} \frac{1}{\sqrt{2\pi}} e^{-(a^2/2)} \, da,$$

where $H_\lambda(a) = H(a+\lambda)$, $\widetilde{F}_\lambda(a) = F(a+\lambda) + \log \widetilde{Z}(\lambda)$ and $\widetilde{Z}(\lambda)$ is a normalizing constant. It is easy to check that $\|\exp\{\pm\widetilde{F}_\lambda\}\|_\infty \leqslant \exp 2\|F\|_\infty$. In particular, by Corollary 6.2.45 in [7], the previous expression is bounded above by

$$2e^{4\|F\|_\infty} \int [H'_\lambda(a)]^2 \, e^{-\widetilde{F}_\lambda(a)} \frac{1}{\sqrt{2\pi}} e^{-(a^2/2)} \, da = 2e^{4\|F\|_\infty} \int [H'(a)]^2 \bar\nu^1_\lambda(da).$$

This proves the lemma in the case of grand canonical measures with $C_0 = 2\exp\{4\|F\|_\infty\}$.

For canonical measures, we just need to use the local central limit theorem for large values of L and explicit computations for small values of L. We start with the case of large values of L. Fix a smooth function $H:\mathbb{R}\to\mathbb{R}$ with $\langle H^2\rangle_{\nu^1_{\Lambda_{L+1},M}} = 1$ and recall the notation introduced in the proof of Lemma 3.1. The left hand side of (4.1) can be written as

$$\frac{\sqrt{L+1}}{\sqrt{L}} \int H(a)^2 \log H(a)^2\, g_\lambda(a) \frac{f_{\lambda,L}([\gamma_1 - a]/\sigma\sqrt{L})}{f_{\lambda,L+1}(0)}\, da,$$

where g_λ stands for the density $Z(\lambda)^{-1}\exp\{\lambda x - V(x)\}$ and $\lambda = \Phi(M/[L+1])$.

We base our proof on two facts. First, that if a function W is strictly convex then the measure $\mu_W(dx) = Z^{-1}\exp\{-W(x)\}\,dx$ associated to the potential W satisfies a logarithmic Sobolev inequality. Secondly, if $\mu(dx)$ satisfies a logarithmic Sobolev inequality, and f is a density with respect to μ, which is bounded below and above $(0 < C_1 \leqslant f \leqslant C_1^{-1})$, then $f\,d\mu$ satisfies a logarithmic Sobolev inequality. The proof of these two well known sentences can be found, for instance, in [13].

In view of these statements, we just need to show that the above density is equivalent to the density of a measure associated to a convex potential. Here and below two functions g, f are said to be equivalent if there exists a finite, strictly positive constant C_0 depending only on V (and not on M, λ or L) such that $C_0 g \leqslant f \leqslant C_0^{-1} g$. We shall rely on the local central limit theorem to show the equivalence of the above density with some density associated to a convex potential.

By Theorem 5.2, for L large enough $f_{\lambda,L+1}(0)$ is bounded above and below by a constant. We may therefore ignore the denominator in the previous integral. Recall from the previous section that we denote by $g_{\lambda,L}$ the density of the random variable $\sum_{1\leqslant j\leqslant L} X_j^\lambda$. An elementary computation, already mentioned in the proof of Lemma 3.1, gives that

$$g_{\lambda,L}(a) = \mathrm{e}^{(\lambda-\mu)a}\left(\frac{Z(\mu)}{Z(\lambda)}\right)^L g_{\mu,L}(a)$$

for all λ, μ in $\mathbb{R}$. In particular, writing $f_{\lambda,L}$ in terms of $g_{\lambda,L}$, we get that

$$f_{\lambda,L}\left(\frac{\gamma_1(\lambda) - x}{\sigma(\lambda)\sqrt{L}}\right) = \frac{\sigma(\lambda)}{\sigma(\mu)}\left(\frac{Z(\mu)}{Z(\lambda)}\right)^L \exp\{(\lambda-\mu)(\gamma_1(\lambda) - x) + L(\lambda-\mu)\gamma_1(\lambda)\}$$
$$\times f_{\mu,L}\left(\frac{\gamma_1(\lambda) - x}{\sigma(\mu)\sqrt{L}} + \sqrt{L}\,\frac{\gamma_1(\lambda) - \gamma_1(\mu)}{\sigma(\mu)}\right).$$

We will now choose μ for the variable on the right hand side to vanish. In this case, we will be able to apply the local central limit theorem to claim that $f_{\mu,L}(0)$ is bounded above and below by positive finite constants. Set

$$\mu(x) = \Phi\left(\left\{1 + \frac{1}{L}\right\}\gamma_1(\lambda) - \frac{x}{L}\right). \tag{4.3}$$

With this choice, $\gamma_1(\mu) = (1 + L^{-1})\gamma_1(\lambda) - (x/L)$ so that the right hand side of the previous formula becomes

$$\frac{\sigma(\lambda)}{\sigma(\mu)}\left(\frac{Z(\mu)}{Z(\lambda)}\right)^L \mathrm{e}^{L(\lambda-\mu)\gamma_1(\mu)} f_{\mu,L}(0).$$

By Lemma 5.1, $\sigma(\cdot)$ is a function bounded below and above by strictly positive finite constants. By Theorem 5.2, $f_{\mu,L}(0)$ is bounded below and above by strictly positive finite constants. It follows from this observation and from the previous estimate on $f_{\lambda,L+1}(0)$ that the density of the measure $\nu^1_{\Lambda_{L+1},M}$ with respect to the Lebesgue measure, denoted by $R_{L+1,M}(x)$, is equivalent in the sense defined above to the function

$$\exp -\left\{(1/2)(x-\lambda)^2 + L\left[\log Z(\lambda) - \log Z(\mu) - (\lambda-\mu)\gamma_1(\mu)\right]\right\}$$

because, by (5.4), $g_\lambda(x)$ is equivalent to $\exp\{-(1/2)(x-\lambda)^2\}$. In this formula $\mu = \mu(x)$ is defined by (4.3).

It remains to show that the function inside braces, denoted by $\Theta(x) = \Theta_\lambda(x)$, is convex. Straightforward computations show that

$$(\partial_x \Theta)(x) = x + \gamma_1(\mu(x))\left(1 - \frac{1}{\sigma(\mu(x))^2}\right) - \mu(x),$$
$$(\partial_x^2 \Theta)(x) = 1 + \frac{1}{L}\left\{-1 + \frac{2}{\sigma(\mu(x))^2} + \frac{\gamma_1(\mu(x))\gamma_3(\mu(x))}{\sigma(\mu(x))^6}\right\}.$$

It follows from Lemma 5.1 that Θ is strictly convex for for L large enough. This proves the lemma in the canonical case for large values of L.

We now turn to the case of small values of L. Recall the notation introduced just before Lemma 5.6. The density $R_{L+1,M}(x)$ can be written as

$$\frac{\sqrt{L+1}}{\sqrt{L}} g_\lambda(x) \frac{\tilde f_{\lambda,L}(L^{-1/2}[\lambda - x])}{\tilde f_{\lambda,L+1}(0)}.$$

Here $\lambda = M/(L+1)$. By Lemma 5.6 this expression is bounded above (and below by an expression with C_0^L replaced by C_0^{-L})

$$C_0^L \exp -\frac{1}{2}\left\{(1+L^{-1})(x-\lambda)^2\right\},$$

where C_0 depends on $\|F\|_\infty$ only. Since $L \leqslant L_0$, this proves that the density $R_{L+1,M}$ is equivalent to a Gaussian density, which proves the lemma in the canonical case for small values of L. □

We now obtain a recursive formula for $\theta(L)$ in terms of $\theta(L-1)$, L. Assume that $\theta(K) < \infty$ for $2 \leqslant K \leqslant L-1$.

Step 2. Decomposition of the entropy. Use an elementary property of the conditional expectation to decompose the entropy as

$$S_{\Lambda_L}(\nu_{\Lambda_L,M}, f) = \int f^2 \log \frac{f^2}{E_{\Lambda_L,M}[f^2 \mid \eta_L]} \, d\nu_{\Lambda_L,M}$$
$$+ \int E_{\Lambda_L,M}[f^2 \mid \eta_L] \log E_{\Lambda_L,M}[f^2 \mid \eta_L] \, \nu^1_{\Lambda_L,M}(d\eta_L). \qquad (4.4)$$

The first term on the right hand side of (4.4) is estimated through the induction assumption. Indeed, taking conditional expectation with respect to η_L, we may rewrite this integral as

$$\int E_{\Lambda_{L-1},M-\eta_L}\left[\frac{f^2}{E_{\Lambda_L,M}[f^2 \mid \eta_L]}\log\frac{f^2}{E_{\Lambda_L,M}[f^2 \mid \eta_L]}\right]E_{\Lambda_L,M}[f^2 \mid \eta_L]\,\nu^1_{\Lambda_L,M}(d\eta_L).$$

Since the integral of $f^2/E_{\Lambda_L,M}[f^2 \mid \eta_L]$ with respect to $\nu_{\Lambda_{L-1},M-\eta_L}$ is equal to 1, the previous expression is bounded above by

$$\theta(L-1)\int D_{\Lambda_{L-1}}\bigl(\nu_{\Lambda_{L-1},M-\eta_L}, f/E_{\Lambda_L,M}[f^2 \mid \eta_L]^{1/2}\bigr)E_{\Lambda_L,M}[f^2 \mid \eta_L]\,d\nu^1_{\Lambda_L,M}(\eta_L) \\ \leqslant \theta(L-1)D_{\Lambda_L}(\nu_{\Lambda_L,M}, f). \tag{4.5}$$

The last inequality follows from a direct computation.

The second term in (4.4) is estimated through Lemma 4.1. Let $H(\eta_L) = E_{\Lambda_L,M}[f^2 \mid \eta_L]^{1/2}$. By Lemma 4.1, the second term on the right hand side of (4.4) is bounded above by

$$C_0 E_{\nu^1_{\Lambda_L,M}}\left[\left(\frac{\partial E_{\Lambda_L,M}[f^2 \mid \eta_L]^{1/2}}{\partial\eta_L}\right)^2\right].$$

A computation, similar to the one performed in (3.4), shows that $(\partial H/\partial\eta_L)^2$ is equal to

$$\frac{1}{4E_{\Lambda_L,M}[f^2 \mid \eta_L]}\left\{\frac{1}{L-1}\sum_{x=1}^{L-1}E_{\Lambda_L,M}\left[\frac{\partial f^2}{\partial\eta_L}-\frac{\partial f^2}{\partial\eta_x}\,\middle|\,\eta_L\right]\right. \\ \left.-E_{\Lambda_L,M}\left[f^2;\frac{1}{L-1}\sum_{x=1}^{L-1}V'(\eta_x)\,\middle|\,\eta_L\right]\right\}^2. \tag{4.6}$$

Following the computation presented just after (3.4), we obtain by Schwarz inequality, that

$$\frac{1}{4E_{\Lambda_L,M}[f^2 \mid \eta_L]}\left\{\frac{1}{L-1}\sum_{x=1}^{L-1}E_{\Lambda_L,M}\left[\frac{\partial f^2}{\partial\eta_L}-\frac{\partial f^2}{\partial\eta_x}\,\middle|\,\eta_L\right]\right\}^2 \\ \leqslant C_0 L\sum_{x=1}^{L-1}E_{\Lambda_L,M}[(T^{x,x+1}f)^2 \mid \eta_L] \tag{4.7}$$

for some finite universal constant C_0. We have thus a bound on the first term in (4.6).

The analysis of the second term on the right hand side of (4.6) is more demanding and is the main goal of Steps 3 and 4.

Step 3. Bounds on the Glauber dynamics, small values of *L*. We first replace $V'(\eta_x)$ by $F'(\eta_x)$ because $\sum_{1\leqslant y\leqslant L-1}\eta_y$ is fixed for the measure $E_{\Lambda_L,M}[\cdot \mid \eta_L]$. The following lemma will be particularly useful.

LEMMA 4.2. – *There exists a finite constant C_0 depending only on $\|F''\|_\infty$ such that*

$$E_{\Lambda_L,M}\left[g^2;\frac{1}{L}\sum_{x=1}^{L}F'(\eta_x)\right]^2 \leqslant \frac{C_0\theta(L)}{L}\sum_{x=1}^{L-1}E_{\Lambda_L,M}[(T^{x,x+1}g)^2] \tag{4.8}$$

for all $L \geqslant 2$, *all* M *in* $\mathbb{R}$ *and all smooth functions* g *in* $L^2(\nu_{\Lambda_L,M})$ *such that* $\langle g^2\rangle_{\nu_{\Lambda_L,M}} = 1$.

Proof. – Denote by $\widetilde{F}_{L,M}(\eta_x)$ the function $F'(\eta_x) - E_{\Lambda_L,M}[F'(\eta_x)]$. With this notation,

$$E_{\Lambda_L,M}\left[g^2; \frac{1}{L}\sum_{x=1}^{L} F'(\eta_x)\right] = E_{\Lambda_L,M}\left[g^2 \frac{1}{L}\sum_{x=1}^{L} \widetilde{F}_{L,M}(\eta_x)\right].$$

By the entropy inequality, this expression is bounded above by

$$\frac{1}{\beta L}\log\int \exp\left\{\beta\sum_{x=1}^{L}\widetilde{F}_{L,M}(\eta_x)\right\} d\nu_{\Lambda_L,M} + \frac{1}{\beta L} S_{\Lambda_L}(\nu_{\Lambda_L,M}, g)$$

for every $\beta > 0$. By Lemma 6.1, the first term is bounded above by $C_0\beta$ for some finite constant C_0 that depends only on $\|F''\|_\infty$. Minimizing over $\beta > 0$ we obtain that the left hand side of (4.8) is bounded above by $C_0 L^{-1} S_{\Lambda_L}(\nu_{\Lambda_L,M}, g)$. By definition of $\theta(L)$, this expression is less than or equal to the right hand side of (4.8). □

It follows from Lemma 4.2 applied to the measure $\nu_{L-1,M-\eta_L}$ and to the function $g^2 = f^2/E_{\Lambda_L,M}[f^2 \mid \eta_L]$ that the second term of (4.6) is bounded above by

$$\frac{C_0\theta(L-1)}{L}\sum_{x=1}^{L-2} E_{\Lambda_L,M}[(T^{x,x+1}f)^2 \mid \eta_L].$$

Taking expectation with respect to $\nu_{\Lambda_L,M}$ in this formula and in (4.7), we obtain that the expectation of (4.6) is less than or equal to

$$C_0\{L + L^{-1}\theta(L-1)\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

The second term of (4.4), which is bounded by the expectation with respect to $\nu_{\Lambda_L,M}$ of (4.6), is less than or equal to the same expression. Therefore, in view of (4.5),

$$S_{\Lambda_L}(\nu_{\Lambda_L,M}, f) \leqslant \{C_0 L + (1 + C_0 L^{-1})\theta(L-1)\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

In particular, by definition of $\theta(L)$,

$$\theta(L) \leqslant C_0 L + (1 + C_0 L^{-1})\theta(L-1).$$

This relation together with the fact that $\theta(2) \leqslant C_0$, which was proved in the first step, gives that $\theta(L) < C_1^L$, $\theta(L) \leqslant C_1\theta(L-1)$ for some finite constant C_1 depending only on $\|F\|_\infty$, $\|F'\|_\infty$, $\|F''\|_\infty$.

Step 4. Bounds on the Glauber dynamics, large values of *L*. We now give an alternative estimate of the second term of (4.6) that we shall use for large values of L.

PROPOSITION 4.3. – *Fix $\delta > 0$. There exist $L_0 \geqslant 2$ and a finite constant $C_0 = C_0(\delta, \|F\|_\infty, \|F'\|_\infty)$ such that*

$$\left(E_{\Lambda_L,M}\left[g^2; \frac{1}{L}\sum_{x=1}^{L} F'(\eta_x)\right]\right)^2 \leqslant \left\{C_0 L + \frac{\delta\theta(L)}{L}\right\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, g) \tag{4.9}$$

for all $L \geqslant L_0$, M in $\mathbb{R}$ and functions g in $L^2(\nu_{\Lambda_L,M})$ such that $\langle g^2\rangle_{\nu_{\Lambda_L,M}} = 1$.

We first assume this result to conclude the proof of Theorem 2.2. Recall the decomposition (4.4) of the entropy and the estimate (4.5). The second term on the right hand side of (4.4) was estimated by Lemma 4.1, giving (4.6). The first term of (4.6) was bounded by (4.7). Fix $\delta < 2$. It follows from Proposition 4.3 applied to the measure $\nu_{L-1,M-\eta_L}$ and the function $g^2 = f^2/E_{\Lambda_L,M-\eta_L}[f^2 \mid \eta_L]$ that the second term in (4.6) is bounded above by

$$\left\{C_0 L + \frac{\delta\theta(L-1)}{L-1}\right\} D_{\Lambda_{L-1}}\big(\nu_{\Lambda_{L-1},M-\eta_L}, f/E_{\Lambda_L,M-\eta_L}[f^2 \mid \eta_L]^{1/2}\big)$$

provided that L is large enough. Taking expectations with respect to $\nu_{L,M}$ in (4.6), we obtain that the second term in (4.4) is less than or equal to

$$\left\{C_1 L + \frac{\delta\theta(L-1)}{L-1}\right\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, f).$$

In particular, by (4.5) and (4.4),

$$S_{\Lambda_L}(\nu_{\Lambda_L,M}, f) \leqslant \left\{C_2 L + \left(1 + \frac{\delta}{L-1}\right)\theta(L-1)\right\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, f)$$

or, by definition of $\theta(L)$,

$$\theta(L) \leqslant \left\{C_2 L + \left(1 + \frac{\delta}{L-1}\right)\theta(L-1)\right\}.$$

It is easy to derive form this inequality the existence of a finite constant C such that $\theta(L) \leqslant CL^2$ for all $L \geqslant 2$. This concludes the proof of Theorem 2.2. □

We now turn to the proof of Proposition 4.3. For clarity reasons, we divide it in several lemmas. We first repeat the procedure presented in Step 4 of the previous section. Fix $K \geqslant 1$ and divide the interval $\{1, \ldots, L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$. Denote by I_j the jth interval, by M_j the total spin on I_j: $M_j = \sum_{x \in I_j} \eta_x$ and by E_{I_j,M_j} the expectation with respect to the canonical measure ν_{I_j,M_j}. The left hand side of (4.9) is bounded above by

$$2\left(E_{\Lambda_L,M}\left[g^2;\frac{1}{L}\sum_{j=1}^{\ell}\sum_{x\in I_j}\{F'(\eta_x)-E_{I_j,M_j}[F']\}\right]\right)^2$$
$$+2\left(E_{\Lambda_L,M}\left[g^2;\frac{1}{L}\sum_{j=1}^{\ell}|I_j|E_{I_j,M_j}[F']\right]\right)^2. \tag{4.10}$$

LEMMA 4.4. – *Fix $2\leqslant K\leqslant L$ and M in $\mathbb{R}$. There exists a finite constant C_0 depending only on K such that*

$$\left(E_{\Lambda_L,M}\left[g^2;\frac{1}{L}\sum_{j=1}^{\ell}\sum_{x\in I_j}\{F'(\eta_x)-E_{I_j,M_j}[F']\}\right]\right)^2\leqslant\frac{C_0}{L}D_{\Lambda_L}(\nu_{\Lambda_L,M},g)$$

for all smooth functions g in $L^2(\nu_{\Lambda_L,M})$ such that $\langle g^2\rangle_{\nu_{\Lambda_L,M}}=1$.

Proof. – Taking conditional expectation with respect to M_j, we rewrite the left hand side of the statement of the lemma as

$$\left(\frac{1}{L}\sum_{j=1}^{\ell}E_{\Lambda_L,M}\left[E_{I_j,M_j}[g^2]\,E_{I_j,M_j}\left[g_j^2;\sum_{x\in I_j}F'(\eta_x)\right]\right]\right)^2$$
$$\leqslant\frac{\ell}{L^2}\sum_{j=1}^{\ell}E_{\Lambda_L,M}\left[E_{I_j,M_j}[g^2]\left(E_{I_j,M_j}\left[g_j^2;\sum_{x\in I_j}F'(\eta_x)\right]\right)^2\right], \tag{4.11}$$

where $g_j^2=g^2/E_{I_j,M_j}[g^2]$ has mean one with respect to ν_{I_j,M_j}. In the last step we used Schwarz inequality and the fact $E_{\Lambda_L,M}[E_{I_j,M_j}[g^2]]=1$. Fix $1\leqslant j\leqslant\ell$. By the entropy inequality, $E_{I_j,M_j}[g_j^2;\sum_{x\in I_j}F'(\eta_x)]$ is bounded above by

$$\frac{1}{\beta}\log\int e^{\beta\sum_{x\in I_j}F_j(\eta_x)}\,d\nu_{I_j,M_j}+\frac{1}{\beta}S_{I_j}(\nu_{I_j,M_j},g_j)$$

for every $\beta>0$. Here, $F_j(\eta_x)=F'(\eta_x)-E_{I_j,M_j}[F']$. By definition of $\theta(|I_j|)$, the second term is bounded above by $\theta(|I_j|)\beta^{-1}D_{I_j}(\nu_{I_j,M_j},g_j)$. On the other hand, by Lemma 6.1, the first one is bounded above by $C_0\beta K$ for some finite constant C_0. Minimizing over β and summing over j, since $\theta(K+1)\leqslant C\theta(K)$, we get that (4.11) is less than or equal to

$$\frac{C_0\,\theta(K)}{L}\sum_{j=1}^{\ell}E_{\Lambda_L,M}\big[E_{I_j,M_j}[g^2]\,D_{I_j}(\nu_{I_j,M_j},g_j)\big]\leqslant\frac{C_0\theta(K)}{L}D_{\Lambda_L}(\nu_{\Lambda_L,M},g). \tag{4.12}$$

This concludes the proof of the lemma. □

We turn now to the second term of (4.10). Recall that m, m_j stand for M/L, $M_j/|I_j|$, respectively. Let $G(m_j)=E_{I_j,M_j}[F']-A(m)-A'(m)[m_j-m]$, where $A(m)=E_{\nu_m}[F']$. Since we may add constants in a covariance, the expectation in the second term of (4.10) is equal to

$$E_{\Lambda_L,M}\left[g^2; \frac{1}{L}\sum_{j=1}^{\ell}|I_j|G(m_j)\right]. \tag{4.13}$$

To estimate this covariance we need to consider two cases. Let β_0, be the constant given by Lemma 6.5 and fix $0<\delta<2$. By Lemma 6.5, there exists K_0 for which the left hand side of (6.10) is bounded by $\delta\beta$ for all $\beta \leqslant \beta_0$ and all $L \geqslant 2K \geqslant 2K_0$.

LEMMA 4.5. – *Fix $L \geqslant 2K \geqslant 2K_0$, M in $\mathbb{R}$ and a smooth function g in $L^2(\nu_{\Lambda_L,M})$ such that $\langle g^2\rangle_{\nu_{\Lambda_L,M}} = 1$. Assume that $\theta(L)L^{-1}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g) < \delta\beta_0^2$. Then,*

$$\left(E_{\Lambda_L,M}\left[g^2; \frac{1}{L}\sum_{j=1}^{\ell}|I_j|G(m_j)\right]\right)^2 \leqslant \frac{\delta\theta(L)}{L}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g).$$

Proof. – Fix a density g^2 satisfying the assumptions. By the entropy inequality, the expectation in the statement of the lemma is bounded by

$$\frac{1}{\beta L}\log E_{\Lambda_L,M}\left[\exp\left\{\beta\sum_{j=1}^{\ell}|I_j|G(m_j)\right\}\right] + \frac{1}{\beta L}S_{\Lambda_L}(\nu_{\Lambda_L,M}, g) \tag{4.14}$$

for every $\beta > 0$. By Lemma 6.5 and our choice of K, L, the first term is bounded above by $\delta\beta$ for all $\beta < \beta_0$. The second one, by definition of θ, is bounded above by $(\theta(L)/\beta L)D_{\Lambda_L}(\nu_{\Lambda_L,M}, g)$. Therefore, (4.13) is less than or equal to

$$\delta\beta + \frac{\theta(L)}{\beta L}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g).$$

The value of β that minimizes this expression is

$$\beta_1^2 = \frac{\theta(L)}{\delta L}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g).$$

By hypothesis, $\beta_1 < \beta_0$ and we may therefore minimize in $\beta < \beta_0$ to obtain that the square of (4.13) is bounded above by

$$\frac{\delta\theta(L)}{L}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g),$$

which concludes the proof of the lemma. □

LEMMA 4.6. – *Fix $L \geqslant 2K \geqslant 2K_0$, M in $\mathbb{R}$ and a smooth function g in $L^2(\nu_{\Lambda_L,M})$ such that $\langle g^2\rangle_{\nu_{\Lambda_L,M}} = 1$. Assume that $\theta(L)L^{-1}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g) \geqslant \delta\beta_0^2$. Then, there exists a finite constant C such that*

$$\left(E_{\Lambda_L,M}\left[g^2; \frac{1}{L}\sum_{j=1}^{\ell}|I_j|G(m_j)\right]\right)^2 \leqslant \left\{CL + \frac{\delta\theta(L)}{L}\right\}D_{\Lambda_L}(\nu_{\Lambda_L,M}, g). \tag{4.15}$$

Proof. – The covariance $E_{\Lambda_L,M}[g^2; \sum_{1\leqslant j\leqslant \ell} |I_j| G(m_j)]$ is equal to the covariance of g^2 and $\sum_{1\leqslant j\leqslant \ell} |I_j| \widetilde{H}_K(m_j)$, where $\widetilde{H}_K(m_j) = E_{\Lambda_{I_j},M_j}[F']$. Since g^2 is a density with respect to $\nu_{\Lambda_L,M}$, by Schwarz inequality, the left hand side of (4.15) is bounded above by

$$2\left(\frac{1}{L}\sum_{j=1}^{\ell} |I_j| E_{\Lambda_L,M}\left[g^2(\widetilde{H}_K(m_j) - \widetilde{H}_K(m))\right]\right)^2 + 2\left(\frac{1}{L}\sum_{j=1}^{\ell} |I_j| E_{\Lambda_L,M}[\widetilde{H}_K(m_j) - \widetilde{H}_K(m)]\right)^2. \tag{4.16}$$

By Lemma 6.6, $\widetilde{H}_K$ is uniformly Lipschitz. In particular, since g^2 is a density, by Schwarz inequality the first term is bounded above by

$$\frac{C}{L}\sum_{j=1}^{\ell} |I_j| E_{\Lambda_L,M}\left[g^2[m_j - m]^2\right] \leqslant \frac{CK}{\ell L}\sum_{1\leqslant i\neq j\leqslant \ell} E_{\Lambda_L,M}[g^2[m_j - m_i]^2]$$

for some finite constant C because m is just the average of the densities m_i. By Lemma 4.7 below, each expectation is bounded by

$$C_1(K) + C_2(K)\left\{D_{I_i}(\nu_{I_i,M_i}, g) + D_{I_j}(\nu_{I_j,M_j}, g) + E_{\Lambda_L,M}\left[\left\{\frac{\partial g}{\partial \eta_{y_i}} - \frac{\partial g}{\partial \eta_{x_j}}\right\}^2\right]\right\},$$

where $C_2(K)$ is a finite constant and $C_1(K)$ is a constant that can be made as small as one wishes by letting $K \uparrow \infty$. Here we are assuming that the cubes I_j are ordered, that $i < j$ and that y_i is the rightmost site in I_i and x_j is the leftmost site in I_j. An elementary computation shows that the expectation in the previous formula is bounded above by $LD_{\Lambda_L}(\nu_{\Lambda_L,M}, g)$. Therefore, the first term in (4.16) is less than or equal to

$$C_1(K) + C_2(K) L D_{\Lambda_L}(\nu_{\Lambda_L,M}, g).$$

The second term in (4.16) is easy to estimate. Since $\widetilde{H}_K$ is uniformly Lipschitz, by Schwarz inequality, this term is bounded by

$$\frac{CK}{L}\sum_{j=1}^{\ell} E_{\Lambda_L,M}[(m_j - m)^2]$$

for some finite constant C. By Corollary 5.5, this term is bounded above by CK^{-1}. In conclusion, we proved that (4.16) is bounded above by

$$C_1(K) + C_2(K) L D_{\Lambda_L}(\nu_{\Lambda_L,M}, g),$$

where $C_1(K)$ is a constant that can be made as small as one wishes by letting $K \uparrow \infty$. In particular, choosing K large enough for $C_1(K) \leqslant \delta^2\beta_0^2$, by assumption, the previous term is less than or equal to

$$\left\{\frac{\delta\theta(L)}{L} + CL\right\} D_{\Lambda_L}(\nu_{\Lambda_L,M}, g)$$

for some finite constant C depending on δ. This concludes the proof of the lemma. □

Proposition 4.3 follows from the decomposition (4.10) and Lemmas 4.4–4.6.

We conclude this section with a technical result needed above. Consider the cube Λ_{2K}. Denote by m_i, $i = 1, 2$, the average spin over the first and second half: $m_1 = K^{-1}\sum_{1\leqslant x\leqslant K}\eta_x$, $m_2 = K^{-1}\sum_{K<x\leqslant 2K}\eta_x$.

LEMMA 4.7. – *There exist finite constants $C_1(K)$, $C_2(K)$ such that*

$$E_{\Lambda_{2K},M}\left[g^2(m_1 - m_2)^2\right] \leqslant C_1(K) + C_2(K) D_{\Lambda_{2K}}(\nu_{\Lambda_{2K},M}, g) \tag{4.17}$$

for all densities g^2 with respect to $\nu_{\Lambda_{2K},M}$. Moreover, $\lim_{K\to\infty} C_1(K) = 0$.

Proof. – By the entropy inequality and by definition of θ, the left hand side of (4.17) is bounded above by

$$\frac{1}{\beta}\log E_{\Lambda_{2K},M}\left[\exp\{\beta(m_1 - m_2)^2\}\right] + \frac{\theta(2K)}{\beta} D_{\Lambda_{2K}}(\nu_{\Lambda_{2K},M}, g).$$

We now recall that $\mathrm{e}^x \leqslant 1 + x + x^2\mathrm{e}^x$ for $x > 0$ and that $\log(1+x) \leqslant x$ to estimate the first term by

$$\frac{1}{\beta}\left\{4\beta E_{\Lambda_{2K},M}\left[(m_1 - m)^2\right] + 16\beta^2 E_{\Lambda_{2K},M}\left[(m_1 - m)^4\exp\{4\beta(m_1 - m)^2\}\right]\right\}$$

because $m_1 - m_2 = 2(m_1 - m)$. By Corollary 5.5, we may replace the expectation with respect to canonical measures by expectation with respect to grand canonical measures, paying the price of a finite constant. Since the grand canonical measures are product measures, by Schwarz inequality, the previous expression is bounded above by

$$\frac{C}{K} + C\beta E_{\nu_m}\left[(m_1 - m)^8\right]^{1/2} E_{\nu_m}\left[\exp\{8\beta(m_1 - m)^2\}\right]^{1/2}.$$

Since $\exp\{ax^2\}$ is a convex function for $a > 0$ and since ν_m is a product measure, this sum is less than or equal to

$$\frac{C}{K} + \frac{C\beta}{K^2} E_{\nu_m}\left[\exp\{8\beta(\eta_1 - m)^2\}\right]^{1/2}.$$

For β small enough, the previous expectation is bounded, uniformly in m. This proves the lemma. □

5. Local central limit theorem

We prove in this section some estimates that follow from the local central limit theorem and play a central role in the proof of the spectral gap and the logarithmic Sobolev inequality.

For λ in $\mathbb{R}$, denote by P_λ the probability measure on the product space $\mathbb{R}^{\mathbb{N}}$ that makes the coordinates $\{X_k,\ k \geqslant 1\}$ independent random variables with marginal density $Z(\lambda)^{-1}\exp\{\lambda x - V(x)\}$. Denote by E_λ expectation with respect to P_λ. Recall that $\gamma_1(\lambda)$, $\sigma(\lambda)^2$, $\{\gamma_k(\lambda),\ k \geqslant 3\}$ stand for the expectation, the variance and the kth truncated moment of the coordinate variables under the distribution P_λ:

$$\gamma_1(\lambda) = E_\lambda[X_1], \qquad \sigma(\lambda)^2 = E_\lambda\big[(X_1 - \gamma_1(\lambda))^2\big], \qquad \gamma_k(\lambda) = E_\lambda\big[(X_1 - \gamma_1(\lambda))^k\big].$$

For $N \geqslant 1$, denote by $f_{\lambda,N}$ the density of the random variable $(\sigma(\lambda)^2 N)^{-1/2}\sum_{1\leqslant j\leqslant N}(X_j - \gamma_1(\lambda))$.

LEMMA 5.1. – *Assume that $\|F\|_\infty < \infty$. Then, there exist finite constants $\{C_j,\ j \geqslant 1\}$, depending only on j and $\|F\|_\infty$, such that*

$$0 < C_1^{-1} < \sigma(\lambda)^2 < C_1, \qquad 0 < C_j^{-1} < \gamma_{2j}(\lambda) < C_j$$

for all λ in $\mathbb{R}$.

Proof. – We first claim that $Z(\lambda)\exp\{-\lambda^2/2\}$ is bounded above an below by finite positive constants. Indeed, by definition,

$$Z(\lambda) = e^{\lambda^2/2}\int da\, e^{-(1/2)(a-\lambda)^2 - F(a)} = e^{\lambda^2/2}\int da\, e^{-(1/2)a^2 - F_\lambda(a)},$$

where $F_\lambda(a) = F(a+\lambda)$. Since F is absolutely bounded, this expression is bounded below and above by $\sqrt{2\pi}\exp\{\lambda^2/2\}\exp\{\pm\|F\|_\infty\}$, proving the claim.

We now claim that $|\gamma_1(\lambda) - \lambda|$ is bounded by $\|F\|_\infty\exp\{2\|F\|_\infty\}$. Indeed, by definition, the difference $\gamma_1(\lambda) - \lambda$ is equal to

$$\frac{1}{Z(\lambda)}\int_{\mathbb{R}}(x-\lambda)e^{\lambda x - (x^2/2) - F(x)}\,dx.$$

Changing variables, we may rewrite this integral as

$$\int_{\mathbb{R}} x e^{-(x^2/2) - F_\lambda(x)}\,dx \Big/ \int_{\mathbb{R}} e^{-(x^2/2) - F_\lambda(x)}\,dx.$$

Since $\int dx\, x\exp\{-(1/2)x^2\}$ vanishes, by Schwarz inequality, the absolute value of this expression is bounded above by

$$e^{\|F\|_\infty}\frac{1}{\sqrt{2\pi}}\bigg|\int_{\mathbb{R}} x e^{-(1/2)x^2}\big(e^{-F_\lambda(x)} - 1\big)\,dx\bigg| \leqslant \|F\|_\infty e^{2\|F\|_\infty},$$

which proves the claim.

We now prove a lower bound for $\sigma(\lambda)^2$. The same ideas permit to derive an upper bound for $\sigma(\lambda)^2$ or upper and lower bounds for the truncated moments $\{\gamma_{2j}(\lambda),\ j \geqslant 2\}$. A change of variables and the estimate on $Z(\lambda)\exp\{-\lambda^2/2\}$ gives that

$$\sigma(\lambda)^2 \geqslant \mathrm{e}^{-2\|F\|_\infty} \frac{1}{\sqrt{2\pi}} \int da\, [a + \lambda - \gamma_1(\lambda)]^2 \mathrm{e}^{-a^2/2}$$
$$\geqslant \mathrm{e}^{-2\|F\|_\infty} \inf_{\beta,\, |\beta| \leqslant \|\lambda - \gamma_1(\lambda)\|_\infty} \frac{1}{\sqrt{2\pi}} \int da\, [a + \beta]^2 \mathrm{e}^{-a^2/2} \geqslant C_1 > 0,$$

where C_1 depends only on $\|F\|_\infty$. This concludes the proof of the lemma. □

It follows from this lemma that

$$\sup_{\lambda \in \mathbb{R}} \left| \frac{\gamma_j(\lambda)}{\sigma(\lambda)^j} \right| \leqslant \widetilde{C}_j \tag{5.1}$$

for all $j \geqslant 3$, which is the estimate needed in order to prove the uniform local central limit theorem.

THEOREM 5.2. – *Assume that $\|F\|_\infty < \infty$. There exists $N_0 \geqslant 1$ and a finite constant C depending only on $\|F\|_\infty$ such that*

$$\left| f_{\lambda,N}(x) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \left\{ 1 - \frac{\gamma_3(\lambda) x}{6\sigma(\lambda)^3 N^{1/2}} \right\} \right| \leqslant \frac{C}{N}$$

for all $N \geqslant N_0$, x in $\mathbb{R}$ and λ in $\mathbb{R}$.

For a fixed parameter λ this is just the usual statement of the local central limit theorem for i.i.d. random variables with finite fourth moments. The important point here is the uniformity over the parameter λ. This uniformity can be obtained in virtue of (5.1) and the estimates presented in the Lemma 5.1.

The local central limit theorem gives asymptotic expansions of the expectation of a function with respect to a canonical measure. This is the content of the next result.

COROLLARY 5.3. – *Fix $\ell \geqslant 1$ and fix a function $G : \mathbb{R}^\ell \to \mathbb{R}$. There exist $N_0 \geqslant 1$ and a finite constant C depending only on $\|F\|_\infty$ such that for all $N \geqslant N_0$ and all M in $\mathbb{R}$*

$$\left| E_{\Lambda_N,M}[G] - E_{\nu_m}[G] \right| \leqslant \frac{C\ell}{|\Lambda_N|} \|G\|_\infty \quad \textit{if } G \textit{ is bounded and}$$
$$\left| E_{\Lambda_N,M}[G] - E_{\nu_m}[G] \right| \leqslant \frac{C\ell}{|\Lambda_N|} \sqrt{E_{\nu_m}[G; G]}.$$

In these formulas, $m = M/|\Lambda_N|$.

The proof is elementary (cf. Corollary A2.1.4 in [10]). Of course, by changing the value of the constant C, the first inequality remains valid for all values of $N \geqslant \ell$.

COROLLARY 5.4. – *Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth bounded function and let $G_{L,M} = G - E_{\nu_{\Lambda_L,M}}[G]$. There exists a finite constant C_0, depending only on $\|F\|_\infty$ such that*

$$E_{\Lambda_L,M}\left[\left(\frac{1}{L} \sum_{x=1}^{L} G_{L,M}(\eta_x) \right)^2 \right] \leqslant C_0 \frac{\|G\|_\infty^2}{L}$$

for all $L \geqslant 1$ and M in $\mathbb{R}$.

Proof. – The variance is equal to

$$\frac{1}{L}E_{\Lambda_L,M}\big[(G_{L,M}(\eta_1))^2\big] + \left(1-\frac{1}{L}\right)E_{\Lambda_L,M}[G_{L,M}(\eta_1)G_{L,M}(\eta_2)].$$

The first expression is bounded by $4\|G\|_\infty^2 L^{-1}$ for all $L \geqslant 1$ and $M \in \mathbb{R}$. The second one, by definition of $G_{L,M}$ is equal to

$$\left(1-\frac{1}{L}\right)\big\{E_{\Lambda_L,M}[G(\eta_1)G(\eta_2)] - E_{\Lambda_L,M}[G(\eta_L)]^2\big\}.$$

By Corollary 5.3, since ν_α is a product measure, the first term of the previous expression is equal to $E_\alpha[G(\eta_L)]^2 \pm CL^{-1}\|G\|_\infty$, where C is a finite constant depending only on $\|F\|_\infty$. By the same result, the second term is equal to $E_\alpha[G(\eta_L)]^2 \pm CL^{-1}\|G\|_\infty^2$, which concludes the proof. □

For $1 \leqslant K < L$, denote by $\nu^K_{\Lambda_L,M}$ the marginal on $\mathbb{R}^{\Lambda_K}$ of the canonical measure $\nu_{\Lambda_L,K}$. An elementary computation shows that $\nu^K_{\Lambda_L,M}$ is absolutely continuous with respect to the Lebesgue measure and that its Radon–Nikodym derivative $R_{K,L,M}(\mathbf{x}_K)$ is given by

$$L^{1/2}(L-K)^{-1/2} g_\lambda^K(\mathbf{x}_K) f_{\lambda,L-K}\left(\big(\sigma\sqrt{L-K}\,\big)^{-1}\sum_{1\leqslant i\leqslant K}[\gamma_1 - x_i]\right) f_{\lambda,L}(0)^{-1}\, d\mathbf{x}_K,$$

where $\mathbf{x}_K = (x_1,\dots,x_K)$, g_λ^K stands for the density $Z(\lambda)^{-K}\exp\{\sum_{1\leqslant i\leqslant K}\lambda x_i - V(x_i)\}$ and $\lambda = \Phi(M/L)$. The next result shows that the ratio $R^K_{L,M}(\mathbf{x}_K)/g^K_\lambda(\mathbf{x}_K)$ is bounded above, uniformly over λ, provided K/L is bounded away from 1. [4] has obtained the same result in the case of lattice gases under strong mixing assumptions.

COROLLARY 5.5. – *There exists a finite constant C_0 depending only on $\|F\|_\infty$, such that*

$$\frac{R_{K,L,M}(\mathbf{x}_K)}{g_\lambda^K(\mathbf{x}_K)} \leqslant C_0$$

for all $L/2 \geqslant K \geqslant 1$ and $\mathbf{x}_K$ in $\mathbb{R}^{\Lambda_K}$. In this formula, $\lambda = \Phi(M/L)$. In particular, if $K \leqslant L/2$, for any local function $H : \mathbb{R}^{\Lambda_K} \to \mathbb{R}$,

$$E_{\Lambda_L,M}[H(\eta_1,\dots,\eta_K)] \leqslant C_0 E_{\nu_{M/|\Lambda_L|}}[\,|H|\,]. \tag{5.2}$$

Proof. – In view of the explicit formula for the density $R_{K,L,M}$ and since $K \leqslant L/2$, we only have to show that

$$\frac{f_{\lambda,L-K}(\{\sigma\sqrt{L-K}\}^{-1}\sum_{1\leqslant i\leqslant K}[\gamma_1 - x_i])}{f_{\lambda,L}(0)} \leqslant C_0. \tag{5.3}$$

We prove separately that the numerator is bounded and that the denominator is bounded below by a strictly positive constant. Consider, for instance the denominator. For L large enough, the lower bound follows from Theorem 5.2. For L small, it follows by inspection. The same argument applies to the numerator with $L-K$ in place of L. This proves the corollary since (5.2) follows at once form (5.3). □

Theorem 5.2 and its corollaries permit to estimate expectation with respect to a canonical measure $\mu_{\Lambda_L,M}$, provided L is large. The next result provides an estimate for small values of L. The important point in this result is once again the uniformity over the parameter λ. Denote by $\tilde{f}_{\lambda,N}$ the density of the random variable $N^{-1/2}\sum_{1\leqslant j\leqslant N}\{X_j-\lambda\}$ under the measure P_λ. Note that we are not renormalizing by $\sigma(\lambda)$ and that we are subtracting λ instead of $\gamma_1(\lambda)$.

LEMMA 5.6. – *There exists a positive and finite constant C_1, depending only on $\|F\|_\infty$, such that*

$$C_1^{-N}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\leqslant \tilde{f}_{\lambda,N}(x)\leqslant C_1^N\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$$

for every λ in $\mathbb{R}$.

Proof. – The proof is elementary. We present the upper bound. For $N\geqslant 1$, let $g_{\lambda,N}$ be the density with respect to the Lebesgue measure of the random variable $\sum_{1\leqslant j\leqslant N}X_j$ under P_λ. By the estimate on $Z(\lambda)\exp\{\lambda^2/2\}$ obtained in the proof of Lemma 5.1 and by the explicit formula for $g_{\lambda,N}$, we have that $g_{\lambda,N}(x)$ is bounded by

$$C_1^N\mathrm{e}^{\lambda x-(\lambda^2N/2)}\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^{N-1}}dx_1\ldots dx_{N-1}\,\exp\left\{-\frac{1}{2}\sum_{i=1}^{N-1}x_i^2-\frac{1}{2}\left(x-\sum_{i=1}^{N-1}x_i\right)^2\right\}$$

for some constant C_1 depending only on $\|F\|_\infty$. Since the integral with the renormalization factor in front is equal to $(2\pi)^{-1/2}\exp\{-x^2/2N\}$, the previous expression is equal to $C_1^N(2\pi)^{-1/2}\exp\{-(x-\lambda N)^2/2N\}$. To conclude the proof of the lemma, it remains to express $\tilde{f}_{\lambda,N}$ in terms of $g_{\lambda,N}(x)$. □

The same argument shows that $g_\lambda(x)=Z(\lambda)^{-1}\exp\{\lambda x-V(x)\}$ is bounded above and below by a Gaussian density. More precisely, there exists a finite, strictly positive constant C_0 depending only on $\|F\|_\infty$, such that

$$C_0\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(x-\lambda)^2/2}\leqslant g_\lambda(x)\leqslant C_0^{-1}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(x-\lambda)^2/2}\tag{5.4}$$

for every λ in $\mathbb{R}$.

LEMMA 5.7. – *There exists $\beta_0>0$ and a finite constant C_0 such that*

$$E_{\nu_\alpha}\left[\exp\{\beta_0|\Lambda_L|\{m_{\Lambda_L}-\alpha\}^2\}\right]\leqslant C_0$$

for every α in $\mathbb{R}$ and $L\geqslant 1$. In this formula, $m_\Lambda=|\Lambda_L|^{-1}\sum_{x\in\Lambda_L}\eta_x$.

Proof. – For small values of L this statement is a straightforward consequence of the previous lemma, the fact that $\gamma_1(\lambda)-\lambda$ is absolutely bounded, proved in Lemma 5.2, and the fact that the statement holds for Gaussian distributions.

For large values of L, with the notation introduced in the beginning of this section, the expectation can be written as

$$\int_{\mathbb{R}} e^{\beta_0\sigma(\lambda)^2x^2} f_{\lambda,L}(x)\,dx$$

for some appropriate choice of λ. Notice that the local central limit theorem, stated in Theorem 5.1, gives a good bound only for small values of x. The idea is therefore to replace in the previous formula λ by a variable μ which makes x a typical value. By (4.3) or a direct computation,

$$f_{\lambda,L}(x) = \frac{\sigma_\lambda}{\sigma_\mu}\left(\frac{Z_\mu}{Z_\lambda}\right)^L e^{(\lambda-\mu)[x\sigma_\lambda\sqrt{L}+L\gamma_1(\lambda)]} f_{\mu,L}\left(\frac{x\sigma_\lambda}{\sigma_\mu} + \frac{\sqrt{L}(\gamma_1(\lambda)-\gamma_1(\mu))}{\sigma_\mu}\right).$$

Choose μ for the expression inside $f_{\mu,L}$ to be small (in order to be able to use the local central limit estimate):

$$x\sigma_\lambda\sqrt{L} = L[\gamma_1(\mu) - \gamma_1(\lambda)].$$

With this choice, since by Theorem 5.1 $C_1^{-1} \leqslant f_\mu(0) \leqslant C_1$ for some universal constant C_1, and since by Lemma 5.2 σ_λ is bounded,

$$f_{\lambda,L}(x) \sim \exp\bigl\{L\log\{Z_\mu/Z_\lambda\} + (\lambda-\mu)[x\sigma_\lambda\sqrt{L} + L\gamma_1(\lambda)]\bigr\},$$

where $\sim$ means that the left hand side is bounded above and below by the right hand side multiplied by finite positive constants. The expression inside the exponential vanishes at $x = 0$. It is also not difficult to show that it is strictly concave in x (cf. computation right after (4.3)). In particular,

$$f_{\lambda,L}(x) \sim e^{-C_2x^2}$$

for some finite constant C_2 and we are back to the Gaussian case.

6. Large deviations estimates

Fix a differentiable function $R:\mathbb{R}\to\mathbb{R}$ with bounded derivative: $\|R'\|_\infty < \infty$. Let $R_\alpha^g(a) = R(a) - \langle R\rangle_{\nu_\alpha}$ in the case of grand canonical measures and let $R_\alpha(a) = R(a) - \langle R\rangle_{\nu_{\Lambda_L,M}}$ in the case of canonical measures. Notice that $R_\alpha(\eta_x) - R_\alpha^g(\eta_x) = E_{\nu_\alpha}[R(\eta_x)] - E_{\Lambda_L,M}[R(\eta_x)]$. It follows from Corollary 5.3 that

$$\bigl|E_{\nu_\alpha}[R(\eta_x)] - E_{\Lambda_L,M}[R(\eta_x)]\bigr| \leqslant \frac{C\|R'\|_\infty}{|\Lambda_L|} \tag{6.1}$$

for some finite constant C depending only on $\|F\|_\infty$ because $E_{\nu_\alpha}[R;R] \leqslant E_{\nu_\alpha}[\{R(\eta_1) - R(\alpha)\}^2] \leqslant \|R'\|_\infty^2\sigma(\Phi(\alpha))^2$.

We claim that there exists a finite constant C_0 depending only on $\|F\|_\infty$ for which

$$|R_\alpha^g(a)| \leqslant C_0\|R'\|_\infty(1+|a-\alpha|), \qquad |R_\alpha(a)| \leqslant C_0\|R'\|_\infty(1+|a-\alpha|) \tag{6.2}$$

for all a, α in $\mathbb{R}$ (in the canonical case for all $L \geqslant 1$, M in $\mathbb{R}$). Consider first the grand canonical case. Notice that

$$|R^g_\alpha(a)| \leqslant E_{\nu_\alpha}[|R(a) - R(\eta_1)|] \leqslant \|R'\|_\infty E_{\nu_\alpha}[|a-\eta_1|] \\ \leqslant \|R'\|_\infty \{|a-\alpha| + E_{\nu_\alpha}[(\eta_1-\alpha)^2]^{1/2}\}.$$

By Lemma 5.1 the second term inside braces in the last expression is bounded above by some finite constant C_1 that depends on $\|F\|_\infty$ only. This proves the claim in the grand canonical case. The same arguments apply to the canonical case provide we show that $E_{\Lambda_L,M}[(\eta_1-\alpha)^2]$ is uniformly bounded. But this is part of the content of Corollary 5.5.

LEMMA 6.1. – *Fix a differentiable function $R:\mathbb{R}\to\mathbb{R}$ with bounded derivative and $L \geqslant 2$. There exists a constant C, depending only on $\|F\|_\infty$, such that*

$$\frac{1}{\beta|\Lambda_L|} \log \int \exp\left\{\beta \sum_{x\in\Lambda_L} R_\alpha(\eta_x)\right\} d\nu_{\Lambda_L,M} \leqslant C\|R'\|^2_\infty \beta \tag{6.3}$$

for all $\beta > 0$ and all M in $\mathbb{R}$. Here $R_\alpha = R - \langle R\rangle_{\nu_{\Lambda_L,M}}$.

Proof. – We first prove this result for the grand canonical measure in place of the canonical measure. In this case we replace R_α by R^g_α and we only need to show that

$$\frac{1}{\beta} \log \int \exp\{\beta R^g_\alpha(\eta_1)\}\, d\nu_\alpha \leqslant C\|R'\|^2_\infty \beta \tag{6.4}$$

for all $\beta > 0$ because ν_α is a product measure.

We consider first the case of β small. By the spectral gap for the Glauber dynamics (Lemma 3.1), there exists a universal constant C_0 such that

$$\langle f^2\rangle_{\nu_\alpha} - \langle f\rangle^2_{\nu_\alpha} \leqslant C_0 \langle (\partial_{\eta_1} f)^2\rangle_{\nu_\alpha}$$

for all smooth functions f in $L^2(\nu^1_\alpha)$. Let $C_1 = C_0\|R'\|^2_\infty$ and assume that $\beta < C_1^{-1/2}$. Applying this inequality to the function $f = \exp\{(\beta/2)\, R^g_\alpha\}$, we obtain that

$$E_{\nu_\alpha}\big[e^{\beta R^g_\alpha}\big] \leqslant \big\{E_{\nu_\alpha}\big[e^{(\beta/2)R^g_\alpha}\big]\big\}^2 + C_0\left(\frac{\beta}{2}\right)^2 \|R'\|^2_\infty E_{\nu_\alpha}\big[e^{\beta R^g_\alpha}\big]$$

so that

$$E_{\nu_\alpha}\big[e^{\beta R^g_\alpha}\big] \leqslant \frac{1}{1 - C_0\|R'\|^2_\infty(\beta/2)^2} \big\{E_{\nu_\alpha}\big[e^{(\beta/2)R^g_\alpha}\big]\big\}^2 \\ \leqslant e^{(1/2)C_0\|R'\|^2_\infty\beta^2} \big\{E_{\nu_\alpha}\big[e^{(\beta/2)R^g_\alpha}\big]\big\}^2$$

because $(1-x)^{-1} \leqslant e^{2x}$ for $0 \leqslant x < 1/2$. Iterating this estimate $n-1$ times we obtain that

$$E_{\nu_\alpha}\big[e^{\beta R^g_\alpha}\big] \leqslant \exp\left\{C_1\beta^2 \sum_{j=1}^n 2^{-j}\right\} \big\{E_{\nu_\alpha}\big[e^{(\beta/2^n)R^g_\alpha}\big]\big\}^{2^n}.$$

The exponential is obviously bounded by $\exp\{C_1\beta^2\}$. On the other hand, we claim that

$$\lim_{n\to\infty} n \log E_{\nu_\alpha}\left[e^{(1/n)R_\alpha^g}\right] = 0, \tag{6.5}$$

showing that the left hand side of (6.4) is bounded above by $C_1\beta = C_0\beta\|R'\|_\infty^2$ provided $\beta < C_1^{-1/2}$.

To prove (6.5), just notice that $\exp\{(1/n)R_\alpha^g\}$ is bounded above by $1 + (1/n)R_\alpha^g + (1/n^2)(R_\alpha^g)^2 \exp\{(1/n)|R_\alpha^g|\}$. Since $\log(1+x) \leqslant x$ and since R_α^g has mean zero with respect to ν_α, we obtain that

$$n \log E_{\nu_\alpha}\left[e^{(1/n)R_\alpha^g}\right] \leqslant \frac{1}{n} E_{\nu_\alpha}\left[(R_\alpha^g)^2 \exp\{(1/n)|R_\alpha^g|\}\right].$$

By (6.2), the right hand side is bounded above by

$$\frac{C}{n} E_{\nu_\alpha}\left[\{1 + (\eta_1 - \alpha)^2\} \exp\{C|\eta_1 - \alpha|/n\}\right]$$

for some finite constant C depending only on $\|F\|_\infty$, $\|R'\|_\infty$. The expectation is bounded for all $n \geqslant 1$ because ν_α has Gaussian tails. This proves (6.4) for $\beta < C_1^{-1/2}$.

We now turn to the case of large β, which is simpler. Assume that $\beta \geqslant C_1^{-1/2}$. It follows from (6.2) that the left hand side of (6.4) is bounded above by

$$C_2\|R'\|_\infty + \beta^{-1} \log E_{\nu_\alpha}\left[e^{\beta\|R'\|_\infty C_2|\eta_1-\alpha|}\right]. \tag{6.6}$$

Since $e^{|x|} \leqslant e^x + e^{-x}$, we need only to estimate $E_{\nu_\alpha}[\exp\{\beta\|R'\|_\infty C_2(\eta_1 - \alpha)\}]$ for β and $-\beta$. Recall the definition of the partition function Z given in Eq. (2.1). The logarithm of the previous expectation is equal to $\log Z(\Phi(\alpha) + \beta\|R'\|_\infty C_2) - \log Z(\Phi(\alpha)) - \beta\|R'\|_\infty C_2\alpha$. An elementary computation gives that $(\log Z)'(\Phi(\alpha)) = \alpha$ so that the previous difference can be written as

$$\log Z\bigl(\Phi(\alpha) + \beta\|R'\|_\infty C_2\bigr) - \log Z\bigl(\Phi(\alpha)\bigr) - (\log Z)'\bigl(\Phi(\alpha)\bigr)\beta\|R'\|_\infty C_2.$$

By Taylor's expansion, this difference is bounded by $(1/2)(\beta\|R'\|_\infty C_2)^2(\log Z)''(\lambda)$ for some λ between $\Phi(\alpha)$ and $\Phi(\alpha) + \beta\|R'\|_\infty C_2$. Since $(\log Z)''(\lambda) = \sigma^2(\lambda)$ and since, by Lemma 5.1, $\sigma^2(\lambda)$ is bounded uniformly in λ, we have that

$$\log E_{\nu_\alpha}\left[\exp\{\beta\|R'\|_\infty C_2(\eta_1 - \alpha)\}\right] \leqslant C\|R'\|_\infty^2\beta^2$$

for some constant depending only on $\|F\|_\infty$. Since $\log\{a + b\} \leqslant \log 2 + \max\{\log a, \log b\}$, (6.6) is bounded above by

$$C_2\|R'\|_\infty + \frac{\log 2}{\beta} + C_3\|R'\|_\infty^2\beta,$$

which is obviously bounded above by $C_4\|R'\|_\infty^2\beta$ because $\beta \geqslant C_1^{-1/2}$. This concludes the proof of the lemma in the case of the grand canonical measure.

We now turn to the canonical measure. We need to consider two cases. Assume first that $\beta\|R'\|_\infty \leqslant |\Lambda_L|^{-1}$. By Schwarz inequality, the left hand side of (6.3) is bounded above by

$$\frac{1}{\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1\leqslant x\leqslant L/2}R_\alpha(\eta_x)\Big\}\,d\nu_{\Lambda_L,M}.$$

The difference is that we are now summing only over one half of the cube and that we had to pay a factor 2 in the exponential to do it. Since $\mathrm{e}^x\leqslant 1+x+x^2\mathrm{e}^{|x|}$, since $\log(1+x)\leqslant x$ and since R_α has mean zero, the previous expression is bounded above by

$$\frac{4\beta}{|\Lambda_L|}\int\Big\{\sum_{1\leqslant x\leqslant L/2}R_\alpha(\eta_x)\Big\}^2\exp\Big\{2\beta\Big|\sum_{1\leqslant x\leqslant L/2}R_\alpha(\eta_x)\Big|\Big\}\,d\nu_{\Lambda_L,M}. \tag{6.7}$$

Since $\mathrm{e}^{|x|}\leqslant \mathrm{e}^x+\mathrm{e}^{-x}$, we may remove the absolute value in the exponential, provide we estimate the expression for R_α, as well as for $-R_\alpha$. Moreover, by Corollary 5.5, we may replace the canonical measure by the grand canonical one paying the price of a finite constant and turning R_α into a *non*-mean-zero function. At this point, we need to estimate

$$\frac{C_0\beta}{|\Lambda_L|}\int\Big\{\sum_{1\leqslant x\leqslant L/2}R_\alpha(\eta_x)\Big\}^2\exp\Big\{2\beta\sum_{1\leqslant x\leqslant L/2}R_\alpha(\eta_x)\Big\}\,d\nu_\alpha,$$

with $\alpha=M/|\Lambda_L|$. Since ν_α is a product measure, expanding the square, we get that the previous integral is less than or equal to

$$\begin{aligned}&C_0\beta E_{\nu_\alpha}\big[R_\alpha(\eta_1)^2\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]^{(L/2)-1}\\&\quad+C_0\beta|\Lambda_L|\big(E_{\nu_\alpha}\big[R_\alpha(\eta_1)\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]\big)^2E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]^{(L/2)-2}.\end{aligned} \tag{6.8}$$

There are three different types of terms in the previous formula and we estimate them separately. We first examine the exponentials. By (6.1),

$$E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]^{(L/2)}\leqslant \mathrm{e}^{C\beta\|R'\|_\infty}E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big]^{(L/2)}.$$

On the range considered $\beta\|R'\|_\infty\leqslant 1$, so that the exponential term is less than some finite constant C. On the other hand, since R^g_α has mean zero with respect to ν_α, since $\mathrm{e}^x\leqslant 1+x+x^2\mathrm{e}^{|x|}$, since by (6.2) $|R^g_\alpha(a)|\leqslant C_0\|R'\|_\infty[1+|a-\alpha|]$ and since $\beta^2\|R'\|^2_\infty\leqslant 1$,

$$E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big]\leqslant 1+C_0\beta^2\|R'\|^2_\infty.$$

Here we took advantage of the fact that there exists some finite constant C_2 depending only on $\|F\|_\infty$ such that

$$E_{\nu_\alpha}\big[\{1+|\eta_1-\alpha|^2\}\mathrm{e}^{2|\eta_1-\alpha|}\big]\leqslant C_2$$

for all α in $\mathbb{R}$ because ν_α has uniform Gaussian tails. Since $1+x\leqslant \mathrm{e}^x$,

$$\big(E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]\big)^L\leqslant\exp\{C_0\beta^2\|R'\|^2_\infty L\}\leqslant C_2$$

because $\beta^2\|R'\|^2_\infty\leqslant L^{-1}$.

We now turn to the remaining expectations in (6.8). By (6.1),

$$E_{\nu_\alpha}\big[R_\alpha(\eta_1)^2 \mathrm{e}^{2\beta R_\alpha(\eta_1)}\big] \leqslant C E_{\nu_\alpha}\big[R_\alpha(\eta_1)^2 \mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big]$$

because $\beta\|R'\|_\infty \leqslant 1$. The same estimate (6.1) gives that the previous expression is less than or equal to

$$\frac{C\|R'\|^2_\infty}{|\Lambda_L|^2} E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big] + C E_{\nu_\alpha}\big[R^g_\alpha(\eta_1)^2 \mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big].$$

By (6.2), $|R^g_\alpha(\eta_1)| \leqslant C_0\|R'\|_\infty(1+|\eta_1-\alpha|)$. The previous sum is thus bounded by

$$C\|R'\|^2_\infty + C\|R'\|^2_\infty E_{\nu_\alpha}\big[\{1+|\eta_1-\alpha|\}^2 \mathrm{e}^{2C_0|\eta_1-\alpha|}\big].$$

This expression is less than $C\|R'\|^2_\infty$ because ν_α has uniform exponential tails.

It remains to estimate

$$|\Lambda_L|\big(E_{\nu_\alpha}\big[R_\alpha(\eta_1)\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]\big)^2.$$

As before, we may replace R_α by R^g_α in the exponential. After this replacement, applying (6.1), we bound the previous expression by

$$C|\Lambda_L|\big(E_{\nu_\alpha}\big[R^g_\alpha(\eta_1)\mathrm{e}^{2\beta R_\alpha(\eta_1)}\big]\big)^2 + C\frac{\|R'\|^2_\infty}{|\Lambda_L|}\big(E_{\nu_\alpha}\big[\mathrm{e}^{2\beta R^g_\alpha(\eta_1)}\big]\big)^2.$$

The second term is seen to be less than or equal to $C\|R'\|^2_\infty/|\Lambda_L|$, while the first, since $a\mathrm{e}^b \leqslant a+|ab|\mathrm{e}^{|b|}$ and since R^g_α has mean zero, is bounded by

$$\begin{aligned} &C|\Lambda_L|\beta^2\big(E_{\nu_\alpha}\big[R^g_\alpha(\eta_1)^2\mathrm{e}^{2\beta|R^g_\alpha(\eta_1)|}\big]\big)^2 \\ &\quad\leqslant C|\Lambda_L|\beta^2\|R'\|^4_\infty\big(E_{\nu_\alpha}\big[\{1+|\eta_1-\alpha|^2\}\mathrm{e}^{2C_0|\eta_1-\alpha|}\big]\big)^2. \end{aligned}$$

This expression is bounded by $C\|R'\|^2_\infty$ because ν_α has uniform exponential tails and because $\beta^2\|R'\|^2_\infty \leqslant |\Lambda_L|^{-1}$. This proves the lemma in the case of small β.

We now turn to the case of large β. Assume that $\beta^2\|R'\|^2_\infty > |\Lambda_L|^{-1}$. We first replace R_α by R^g_α. By (6.1), the left hand side of (6.3) is bounded above by

$$\frac{1}{\beta|\Lambda_L|}\log\int\exp\Big\{\beta\sum_{x\in\Lambda_L}R^g_\alpha(\eta_x)\Big\}\,d\nu_{\Lambda_L,M} + \frac{C_0\|R'\|_\infty}{|\Lambda_L|}.$$

Since $|\Lambda_L|^{-2} \leqslant |\Lambda_L|^{-1} < \beta^2\|R'\|^2_\infty$, $|\Lambda_L|^{-1} \leqslant \beta\|R'\|_\infty$. In particular, the second term is less than or equal to $C_0\beta\|R'\|^2_\infty$.

It remains to estimate the first term. By Schwarz inequality, this expression is bounded above by

$$\frac{1}{\beta|\Lambda_L|}\log\int\exp\Big\{2\beta\sum_{1\leqslant x\leqslant L/2}R^g_\alpha(\eta_x)\Big\}\,d\nu_{\Lambda_L,M}.$$

By Corollary 5.5, this expression is less than or equal to

$$\frac{\log C}{\beta|\Lambda_L|} + \frac{1}{\beta|\Lambda_L|}\log\int \exp\Big\{2\beta \sum_{1\leqslant x\leqslant L/2} R^g_\alpha(\eta_x)\Big\}\, d\nu_\alpha,$$

where $\alpha = M/|\Lambda_L|$. Since $\beta^2 > C_1\|R'\|_\infty^{-2}|\Lambda_L|^{-1}$, the first term is bounded by $C\beta\|R'\|^2_\infty$. It remains to consider the second one which is equal to

$$\frac{1}{2\beta}\log\int \exp\{2\beta R^g_\alpha(\eta_1)\}\, d\nu_\alpha \tag{6.9}$$

because ν_α is a product measure. This expression is just (6.4) and we proved in the first part of the lemma that it is bounded by $C\beta\|R'\|^2_\infty$. This concludes the proof. □

The same proof gives the following estimate that we state for further use.

LEMMA 6.2. – *Fix a differentiable function $R:\mathbb{R}\to\mathbb{R}$ with bounded derivative: $\|R'\|_\infty < \infty$ and $L \geqslant 2$. There exists a constant C, depending only on $\|F\|_\infty$, such that*

$$\frac{1}{\beta}\log\int \exp\{\beta R_\alpha(\eta_1)\}\, d\nu_{\Lambda_L,M} \leqslant C\|R'\|^2_\infty\beta$$

for all $\beta > 0$ and all M in $\mathbb{R}$.

Lemma 6.1 provides an estimate, uniform over the charge M, on the expectation of $|\Lambda_L|^{-1}\sum_{x\in\Lambda_L} R_\alpha(\eta_x)$ with respect to some measure $f\, d\nu_{\Lambda_L,M}$ in terms of the entropy of this measure.

COROLLARY 6.3. – *Fix $L \geqslant 2$, M in $\mathbb{R}$, a differentiable function $R:\mathbb{R}\to\mathbb{R}$ with bounded derivative and a density f with respect to $\nu_{\Lambda_L,M}$. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\left(\int\Big\{\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} R_\alpha(\eta_x)\Big\} f\, d\nu_{\Lambda_L,M}\right)^2 \leqslant C_0\frac{\|R'\|^2_\infty}{|\Lambda_L|} S_{\Lambda_L}\big(\nu_{\Lambda_L,M},\sqrt{f}\,\big).$$

Proof. – By the entropy inequality, the integral on the left hand side of the statement of the lemma is bounded above by

$$\frac{1}{\beta|\Lambda_L|}\log\int \exp\Big\{\beta\sum_{x\in\Lambda_L} R_\alpha(\eta_x)\Big\}\, d\nu_{\Lambda_L,M} + \frac{S_{\Lambda_L}(\nu_{\Lambda_L,M},\sqrt{f})}{\beta|\Lambda_L|}$$

for all $\beta > 0$. By Lemma 6.1, the first term is bounded above by $C_0\|R'\|^2_\infty\beta$ for some finite constant depending only on $\|F\|_\infty$. Minimizing in β we conclude the proof of the lemma. □

Lemma 6.2 provides a similar estimate in the case of a one-site function:

COROLLARY 6.4. – *Fix $L \geqslant 2$, M in $\mathbb{R}$, a differentiable function $R:\mathbb{R}\to\mathbb{R}$ with bounded derivative and a density f with respect to $\nu^1_{\Lambda_L,M}$. There exists a constant C_0,* *depending only on* $\|F\|_\infty$*, such that*

$$\left(\int R_\alpha(\eta_1)\, f(\eta_1)\, d\nu^1_{\Lambda_L,M}\right)^2 \leqslant C_0 \|R'\|^2_\infty S_{\{0\}}\big(\nu^1_{\Lambda_L,M}, \sqrt{f}\big).$$

The proof is the same as the one of Corollary 6.3.

Fix $K \geqslant 1$, $L \geqslant K^2$ and divide the interval $\{1,\dots,L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a \rfloor$ represents the integer part of a. Denote by I_j the jth interval, by M_j the total spin on I_j: $M_j = \sum_{x\in I_j} \eta_x$ and by E_{I_j,M_j} the expectation with respect to the canonical measure ν_{I_j,M_j}. Let m, m_j stand for M/L, $M_j/|I_j|$, respectively and let $G(m_j) = E_{I_j,M_j}[F'] - E_{\Lambda_L,M}[F'] - A'(m)[m_j - m]$, where $A(m) = E_{\nu_m}[F']$.

LEMMA 6.5. – *There exist* $\beta_0 > 0$ *and a finite constant* C_0 *depending only on* $\|F\|_\infty$, $\|F''\|_\infty$ *such that*

$$\frac{1}{\beta L} \log E_{\Lambda_L,M}\left[\exp\left\{\beta \sum_{j=1}^{\ell} |I_j| G(m_j)\right\}\right] \leqslant \frac{C_0 \beta}{K} \tag{6.10}$$

for all $\beta \leqslant \beta_0$*, all* $L \geqslant K^2$ *and all* M *in* $\mathbb{R}$*.*

Proof. – We first prove the lemma in the grand canonical case with G replaced by the mean-zero function $\widetilde{G}$ given by:

$$\widetilde{G}(m_j) = E_{I_j,M_j}[F'] - E_{\nu_m}[F'] - A'(m)[m_j - m].$$

Fix a density m. To keep notation simple, assume that all cubes I_j have the same length K. Since ν_m is a product measure, the left hand side of (6.10) is equal to

$$\frac{1}{\beta K} \log E_{\nu_m}[\exp\{\beta K \widetilde{G}(m_1)\}].$$

Since $e^x \leqslant 1 + x + x^2 e^{|x|}$, since $\log(1+x) \leqslant x$ and since $E_{\nu_m}[\widetilde{G}] = 0$, the previous expression is less than or equal to

$$\frac{\beta}{K} E_{\nu_m}\big[\{K\widetilde{G}(m_1)\}^2 \exp\{\beta K |\widetilde{G}(m_1)|\}\big].$$

We claim that there exists β_1 and a finite constant C_0 such that

$$E_{\nu_m}\big[\{K\widetilde{G}(m_1)\}^2 \exp\{\beta K |\widetilde{G}(m_1)|\}\big] \leqslant C_0 \tag{6.11}$$

for all m in $\mathbb{R}$, all $K \geqslant 1$ and all $\beta \leqslant \beta_1$. Indeed, let $A(\alpha) = E_{\nu_\alpha}[F']$. Since $\widetilde{G}(m_1) = \{E_{I_1,M_1}[F'] - E_{\nu_{m_1}}[F']\} + A(m_1) - A(m) - A'(m)[m_1 - m]$, by Lemma 3.3 and Corollary 5.3, $\widetilde{G}$ is bounded in absolute value by $CK^{-1} + C(m_1 - m)^2$ for some finite constant C. In particular, the left hand side of (6.11) is bounded above by

$$\begin{aligned} &C e^{C\beta} E_{\nu_m}\big[\{1 + K^2(m_1 - m)^4\} \exp\{C\beta K (m_1 - m)^2\}\big] \\ &\quad \leqslant C E_{\nu_m}\big[\exp\{C'\beta K (m_1 - m)^2\}\big]. \end{aligned}$$

By Lemma 5.7, there exists $\beta_1 > 0$ such that for $\beta < \beta_1$, the expectation is bounded uniformly in K and m. This proves claim (6.11) and that the left hand side of (6.10) is bounded by $C\beta/K$ for $\beta \leqslant \beta_1$, which concludes the proof of the lemma in the grand canonical case.

We now turn to the canonical measure. Notice first that

$$|G(m_1) - \widetilde{G}(m_1)| \leqslant \frac{C\|F''\|_\infty}{L} \tag{6.12}$$

for some finite constant $C = C(\|F\|_\infty)$.

We now turn to the proof of (6.10). By Schwarz inequality, the left hand side of (6.10) is bounded by

$$\frac{1}{\beta L}\log E_{\Lambda_L,M}\left[\exp\left\{2\beta\sum_{j=1}^{\ell/2}|I_j|G(m_j)\right\}\right].$$

The difference is that we are now summing only over the first $\ell/2$ cubes of length K so that we can use Corollary 5.5 to estimate the expectation with respect to canonical measure by the expectation with respect to grand canonical measure. Assume that $K/L \leqslant \beta^2 \leqslant \beta_0^2 = \beta_1^2/4$. By (6.12), the previous expression is bounded by

$$\frac{1}{\beta L}\log E_{\Lambda_L,M}\left[\exp\left\{2\beta\sum_{j=1}^{\ell/2}|I_j|\widetilde{G}(m_j)\right\}\right] + \frac{C}{L}$$

for some finite constant $C = C(\|F\|_\infty, \|F''\|_\infty)$. In the range considered, $L^{-1} \leqslant \beta^2/K \leqslant C\beta/K$ because $\beta \leqslant \beta_0$. The remainder term is thus bounded by $C\beta/K$ for some finite constant $C = C(\|F\|_\infty, \|F''\|_\infty)$. On the other hand, by Corollary 5.5, the previous expression is bounded above by

$$\frac{C_0}{\beta L} + \frac{1}{\beta L}\log E_{\nu_m}\left[\exp\left\{2\beta\sum_{j=1}^{\ell/2}|I_j|\widetilde{G}(m_j)\right\}\right].$$

Since $\beta^2 \geqslant K/L$, the first term is bounded by $C_0\beta/K$. On the other hand, by the first part of the proof, the second term is bounded by $C_0\beta/K$ because $2\beta \leqslant \beta_1$. This proves (6.10) provided $K/L \leqslant \beta^2 \leqslant \beta_0^2$.

Assume now that $\beta^2 \leqslant \min\{K/L, \beta_0^2\}$. In this case, since $\exp\{x\} \leqslant 1 + x + x^2\exp\{|x|\}$, since $\log(1+x) \leqslant x$ and since the sum that appears in the exponential of (6.10) has mean zero with respect to the canonical measure, the left hand side in (6.10) is bounded above by

$$\frac{4\beta}{L}E_{\Lambda_L,M}\left[\left(\sum_{j=1}^{\ell/2}|I_j|G(m_j)\right)^2\exp\left\{2\beta\left|\sum_{j=1}^{\ell/2}|I_j|G(m_j)\right|\right\}\right].$$

Since $e^{|x|} \leqslant e^x + e^{-x}$, we may remove the absolute value in the exponential provide we estimate the previous expression with $-\beta$ in place of β in the exponential. Consider the case with β. By Corollary 5.5, the previous expression without the absolute value in the exponential is less than or equal to

$$\frac{C\beta}{L} E_{\nu_m}\left[\left(\sum_{j=1}^{\ell/2} |I_j| G(m_j)\right)^2 \exp\left\{2\beta \sum_{j=1}^{\ell/2} |I_j| G(m_j)\right\}\right].$$

Since ν_m is a product measure, expanding the square we obtain that this term is equal to

$$\frac{C\beta}{K} E_{\nu_m}\left[\{KG(m_1)\}^2 \mathrm{e}^{2\beta KG(m_1)}\right] E_{\nu_m}\left[\mathrm{e}^{2\beta KG(m_1)}\right]^{(\ell/2)-1} + \frac{C\beta L}{K^2} E_{\nu_m}\left[KG(m_1)\mathrm{e}^{2\beta KG(m_1)}\right]^2 E_{\nu_m}\left[\mathrm{e}^{2\beta KG(m_1)}\right]^{(\ell/2)-2}. \tag{6.13}$$

We estimate separately each of the expectations appearing in this formula.

We start examining the exponential terms. By (6.12), we have that

$$E_{\nu_m}\left[\mathrm{e}^{2\beta KG(m_1)}\right]^{(\ell/2)} \leqslant \mathrm{e}^{C\beta} E_{\nu_m}\left[\mathrm{e}^{2\beta K\widetilde{G}(m_1)}\right]^{(\ell/2)}$$

for some finite constant $C = C(\|F\|_\infty, \|F''\|_\infty)$. Since $\beta \leqslant \beta_0$, $\exp\{C\beta\} \leqslant C$. Since $\widetilde{G}(m_1)$ has mean zero with respect to ν_m, expanding the exponential up to the second order, we get that $E_{\nu_m}[\exp\{2\beta K\widetilde{G}(m_1)\}]$ is bounded above by

$$1 + 4\beta^2 E_{\nu_m}\left[\{K\widetilde{G}(m_1)\}^2 \mathrm{e}^{2\beta K|\widetilde{G}(m_1)|}\right].$$

Since $\beta \leqslant \beta_0$, by (6.11) the previous expression is less than or equal to $1 + C\beta^2 \leqslant \exp\{C\beta^2\}$. Therefore,

$$E_{\nu_m}\left[\mathrm{e}^{2\beta K\widetilde{G}(m_1)}\right]^{\ell} \leqslant \mathrm{e}^{C\beta^2\ell} \leqslant C_1$$

because $\beta^2 \leqslant K/L = \ell^{-1}$.

We now estimate $E_{\nu_m}[\{KG(m_1)\}^2 \exp\{2\beta KG(m_1)\}]$. Here again we first replace $G(m_1)$ by $\widetilde{G}(m_1)$. By (6.12), this expression is bounded above by

$$CE_{\nu_m}\left[\{K\widetilde{G}(m_1)\}^2 \exp\{2\beta K\widetilde{G}(m_1)\}\right] + CE_{\nu_m}\left[\exp\{2\beta K\widetilde{G}(m_1)\}\right]$$

for some finite constant $C = C(\|F\|_\infty, \|F''\|_\infty)$ because $\beta \leqslant \beta_0$. We have already seen that the exponential term is bounded. On the other hand, by (6.11) the first expectation is bounded by a constant because $\beta \leqslant \beta_0 \leqslant \beta_1/2$.

It remains to estimate

$$\frac{L}{K} E_{\nu_m}\left[KG(m_1)\mathrm{e}^{2\beta KG(m_1)}\right]^2.$$

Here again we start replacing G by the mean-zero function $\widetilde{G}$. By (6.12) the previous expression is less than or equal to

$$\frac{CL}{K} E_{\nu_m}\left[K\widetilde{G}(m_1)\mathrm{e}^{2\beta K\widetilde{G}(m_1)}\right]^2 + \frac{CK}{L} E_{\nu_m}\left[\mathrm{e}^{2\beta K\widetilde{G}(m_1)}\right]^2$$

for some finite constant $C = C(\|F\|_\infty, \|F''\|_\infty)$ because $\beta \leqslant \beta_0$. We have seen that the expectation of the exponential term is bounded. On the other hand, since $a\exp\{a\} \leqslant$

$a + a^2 \exp\{|a|\}$ and since $\widetilde{G}(m_1)$ has mean zero with respect to ν_m,

$$E_{\nu_m}\big[K\widetilde{G}(m_1)\mathrm{e}^{2\beta K\widetilde{G}(m_1)}\big] \leqslant 2\beta E_{\nu_m}\big[\{K\widetilde{G}(m_1)\}^2\mathrm{e}^{2\beta K|\widetilde{G}(m_1)|}\big].$$

Since $\beta \leqslant \beta_0$, by (6.11) the previous expression is bounded by $C\beta$. In view of the previous estimates, (6.13) is bounded above by

$$\frac{C\beta}{K} + \frac{C\beta^3 L}{K^2} \leqslant \frac{C\beta}{K}$$

because $\beta^2 \leqslant K/L$. This proves (6.10) in the case where $\beta^2 \leqslant \min\{K/L, \beta_0^2\}$ and concludes the proof of the lemma. □

LEMMA 6.6. – *Fix a bounded function $H:\mathbb{R}\to\mathbb{R}$ and $L \geqslant 2$. The function $\widetilde{H}_L:\mathbb{R}\to\mathbb{R}$ defined by $\widetilde{H}_L(m) = E_{\Lambda_L,M}[H(\eta_1)]$ is Lipschitz continuous on $\mathbb{R}$ and the Lipschitz constant does not depend on L.*

Proof. – An elementary computation shows that

$$\partial_M E_{\Lambda_L,M}[H(\eta_1)] = -E_{\Lambda_L,M}[\eta_2; H(\eta_1)] = -E_{\Lambda_L,M}[H(\eta_1)\{\eta_2 - m\}].$$

By Corollary 5.3, the absolute value of the previous expression is bounded above by $C_0 L^{-1}\sigma(\Phi(m))$ for some finite constant C_0 depending on $\|H\|_\infty$ because ν_m is a product measure. Since $\widetilde{H}_L' = L\partial_M E_{\Lambda_L,M}[H(\eta_1)]$, it remains to recall the statement of Lemma 5.1 to conclude the proof of the lemma. □

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