Ann. I. H. Poincaré – PR 38, 4 (2002) 507–556

S0246-0203(01)01095-0/FLA

CONVERGENCE OF LOCAL TYPE DIRICHLET FORMS TO A NON-LOCAL TYPE ONE

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Received 4 December 2000, revised 16 May 2001

ABSTRACT. – Convergence of Dirichlet forms of diffusion processes is investigated without assuming that the underlying measures are fixed or compatible with a fixed one. Here we treat the case where the basic processes are skew product of finite dimensional diffusions and one-dimensional ones. We note the corresponding diffusions to the Dirichlet forms can be represented as time changed processes of the basic processes, where the time change is given by the additive functional associated with the underlying measure. Then the convergence of the Markov semigroups of the obtained processes and the Feller property of the limit process are proved by providing some convergence properties on additive functionals. The concrete expression on a core for the limit Dirichlet form is also obtained, which may be of non-local type due to the degeneracy of the underlying measure. Finally, under some regularity assumption, the partial differential equation associated with the limit process is given, which is elliptic on infinitely many disjoint strips with the non-local boundary condition including the boundary values on the neighboring strips. © 2002 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On étudie la convergence de la forme de Dirichlet des processus de diffusion sans l'hypothèse que les mesures de base sont fixées ou compatibles avec une mesure fixée. Nous traitons ici le cas où chaque processus fondamental est le produit semi-direct d'un processus de diffusion de dimension finie et d'un processus de diffusion unidimensionnel. Nous notons que les processus correspondants à la forme de Dirichlet peuvent être représentés comme des processus changés de temps des processus fondamentaux. Le changement de temps est donné par les fonctionnelles additives associées aux mesures de base. Nous démontrons la convergence des semi-groupes des processus markoviens ainsi obtenus et la propriété Fellerienne du processus limite en vérifiant la convergence des fonctionnelles additives. L'expression concrète de la forme de Dirichlet sur un domaine dense est aussi obtenue. Elle peut être de type non local du fait de la dégénérescence de la mesure de base du processus limite. Enfin, sous une certaine hypothèse sur

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[1] Research supported by Grant-in-Aid for Scientific Research No. 11304003.
[2] Research supported by Grants-in-Aid for Scientific Research No. 11304003 and 12640125.
[3] Research supported by Grant-in-Aid for Scientific Research No. 10640112.

régularité, nous présentons l'équation aux derivées partielles associée au processus limite. Il est elliptique sur une infinité de bandes deux à deux disjointes et satisfait à une condition au bord sur la frontière de chaque bande, en fonction des valeurs sur les bandes voisines.

1. Introduction

In the study of convergence of a sequence of Markov processes, the theory of Dirichlet forms is one of the most useful tools. In fact, the notion of Γ-convergence and Mosco convergence as well as a monotone convergence theorem derive a lot of interesting results (see [2,15,17,12,10,14,20], etc.). However, they are basically for a sequence of Markov processes associated with Dirichlet forms on a fixed underlying Hilbert space, which means that the underlying measures are fixed or at least compatible with a fixed one.

In this article, we will be concerned with a sequence of diffusion processes whose underlying measures of Dirichlet forms converge to a degenerate one. Especially, we are interested in the case where a sequence of diffusion processes converges to a non-local Markov process, which never happens unless the underlying measures degenerate. As a prototype of such sequences, we adopt that of diffusion processes which are skew products of finite dimensional diffusion processes and one-dimensional ones. This enables us to express the semigroups as time changed processes of product diffusions and obtain some properties such as convergence of semigroups and Feller property of the limit process. The feature of the Dirichlet forms with degenerate underlying measures and Dirichlet energy are already given in [7] using harmonic operators. We would notice that those are actually expressed as a sum of local Dirichlet forms on the support of the underlying measure and non-local Dirichlet forms on its boundaries. We will give the explicit form of the Lévy measure for the Dirichlet form obtained as a limit of diffusion processes in our setting (see Theorem 5.1 below). As is noted in [18] and [19], the Dirichlet form of the type thus obtained corresponds to the second order differential equation with non-local type boundary condition. Our simpler proof of Feller property here offers another approach to this problem, which is actually an extension of the results in [8]. However we must confess that this is fully based on the skew product structure, and the rather tedious real analysis might be on stage if one deal with more general processes. Finally, we would say that the results even in our case give a new type of phenomena of the convergence of Dirichlet forms and well provide the rough shape of formulas in forthcoming general settings.

This article is organized as follows. In the next Section 2, we set up our problem in terms of Dirichlet forms and give an intuitive scope of our limit theorem. We establish our limit theorem in Section 3, where we express the processes in terms of time change by additive functionals and prove their convergence. In Section 4, we show the Feller property of the semigroup of the limit process, using the expression of one-dimensional diffusion processes by a Brownian motion and its local time. In Section 5, we give the explicit form of the Dirichlet form associated with the limit process as a sum of local and non-local Dirichlet forms under a little stronger conditions. We also give several examples, in which we can compute the explicit formulas of Lévy measures and determine the domain of the Dirichlet forms except for the last one. In the final Section 6, we study the partial differential equations for the resolvent of the limit process. The obtained one is elliptic on infinitely many disjoint strips with the non-local boundary condition including the boundary values on the neighboring strips.

2. Preliminaries

For $E = \mathbb{R}^d$ or $E = \mathbb{R}^{d-1}$, we denote by $C(E)$ the space of all bounded continuous functions on E, and by $\widehat{C}(E)$ the space of those functions in $C(E)$ which vanish at infinity. The set of all infinitely differentiable functions on E with compact support is denoted by $C_0^\infty(E)$, while the Sobolev space of order one by $H^1(E)$. We also write as $x = (x^1, \ldots, x^{d-1}, x^d) = (x', x^d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and denote by $\mathrm{d}x$, $\mathrm{d}x'$, $\mathrm{d}\xi$ the d-dimensional, $(d-1)$-dimensional, one-dimensional Lebesgue measure respectively.

Let $a^{ij} = a^{ij}(x')$, $i, j = 1, 2, \ldots, d-1$, be a system of bounded measurable functions on $\mathbb{R}^{d-1}$ which compose a symmetric and uniformly elliptic matrix $(a^{ij})_{i,j=1,2,\ldots,d-1}$. Let also $a^{dd} = a^{dd}(\xi)$ be a bounded measurable function which is bounded from below by a positive constant.

For each $n \in \mathbb{N}$, let $\rho_0^{(n)} = \rho_0^{(n)}(\xi)$ be a nonnegative bounded measurable function on $\mathbb{R}$ and $\mu_0^{(n)} = \mu_0^{(n)}(x)$ a bounded measurable function on $\mathbb{R}^d$ which are bounded from below by a positive constant. We now let $\mu^{(n)}(\mathrm{d}x) = \mu_0^{(n)}(x)\,\mathrm{d}x$ and consider the Dirichlet form $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ on $L^2(\mathbb{R}^d, \mu^{(n)})$ given by

$$\mathcal{E}^{(n)}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d} \Bigg\{ \big(1+\rho_0^{(n)}(x^d)\big) \sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}u(x)\partial_{x^j}v(x) + a^{dd}(x^d)\,\partial_{x^d}u(x)\partial_{x^d}v(x) \Bigg\}\,\mathrm{d}x, \tag{2.1}$$

$$\mathcal{F}^{(n)} = H^1(\mathbb{R}^d), \tag{2.2}$$

where the derivatives $\partial_{x^i}u := \partial u/\partial x^i$, $1 \leqslant i \leqslant d$, are taken in the distribution sense. It then follows that the Dirichlet form $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ on $L^2(\mathbb{R}^d, \mu^{(n)})$ is regular, so that there exists an associated Hunt process $\mathbf{X}^{(n)}$ on $\mathbb{R}^d$ (see [7]). It is also known that the associated semigroup $\{p_t^{(n)}\colon t>0\}$ on $L^2(\mathbb{R}^d, \mu^{(n)})$ is Markovian in the sense that $0 \leqslant p_t^{(n)} f \leqslant 1$, a.e. whenever $0 \leqslant f \leqslant 1$, a.e. ([7]).

PROPOSITION 2.1. – *For each $n \in \mathbb{N}$, $\{p_t^{(n)}\colon t > 0\}$ is a Feller semigroup on $\widehat{C}(\mathbb{R}^d)$. Namely,*

$$p_t^{(n)}\big(\widehat{C}(\mathbb{R}^d)\big) \subset \widehat{C}(\mathbb{R}^d), \quad t > 0,$$

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\big|p_t^{(n)}f(x) - f(x)\big| = 0, \quad f \in \widehat{C}(\mathbb{R}^d).$$

There exists a conservative diffusion process $\mathbf{X}^{(n)} = (X_t^{(n)}, P_x^{(n)})$ on $\mathbb{R}^d$ which has $\{p_t^{(n)}, t > 0\}$ as the transition operator, that is,

$$p_t^{(n)} f(x) = E_x^{(n)}\left[f\left(X_t^{(n)}\right)\right], \quad t > 0,\ x \in \mathbb{R}^d,\ f \in \widehat{C}\left(\mathbb{R}^d\right),$$

where $E_x^{(n)}$ stands for the expectation with respect to $P_x^{(n)}$.

Proof. – Under the assumptions, $L^2(\mathbb{R}^d, \mu^{(n)}) = L^2(\mathbb{R}^d, \mathrm{d}x)$ and the space $\{u \in L^2(\mathbb{R}^d, \mu^{(n)})\colon \partial_{x^i} u \in L^2(\mathbb{R}^d, \mathrm{d}x),\ 1 \leqslant i \leqslant d\}$ coincides with the Sobolev space $H^1(\mathbb{R}^d)$. Further, the norm $\mathcal{E}_1^{(n)}(\ ,\ )^{1/2}$ is equivalent to the norm $\|\ \|_{H^1(\mathbb{R}^d)}$, where $\mathcal{E}_\lambda^{(n)}(u, v) = \mathcal{E}^{(n)}(u, v) + \lambda(u, v)_{L^2(\mathbb{R}^d, \mu^{(n)})}$, $\lambda > 0$. Therefore we get the proposition by means of results due to Nash [16], Stampacchia [21] and Kunita [11]. □

In what follows, we consider the convergence of the diffusion processes $\mathbf{X}^{(n)}$ when $\rho_0^{(n)}(\xi)\,\mathrm{d}\xi$ converge to a measure $\rho(\mathrm{d}\xi)$ and $\mu_0^{(n)}(x)\,\mathrm{d}x$ to $\mu(\mathrm{dx})$ in some sense. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of the limit process could be supposed to be

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^{d-1} a^{ij}(x') \partial_{x^i} u(x) \partial_{x^j} v(x)\,\mathrm{d}x' \left(\mathrm{d}x^d + \rho\left(\mathrm{d}x^d\right)\right) + \frac{1}{2} \int_{\mathbb{R}^d} a^{dd}\left(x^d\right) \partial_{x^d} u(x) \partial_{x^d} v(x)\,\mathrm{d}x. \tag{2.3}$$

However this is not right as it stands, since we must consider it on the space $L^2(\mathbb{R}^d, \mu)$ and the support of the measure μ is not necessarily equal to the whole space $\mathbb{R}^d$. Actually this is the very case we are interested in, and we will give the right answer in Section 5 below. To do this we study the rigorous form of the limit theorem and analyze the limit process $\mathbf{X}$. We also show the Feller property of $\mathbf{X}$ under additional assumptions.

3. Limit theorem

In this section, we will give the limit theorem for the diffusion processes in the previous section. Our assumptions are the following:

(A.1) The measures $\rho_0^{(n)}(\xi)\,\mathrm{d}\xi$ converge vaguely to a Radon measure $\rho(\mathrm{d}\xi)$ on $\mathbb{R}$.

(A.2) The functions $\mu_0^{(n)}$ are decomposed as $\mu_0^{(n)}(x) = \mu_1^{(n)}(x')\mu_2^{(n)}(x^d)$, and $\mu_1^{(n)}(x')$ converge uniformly to 1 and the measures $\mu_2^{(n)}(\xi)\,\mathrm{d}\xi$ converge vaguely to a non-zero Radon measure m on $\mathbb{R}$.

In order to get our limit theorem, we will express the approximating process $\mathbf{X}^{(n)}$ by means of time change. Let $\mathbf{X}' = (X_t', P_{x'}')$ be the diffusion process on $\mathbb{R}^{d-1}$ associated with the following Dirichlet form $(\mathcal{E}', \mathcal{F}')$ on $L^2(\mathbb{R}^{d-1}, \mathrm{d}x')$:

$$\mathcal{E}'(u, v) = \frac{1}{2} \int_{\mathbb{R}^{d-1}} \sum_{i,j=1}^{d-1} a^{ij}(x') \partial_{x^i} u(x') \partial_{x^j} v(x')\,\mathrm{d}x', \tag{3.1}$$

$$\mathcal{F}' = H^1\left(\mathbb{R}^{d-1}\right). \tag{3.2}$$

We note that the transition operator $\{p_t', t > 0\}$ of $\mathbf{X}'$ has the Feller property as in Proposition 2.1. Also let $\Xi = (\xi_t, P_\xi^\Xi)$ be the one-dimensional diffusion process associated with the following Dirichlet form $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$ on $L^2(\mathbb{R}, \mathrm{d}\xi)$:

$$\mathcal{E}^{\Xi}(u,v)=\frac{1}{2}\int_{\mathbb{R}} u'(\xi)v'(\xi)a^{dd}(\xi)\,\mathrm{d}\xi, \quad \mathcal{F}^{\Xi}=H^1(\mathbb{R}).$$

In the following the sample paths $X'_t,\ \xi_t$, etc. are sometimes denoted by $X'(t),\ \xi(t)$, etc. It is known that the diffusion process Ξ has the local time $\ell^{\Xi}(t,\xi)$ which is continuous with respect to $(t,\xi)\in[0,\infty)\times\mathbb{R}$ and satisfies $2\int_E \ell^{\Xi}(t,\xi)\,\mathrm{d}\xi=\int_0^t I_E(\xi_s)\,\mathrm{d}s,\ t>0$, for every measurable set $E\subset\mathbb{R}$ ([9]), where I_E is the indicator for a set E. We set

$$\mathbf{f}^{(n)}(t)=\int_0^t \bigl\{1+\rho_0^{(n)}(\xi_s)\bigr\}\,\mathrm{d}s=t+2\int_{\mathbb{R}} \ell^{\Xi}(t,\xi)\rho_0^{(n)}(\xi)\,\mathrm{d}\xi, \quad t\geqslant 0, \tag{3.3}$$

$$\mathbf{f}(t)=t+2\int_{\mathbb{R}} \ell^{\Xi}(t,\xi)\,\rho(\mathrm{d}\xi), \quad t\geqslant 0, \tag{3.4}$$

$$A^{(n)}(t)=\int_0^t \mu_1^{(n)}\bigl(X'\bigl(\mathbf{f}^{(n)}(s)\bigr)\bigr)\mu_2^{(n)}(\xi_s)\,\mathrm{d}s, \quad t\geqslant 0, \tag{3.5}$$

$$A(t)=2\int_{\mathbb{R}} \ell^{\Xi}(t,\xi)\,m(\mathrm{d}\xi), \quad t\geqslant 0. \tag{3.6}$$

Denote the right continuous inverses of $A^{(n)}(t)$ and $A(t)$ by $\tau_t^{(n)}$ and τ_t respectively.

Let us consider the time changed process $\widetilde{\mathbf{X}}^{(n)}$ defined by

$$\widetilde{\mathbf{X}}^{(n)}=\bigl[\bigl(X'\bigl(\mathbf{f}^{(n)}\bigl(\tau_t^{(n)}\bigr)\bigr),\xi\bigl(\tau_t^{(n)}\bigr)\bigr),\ P_x=P'_{x'}\otimes P^{\Xi}_{x^d},\ x=\bigl(x',x^d\bigr)\in\mathbb{R}^d\bigr]. \tag{3.7}$$

LEMMA 3.1. – *$\widetilde{\mathbf{X}}^{(n)}$ is equivalent to the diffusion process $\mathbf{X}^{(n)}$ defined in Section* 2.

Proof. – Since $P^{\Xi}_{\xi}(\lim_{t\to\infty}\mathbf{f}^{(n)}(t)=\infty)=1,\ \xi\in\mathbb{R}$, by virtue of [6], the Dirichlet form of the skew product $\mathbf{Y}^{(n)}:=[(X'(\mathbf{f}^{(n)}(t)),\xi(t)),\ P_x=P'_{x'}\otimes P^{\Xi}_{x^d},\ x=(x',x^d)\in\mathbb{R}^d]$ is given by $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$, where the underlying measure is $\mathrm{d}x$. $A^{(n)}(t)$ is a positive continuous additive functional of $\mathbf{Y}^{(n)}$ in the strict sense with Revuz measure $\mu^{(n)}$, which has full quasi support. Therefore, in view of Theorem 6.2.1 in [7], the time changed process $\widetilde{\mathbf{X}}^{(n)}$ is a Hunt process on $\mathbb{R}^d$ and the Dirichlet form coincides with $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$ defined by (2.1), (2.2) with the underlying measure $\mu^{(n)}$. Thus we get the conclusion. □

Let $\mathbf{X}$ be the time changed process defined by

$$\mathbf{X}=\bigl[\bigl(X'\bigl(\mathbf{f}(\tau_t)\bigr),\xi(\tau_t)\bigr),\ P_x=P'_{x'}\otimes P^{\Xi}_{x^d},\ x=\bigl(x',x^d\bigr)\in\mathbb{R}^d\bigr]. \tag{3.8}$$

Now our limit theorem is

THEOREM 3.2. – *The diffusion processes $\widetilde{\mathbf{X}}^{(n)}$ converge to $\mathbf{X}$ in the sense that for the transition operators $p_t^{(n)}$,*

$$\lim_{n\to\infty} p_t^{(n)}f(x)=E_x\bigl[f\bigl(X'\bigl(\mathbf{f}(\tau_t)\bigr),\xi(\tau_t)\bigr)\bigr], \quad t>0,\ x\in\mathbb{R}^d,\ f\in C\bigl(\mathbb{R}^d\bigr), \tag{3.9}$$

where E_x stands for the expectation with respect to P_x.

Proof. – First we note that the set $E_t := [\min_{0\leqslant s\leqslant t}\xi_s, \max_{0\leqslant s\leqslant t}\xi_s]$ is compact and $\ell^{\Xi}(t,\xi)=0$ for $\xi\notin E_t$, P_ξ^{Ξ}-a.s., $\xi\in\mathbb{R}$. Hence by means of (A.1) and (A.2),

$$\lim_{n\to\infty}\mathbf{f}^{(n)}(t)=\mathbf{f}(t),\ t\geqslant 0,\quad P_\xi^{\Xi}\text{-a.s.},\ \xi\in\mathbb{R},\tag{3.10}$$

$$\lim_{n\to\infty}A^{(n)}(t)=A(t),\ t\geqslant 0,\quad P_x\text{-a.s.},\ x\in\mathbb{R}^d.\tag{3.11}$$

By virtue of (3.11), it holds that

$$\lim_{n\to\infty}\tau_t^{(n)}=\tau_t\quad\text{for } t>0 \text{ with } \tau_{t-}=\tau_t,\quad P_x\text{-a.s. } x\in\mathbb{R}^d.$$

We should notice that

$$P_x\big(\lim_{n\to\infty}\tau_t^{(n)}=\tau_t\big)=1,\quad t>0,\ x\in\mathbb{R}^d.\tag{3.12}$$

To this end, it is enough to show

$$P_\xi^{\Xi}(\tau_{t-}<\tau_t)=0,\quad t>0,\ \xi\in\mathbb{R}.\tag{3.13}$$

In the case where $\operatorname{supp}[m]=\mathbb{R}$, τ_t is continuous and hence (3.13) is obvious. Let us consider the case where $S_m:=\operatorname{supp}[m]\neq\mathbb{R}$. Then $\mathbb{R}\setminus S_m$ is denoted by a finite or a countable disjoint union of open intervals $I_k=(a_k,b_k)$, $k\in K$ with the end points belonging to $S_m\cup\{-\infty,\ \infty\}$. Since at least one of a_k, b_k belongs to S_m and τ_{t-} is a Markov time of Ξ, we find that for $\varepsilon>0$ and $\xi\in\mathbb{R}$,

$$\begin{aligned}P_\xi^{\Xi}(\tau_t-\tau_{t-}>\varepsilon)&\leqslant\sum_{k\in K}P_\xi^{\Xi}\big(\xi(\tau_{t-})\in\{a_k,b_k\}\cap S_m,\ \xi_u\in I_k\text{ for }\tau_{t-}<{}^\forall u<\tau_{t-}+\varepsilon\big)\\&=\sum_{k\in K}E_\xi^{\Xi}\big[P^{\Xi}_{\xi(\tau_{t-})}(\xi_u\in I_k\text{ for }0<{}^\forall u<\varepsilon);\ \xi(\tau_{t-})\in\{a_k,b_k\}\cap S_m\big].\end{aligned}$$

The probability in the last term is zero because of $P_\xi^{\Xi}(\xi_u<\xi\text{ for }0<{}^\forall u<\varepsilon)=P_\xi^{\Xi}(\xi_u>\xi\text{ for }0<{}^\forall u<\varepsilon)=0$, $\xi\in\mathbb{R}$, $\varepsilon>0$. Thus we obtain (3.13).

We next note that

$$P_x\big(\lim_{n\to\infty}\mathbf{f}^{(n)}\big(\tau_t^{(n)}\big)=\mathbf{f}(\tau_t)\big)=1,\quad x\in\mathbb{R}^d,\ t>0.\tag{3.14}$$

Indeed,

$$\begin{aligned}\big|\mathbf{f}^{(n)}\big(\tau_t^{(n)}\big)-\mathbf{f}(\tau_t)\big|&\leqslant\big|\tau_t^{(n)}-\tau_t\big|+2\int_{\mathbb{R}}\big|\ell^{\Xi}\big(\tau_t^{(n)},\xi\big)-\ell^{\Xi}(\tau_t,\xi)\big|\rho_0^{(n)}(\xi)\,\mathrm{d}\xi\\&\quad+2\bigg|\int_{\mathbb{R}}\ell^{\Xi}(\tau_t,\xi)\rho_0^{(n)}(\xi)\,\mathrm{d}\xi-\int_{\mathbb{R}}\ell^{\Xi}(\tau_t,\xi)\,\rho(\mathrm{d}\xi)\bigg|\\&\equiv I+II+III.\end{aligned}$$

By virtue of (3.12), there is an $n_o \in \mathbb{N}$ such that $\tau_t^{(n)} \leqslant \tau_t + 1,\ n \geqslant n_o$, and $I \to 0$ as $n \to \infty$. Since $\ell^{\Xi}(t,\xi)$ is continuous in $(t,\xi) \in [0,\infty) \times \mathbb{R}$, (3.12) and (A.1) imply that

$$II \leqslant 2 \sup_{\xi \in E_{\tau_t+1}} \left|\ell^{\Xi}(\tau_t^{(n)},\xi) - \ell^{\Xi}(\tau_t,\xi)\right| \int_{E_{\tau_t+1}} \rho_0^{(n)}(\xi)\,\mathrm{d}\xi \to 0 \quad \text{as } n \to \infty.$$

Since the set E_{τ_t} is compact and $\ell^{\Xi}(\tau_t,\xi) = 0$ for $\xi \notin E_{\tau_t}$, $III \to 0$ as $n \to \infty$. Accordingly (3.14) follows.

By virtue of Lemma 3.1,

$$p_t^{(n)} f(x) = E_x\left[f\left(X'\left(\mathbf{f}^{(n)}\left(\tau_t^{(n)}\right)\right),\ \xi\left(\tau_t^{(n)}\right)\right)\right]. \tag{3.15}$$

Combining this with (3.12) and (3.14),

$$\lim_{n\to\infty} p_t^{(n)} f(x) = E_x\left[f\left(X'\left(\mathbf{f}(\tau_t)\right),\ \xi(\tau_t)\right)\right], \tag{3.16}$$

because $f \in C(\mathbb{R}^d)$, and $X'(t)$ and $\xi(t)$ are continuous in t. □

Remark 3.3. – By means of (3.12), (3.14) and (3.15), we also see that

$$\lim_{n\to\infty} E_x\left[\prod_{i=1}^{k} f_i\left(X'\left(\mathbf{f}^{(n)}\left(\tau_{t_i}^{(n)}\right)\right),\ \xi\left(\tau_{t_i}^{(n)}\right)\right)\right] = E_x\left[\prod_{i=1}^{k} f_i\left(X'\left(\mathbf{f}(\tau_{t_i})\right),\ \xi(\tau_{t_i})\right)\right],$$

for $x \in \mathbb{R}^d,\ 0 < t_1 < t_2 < \cdots < t_k,\ f_i \in C(\mathbb{R}^d)\ (i = 1,2,\ldots,k),\ k \in \mathbb{N}$, which shows the convergence of finite dimensional distributions of $\widetilde{\mathbf{X}}^{(n)}$ to those of $\mathbf{X}$.

4. Semigroup of the limit process

In this section we will show some properties of semigroup of the limit process obtained in the previous section. Let $\mu(\mathrm{d}x)$ be the product measure defined by $\mu(\mathrm{d}x) = \mathrm{d}x' \times m(\mathrm{d}x^d)$, whose support Ω_μ is given by $\mathbb{R}^{d-1} \times S_m$. Since $P_\xi^{\Xi}(\xi(\tau_t) \in S_m,\ t \geqslant 0) = 1,\ \xi \in \mathbb{R}$, it is enough to consider the restrictions of f to Ω_μ for the right hand side of (3.9). We denote them by $p_t f(x)$, that is,

$$p_t f(x) = E_x\left[f\left(X'\left(\mathbf{f}(\tau_t)\right), \xi(\tau_t)\right)\right], \quad t > 0,\ x \in \mathbb{R}^d,\ f \in C(\Omega_\mu). \tag{4.1}$$

Furthermore we note that $p_t f(x)$ is essentially a function of $x \in \Omega_\mu$. Let $p'(t,x',y')$ be the transition probability density of the diffusion process $\mathbf{X}'$ and σ_E^{Ξ} the first hitting time of the one-dimensional diffusion process Ξ for a set E: $\sigma_E^{\Xi} = \inf\{t > 0:\ \xi_t \in E\}$.

PROPOSITION 4.1. – *Let $f \in C(\Omega_\mu)$, $t > 0$ and $x = (x', x^d) \in \mathbb{R}^d \setminus \Omega_\mu \neq \emptyset$. Then*

$$\begin{aligned} p_t f(x) = \int_{(y',y^d,s)\in\mathbb{R}^{d-1}\times\partial S_m\times(0,\infty)} & p_t f\left(y', y^d\right) p'(s,x',y')\,\mathrm{d}y' \\ & \times P_{x^d}^{\Xi}\left(\mathbf{f}\left(\sigma_{S_m}^{\Xi}\right) \in \mathrm{d}s,\ \xi\left(\sigma_{S_m}^{\Xi}\right) \in \mathrm{d}y^d\right), \end{aligned} \tag{4.2}$$

where ∂S_m stands for the boundary of S_m.

Proof. – Let $\mathbf{Y} = (Y_t, P_x)$ be the skew product of $\mathbf{X}'$ and Ξ with respect to $\mathbf{f}(t)$ defined by (3.4), that is,

$$\mathbf{Y} = \left[Y_t = \left(X'\left(\mathbf{f}(t)\right),\ \xi(t)\right),\ P_x = P'_{x'} \otimes P^{\Xi}_{x^d},\ x = \left(x', x^d\right) \in \mathbb{R}^d\right]. \tag{4.3}$$

We note that $\mathbf{Y}$ is a diffusion process on $\mathbb{R}^d$. Since τ_t is a Markov time of $\mathbf{Y}$ and $\tau_{s+t}(\omega) = \tau_s(\omega) + \tau_t(\omega^+_{\tau_s(\omega)})$ for $0 \leqslant s \leqslant t$, where $\omega^+_{\cdot}$ stands for the shifted path of ω, by means of the strong Markov property of $\mathbf{Y}$,

$$E_x\left[f\left(Y(\tau_{s+t})\right)\right] = E_x\left[E_{Y(\tau_s)}\left[f\left(Y(\tau_t)\right)\right]\right], \quad 0 \leqslant s \leqslant t,\ x \in \mathbb{R}^d. \tag{4.4}$$

We also note that, for $x \in \mathbb{R}^d \setminus \Omega_\mu$,

$$P_x\left(\tau_0 = \sigma^{\Xi}_{S_m} > 0\right) = 1,$$
$$P_x\left(Y(\tau_0) = \left(X'\left(\mathbf{f}(\sigma^{\Xi}_{S_m})\right), \xi\left(\sigma^{\Xi}_{S_m}\right)\right) \in \mathbb{R}^{d-1} \times \partial S_m\right) = 1.$$

Hence putting $s = 0$ in (4.4), we get for $x \in \mathbb{R}^d \setminus \Omega_\mu$,

$$\begin{aligned}
&E_x\left[f\left(X'\left(\mathbf{f}(\tau_t)\right), \xi(\tau_t)\right)\right] = E_x\left[E_{Y(\tau_0)}\left[f\left(Y(\tau_t)\right)\right]\right] \\
&\quad = \int_{\mathbb{R}^{d-1}\times\partial S_m} p_t f(y) P_x\left(Y(\tau_0) \in \mathrm{d}y\right) \\
&\quad = \int_{(y', y^d)\in\mathbb{R}^{d-1}\times\partial S_m} p_t f(y) P_x\left(X'\left(\mathbf{f}(\sigma^{\Xi}_{S_m})\right) \in \mathrm{d}y',\ \xi\left(\sigma^{\Xi}_{S_m}\right) \in \mathrm{d}y^d\right) \\
&\quad = \int_{(y', y^d, s)\in\mathbb{R}^{d-1}\times\partial S_m\times(0,\infty)} p_t f\left(y', y^d\right) P'_{x'}(X'_s \in \mathrm{d}y') \\
&\qquad \times P^{\Xi}_{x^d}\left(\mathbf{f}(\sigma^{\Xi}_{S_m}) \in \mathrm{d}s,\ \xi\left(\sigma^{\Xi}_{S_m}\right) \in \mathrm{d}y^d\right) \\
&\quad = \int_{(y', y^d, s)\in\mathbb{R}^{d-1}\times\partial S_m\times(0,\infty)} p_t f\left(y', y^d\right) p'(s, x', y')\mathrm{d}y' \\
&\qquad \times P^{\Xi}_{x^d}\left(\mathbf{f}(\sigma^{\Xi}_{S_m}) \in \mathrm{d}s,\ \xi\left(\sigma^{\Xi}_{S_m}\right) \in \mathrm{d}y^d\right).
\end{aligned}$$

This shows (4.2). □

Remark 4.2. – If $\mathbb{R} \setminus S_m \neq \emptyset$, then $\mathbb{R} \setminus S_m$ is expressed by a finite or a countable disjoint union of open intervals I_k with the end points belonging to $S_m \cup \{-\infty,\ \infty\}$. Suppose that $x^d \in I_k \equiv (a, b)$. Then

$$\sigma^{\Xi}_{S_m} = \begin{cases} \sigma^{\Xi}_b, & -\infty = a < b < \infty, \\ \min\{\sigma^{\Xi}_a,\ \sigma^{\Xi}_b\}, & -\infty < a < b < \infty, \\ \sigma^{\Xi}_a, & -\infty < a < b = \infty, \end{cases} \qquad P^{\Xi}_{x^d}\text{-a.s.},$$

where σ^{Ξ}_p is the first hitting time to a point p. Hence

$$P^{\Xi}_{x^d}\left(\mathbf{f}(\sigma^{\Xi}_{S_m}) \in \mathrm{d}s,\ \xi\left(\sigma^{\Xi}_{S_m}\right) \in \mathrm{d}y^d\right)$$

$$= \begin{cases} P^{\Xi}_{x^d}\big(\mathbf{f}(\sigma^{\Xi}_b) \in \mathrm{d}s\big)\,\delta_b(\mathrm{d}y^d), & -\infty = a < b < \infty, \\ P^{\Xi}_{x^d}\big(\mathbf{f}(\sigma^{\Xi}_a) \in \mathrm{d}s,\ \sigma^{\Xi}_a < \sigma^{\Xi}_b\big)\,\delta_a\big(\mathrm{d}y^d\big) & \\ \quad + P^{\Xi}_{x^d}\big(\mathbf{f}(\sigma^{\Xi}_b) \in \mathrm{d}s,\ \sigma^{\Xi}_b < \sigma^{\Xi}_a\big)\,\delta_b\big(\mathrm{d}y^d\big), & -\infty < a < b < \infty, \\ P^{\Xi}_{x^d}\big(\mathbf{f}(\sigma^{\Xi}_a) \in \mathrm{d}s\big)\,\delta_a\big(\mathrm{d}y^d\big), & -\infty < a < b = \infty, \end{cases}$$

where $\delta_p(\mathrm{d}\xi)$ is the unit measure concentrated at a point p.

We next show that $\{p_t,\ t > 0\}$ is a Feller semigroup. Put $l_1 = \inf S_m,\ l_2 = \sup S_m$.

THEOREM 4.3. – *Let us assume that*

$$\int_{-\infty}^{0} |\xi| m(\mathrm{d}\xi) = \infty \quad \text{if } l_1 = -\infty, \tag{4.5}$$

$$\int_{0}^{\infty} \xi m(\mathrm{d}\xi) = \infty \quad \text{if } l_2 = \infty. \tag{4.6}$$

Then it holds that

$$p_t f \in C\big(\mathbb{R}^d\big) \cap \widehat{C}(\Omega_\mu) \quad \text{for } t > 0,\ f \in \widehat{C}(\Omega_\mu), \tag{4.7}$$

$$\limsup_{t\downarrow 0}\sup_{x\in\Omega_\mu} \big|p_t f(x) - f(x)\big| = 0 \quad \text{for } f \in \widehat{C}(\Omega_\mu). \tag{4.8}$$

For the proof of Theorem 4.3, we prepare some lemmas.

First of all we give an expression of $p_t f(x)$ in terms of the one-dimensional Brownian motion $\mathbf{B} = (B_t, P^B_\xi)$. We set

$$s(\xi) = \int_0^\xi \big\{1/a^{dd}(\eta)\big\}\,\mathrm{d}\eta, \quad \xi \in \mathbb{R}.$$

Obviously $s(\xi)$ is a strictly increasing continuous function on $\mathbb{R}$ and $s(-\infty) = -\infty,\ s(\infty) = \infty$. Since $\Lambda^{-1} \leqslant a^{dd}(\xi) \leqslant \Lambda,\ \xi \in \mathbb{R}$, for some positive constant $\Lambda > 1$, we see that

$$\Lambda^{-1}|\xi| \leqslant |s(\xi)| \leqslant \Lambda|\xi|, \quad \xi \in \mathbb{R}. \tag{4.9}$$

Denote by $\gamma(\xi)$ the inverse of $s(\xi)$. Note that $\gamma'(\xi) = a^{dd}(\gamma(\xi))$, a.e. $\xi \in \mathbb{R}$, and

$$\Lambda^{-1}|\xi| \leqslant |\gamma(\xi)| \leqslant \Lambda|\xi|, \quad \xi \in \mathbb{R}.$$

For each $a \in \mathbb{R}$, we set

$$\Phi(t;a) = 2\int_{\mathbb{R}} \ell^B\big(t, s(\xi) - s(a)\big)\,\mathrm{d}\xi, \quad t \geqslant 0,$$

where $\ell^B(t,\xi)$ is the local time of the Brownian motion $\mathbf{B}$, that is, $\int_0^t I_E(B_s)\,\mathrm{d}s = 2\int_E \ell^B(t,\xi)\,\mathrm{d}\xi$, for measurable sets $E \subset \mathbb{R}$. We denote by $\phi(t;a)$ the inverse of

$t \longmapsto \Phi(t;a)$. $\phi(t;a)$ is a strictly increasing continuous function in t. We note that

$$\Phi(t;a) = 2\int_{\mathbb{R}} \ell^B(t,\eta)\gamma'\big(\eta + s(a)\big)\,\mathrm{d}\eta, \quad t \geqslant 0. \tag{4.10}$$

Therefore

$$t/\Lambda \leqslant \Phi(t;a) \leqslant \Lambda t, \quad t \geqslant 0,\ a \in \mathbb{R},$$

from which

$$t/\Lambda \leqslant \phi(t;a) \leqslant \Lambda t, \quad t \geqslant 0,\ a \in \mathbb{R}. \tag{4.11}$$

Further we put

$$\mathbf{f}(t;a) = t + 2\int_{\mathbb{R}} \ell^B\big(\phi(t;a), s(\xi) - s(a)\big)\rho(\mathrm{d}\xi),$$

$$A(t;a) = 2\int_{\mathbb{R}} \ell^B\big(\phi(t;a), s(\xi) - s(a)\big)m(\mathrm{d}\xi),$$

$$\tau(t;a) = \text{the right continuous inverse of } t \longmapsto A(t;a).$$

LEMMA 4.4. – *It holds that for every $t > 0$, $x \in \mathbb{R}^d$, $f \in C(\Omega_\mu)$,*

$$p_t f(x) = E_0^B\big[E'_{x'}\big[f\big(X'_{\mathbf{f}(\tau(t;x^d);x^d)}, \gamma\big(B_{\phi(\tau(t;x^d);x^d)} + s\big(x^d\big)\big)\big)\big]\big], \tag{4.12}$$

where $E'_{x'}$, $E^B_{x^d}$ stand for the expectations with respect to $P'_{x'}$, $P^B_{x^d}$ respectively, and the right hand side of (4.12) *should be read as*

$$E_0^B\big[E'_{x'}[f(X'_u,\xi)]|_{u=\mathbf{f}(\tau(t;x^d);x^d),\ \xi=\gamma(B_{\phi(\tau(t;x^d);x^d)}+s(x^d))}\big].$$

Proof. – Following the terminology in [9], Ξ is a diffusion process on $\mathbb{R}$ with the *scale function* $s(\xi)$ and the *speed measure* $2\,\mathrm{d}\xi$. By virtue of argument in [9, Ch. 5], $\Xi = (\xi_t, P_\xi^\Xi)$ is equivalent to the time changed process $(\gamma(B_{\phi_t}), P^B_{s(\xi)})$, where

$$\Phi(t) = 2\int_{\mathbb{R}} \ell^B(t,\xi)\gamma'(\xi)\,\mathrm{d}\xi = 2\int_{\mathbb{R}} \ell^B\big(t, s(\xi)\big)\,\mathrm{d}\xi,$$

$$\phi_t = \text{the inverse of } t \longmapsto \Phi(t).$$

Since $(B_t, P^B_{s(\xi)})$ is also equivalent to $(B_t + s(\xi), P_0^B)$, the process $(\ell^\Xi(t,\cdot), P_\xi^\Xi)$ is equivalent to $(\ell^B(\phi(t;\xi), s(\cdot) - s(\xi)), P_0^B)$ and (ξ_t, P_ξ^Ξ) is equivalent to $(\gamma(B_{\phi(t;\xi)} + s(\xi)), P_0^B)$. Combining these facts with (4.1) and using of Fubini's theorem, we get the assertion of the lemma. □

Remark 4.5. – It should be noted that $D := (\gamma(B_{\phi(\tau(t;\xi);\xi)} + s(\xi)), P_0^B)$ is a one-dimensional *generalized* diffusion process having the scale function s and the speed measure $2m$ (cf. [9]). D is conservative because m is a Radon measure on $\mathbb{R}$. When $l_1 > -\infty$ $[l_2 < \infty]$, the boundary l_1 $[l_2]$ is *regular reflecting* in the sense of Feller [5]. When $l_1 = -\infty$ $[l_2 = \infty]$, the boundary l_1 $[l_2]$ is *entrance* or *natural* in the sense of Feller [5] according as the integral $\int_{-\infty}^{0} |s(\xi)|\, m(\mathrm{d}\xi)[\int_{0}^{\infty} s(\xi)\, m(\mathrm{d}\xi)]$ converges or diverges. The transition operator $\{p_t^D\}$ possesses the following properties:

$$p_t^D\big(B(S_m)\big) \subset C(S_m), \tag{4.13}$$

$$\lim_{\xi\to l_i,\, \xi\in S_m} p_t^D f(\xi) = 0 \quad \text{if } l_i \text{ is natural and } \lim_{\xi\to l_i,\, \xi\in S_m} f(\xi) = 0, \quad \text{for each } i = 1,\ 2, \tag{4.14}$$

$$\lim_{t\downarrow 0} \sup_{\xi\in S_m} \big|p_t^D f(\xi) - f(\xi)\big| = 0, \quad f \in \widehat{C}(S_m), \tag{4.15}$$

where $B(S_m)$ stands for the space of all bounded measurable functions on S_m. In the case where l_1 $[l_2]$ is entrance, it happens that $\lim_{\xi\to l_1 [l_2],\, \xi\in S_m} p_t^D f(\xi) \neq 0$ even if $\lim_{\xi\to l_1 [l_2],\ \xi\in S_m} f(\xi) = 0$.

LEMMA 4.6. – *Let $\{a_n\}$ be a sequence converging to $a \in \mathbb{R}$. Then*

$$P_0^B\big(\lim_{n\to\infty} \Phi(t; a_n) = \Phi(t; a),\ t \geqslant 0\big) = 1, \tag{4.16}$$

$$P_0^B\big(\lim_{n\to\infty} \phi(t; a_n) = \phi(t; a),\ t \geqslant 0\big) = 1, \tag{4.17}$$

$$P_0^B\big(\lim_{n\to\infty} \mathbf{f}(t; a_n) = \mathbf{f}(t; a),\ t \geqslant 0\big) = 1, \tag{4.18}$$

$$P_0^B\big(\lim_{n\to\infty} A(t; a_n) = A(t; a),\ t \geqslant 0\big) = 1, \tag{4.19}$$

$$P_0^B\big(\lim_{n\to\infty} \tau(t; a_n) = \tau(t; a)\big) = 1, \quad t > 0, \tag{4.20}$$

$$P_0^B\big(\lim_{n\to\infty} \phi\big(\tau(t; a_n); a_n\big) = \phi\big(\tau(t; a); a\big)\big) = 1, \quad t > 0, \tag{4.21}$$

$$P_0^B\big(\lim_{n\to\infty} \mathbf{f}\big(\tau(t; a_n); a_n\big) = \mathbf{f}\big(\tau(t; a); a\big)\big) = 1, \quad t > 0. \tag{4.22}$$

Proof. – We set $\gamma_a(\mathrm{d}\eta) = \gamma'(\eta + s(a))\,\mathrm{d}\eta$. The Radon measures γ_{a_n} converge to the Radon measure γ_a vaguely as $a_n \to a$. Therefore (4.16) follows from (4.10) by the same reason as for (3.10). (4.17) is a direct consequence of (4.16). We set $\widetilde{\rho}(\xi) = \rho([0, \xi]),\ \xi \geqslant 0,\ = -\rho((\xi, 0)),\ \xi < 0$, and further $\widetilde{\rho}_a(\xi) = \widetilde{\rho}(\gamma(\xi + s(a)))$. Then it holds that

$$\mathbf{f}(t; a) = t + 2 \int_{\mathbb{R}} \ell^B\big(\phi(t; a), \xi\big)\, \widetilde{\rho}_a(\mathrm{d}\xi).$$

Since the measures $\widetilde{\rho}_{a_n}$ converge to the measure $\widetilde{\rho}_a$ vaguely as $a_n \to a$, we get (4.18) by the same argument as for (3.14). In a similar way we obtain (4.19). We can derive (4.20) from (4.19) by the same argument as for (3.12). It is easy to see that

$$\begin{aligned}
\lim_{n\to\infty} A\big(\Phi(t; a_n); a_n\big) &= \lim_{n\to\infty} 2 \int_{\mathbb{R}} \ell^B\big(t, s(\xi) - s(a_n)\big) m(\mathrm{d}\xi) \\
&= 2 \int_{\mathbb{R}} \ell^B\big(t, s(\xi) - s(a)\big) m(\mathrm{d}\xi) \\
&= A(\Phi(t; a); a), \quad t \geqslant 0, \quad P_0^B\text{-a.s.}
\end{aligned}$$

Since $\phi(\tau(t;a);a)$ is the right continuous inverse of $t \longmapsto A(\Phi(t;a);a)$, we get (4.21) by the same reason as for (3.12). We obtain (4.22) from (4.18) and (4.20) by the same method as for (3.14). □

PROPOSITION 4.7. – *It holds that*

$$p_t\big(\widehat{C}(\Omega_\mu)\big) \subset C\big(\mathbb{R}^d\big), \quad t > 0.$$

Proof. – Let us fix $t > 0$ and $f \in \widehat{C}(\Omega_\mu)$ arbitrarily and let $\{x_n\}_{n=1}^\infty$ be a sequence converging to $x_0 \in \mathbb{R}^d$. We set

$$u_n = \mathbf{f}\big(\tau\big(t; x_n^d\big); x_n^d\big), \quad \xi_n = \gamma\left(B_{\phi(\tau(t;x_n^d);x_n^d)} + s\big(x_n^d\big)\right), \quad n \in \mathbb{N} \cup \{0\}.$$

By means of Lemma 4.4,

$$\begin{aligned}
|p_t f(x_n) - p_t f(x_0)| &\leqslant E_0^B\big[\big|E'_{x'_n}\big[f\big(X'(u_n), \xi_n\big)\big] - E'_{x'_0}\big[f\big(X'(u_0), \xi_0\big)\big]\big|\big] \\
&\leqslant E_0^B\big[E'_{x'_n}\big[\big|f\big(X'(u_n), \xi_n\big) - f\big(X'(u_n), \xi_0\big)\big|\big]\big] \\
&\quad + E_0^B\big[\big|E'_{x'_n}\big[f\big(X'(u_n), \xi_0\big)\big] - E'_{x'_0}\big[f\big(X'(u_0), \xi_0\big)\big]\big|\big] \\
&\leqslant E_0^B\Big[\sup_{y' \in \mathbb{R}^{d-1}} \big|f(y', \xi_n) - f(y', \xi_0)\big|\Big] \\
&\quad + E_0^B\big[\big|p'_{u_n} f(\cdot, \xi_0)(x'_n) - p'_{u_0} f(\cdot, \xi_0)(x'_0)\big|\big] \\
&\equiv I_n + II_n.
\end{aligned}$$

Since $P_0^B(\lim_{n\to\infty} \xi_n = \xi_0) = 1$ by virtue of Lemma 4.6,

$$\limsup_{n\to\infty} I_n \leqslant E_0^B\Big[\limsup_{n\to\infty} \sup_{y' \in \mathbb{R}^{d-1}} \big|f(y', \xi_n) - f(y', \xi_0)\big|\Big] = 0.$$

Noting that $\{p'_t, t > 0\}$ is a Feller semigroup, and $P_0^B(\lim_{n\to\infty} u_n = u_0) = 1$ by Lemma 4.6, we have

$$\limsup_{n\to\infty} II_n \leqslant E_0^B\Big[\limsup_{n\to\infty} \big|p'_{u_n} f(\cdot, \xi_0)(x'_n) - p'_{u_0} f(\cdot, \xi_0)(x'_0)\big|\Big] = 0.$$

We thus get the conclusion of the proposition. □

LEMMA 4.8. – *For each $K > 0$, let*

$$\mathbf{f}_K(t) = \sup_{|a| \leqslant K} \mathbf{f}(t;a), \qquad A_K(t) = \inf_{|a| \leqslant K,\ a \in S_m} A(t;a).$$

Then it holds that

$$P_0^B(\mathbf{f}_K(0+) = 0) = 1, \tag{4.23}$$

$$P_0^B\big(\mathbf{f}_K(t) < \infty,\ t \geqslant 0\big) = 1, \tag{4.24}$$

$$P_0^B\big(A_K(t) > 0,\ t > 0\big) = 1 \quad \textit{if } S_m \cap [-K, K] \neq \emptyset, \tag{4.25}$$

$$P_0^B\Big(\lim_{t\to\infty} A_K(t) = \infty\Big) = 1. \tag{4.26}$$

Proof. – Let $a \in \mathbb{R}$, $-\infty < \alpha \leqslant 0 \leqslant \beta < \infty$ and $I = [\gamma(s(a)+\alpha), \gamma(s(a)+\beta)]$. We then note that

$$s(\xi) - s(a) \in [\alpha, \beta] \quad \text{if and only if} \quad \xi \in I, \tag{4.27}$$

$$\left[a + \Lambda^{-1}\alpha, a + \Lambda^{-1}\beta\right] \subset I \subset [a + \Lambda\alpha, a + \Lambda\beta]. \tag{4.28}$$

Indeed,

$$\gamma\left(s(a)+\beta\right) - a = \gamma\left(s(a)+\beta\right) - \gamma\left(s(a)\right) = \int_{s(a)}^{s(a)+\beta} \gamma'(\eta)\,\mathrm{d}\eta,$$

and hence $a + \Lambda^{-1}\beta \leqslant \gamma(s(a)+\beta) \leqslant a + \Lambda\beta$. In the same way, $a + \Lambda\alpha \leqslant \gamma(s(a)+\alpha) \leqslant a + \Lambda^{-1}\alpha$. Therefore we get (4.28).

We set $\alpha_t = \min_{0 \leqslant s \leqslant t} B_s$, $\beta_t = \max_{0 \leqslant s \leqslant t} B_s$, and $E_t = [\alpha_t, \beta_t]$. By means of (4.11), (4.27) and (4.28),

$$\begin{aligned}
\int_{\mathbb{R}} \ell^B\left(\phi(t;a), s(\xi) - s(a)\right)\rho(\mathrm{d}\xi) &\leqslant \int_{\mathbb{R}} \ell^B\left(\Lambda t, s(\xi) - s(a)\right)\rho(\mathrm{d}\xi) \\
&\leqslant \max_{\eta \in E_{\Lambda t}} \ell^B(\Lambda t, \eta)\rho\left(\{\xi\colon\, s(\xi) - s(a) \in E_{\Lambda t}\}\right) \\
&\leqslant \max_{\eta \in E_{\Lambda t}} \ell^B(\Lambda t, \eta)\rho\left([a + \Lambda\alpha_{\Lambda t}, a + \Lambda\beta_{\Lambda t}]\right) \\
&\leqslant \max_{\eta \in E_{\Lambda t}} \ell^B(\Lambda t, \eta)\rho\left([-K + \Lambda\alpha_{\Lambda t}, K + \Lambda\beta_{\Lambda t}]\right),
\end{aligned}$$

from which (4.23) follows.

By means of (4.18), $\mathbf{f}(t,a)$ is continuous in $a \in \mathbb{R}$, which implies (4.24).

By using (4.11), (4.27) and (4.28) again, and noting that $\Lambda^{-1} < 1$,

$$\begin{aligned}
A(t;a) &\geqslant 2 \int_{\xi\colon\, s(\xi) - s(a) \in E_{t/\Lambda}} \ell^B\left(t/\Lambda, s(\xi) - s(a)\right) m(\mathrm{d}\xi) \\
&\geqslant 2 \int_{\xi\colon\, s(\xi) - s(a) \in [\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}]} \ell^B\left(t/\Lambda, s(\xi) - s(a)\right) m(\mathrm{d}\xi) \\
&\geqslant 2 \min_{\eta \in [\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}]} \ell^B(t/\Lambda, \eta) \\
&\quad \times m\left(\left\{\xi\colon\, s(\xi) - s(a) \in \left[\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}\right]\right\}\right) \\
&\geqslant 2 \min_{\eta \in [\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}]} \ell^B(t/\Lambda, \eta) m\left(\left[a + \Lambda^{-2}\alpha_{t/\Lambda}, a + \Lambda^{-2}\beta_{t/\Lambda}\right]\right).
\end{aligned}$$

Hence

$$A_K(t) \geqslant 2 \min_{\eta \in [\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}]} \ell^B(t/\Lambda, \eta) \inf_{|a| \leqslant K,\ a \in S_m} m\left(\left[a + \Lambda^{-2}\alpha_{t/\Lambda}, a + \Lambda^{-2}\beta_{t/\Lambda}\right]\right).$$

We notice that

$$\min_{\eta \in [\Lambda^{-1}\alpha_{t/\Lambda}, \Lambda^{-1}\beta_{t/\Lambda}]} \ell^B(t/\Lambda, \eta) > 0, \quad t > 0,$$

because of $\Lambda^{-1} < 1$. We next note that

$$\inf_{|a|\leqslant K,\ a\in S_m} m\big(\big[a+\Lambda^{-2}\alpha_{t/\Lambda}, a+\Lambda^{-2}\beta_{t/\Lambda}\big]\big) > 0, \quad t>0.$$

For, if not, there exists an a such that $|a|\leqslant K,\ a\in S_m$ and

$$m\big(\big(a+\Lambda^{-2}\alpha_{t/\Lambda}, a+\Lambda^{-2}\beta_{t/\Lambda}\big)\big) = 0,$$

which contradicts that $a\in S_m$. Thus we get (4.25).

We take an $R>0$ such that $m([-R,R])>0$. Then

$$\begin{aligned} A(t;a) &\geqslant 2\int_{|\xi|\leqslant R} \ell^B\big(t/\Lambda, s(\xi)-s(a)\big)m(\mathrm{d}\xi) \\ &\geqslant 2\min_{\eta=s(\xi)-s(a),\ |\xi|\leqslant R} \ell^B(t/\Lambda,\eta)m([-R,R]) \\ &\geqslant 2\min_{|\eta|\leqslant\Lambda(R+K)} \ell^B(t/\Lambda,\eta)m([-R,R]). \end{aligned}$$

In order to get (4.26), it is enough to show that

$$P_0^B\big(\lim_{t\to\infty}\min_{|\xi|\leqslant L}\ell^B(t,\xi)=\infty\big)=1, \quad L>0.$$

Since $(B(t),\ P_0^B)$ is equivalent to $(c^{-1}B(c^2t),\ P_0^B)$ and hence $(\ell^B(t,\xi),\ P_0^B)$ is equivalent to $(c^{-1}\ell^B(c^2t,c\xi),\ P_0^B)$ for $c>0$, we see that, for every $r>0$,

$$\begin{aligned} P_0^B\big(\lim_{k\to\infty}\min_{|\xi|\leqslant L}\ell^B(k,\xi)>r\big) &= P_0^B\Big(\bigcup_{k\in\mathbb{N}}\big\{\min_{|\xi|\leqslant L}\ell^B(k,\xi)>r\big\}\Big) \\ &= \lim_{k\to\infty} P_0^B\big(\sqrt{k}\min_{|\xi|\leqslant L}\ell^B(1,\xi/\sqrt{k})>r\big) \\ &= P_0^B\Big(\bigcup_{k\in\mathbb{N}}\big\{\min_{|\xi|\leqslant L/\sqrt{k}}\ell^B(1,\xi)>r/\sqrt{k}\big\}\Big) \\ &= P_0^B\big(\ell^B(1,0)>0\big)=1. \qquad \square \end{aligned}$$

We denote by $\tau_K(t)$ the right continuous inverse of $t\longmapsto A_K(t)$. By virtue of Lemma 4.8,

$$\sup_{|a|\leqslant K,\,a\in S_m}\mathbf{f}\big(\tau(t;a);a\big)\leqslant \mathbf{f}_K\big(\tau_K(t)\big)<\infty, \quad t>0, \quad P_0^B\text{-a.s.}, \tag{4.29}$$

$$\mathbf{f}_K\big(\tau_K(0+)\big)=0, \quad P_0^B\text{-a.s.}, \tag{4.30}$$

for every $K>0$.

PROPOSITION 4.9. – *It holds that*

$$\lim_{|x'|\to\infty}\sup_{|x^d|\leqslant K,\,x^d\in S_m}|p_tf(x)|=0, \quad f\in\widehat{C}(\Omega_\mu),\ K>0.$$

Proof. – We set $h(y') = \sup_{y^d \in \mathbb{R}} |f(y', y^d)|$ $(\in \widehat{C}(\mathbb{R}^{d-1}))$ and $u = \mathbf{f}(\tau(t; x^d); x^d)$. By virtue of Lemma 4.4,

$$|p_t f(x)| \leqslant E_0^B\big[E'_{x'}[h(X'_u)]\big] = E_0^B\bigg[\int_{\mathbb{R}^{d-1}} p'(u, x', y')h(y')\,\mathrm{d}y'\bigg].$$

It is well known that

$$C_1 t^{-(d-1)/2} \mathrm{e}^{-C_2|x'-y'|^2/t} \leqslant p'(t, x', y') \leqslant C_3 t^{-(d-1)/2} \mathrm{e}^{-C_4|x'-y'|^2/t},\quad t > 0,\ x', y' \in \mathbb{R}^{d-1}, \tag{4.31}$$

with some positive constants C_i $(i = 1, 2, 3, 4)$ ([3,4]), from which

$$\begin{aligned}
&\limsup_{|x'|\to\infty} \sup_{|x^d|\leqslant K,\, x^d\in S_m} |p_t f(x)| \\
&\quad \leqslant E_0^B\bigg[\limsup_{|x'|\to\infty} \sup_{|x^d|\leqslant K,\, x^d\in S_m} \int_{\mathbb{R}^{d-1}} p'(u, x', y')h(y')\,\mathrm{d}y'\bigg] \\
&\quad \leqslant C_3 E_0^B\bigg[\limsup_{|x'|\to\infty} \sup_{|x^d|\leqslant K,\, x^d\in S_m} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-C_4|y'|^2} h\big(x' + \sqrt{u}\,y'\big)\,\mathrm{d}y'\bigg] \\
&\quad \leqslant C_3 E_0^B\bigg[\int_{\mathbb{R}^{d-1}} \mathrm{e}^{-C_4|y'|^2} \limsup_{|x'|\to\infty} \sup_{|x^d|\leqslant K,\, x^d\in S_m} h\big(x' + \sqrt{u}\,y'\big)\,\mathrm{d}y'\bigg].
\end{aligned}$$

Noting (4.29), we see that

$$\sup_{|x'|\to\infty} \sup_{|x^d|\leqslant K,\, x^d\in S_m} h\big(x' + \sqrt{u}\,y'\big) = 0, \qquad y' \in \mathbb{R}^{d-1}.$$

We thus obtain the conclusion of the proposition. □

PROPOSITION 4.10. – *If* (4.5) *is satisfied, then*

$$\lim_{x^d\to-\infty,\, x^d\in S_m} \sup_{x'\in\mathbb{R}^{d-1}} |p_t f(x)| = 0, \qquad f \in \widehat{C}(\Omega_\mu). \tag{4.32}$$

If (4.6) *is satisfied, then*

$$\lim_{x^d\to\infty,\, x^d\in S_m} \sup_{x'\in\mathbb{R}^{d-1}} |p_t f(x)| = 0, \qquad f \in \widehat{C}(\Omega_\mu). \tag{4.33}$$

Proof. – We set $h(y^d) = \sup_{y'\in\mathbb{R}^{d-1}} |f(y', y^d)|$, which belongs to $\widehat{C}(S_m)$. By virtue of Lemma 4.4 and Remark 4.5,

$$\sup_{x'\in\mathbb{R}^{d-1}} |p_t f(x)| \leqslant E_0^B\big[h\big(\gamma\big(B_{\phi(\tau(t;x^d);x^d)} + s\big(x^d\big)\big)\big)\big] = p_t^D h\big(x^d\big). \tag{4.34}$$

By means of (4.9) and Remark 4.5, the assumption (4.5) [(4.6)] implies that if $l_1 = -\infty$ $[l_2 = \infty]$, it is a natural boundary. Accordingly, (4.32) and (4.33) immediately follow from (4.14). □

PROPOSITION 4.11. – *Suppose that* (4.5) *and* (4.6) *are satisfied. Then it holds that*

$$\limsup_{t\downarrow 0}\sup_{x\in\Omega_\mu}\left|p_t f(x)-f(x)\right|=0,\qquad f\in\widehat{C}(\Omega_\mu). \tag{4.35}$$

Proof. – Let $f\in\widehat{C}(\Omega_\mu)$ and put $h(y^d)=\sup_{y'\in\mathbb{R}^{d-1}}|f(y',y^d)|$. Let us fix an $\varepsilon>0$ arbitrarily and take a $K>0$ satisfying $\sup_{|y^d|\geqslant K,\,y^d\in S_m}h(y^d)<\varepsilon$ and $[-K,K]\cap S_m\neq\emptyset$. Then

$$\begin{aligned}&\sup_{x\in\Omega_\mu}\left|p_t f(x)-f(x)\right|\\&\quad\leqslant\sup_{x'\in\mathbb{R}^{d-1},\,|x^d|\geqslant K,\,x^d\in S_m}\left|p_t f(x)-f(x)\right|+\sup_{x'\in\mathbb{R}^{d-1},\,|x^d|\leqslant K,\,x^d\in S_m}\left|p_t f(x)-f(x)\right|\\&\quad\equiv I_t+II_t.\end{aligned}$$

By means of (4.15), we have a $t_o>0$ such that

$$\left|p_t^D h\left(x^d\right)-h\left(x^d\right)\right|<\varepsilon,\qquad x^d\in S_m,\ 0<t\leqslant t_o,$$

from which

$$p_t^D h\left(x^d\right)\leqslant h\left(x^d\right)+\varepsilon<2\varepsilon,\qquad \left|x^d\right|\geqslant K,\ x^d\in S_m,\ 0<t\leqslant t_o.$$

Combining this with (4.34),

$$I_t<3\varepsilon,\qquad 0<t\leqslant t_o. \tag{4.36}$$

Since the diffusion processes $\mathbf{X}'$ and D are conservative, we find by Lemma 4.4 that

$$\begin{aligned}\left|p_t f(x)-f(x)\right|&\leqslant E_0^B\left[E'_{x'}\left[\left|f(X'_u,\xi)-f(x',\xi)\right|\right]\right]+E_0^B\left[\left|f(x',\xi)-f\left(x',x^d\right)\right|\right]\\&\equiv III(t,x)+IV(t,x),\end{aligned}$$

where $u=\mathbf{f}(\tau(t;x^d);x^d)$, $\xi=\gamma(B_{\phi(\tau(t;x^d);x^d)}+s(x^d))$. Noting (4.31), we find that

$$\begin{aligned}III(t,x)&=E_0^B\Bigg[\int_{\mathbb{R}^{d-1}}p'(u,x',y')\left|f(y',\xi)-f(x',\xi)\right|\mathrm{d}y'\Bigg]\\&\leqslant C_3E_0^B\Bigg[\int_{\mathbb{R}^{d-1}}\mathrm{e}^{-C_4|y'|^2}\left|f\left(x'+\sqrt{u}y',\xi\right)-f(x',\xi)\right|\mathrm{d}y'\Bigg].\end{aligned}$$

We note that $u\to 0$ as $t\downarrow 0$ uniformly in $x^d\in[-K,K]\cap S_m$ by virtue of (4.29) and (4.30). Hence

$$\limsup_{t\downarrow 0}\sup_{x'\in\mathbb{R}^{d-1},\,|x'|\leqslant K,\,x^d\in S_m,\,\eta\in S_m}\left|f\left(x'+\sqrt{u}y',\eta\right)-f(x',\eta)\right|=0,\qquad y'\in\mathbb{R}^{d-1},$$

from which

$$\limsup_{t\downarrow 0}\sup_{x'\in\mathbb{R}^{d-1},\,|x'|\leqslant K,\,x^d\in S_m}III(t,x)$$

$$
\begin{aligned}
&\leqslant C_3 E_0^B\Bigg[\int\limits_{\mathbb{R}^{d-1}} \mathrm{e}^{-C_4|y'|^2} \\
&\quad \times \limsup_{t\downarrow 0} \sup_{x'\in\mathbb{R}^{d-1},\,|x'|\leqslant K,\,x^d\in S_m,\,\eta\in S_m} \big|f\big(x'+\sqrt{u}\,y',\eta\big)-f(x',\eta)\big|\,\mathrm{d}y'\Bigg] \\
&= 0. \qquad (4.37)
\end{aligned}
$$

By means of (4.11) and Lemma 4.8,

$$
\begin{aligned}
\sup_{|x^d|\leqslant K,\,x^d\in S_m} \big|\xi - x^d\big| &= \sup_{|x^d|\leqslant K,\,x^d\in S_m} \big|\gamma\big(B_{\phi(\tau(t;x^d);x^d)} + s\big(x^d\big)\big) - \gamma\big(s\big(x^d\big)\big)\big| \\
&\leqslant \Lambda \sup_{|x^d|\leqslant K,\,x^d\in S_m} |B_{\phi(\tau(t;x^d);x^d)}| \leqslant \Lambda \max_{0\leqslant s\leqslant \Lambda\tau_K(t)} |B_s| \to 0 \quad \text{as } t\downarrow 0,
\end{aligned}
$$

from which

$$
\begin{aligned}
&\limsup_{t\downarrow 0} \sup_{x'\in\mathbb{R}^{d-1},\,|x^d|\leqslant K,\,x^d\in S_m} IV(t,x) \\
&\quad \leqslant E_0^B\Big[\limsup_{t\downarrow 0} \sup_{x'\in\mathbb{R}^{d-1},\,|x^d|\leqslant K,\,x^d\in S_m} \big|f(x',\xi) - f\big(x',x^d\big)\big|\Big] = 0. \qquad (4.38)
\end{aligned}
$$

(4.37) and (4.38) yield $\lim_{t\downarrow 0} II_t = 0$. Combining this with (4.36), we obtain (4.35). □

Proof of Theorem 4.3. – Let us fix an $\varepsilon > 0$ arbitrarily. By virtue of Proposition 4.10, there is a $K_1 > 0$ such that

$$
\sup_{x'\in\mathbb{R}^{d-1},\,|x^d|\geqslant K_1,\,x^d\in S_m} |p_t f(x)| < \varepsilon.
$$

Further, by means of Proposition 4.9, there is a $K_2 > K_1$ such that

$$
\sup_{|x'|\geqslant K_2,\,|x^d|\leqslant K_1,\,x^d\in S_m} |p_t f(x)| < \varepsilon.
$$

Consequently

$$
\sup_{|x|\geqslant 2K_2,\,x\in\Omega_\mu} |p_t f(x)| < \varepsilon.
$$

Proposition 4.7 coupled with this implies (4.7). (4.8) is already shown in Proposition 4.11. □

5. Dirichlet form of the limit process

In this section, we derive the Dirichlet form $(\mathcal{E},\mathcal{F})$ of the limit process $\mathbf{X}$ defined by (3.8). $\mathbf{X}$ is a time changed process of the conservative diffusion process $\mathbf{Y}$ defined by (4.3), which is the skew product of $\mathbf{X}'$ and Ξ with respect to $\mathbf{f}$ defined by (3.4). Since $\mathrm{d}\xi + \rho(\mathrm{d}\xi)$ charges no set of zero $\mathcal{E}^{\Xi}$-capacity, by virtue of [6], the Dirichlet space $(\mathcal{E}^Y,\mathcal{F}^Y)$ associated with $\mathbf{Y}$ is regular on $L^2(\mathbb{R}^d,\mathrm{d}x)$ and has $C_0^\infty(\mathbb{R}^d)$ as a core. Further $(\mathcal{E}^Y,\mathcal{F}^Y)$ is given by

$$\mathcal{E}^Y(u,u) = \frac{1}{2}\int_{\mathbb{R}^d} \sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}u(x)\partial_{x^j}u(x)\,dx'\big(\mathrm{d}x^d+\rho\big(\mathrm{d}x^d\big)\big)$$
$$+\frac{1}{2}\int_{\mathbb{R}^d} a^{dd}\big(x^d\big)\partial_{x^d}u(x)\partial_{x^d}u(x)\,\mathrm{d}x,$$

$$\mathcal{F}^Y = \left\{u\in L^2(\mathbb{R}^d,\mathrm{d}x):\ \begin{matrix}\partial_{x^i}u\in L^2(\mathbb{R}^d,\mathrm{d}x'(\mathrm{d}x^d+\rho(\mathrm{d}x^d))),\ 1\leqslant i\leqslant d-1\\ \partial_{x^d}u\in L^2(\mathbb{R}^d,\mathrm{d}x)\end{matrix}\right\}.$$

We next note that $\mu(\mathrm{d}x) = \mathrm{d}x' m(\mathrm{d}x^d)$ charges no set of zero $\mathcal{E}^Y$-capacity. For this, it is enough to show that, for every compact set $B \subset \mathbb{R}^d$, there is a positive constant C such that

$$\int_B |u(x)|\,\mu(\mathrm{d}x) \leqslant C\sqrt{\mathcal{E}_1^Y(u,u)},\quad u\in C_0^\infty(\mathbb{R}^d), \tag{5.1}$$

that is, $I_B(x)\,\mu(\mathrm{d}x)$ is of finite energy integral, where $\mathcal{E}_1^Y(u,u) = \mathcal{E}^Y(u,u) + (u,u)_{L^2(\mathbb{R}^d)}$. Let Φ be an element of $C_0^\infty(\mathbb{R})$ such that $\Phi(\xi) = 1$ for $\xi \in I \equiv \{x^d\colon (x',x^d)\in B$ for some $x'\in\mathbb{R}^{d-1}\}$. Then it is easy to see that

$$\int_B |u|\,\mathrm{d}\mu \leqslant \big\{\|\partial_{x^d}u\|_{L^2(\mathbb{R}^d)}\|\Phi\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R}^d)}\|\Phi'\|_{L^2(\mathbb{R})}\big\}\mu(B)^{1/2}m(I)^{1/2},$$

from which (5.1) follows. Further we note that $A(t)$ defined by (3.6) is a positive continuous additive functional of the diffusion process $\mathbf{Y}$ and $P_x(A(t)>0,\ t>0) = 1,\quad x\in\Omega_\mu$, because of $P^{\Xi}_{x^d}(A(t)>0,\ t>0)=1,\quad x^d\in S_m$. Employing Theorem 6.2.1 in [7], we then see that the Dirichlet space $(\mathcal{E},\mathcal{F})$ is regular on $L^2(\Omega_\mu,\mu)$ and has $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ as a core.

If $S_m = \mathbb{R}$, then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is given by (2.3) with the domain $\mathcal{F} = \{u\in L^2(\mathbb{R}^d,\mu)\colon \partial_{x^i}u\in L^2(\mathbb{R}^d,\mathrm{d}x'(\mathrm{d}x^d+\rho(\mathrm{d}x^d))),\ 1\leqslant i\leqslant d-1,\ \partial_{x^d}u\in L^2(\mathbb{R}^d,\mathrm{d}x)\}$, which is immediately derived from [7, Theorem 6.2.1].

We are thus restricted to the case where $\mathbb{R}\setminus S_m \neq \emptyset$. In this case, $p_t f(x)$ is regarded as a function of $x\in\Omega_\mu$ in the sense of Proposition 4.1. Following [7, Theorem 6.2.1], we see that the restricted transition operator $\{p_t, t>0\}$ to Ω_μ determines a strongly continuous semigroup $\{T_t, t>0\}$ on $L^2(\Omega_\mu,\mu)$, which is associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\Omega_\mu,\mu)$ being regarded as the Dirichlet form of the time changed process $\mathbf{X}$.

We now assume that

$$a^{dd}(\xi) = 1,\quad \xi\in\mathbb{R}\setminus S_m, \tag{5.2}$$

$$\mathrm{supp}[\rho]\subset S_m. \tag{5.3}$$

$\mathbb{R}\setminus S_m$ is expressed as $\mathbb{R}\setminus S_m = \bigcup_{k\in K} I_k$, a finite or a countable disjoint union of open intervals $I_k = (a_k,b_k)$ with the end points belonging to $S_m\cup\{-\infty,\ \infty\}$. We put $\Gamma = \mathbb{R}^{d-1}$ and $\Omega_k = \Gamma\times I_k$. Let $G_k(x,y)$ be the Green function on Ω_k of the Dirichlet boundary value problem corresponding to the operator $\frac{1}{2}\{\sum_{i,j=1}^{d-1}\partial_{x^i}(a^{ij}\partial_{x^j}) + \partial_{x^d}\partial_{x^d}\}$

in the weak sense. Note that $a^{dd}(x^d) = 1$ on Ω_k because of (5.2). We introduce two functions U_k and V_k on $\Gamma \times \Gamma$ accompanied with the interval I_k, where we denote the right and left partial derivatives in x^d by $\partial^+_{x^d}$ and $\partial^-_{x^d}$ respectively.

$$U_k(x', y') = \begin{cases} \partial^+_{x^d}\partial^+_{y^d} G_k((x', a_k), (y', a_k)), & \text{if } a_k > -\infty, \\ \partial^-_{x^d}\partial^-_{y^d} G_k((x', b_k), (y', b_k)), & \text{if } b_k < \infty, \end{cases} \tag{5.4}$$

$$\begin{aligned} V_k(x', y') &= -\partial^+_{x^d}\partial^-_{y^d} G_k((x', a_k), (y', b_k)) \\ &= -\partial^-_{x^d}\partial^+_{y^d} G_k((x', b_k), (y', a_k)), \quad \text{if } -\infty < a_k < b_k < \infty. \end{aligned} \tag{5.5}$$

We notice that the right hand sides of (5.4) are the same whenever $-\infty < a_k < b_k < \infty$ (see (5.13) with (5.16) below).

THEOREM 5.1. – *Assume that $S_m \neq \mathbb{R}$ and* (5.2), (5.3) *hold. Then the Dirichlet space $(\mathcal{E}, \mathcal{F})$ of $\mathbf{X}$ is regular on $L^2(\Omega_\mu, \mu)$ and has $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ as a core. For $u \in C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$, the Dirichlet form $\mathcal{E}(u, u)$ is given by*

$$\begin{aligned} \mathcal{E}(u, u) = {} & \frac{1}{2} \int_{\Omega_\mu} \left\{ \sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i} u(x)\partial_{x^j} u(x) + a^{dd}\big(x^d\big)\partial^*_{x^d} u(x)\partial^*_{x^d} u(x) \right\} \mathrm{d}x \\ & + \frac{1}{2} \int_{\Omega_\mu} \sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i} u(x)\partial_{x^j} u(x)\, \mathrm{d}x' \rho\big(\mathrm{d}x^d\big) \\ & + \frac{1}{8} \sum_{k\in K:\ -\infty<a_k<b_k\leqslant\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_k) - u(y', a_k)\}^2 U_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y' \\ & + \frac{1}{8} \sum_{k\in K:\ -\infty\leqslant a_k<b_k<\infty} \iint_{\Gamma\times\Gamma} \{u(x', b_k) - u(y', b_k)\}^2 U_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y' \\ & + \frac{1}{4} \sum_{k\in K:\ -\infty<a_k<b_k<\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_k) - u(y', b_k)\}^2 V_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y', \end{aligned} \tag{5.6}$$

where the first term of the right hand side vanishes in case where $|S_m| = 0$.[4] *In case where $|S_m| > 0$, $\partial^*_{x^d} u$ is defined by*

$$\partial^*_{x^d} u\big(x', x^d\big) = \lim_{\xi\to x^d,\, \xi\in S_m} \big\{u(x', \xi) - u\big(x', x^d\big)\big\}/\big(\xi - x^d\big), \tag{5.7}$$

which exists for every $x' \in \Gamma$ and $\mathrm{d}x^d$-a.e. $x^d \in S_m$.

In order to prove Theorem 5.1, we prepare some lemmas. We first note that $G_k(x, y)$ is expressed as

$$G_k(x, y) = \int_0^\infty p'(t, x', y') q_k\big(t, x^d, y^d\big)\, \mathrm{d}t, \quad x, y \in \Omega_k, \tag{5.8}$$

[4] $|E|$ stands for the Lebesgue measure of a set E.

where $p'(t,x',y')$ is the transition probability density of $\mathbf{X}'$ and $q_k(t,\xi,\eta)$ is that of the absorbing Brownian motion on I_k, which is given as follows. If $-\infty = a_k < b_k < \infty$,

$$q_k(t,\xi,\eta) = \frac{1}{\sqrt{2\pi t}}\left\{e^{-|\xi-\eta|^2/2t} - e^{-|\xi+\eta-2b_k|^2/2t}\right\}. \tag{5.9}$$

If $-\infty < a_k < b_k = \infty$,

$$q_k(t,\xi,\eta) = \frac{1}{\sqrt{2\pi t}}\left\{e^{-|\xi-\eta|^2/2t} - e^{-|\xi+\eta-2a_k|^2/2t}\right\}. \tag{5.10}$$

In the case where $-\infty < a_k < b_k < \infty$, we have the following two expressions:

$$q_k(t,\xi,\eta) = \frac{2}{b_k - a_k}\sum_{n=1}^{\infty} e^{-(n\pi/(b_k-a_k))^2 t/2} \sin\left(\frac{\xi - a_k}{b_k - a_k} n\pi\right) \sin\left(\frac{\eta - a_k}{b_k - a_k} n\pi\right) \tag{5.11}$$

$$= \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{\infty}\left\{e^{-|\eta-\xi+2n(b_k-a_k)|^2/2t} - e^{-|\eta+\xi-2a_k+2n(b_k-a_k)|^2/2t}\right\}. \tag{5.12}$$

Accordingly,

$$U_k(x',y') = \int_0^\infty p'(t,x',y')\alpha_k(t)\,\mathrm{d}t, \tag{5.13}$$

where

$$\alpha_k(t) = \partial_\xi^-\partial_\eta^- q_k(t,b_k,b_k) = \sqrt{\frac{2}{\pi t^3}}, \quad \text{if } -\infty = a_k < b_k < \infty, \tag{5.14}$$

$$\alpha_k(t) = \partial_\xi^+\partial_\eta^+ q_k(t,a_k,a_k) = \sqrt{\frac{2}{\pi t^3}}, \quad \text{if } -\infty < a_k < b_k = \infty, \tag{5.15}$$

$$\begin{aligned}\alpha_k(t) &= \partial_\xi^+\partial_\eta^+ q_k(t,a_k,a_k) = \partial_\xi^-\partial_\eta^- q_k(t,b_k,b_k)\\ &= \frac{2}{b_k-a_k}\sum_{n=1}^{\infty}\left(\frac{n\pi}{b_k-a_k}\right)^2 e^{-(n\pi/(b_k-a_k))^2 t/2},\\ &\quad \text{if } -\infty < a_k < b_k < \infty.\end{aligned} \tag{5.16}$$

Further, if $-\infty < a_k < b_k < \infty$,

$$V_k(x',y') = \int_0^\infty p'(t,x',y')\beta_k(t)\,\mathrm{d}t, \tag{5.17}$$

where

$$\begin{aligned}\beta_k(t) &= -\partial_\xi^+\partial_\eta^- q_k(t,a_k,b_k) = -\partial_\xi^-\partial_\eta^+ q_k(t,b_k,a_k)\\ &= \frac{2}{b_k-a_k}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n\pi}{b_k-a_k}\right)^2 e^{-(n\pi/(b_k-a_k))^2 t/2}\end{aligned} \tag{5.18}$$

$$= \sqrt{\frac{2}{\pi t^3}} \sum_{n=-\infty}^{\infty} \left\{ \frac{(2n+1)^2(b_k - a_k)^2}{t} - 1 \right\} \mathrm{e}^{-(2n+1)^2(b_k-a_k)^2/2t}. \tag{5.19}$$

We note the following estimates.

LEMMA 5.2. – (i) *Let $-\infty = a_k < b_k < \infty$ or $-\infty < a_k < b_k = \infty$. Then there exist positive constants C_5 and C_6 independent of I_k such that for every $x', y' \in \Gamma$,*

$$C_5|x' - y'|^{-d} \leqslant U_k(x', y') \leqslant C_6|x' - y'|^{-d}. \tag{5.20}$$

(ii) *Let $-\infty < a_k < b_k < \infty$. Then there exist positive constants T_o and C_i $(i = 7, 8, \ldots, 12)$ independent of I_k such that, if $|x' - y'| \leqslant T_o(b_k - a_k)$, then*

$$C_7|x' - y'|^{-d} \leqslant U_k(x', y') \leqslant C_8|x' - y'|^{-d}, \tag{5.21}$$

$$C_9(b_k - a_k)^{-d} \leqslant V_k(x', y') \leqslant C_{10}(b_k - a_k)^{-d}, \tag{5.22}$$

and if $|x' - y'| \geqslant T_o(b_k - a_k)$, then for $W = U_k,\ V_k$,

$$\begin{aligned} &C_{11}(b_k - a_k)^{-d/2-1}|x' - y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_2}\pi|x'-y'|/(b_k-a_k)} \\ &\quad \leqslant W(x', y') \leqslant C_{12}(b_k - a_k)^{-d/2-1}|x' - y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_4}\pi|x'-y'|/(b_k-a_k)}, \end{aligned} \tag{5.23}$$

where C_2 and C_4 are positive constants which appeared in (4.31)*.*

Proof. – (i) Let $-\infty = a_k < b_k < \infty$ or $-\infty < a_k < b_k = \infty$. Then we immediately obtain (5.20) from (4.31) and (5.13)–(5.15).

(ii) Let $-\infty < a_k < b_k < \infty$. We divide the proof into four steps. For simplicity we set $\theta = b_k - a_k$. In the following, c_i $(i = 1, 2, \ldots)$ stand for positive constants independent of t, x', y', θ and k.

(Step 1) First we remark that

$$c_1 t^{-3/2} \leqslant \alpha_k(t) \leqslant c_2 t^{-3/2} \qquad \text{if } 0 < t \leqslant \theta^2/2, \tag{5.24}$$

$$c_3\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2} \leqslant \alpha_k(t) \leqslant c_4\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2} \qquad \text{if } t \geqslant \theta^2/2. \tag{5.25}$$

It follows from (5.16) that

$$2\pi^2\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2} \leqslant \alpha_k(t) \leqslant 2\pi^2\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2}\left\{1 + \sum_{n=2}^{\infty} n^2 \mathrm{e}^{-n^2\pi^2 t/4\theta^2}\right\}, \quad t > 0. \tag{5.26}$$

If $t \geqslant \theta^2/2$, then

$$\sum_{n=2}^{\infty} n^2\mathrm{e}^{-n^2\pi^2 t/4\theta^2} \leqslant \sum_{n=2}^{\infty} n^2\mathrm{e}^{-n^2} \leqslant \int_0^{\infty} \xi^2\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}}{4}.$$

Combining this with (5.26), we get (5.25).

Since

$$\begin{aligned} &(1/4)(n+1)^2\mathrm{e}^{-(n+1)^2\pi^2 t/2\theta^2} \leqslant \xi^2\mathrm{e}^{-\xi^2\pi^2 t/2\theta^2} \leqslant 4n^2\mathrm{e}^{-n^2\pi^2 t/2\theta^2}, \\ &n \leqslant \xi \leqslant n+1,\ n \in \mathbb{N}, \end{aligned}$$

we see that

$$\sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2 t/2\theta^2} \leqslant e^{-\pi^2 t/2\theta^2} + 4\int_1^{\infty} \xi^2 e^{-\xi^2\pi^2 t/2\theta^2}\, d\xi$$

$$\leqslant 1 + \frac{8\sqrt{2}\theta^3}{\pi^3 t^{3/2}} \int_{\pi\sqrt{t}/\sqrt{2}\theta}^{\infty} \xi^2 e^{-\xi^2}\, d\xi$$

$$\leqslant \theta^3 t^{-3/2}\{\theta^{-3}t^{3/2} + 2\sqrt{2}\pi^{-5/2}\},$$

$$\sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2 t/2\theta^2} \geqslant \frac{1}{4}\int_1^{\infty} \xi^2 e^{-\xi^2\pi^2 t/2\theta^2}\, d\xi$$

$$= \frac{\theta^3}{\sqrt{2}\pi^3 t^{3/2}} \int_{\pi\sqrt{t}/\sqrt{2}\theta}^{\infty} \xi^2 e^{-\xi^2}\, d\xi.$$

Therefore, if $t \leqslant \theta^2/2$,

$$\frac{\theta^3}{\sqrt{2}\pi^3 t^{3/2}} \int_{\pi/2}^{\infty} \xi^2 e^{-\xi^2}\, d\xi \leqslant \sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2 t/2\theta^2} \leqslant \left(2^{-3/2} + 2\sqrt{2}\pi^{-5/2}\right)\theta^3 t^{-3/2},$$

from which we get (5.24).

(Step 2) We next note that

$$c_5\theta^2 t^{-5/2} e^{-\theta^2/2t} \leqslant \beta_k(t) \leqslant c_6\theta^2 t^{-5/2} e^{-\theta^2/2t} \quad \text{if } 0 < t \leqslant \theta^2/2, \tag{5.27}$$

$$c_7\theta^{-3} e^{-\pi^2 t/2\theta^2} \leqslant \beta_k(t) \leqslant c_8\theta^{-3} e^{-\pi^2 t/2\theta^2} \quad \text{if } t \geqslant \theta^2/2. \tag{5.28}$$

By means of (5.18),

$$\beta_k(t) = 2\pi^2\theta^{-3}\Bigg[e^{-\pi^2 t/2\theta^2}\{1 - 4e^{-(3/2)\pi^2 t/\theta^2}\}$$

$$+ \sum_{n=1}^{\infty}\{(2n+1)^2 e^{-(2n+1)^2\pi^2 t/2\theta^2} - (2n+2)^2 e^{-(2n+2)^2\pi^2 t/2\theta^2}\}\Bigg]$$

$$\leqslant 2\pi^2\theta^{-3} e^{-\pi^2 t/2\theta^2}\left[1 + \sum_{n=1}^{\infty}(2n+1)^2 e^{-\{(2n+1)^2-1\}\pi^2 t/2\theta^2}\right]$$

$$\leqslant 2\pi^2\theta^{-3} e^{-\pi^2 t/2\theta^2}\left[1 + 4\sum_{n=2}^{\infty} n^2 e^{-n^2\pi^2 t/2\theta^2}\right].$$

If $t \geqslant \theta^2/2$, then

$$\sum_{n=2}^{\infty} n^2 e^{-n^2\pi^2 t/2\theta^2} \leqslant \sum_{n=2}^{\infty} n^2 e^{-n^2} < \sqrt{\pi}/4,$$

from which

$$\beta_k(t) < 2\pi^2(1+\sqrt{\pi})\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2}. \tag{5.29}$$

We also see that, if $t \geqslant \theta^2/2$, then $f(\xi) := \xi \mathrm{e}^{-\xi\pi^2 t/2\theta^2}$ is decreasing in $\xi \in [1,\infty)$, and hence $\sum_{n=1}^{\infty}\{f(2n+1) - f(2n+2)\} > 0$, from which

$$\beta_k(t) \geqslant 2\pi^2\theta^{-3}\mathrm{e}^{-\pi^2 t/2\theta^2}\{1 - 4\mathrm{e}^{-3\pi^2/4}\}.$$

This coupled with (5.29) implies (5.28).

Let $0 < t \leqslant \theta^2/2$. Then

$$\frac{(2n+1)^2\theta^2}{t} - 1 \geqslant \frac{\theta^2}{t} - 1 \geqslant \frac{\theta^2}{2t}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Therefore, by means of (5.19),

$$\beta_k(t) \geqslant \frac{1}{\sqrt{2\pi}}\theta^2 t^{-5/2}\mathrm{e}^{-\theta^2/2t}. \tag{5.30}$$

On the other hand,

$$\beta_k(t) \leqslant 2\sqrt{2/\pi}\,\theta^2 t^{-5/2}\mathrm{e}^{-\theta^2/2t}\left\{1 + \sum_{n=1}^{\infty}(2n+1)^2\mathrm{e}^{-\{(2n+1)^2-1\}\theta^2/2t}\right\}.$$

Since

$$\sum_{n=1}^{\infty}(2n+1)^2\mathrm{e}^{-\{(2n+1)^2-1\}\theta^2/2t} \leqslant 4\sum_{n=2}^{\infty}n^2\mathrm{e}^{-n^2} < \sqrt{\pi},$$

we have

$$\beta_k(t) \leqslant 2\sqrt{2/\pi}\,(1+\sqrt{\pi})\theta^2 t^{-5/2}\mathrm{e}^{-\theta^2/2t}.$$

Combining this with (5.30), we get (5.27).

(Step 3) Let C be a positive number and put

$$J_1 = \int_0^{\theta^2/2} t^{-d/2-1}\mathrm{e}^{-C|x'-y'|^2/t}\,\mathrm{d}t,$$

$$J_2 = \theta^2\int_0^{\theta^2/2} t^{-d/2-2}\mathrm{e}^{-C|x'-y'|^2/t}\mathrm{e}^{-\theta^2/2t}\,\mathrm{d}t,$$

$$J_3 = \theta^{-3}\int_{\theta^2/2}^{\infty} t^{-(d-1)/2}\mathrm{e}^{-C|x'-y'|^2/t}\mathrm{e}^{-\pi^2 t/2\theta^2}\,\mathrm{d}t.$$

Then there is a positive constant T_o independent of θ such that, if $|x'-y'| \leqslant T_o\theta$, then

$$c_9|x'-y'|^{-d} \leqslant J_1 \leqslant c_{10}|x'-y'|^{-d}, \tag{5.31}$$

$$c_{11}\theta^{-d} \leqslant J_2 \leqslant c_{12}\theta^{-d}, \tag{5.32}$$

$$0 < J_3 \leqslant c_{13}\theta^{-d}, \tag{5.33}$$

and if $|x'-y'| \geqslant T_o\theta$, then

$$0 < J_1 \leqslant c_{14}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}, \tag{5.34}$$

$$0 < J_2 \leqslant c_{15}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}, \tag{5.35}$$

$$\begin{aligned} &c_{16}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &\quad\leqslant J_3 \leqslant c_{17}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}. \end{aligned} \tag{5.36}$$

Here is a proof. It is known that

$$\int_0^\infty t^{-\nu-1}\mathrm{e}^{-a^2/4t}\mathrm{e}^{-pt}\,\mathrm{d}t = 2^{\nu+1}p^{\nu/2}a^{-\nu}K_\nu\big(a\sqrt{p}\big),$$

where $a > 0$, $p > 0$, and $K_\nu(\xi)$ is the modified Bessel function. Therefore

$$\begin{aligned} J_4 &\equiv \int_0^\infty t^{-(d-1)/2}\mathrm{e}^{-C|x'-y'|^2/t}\mathrm{e}^{-\pi^2 t/2\theta^2}\,\mathrm{d}t \\ &= c_{18}\theta^{-(d-3)/2}|x'-y'|^{-(d-3)/2}K_{(d-3)/2}\big(\sqrt{2C}\,\pi|x'-y'|/\theta\big). \end{aligned}$$

Since $\lim_{\xi\to\infty}\frac{K_\nu(\xi)}{\mathrm{e}^{-\xi}\sqrt{\pi/2\xi}} = 1$, there is a $T_1 > 0$ such that

$$\frac{1}{2}\sqrt{\frac{\pi}{2\xi}}\,\mathrm{e}^{-\xi} \leqslant K_\nu(\xi) \leqslant 2\sqrt{\frac{\pi}{2\xi}}\,\mathrm{e}^{-\xi}, \quad \xi \geqslant T_1.$$

Accordingly, if $|x'-y'|/\theta \geqslant T_1/\sqrt{2C}\,\pi$, then

$$\begin{aligned} &c_{19}\theta^{-d/2+2}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &\quad\leqslant J_4 \leqslant c_{20}\theta^{-d/2+2}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}. \end{aligned} \tag{5.37}$$

Put

$$J_5 = \int_0^{\theta^2/2} t^{-(d-1)/2}\mathrm{e}^{-C|x'-y'|^2/t}\mathrm{e}^{-\pi^2 t/2\theta^2}\,\mathrm{d}t.$$

Then

$$\begin{aligned} J_5 &\leqslant \mathrm{e}^{-C|x'-y'|^2/\theta^2}\int_0^{\theta^2/2} t^{-(d-1)/2}\mathrm{e}^{-C|x'-y'|^2/2t}\,\mathrm{d}t \\ &= 2^{(d-3)/2}|x'-y'|^{-d+3}\mathrm{e}^{-C|x'-y'|^2/\theta^2}\int_0^{\theta^2/|x'-y'|^2} t^{-(d-1)/2}\mathrm{e}^{-C/t}\,\mathrm{d}t. \end{aligned} \tag{5.38}$$

We take a $T_2 \geqslant \min\{T_1/\sqrt{2C}\,\pi,\ 2\}$ such that for every $\xi \geqslant T_2$,

$$\xi^2 \mathrm{e}^{-C\xi^2} \int_0^{1/\xi^2} t^{-d/2-2} \mathrm{e}^{-C/t}\,\mathrm{d}t \leqslant c_{19} \mathrm{e}^{-\sqrt{2C}\,\pi\xi}. \tag{5.39}$$

Hence, if $|x'-y'|/\theta \geqslant T_2$ ($\geqslant 2$), then by means of (5.38),

$$\begin{aligned} J_5 &\leqslant 2^{(d-3)/2} c_{19} \theta^2 |x'-y'|^{-d+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &= 2^{(d-3)/2} c_{19} (\theta/|x'-y'|)^{d/2} \theta^{-d/2+2} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &< 2^{-3/2} c_{19} \theta^{-d/2+2} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}. \end{aligned}$$

Combining this with (5.37), we find that if $|x'-y'|/\theta \geqslant T_2$, then

$$J_3 = \theta^{-3}(J_4 - J_5) \geqslant 2^{-1} c_{19} \theta^{-d/2-1} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}.$$

Noting that $J_3 \leqslant \theta^{-3} J_4$ and putting $T_o = T_2$, we get (5.36).

We also notice that

$$J_3 \leqslant \theta^{-3} \int_{\theta^2/2}^{\infty} t^{-(d-1)/2} \mathrm{e}^{-\pi^2 t/2\theta^2}\,\mathrm{d}t = \theta^{-d} \int_{1/2}^{\infty} t^{-(d-1)/2} \mathrm{e}^{-\pi^2 t/2}\,\mathrm{d}t,$$

which shows (5.33).

We next notice that

$$J_1 = 2^{d/2} |x'-y'|^{-d} \int_0^{\theta^2/|x'-y'|^2} t^{-d/2-1} \mathrm{e}^{-2C/t}\,\mathrm{d}t,$$

and hence, if $|x'-y'|/\theta \leqslant T_o$, then

$$2^{d/2}|x'-y'|^{-d} \int_0^{1/T_o^2} t^{-d/2-1} \mathrm{e}^{-2C/t}\,\mathrm{d}t \leqslant J_1 \leqslant 2^{d/2}|x'-y'|^{-d} \int_0^{\infty} t^{-d/2-1} \mathrm{e}^{-2C/t}\,\mathrm{d}t.$$

We thus obtain (5.31).

If $|x'-y'|/\theta \geqslant T_o$, by means of (5.39),

$$\begin{aligned} J_1 &\leqslant 2^{d/2} |x'-y'|^{-d} \mathrm{e}^{-C|x'-y'|/\theta^2} \int_0^{\theta^2/|x'-y'|^2} t^{-d/2-1} \mathrm{e}^{-C/t}\,\mathrm{d}t \\ &\leqslant 2^{d/2} c_{19} (|x'-y'|/\theta)^{-2} |x'-y'|^{-d} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &= 2^{d/2} c_{19} (|x'-y'|/\theta)^{-d/2-3} \theta^{-d/2-1} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta} \\ &\leqslant 2^{d/2} c_{19} T_o^{-d/2-3} \theta^{-d/2-1} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\,\pi|x'-y'|/\theta}, \end{aligned}$$

which shows (5.34).

Since

$$J_2 = \theta^{-d} \int_0^{1/2} t^{-d/2-2} \mathrm{e}^{-C|x'-y'|^2/\theta^2 t} \mathrm{e}^{-1/2t} \, \mathrm{d}t,$$

we have always

$$J_2 \leqslant \theta^{-d} \int_0^{1/2} t^{-d/2-2} \mathrm{e}^{-1/2t} \, \mathrm{d}t. \tag{5.40}$$

On the other hand, if $|x'-y'|/\theta \leqslant T_o$, then

$$J_2 \geqslant \theta^{-d} \int_0^{1/2} t^{-d/2-2} \mathrm{e}^{-CT_o^2/t} \mathrm{e}^{-1/2t} \, \mathrm{d}t.$$

This coupled with (5.40) implies (5.32).

If $|x'-y'|/\theta \geqslant T_o$, then by means of (5.39),

$$\begin{aligned}
J_2 &\leqslant \theta^2 \mathrm{e}^{-C|x'-y'|^2/\theta^2} \int_0^{\theta^2/2} t^{-d/2-2} \mathrm{e}^{-C|x'-y'|^2/2t} \, \mathrm{d}t \\
&= 2^{d/2+1} \theta^2 |x'-y'|^{-d-2} \mathrm{e}^{-C|x'-y'|^2/\theta^2} \int_0^{\theta^2/|x'-y'|^2} t^{-d/2-2} \mathrm{e}^{-C/t} \, \mathrm{d}t \\
&\leqslant 2^{d/2+1} c_{19} \theta^{-d/2-1} |x'-y'|^{-d/2+1} (|x'-y'|/\theta)^{-d/2-5} \mathrm{e}^{-\sqrt{2C}\pi|x'-y'|/\theta} \\
&\leqslant 2^{d/2+1} c_{19} T_o^{-d/2-5} \theta^{-d/2-1} |x'-y'|^{-d/2+1} \mathrm{e}^{-\sqrt{2C}\pi|x'-y'|/\theta}.
\end{aligned}$$

This shows (5.35).

(Step 4) We give a proof of (5.21), (5.22) and (5.23). By virtue of (4.31), (5.13), (5.24) and (5.25),

$$\begin{aligned}
&c_{20} \left\{ \int_0^{\theta^2/2} t^{-d/2-1} \mathrm{e}^{-C_2|x'-y'|^2/t} \, \mathrm{d}t + \theta^{-3} \int_{\theta^2/2}^{\infty} t^{-(d-1)/2} \mathrm{e}^{-C_2|x'-y'|^2/t} \mathrm{e}^{-\pi^2 t/2\theta^2} \, \mathrm{d}t \right\} \\
&\leqslant U_k(x',y') \\
&\leqslant c_{21} \left\{ \int_0^{\theta^2/2} t^{-d/2-1} \mathrm{e}^{-C_4|x'-y'|^2/t} \, \mathrm{d}t + \theta^{-3} \int_{\theta^2/2}^{\infty} t^{-(d-1)/2} \mathrm{e}^{-C_4|x'-y'|^2/t} \mathrm{e}^{-\pi^2 t/2\theta^2} \, \mathrm{d}t \right\}.
\end{aligned}$$

If $|x'-y'| \leqslant T_o\theta$, then by means of (5.31) and (5.33),

$$c_{22}|x'-y'|^{-d} \leqslant U_k(x',y') \leqslant c_{23}|x'-y'|^{-d},$$

which shows (5.21). If $|x'-y'| \geqslant T_o\theta$, then by means of (5.34) and (5.36),

$$c_{24}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_2}\pi|x'-y'|/\theta}$$
$$\leqslant U_k(x',y') \leqslant c_{25}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_4}\pi|x'-y'|/\theta}.$$

This shows (5.23) with $W=U_k$.

By virtue of (4.31), (5.17), (5.27) and (5.28),

$$\begin{aligned} c_{26}&\left\{\theta^2\int_0^{\theta^2/2} t^{-d/2-2}\mathrm{e}^{-C_2|x'-y'|^2/t}\mathrm{e}^{-\theta^2/2t}\,\mathrm{d}t \right.\\ &\left.+\theta^{-3}\int_{\theta^2/2}^{\infty} t^{-(d-1)/2}\mathrm{e}^{-C_2|x'-y'|^2/t}\mathrm{e}^{-\pi^2t/2\theta^2}\,\mathrm{d}t\right\}\\ &\leqslant V_k(x',y')\\ &\leqslant c_{27}\left\{\theta^2\int_0^{\theta^2/2} t^{-d/2-2}\mathrm{e}^{-C_4|x'-y'|^2/t}\mathrm{e}^{-\theta^2/2t}\,\mathrm{d}t\right.\\ &\left.+\theta^{-3}\int_{\theta^2/2}^{\infty} t^{-(d-1)/2}\mathrm{e}^{-C_4|x'-y'|^2/t}\mathrm{e}^{-\pi^2t/2\theta^2}\,\mathrm{d}t\right\}. \end{aligned}$$

If $|x'-y'|\leqslant T_o\theta$, then by means of (5.32) and (5.33),

$$c_{28}\theta^{-d}\leqslant V_k(x',y')\leqslant c_{29}\theta^{-d},$$

which is (5.22). If $|x'-y'|\geqslant T_o\theta$, then by means of (5.35) and (5.36),

$$c_{30}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_2}\pi|x'-y'|/\theta}$$
$$\leqslant V_k(x',y')\leqslant c_{31}\theta^{-d/2-1}|x'-y'|^{-d/2+1}\mathrm{e}^{-\sqrt{2C_2}\pi|x'-y'|/\theta}.$$

We thus get (5.23) with $W=V_k$. □

Since $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ is a core of $(\mathcal{E},\mathcal{F})$, we fix a $u\in C_0^\infty(\mathbb{R}^d)$ and set $\varphi=u|_{\Omega_\mu}$. Then $\varphi\in\mathcal{F}$ and

$$\mathcal{E}(\varphi,\varphi)=\mathcal{E}^Y(H_{\Omega_\mu}u,H_{\Omega_\mu}u), \tag{5.41}$$

where $H_Eu(x)=E_x[u(Y_{\sigma_E^Y})\colon \sigma_E^Y<\infty]$, $x\in\mathbb{R}^d$, σ_E^Y is the first hitting time of $\mathbf{Y}$ for a set E: $\sigma_E^Y=\inf\{t>0\colon Y_t\in E\}$. Note that $H_{\Omega_\mu}u\in\mathcal{F}^Y$. We put

$$\begin{aligned}\mathcal{E}_{\Omega_\mu}(v,v)&=\frac12\int_{\Omega_\mu}\sum_{i,j=1}^{d-1}a^{ij}(x')\partial_{x^i}v(x)\partial_{x^j}v(x)\,dx'\big(\mathrm{d}x^d+\rho\big(\mathrm{d}x^d\big)\big)\\ &+\frac12\int_{\Omega_\mu}a^{dd}\big(x^d\big)\partial_{x^d}v(x)\partial_{x^d}v(x)\,\mathrm{d}x,\end{aligned}$$

$$\mathcal{E}_{\Omega_k}(v,v)=\frac12\int_{\Omega_k}\sum_{i,j=1}^{d-1}a^{ij}(x')\partial_{x^i}v(x)\partial_{x^j}v(x)\,\mathrm{d}x+\frac12\int_{\Omega_k}\partial_{x^d}v(x)\partial_{x^d}v(x)\,\mathrm{d}x.$$

Noting (5.2) and (5.3), we see that

$$\mathcal{E}(\varphi,\varphi)=\mathcal{E}_{\Omega_\mu}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)+\sum_{k\in K}\mathcal{E}_{\Omega_k}(H_{\Omega_\mu}u,H_{\Omega_\mu}u). \tag{5.42}$$

LEMMA 5.3. – *It holds that*

$$\begin{aligned}&\mathcal{E}_{\Omega_\mu}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)\\&\quad=\frac{1}{2}\int\limits_{\Omega_\mu}\left\{\sum_{i,j=1}^{d-1}a^{ij}(x')\partial_{x^i}\varphi(x)\partial_{x^j}\varphi(x)+a^{dd}\left(x^d\right)\partial^*_{x^d}\varphi(x)\partial^*_{x^d}\varphi(x)\right\}\mathrm{d}x\\&\qquad+\frac{1}{2}\int\limits_{\Omega_\mu}\sum_{i,j=1}^{d-1}a^{ij}(x')\partial_{x^i}\varphi(x)\partial_{x^j}\varphi(x)\,\mathrm{d}x'\rho\left(\mathrm{d}x^d\right),\end{aligned}$$

where $\partial^*_{x^d}$ *is defined by* (5.7)*, and the first term of the right hand side vanishes in case where* $|S_m|=0$.

Proof. – It is obvious that $H_{\Omega_\mu}u(x)=u(x)=\varphi(x)$, $x\in\Omega_\mu$, because of $P_x(\sigma^Y_{\Omega_\mu}=0)=1$ for $x\in\Omega_\mu$. Since $H_{\Omega_\mu}u\in\mathcal{F}^Y$,

$$\begin{aligned}&\partial_{x^i}H_{\Omega_\mu}u(x)=\frac{\partial}{\partial x^i}u(x)=\frac{\partial}{\partial x^i}\varphi(x),\quad \mathrm{d}x'\left(\mathrm{d}x^d+\rho\left(\mathrm{d}x^d\right)\right)\text{-a.e. } x\in\Omega_\mu,\\&\quad 1\leqslant i\leqslant d-1,\\&\partial_{x^d}H_{\Omega_\mu}u(x)=\frac{\partial}{\partial x^d}u(x),\quad \mathrm{d}x'\,\mathrm{d}x^d\text{-a.e. } x\in\Omega_\mu.\end{aligned}$$

Assume that $|S_m|>0$. Then

$$\lim_{\varepsilon\downarrow0}\left|\left(x^d,x^d+\varepsilon\right]\cap S_m\right|/\varepsilon=\lim_{\varepsilon\downarrow0}\left|\left(x^d-\varepsilon,x^d\right]\cap S_m\right|/\varepsilon=1,\quad \mathrm{d}x^d\text{-a.e. } x^d\in S_m,$$

from which

$$\begin{aligned}\frac{\partial}{\partial x^d}u\left(x',x^d\right)&=\lim_{\xi\to x^d,\,\xi\in S_m}\left\{u(x',\xi)-u\left(x',x^d\right)\right\}/\left(\xi-x^d\right)\\&=\lim_{\xi\to x^d,\,\xi\in S_m}\left\{\varphi(x',\xi)-\varphi\left(x',x^d\right)\right\}/\left(\xi-x^d\right)=\partial^*_{x^d}\varphi\left(x',x^d\right),\end{aligned}$$

for every $x'\in\Gamma$ and $\mathrm{d}x^d$-a.e. $x^d\in S_m$. We thus get the conclusion of the lemma. □

We are going to derive an explicit form of $\mathcal{E}_{\Omega_k}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)$.

LEMMA 5.4. – (i) *Let* $-\infty=a_k<b_k<\infty$. *Then*

$$\mathcal{E}_{\Omega_k}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)=\frac{1}{8}\iint\limits_{\Gamma\times\Gamma}\left\{\varphi(x',b_k)-\varphi(y',b_k)\right\}^2U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

(ii) *Let* $-\infty<a_k<b_k=\infty$. *Then*

$$\mathcal{E}_{\Omega_k}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)=\frac{1}{8}\iint\limits_{\Gamma\times\Gamma}\left\{\varphi(x',a_k)-\varphi(y',a_k)\right\}^2U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

Proof. – We assume $b_k < \infty$ and write b in place of b_k. The assumption (5.3) implies that $P_\xi^B(\mathbf{f}(t) = t,\ 0 \leqslant t < \sigma_{S_m}^B) = 1$ for $\xi \in I_k$. We also note that $P_x(\sigma_{\Omega_\mu}^Y = \sigma_{S_m}^B = \sigma_b^B < \infty) = 1$ for $x \in \Omega_k$. Therefore

$$H_{\Omega_\mu} u(x) = E_x\left[\varphi\left(X'\left(\sigma_b^B\right), b\right)\right] = -\frac{1}{2} \int_\Gamma \varphi(y', b)\partial_{y^d}^- G_k\left(x, (y', b)\right) \mathrm{d}y', \quad (5.43)$$

$$P_x\left(X'\left(\sigma_b^B\right) \in \mathrm{d}y'\right) = -\frac{1}{2}\partial_{y^d}^- G_k\left(x, (y', b)\right) \mathrm{d}y', \quad (5.44)$$

for $x \in \Omega_k,\ y' \in \Gamma$. By means of Green's formula and (5.43),

$$\begin{aligned}
\mathcal{E}_{\Omega_k}(H_{\Omega_\mu} u, H_{\Omega_\mu} u) &= \frac{1}{2} \int_\Gamma H_{\Omega_\mu} u(x', b)\partial_{x^d}^- H_{\Omega_\mu} u(x', b)\, \mathrm{d}x' \\
&= \frac{1}{2} \int_\Gamma \varphi(x', b)\partial_{x^d}^- E_x\left[\varphi\left(X'\left(\sigma_b^B\right), b\right)\right]\big|_{x^d=b}\, \mathrm{d}x' \\
&= -\frac{1}{4} \int_\Gamma \partial_{x^d}^- E_x\left[\left\{\varphi(x', b) - \varphi\left(X'\left(\sigma_b^B\right), b\right)\right\}^2\right]\big|_{x^d=b}\, \mathrm{d}x' \\
&\quad + \frac{1}{4} \int_\Gamma \partial_{x^d}^- E_x\left[\varphi(x', b)^2 + \varphi\left(X'\left(\sigma_b^B\right), b\right)^2\right]\big|_{x^d=b}\, \mathrm{d}x' \\
&\equiv J_1 + J_2.
\end{aligned}$$

By virtue of (5.44), (5.4) and (5.20),

$$\begin{aligned}
J_1 &= \frac{1}{8} \iint_{\Gamma\times\Gamma} \left\{\varphi(x', b) - \varphi(y', b)\right\}^2 \partial_{x^d}^- \partial_{y^d}^- G_k\left((x', b), (y', b)\right) \mathrm{d}x'\, \mathrm{d}y' \\
&= \frac{1}{8} \iint_{\Gamma\times\Gamma} \left\{\varphi(x', b) - \varphi(y', b)\right\}^2 U_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y' < \infty.
\end{aligned}$$

Since $H_{\Omega_\mu} 1(x) = P_{x^d}^B(\sigma_b^B < \infty) = 1$ and hence $\partial_{x^d} H_{\Omega_\mu} 1(x) = 0$, by using Green's formula again,

$$J_2 = \frac{1}{4} \int_\Gamma \partial_{x^d}^- E_x\left[\varphi\left(X'\left(\sigma_b^B\right), b\right)^2\right]\big|_{x^d=b}\, \mathrm{d}x' = \frac{1}{2}\mathcal{E}_{\Omega_k}\left(H_{\Omega_\mu} 1, H_{\Omega_\mu}\left(u^2\right)\right) = 0.$$

Thus we arrive at the first assertion. The second assertion is obtained in the same way as above. □

LEMMA 5.5. – *Let* $-\infty < a_k < b_k < \infty$. *Then*

$$\begin{aligned}
\mathcal{E}_{\Omega_k}(H_{\Omega_\mu} u, H_{\Omega_\mu} u) &= \frac{1}{8} \iint_{\Gamma\times\Gamma} \left\{\varphi(x', a_k) - \varphi(y', a_k)\right\}^2 U_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y' \\
&\quad + \frac{1}{8} \iint_{\Gamma\times\Gamma} \left\{\varphi(x', b_k) - \varphi(y', b_k)\right\}^2 U_k(x', y')\, \mathrm{d}x'\, \mathrm{d}y'
\end{aligned}$$

$$+\frac{1}{4}\iint_{\Gamma\times\Gamma}\{\varphi(x',a_k)-\varphi(y',b_k)\}^2 V_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

Proof. – We set $a=a_k$, $b=b_k$. Since $P_x(\sigma^Y_{\Omega_\mu}=\sigma^B_{S_m}=\sigma^B_a\wedge\sigma^B_b<\infty)=1$ for $x\in\Omega_k$, where $r\wedge s=\min\{r,s\}$, we have

$$\begin{aligned}H_{\Omega_\mu}u(x)&=E_x\left[\varphi\left(X'\left(\sigma^B_a\wedge\sigma^B_b\right),B\left(\sigma^B_a\wedge\sigma^B_b\right)\right)\right]\\&=\frac{1}{2}\int_\Gamma\left\{\varphi(y',a)\partial^+_{y^d}G_k\left(x,(y',a)\right)-\varphi(y',b)\partial^-_{y^d}G_k\left(x,(y',b)\right)\right\}\mathrm{d}y',\end{aligned}\tag{5.45}$$

$$P_x\left(X'\left(\sigma^B_a\right)\in\mathrm{d}y',\ \sigma^B_a<\sigma^B_b\right)=\frac{1}{2}\partial^+_{y^d}G_k\left(x,(y',a)\right)\mathrm{d}y',\tag{5.46}$$

$$P_x\left(X'\left(\sigma^B_b\right)\in\mathrm{d}y',\ \sigma^B_b<\sigma^B_a\right)=-\frac{1}{2}\partial^-_{y^d}G_k\left(x,(y',b)\right)\mathrm{d}y',\tag{5.47}$$

for $x\in\Omega_k$ and $y\in\Gamma$. By means of Green's formula and (5.45), we get

$$\begin{aligned}&\mathcal{E}_{\Omega_k}(H_{\Omega_\mu}u,H_{\Omega_\mu}u)\\&\quad=\frac{1}{2}\int_\Gamma\left\{H_{\Omega_\mu}u(x',b)\partial^-_{x^d}H_{\Omega_\mu}u(x',b)-H_{\Omega_\mu}u(x',a)\partial^+_{x^d}H_{\Omega_\mu}u(x',a)\right\}\mathrm{d}x'\\&\quad=\frac{1}{2}\int_\Gamma\Big\{\varphi(x',b)\partial^-_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_a\wedge\sigma^B_b\right),B\left(\sigma^B_a\wedge\sigma^B_b\right)\right)\right]\big|_{x^d=b}\\&\qquad-\varphi(x',a)\partial^+_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_a\wedge\sigma^B_b\right),B\left(\sigma^B_a\wedge\sigma^B_b\right)\right)\right]\big|_{x^d=a}\Big\}\mathrm{d}x'\\&\quad=\frac{1}{2}\int_\Gamma\Big\{\varphi(x',b)\partial^-_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_a\right),a\right);\sigma^B_a<\sigma^B_b\right]\big|_{x^d=b}\\&\qquad+\varphi(x',b)\partial^-_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_b\right),b\right);\sigma^B_b<\sigma^B_a\right]\big|_{x^d=b}\\&\qquad-\varphi(x',a)\partial^+_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_a\right),a\right);\sigma^B_a<\sigma^B_b\right]\big|_{x^d=a}\\&\qquad-\varphi(x',a)\partial^+_{x^d}E_x\left[\varphi\left(X'\left(\sigma^B_b\right),b\right);\sigma^B_b<\sigma^B_a\right]\big|_{x^d=a}\Big\}\mathrm{d}x'\\&\quad=-\frac{1}{4}\int_\Gamma\partial^-_{x^d}E_x\left[\left\{\varphi(x',b)-\varphi\left(X'\left(\sigma^B_a\right),a\right)\right\}^2;\sigma^B_a<\sigma^B_b\right]\big|_{x^d=b}\,\mathrm{d}x'\\&\qquad+\frac{1}{4}\int_\Gamma\partial^-_{x^d}E_x\left[\varphi(x',b)^2+\varphi\left(X'\left(\sigma^B_a\right),a\right)^2;\sigma^B_a<\sigma^B_b\right]\big|_{x^d=b}\,\mathrm{d}x'\\&\qquad-\frac{1}{4}\int_\Gamma\partial^-_{x^d}E_x\left[\left\{\varphi(x',b)-\varphi\left(X'\left(\sigma^B_b\right),b\right)\right\}^2;\sigma^B_b<\sigma^B_a\right]\big|_{x^d=b}\,\mathrm{d}x'\\&\qquad+\frac{1}{4}\int_\Gamma\partial^-_{x^d}E_x\left[\varphi(x',b)^2+\varphi\left(X'\left(\sigma^B_b\right),b\right)^2;\sigma^B_b<\sigma^B_a\right]\big|_{x^d=b}\,\mathrm{d}x'\\&\qquad+\frac{1}{4}\int_\Gamma\partial^+_{x^d}E_x\left[\left\{\varphi(x',a)-\varphi\left(X'\left(\sigma^B_a\right),a\right)\right\}^2;\sigma^B_a<\sigma^B_b\right]\big|_{x^d=a}\,\mathrm{d}x'\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4} \int_{\Gamma} \partial^{+}_{x^d} E_x \left[\varphi(x', a)^2 + \varphi\big(X'(\sigma^B_a), a\big)^2; \sigma^B_a < \sigma^B_b \right]\big|_{x^d = a} \,\mathrm{d}x' \\
&+ \frac{1}{4} \int_{\Gamma} \partial^{+}_{x^d} E_x \left[\big\{ \varphi(x', a) - \varphi\big(X'(\sigma^B_b), b\big) \big\}^2; \sigma^B_b < \sigma^B_a \right]\big|_{x^d = a} \,\mathrm{d}x' \\
&- \frac{1}{4} \int_{\Gamma} \partial^{+}_{x^d} E_x \left[\varphi(x', a)^2 + \varphi\big(X'(\sigma^B_b), b\big)^2; \sigma^B_b < \sigma^B_a \right]\big|_{x^d = a} \,\mathrm{d}x' \\
\equiv{}& J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8.
\end{aligned}
$$

By virtue of (5.46), (5.47), (5.4), (5.5), (5.21), (5.22) and (5.23),

$$
\begin{aligned}
J_1 &= -\frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', b) - \varphi(y', a) \big\}^2 \partial^{-}_{x^d} \partial^{+}_{y^d} G_k\big((x', b), (y', a)\big) \,\mathrm{d}x' \,\mathrm{d}y' \\
&= \frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', b) - \varphi(y', a) \big\}^2 V_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' < \infty, \\
J_3 &= \frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', b) - \varphi(y', b) \big\}^2 \partial^{-}_{x^d} \partial^{-}_{y^d} G_k\big((x', b), (y', b)\big) \,\mathrm{d}x' \,\mathrm{d}y' \\
&= \frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', b) - \varphi(y', b) \big\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' < \infty, \\
J_5 &= \frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', a) - \varphi(y', a) \big\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' < \infty, \\
J_7 &= \frac{1}{8} \iint_{\Gamma \times \Gamma} \big\{ \varphi(x', a) - \varphi(y', b) \big\}^2 V_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' = J_1 < \infty.
\end{aligned}
$$

On the other hand, by means of (5.45) and Green's formula,

$$
\begin{aligned}
&J_2 + J_4 + J_6 + J_8 \\
&\quad = \frac{1}{4} \int_{\Gamma} \Big\{ \partial^{-}_{x^d} E_x \left[\varphi(x', b)^2 + \varphi\big(X'(\sigma^B_a \wedge \sigma^B_b), B(\sigma^B_a \wedge \sigma^B_b)\big)^2 \right]\big|_{x^d = b} \\
&\qquad - \partial^{+}_{x^d} E_x \left[\varphi(x', a)^2 + \varphi\big(X'(\sigma^B_a \wedge \sigma^B_b), B(\sigma^B_a \wedge \sigma^B_b)\big)^2 \right]\big|_{x^d = a} \Big\} \,\mathrm{d}x' \\
&\quad = \frac{1}{4} \int_{\Gamma} \Big\{ \partial^{-}_{x^d} E_x \left[\varphi\big(X'(\sigma^B_a \wedge \sigma^B_b), B(\sigma^B_a \wedge \sigma^B_b)\big)^2 \right]\big|_{x^d = b} \\
&\qquad - \partial^{+}_{x^d} E_x \left[\varphi\big(X'(\sigma^B_a \wedge \sigma^B_b), B(\sigma^B_a \wedge \sigma^B_b)\big)^2 \right]\big|_{x^d = a} \Big\} \,\mathrm{d}x' \\
&\quad = \frac{1}{2} \mathcal{E}_{\Omega_k}\big(H_{\Omega_\mu} 1, H_{\Omega_\mu}(u^2)\big) = 0.
\end{aligned}
$$

We thus get the conclusion of the lemma. □

Proof of Theorem 5.1. – In view of [7, Theorem 6.2.1], $(\mathcal{E},\mathcal{F})$ is regular on $L^2(\Omega_\mu,\mu)$ and $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ is a core of $(\mathcal{E},\mathcal{F})$. We then see that, for $u\in C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$, $\mathcal{E}(u,u)$ is given by (5.6) because of (5.42) and Lemmas 5.3, 5.4, 5.5. □

Example 5.6. – Let $\mathbf{X}'$ be the $(d-1)$-dimensional Brownian motion and Ξ the one-dimensional one, that is, $a^{ij}(x')=\delta^{ij}(x')$, $1\leqslant i,j\leqslant d-1$, and $a^{dd}(\xi)=1$. We exhibit Dirichlet spaces corresponding to some m and ρ.

(i) Let $m(\mathrm{d}\xi)=I_{(a,\infty)}(\xi)\,\mathrm{d}\xi$ and $\rho(\mathrm{d}\xi)=\kappa_a\delta_a(\mathrm{d}\xi)$, where $a\in\mathbb{R}$, κ_a is a nonnegative constant. Then the corresponding Dirichlet space $(\mathcal{E},\mathcal{F})$ in Theorem 5.1 is given by

$$\begin{aligned}\mathcal{E}(u,u)&=\frac{1}{2}\int_{E_a}\sum_{i=1}^{d}(\partial_{x^i}u)^2\,\mathrm{d}x+\frac{\kappa_a}{2}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_a^+u)^2\,\mathrm{d}x'\\&\quad+\frac{1}{8}\iint_{\Gamma\times\Gamma}\bigl\{\gamma_a^+u(x')-\gamma_a^+u(y')\bigr\}^2U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y',\\ \mathcal{F}&=\bigl\{u\in L^2(E_a,\mathrm{d}x):\ \mathcal{E}(u,u)<\infty\bigr\}\\&=\begin{cases}\{u\in H^1(E_a):\ \gamma_a^+u\in H^1(\Gamma)\}, & \text{if } \kappa_a>0,\\ \{u\in H^1(E_a):\ \gamma_a^+u\in H^{1/2}(\Gamma)\}, & \text{if } \kappa_a=0,\end{cases}\end{aligned}\tag{5.48}$$

where $E_a=\Gamma\times(a,\infty)$, $\Gamma_a=\Gamma\times\{a\}$, γ_a^+ is the trace operator on Γ_a from the domain E_a, and

$$\begin{aligned}U_0(x',y')&=\frac{2}{\sqrt{2\pi}}\int_0^\infty(2\pi t)^{-(d-1)/2}\mathrm{e}^{-|x'-y'|^2/2t}t^{-3/2}\,\mathrm{d}t\\&=2\pi^{-d/2}\Gamma(d/2)|x'-y'|^{-d}.\end{aligned}\tag{5.49}$$

$H^{1/2}(\Gamma)$ is the fractional order space defined by $H^{1/2}(\Gamma)=\{u\in L^2(\Gamma):\ \|u\|_{H^{1/2}(\Gamma)}<\infty\}$ with the norm

$$\|u\|_{H^{1/2}(\Gamma)}=\|u\|_{L^2(\Gamma)}+\left\{\iint_{\Gamma\times\Gamma}\bigl(u(x')-u(y')\bigr)^2|x'-y'|^{-d}\,\mathrm{d}x'\,\mathrm{d}y'\right\}^{1/2}.$$

We always identify Γ_a with Γ. In order to see (5.48), it is enough to note that

$$\|\gamma_a^+u\|_{L^2(\Gamma)}\leqslant c_1\|u\|_{H^1(E_a)},\quad u\in H^1(E_a),$$

$$\|v\|_{H^{1/2}(\Gamma)}\leqslant c_2\|v\|_{H^1(\Gamma)},\quad v\in H^1(\Gamma),$$

for some positive constants c_1 and c_2 (cf. [1, Ch. VII]).

(ii) Let $m(\mathrm{d}\xi)=I_{(a,b)}(\xi)\,\mathrm{d}\xi$ and $\rho(\mathrm{d}\xi)=\kappa_a\delta_a(\mathrm{d}\xi)+\kappa_b\delta_b(\mathrm{d}\xi)$, where $-\infty<a<b<\infty$ and κ_a, κ_b are nonnegative constants. Then the corresponding Dirichlet space $(\mathcal{E},\mathcal{F})$ in Theorem 5.1 is given by

$$\mathcal{E}(u,u)=\frac{1}{2}\int_{E_{ab}}\sum_{i=1}^{d}(\partial_{x^i}u)^2\,\mathrm{d}x$$
$$+\frac{\kappa_a}{2}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_a^+u)^2\,\mathrm{d}x'+\frac{\kappa_b}{2}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_b^-u)^2\,\mathrm{d}x'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_a^+u(x')-\gamma_a^+u(y')\}^2U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_b^-u(x')-\gamma_b^-u(y')\}^2U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y',$$

$$\mathcal{F}=\left\{u\in H^1(E_{ab}):\ \begin{matrix}\gamma_a^+u\in H^1(\Gamma) & \text{if } \kappa_a>0,\\ \gamma_a^+u\in H^{1/2}(\Gamma) & \text{if } \kappa_a=0,\\ \gamma_b^-u\in H^1(\Gamma) & \text{if } \kappa_b>0,\\ \gamma_b^-u\in H^{1/2}(\Gamma) & \text{if } \kappa_b=0\end{matrix}\right\},$$

where $E_{ab}=\Gamma\times(a,b)$, $\Gamma_a=\Gamma\times\{a\}$, $\Gamma_b=\Gamma\times\{b\}$, γ_a^+ $[\gamma_b^-]$ stands for the trace operator on Γ_a $[\Gamma_b]$ from the domain E_{ab}, and U_0 is given by (5.49).

(iii) Let $m(\mathrm{d}\xi)=I_{(-\infty,a)}(\xi)\,\mathrm{d}\xi+I_{(b,\infty)}(\xi)\,\mathrm{d}\xi$ and $\rho(\mathrm{d}\xi)=\kappa_a\delta_a(\mathrm{d}\xi)+\kappa_b\delta_b(\mathrm{d}\xi)$, where $-\infty<a<b<\infty$ and κ_a, κ_b are nonnegative constants. Then the corresponding Dirichlet form in Theorem 5.1 is given by

$$\mathcal{E}(u,u)=\frac{1}{2}\int_{E_a\cup E_b}\sum_{i=1}^{d}(\partial_{x^i}u)^2\,\mathrm{d}x$$
$$+\frac{\kappa_a}{2}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_a^-u)^2\,\mathrm{d}x'+\frac{\kappa_b}{2}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_b^+u)^2\,\mathrm{d}x'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_a^-u(x')-\gamma_a^-u(y')\}^2U_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_b^+u(x')-\gamma_b^+u(y')\}^2U_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{4}\iint_{\Gamma\times\Gamma}\{\gamma_a^-u(x')-\gamma_b^+u(y')\}^2V_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y',$$

where $E_a=\Gamma\times(-\infty,a)$, $E_b=\Gamma\times(b,\infty)$, $\Gamma_a=\Gamma\times\{a\}$, $\Gamma_b=\Gamma\times\{b\}$, γ_a^- $[\gamma_b^+]$ stands for the trace operator on Γ_a $[\Gamma_b]$ from the domain E_a $[E_b]$, and $U_{ab}(x',y')$ and $V_{ab}(x',y')$ are given by

$$U_{ab}(x',y')=\int_0^{\infty}(2\pi t)^{-(d-1)/2}\mathrm{e}^{-|x'-y'|^2/2t}\frac{2}{b-a}\sum_{n=1}^{\infty}\left(\frac{n\pi}{b-a}\right)^2\mathrm{e}^{-(n\pi/(b-a))^2t/2}\,\mathrm{d}t$$

$$= 2^{-(d-5)/2}\pi(b-a)^{-(d+3)/2}|x'-y'|^{-(d-3)/2} \times \sum_{n=1}^{\infty} n^{(d+1)/2} K_{(d-3)/2}\big(n\pi|x'-y'|/(b-a)\big), \tag{5.50}$$

$$V_{ab}(x',y') = \int_0^\infty (2\pi t)^{-(d-1)/2} \mathrm{e}^{-|x'-y'|^2/2t} \frac{2}{b-a} \times \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n\pi}{b-a}\right)^2 \mathrm{e}^{-(n\pi/(b-a))^2 t/2}\,\mathrm{d}t$$

$$= 2^{-(d-5)/2}\pi(b-a)^{-(d+3)/2}|x'-y'|^{-(d-3)/2} \times \sum_{n=1}^{\infty} (-1)^{n-1} n^{(d+1)/2} K_{(d-3)/2}\big(n\pi|x'-y'|/(b-a)\big). \tag{5.51}$$

Note that $U_{ab},\ V_{ab}$ satisfy (5.23) with $C_2 = C_4 = 1/2$. Especially, if $d = 2$, $U_{ab},\ V_{ab}$ are reduced to

$$U_{ab}(x',y') = \frac{2\pi}{(b-a)^2} \mathrm{e}^{-|x'-y'|\pi/(b-a)} \big\{1 - \mathrm{e}^{-|x'-y'|\pi/(b-a)}\big\}^{-2},$$

$$V_{ab}(x',y') = \frac{2\pi}{(b-a)^2} \mathrm{e}^{-|x'-y'|\pi/(b-a)} \big\{1 + \mathrm{e}^{-|x'-y'|\pi/(b-a)}\big\}^{-2},$$

because of $K_{-1/2}(\xi) = \sqrt{\pi/2\xi}\,\mathrm{e}^{-\xi}$. By virtue of Lemma 5.2, it holds that for $\phi = \gamma_a^- u$ or $\gamma_b^+ u$ with $u \in H^1(E_a \cup E_b)$,

$$\iint_{\Gamma\times\Gamma} \big|\phi(x') - \phi(y')\big|^2 U_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \leqslant c_1 \|\phi\|^2_{H^{1/2}(\Gamma)}$$

$$\leqslant c_2 \Big\{ \iint_{\Gamma\times\Gamma} \big|\phi(x') - \phi(y')\big|^2 U_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y' + \|\phi\|^2_{L^2(\Gamma)} \Big\},$$

and further,

$$\iint_{\Gamma\times\Gamma} \big|\gamma_a^- u(x') - \gamma_b^+ u(y')\big|^2 V_{ab}(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \leqslant c_3 \big\{ \|\gamma_a^- u\|^2_{L^2(\Gamma)} + \|\gamma_b^+ u\|^2_{L^2(\Gamma)} \big\},$$

where c_i $(i = 1,2,3)$ are positive constants. Therefore the domain $\mathcal{F}$ is given by

$$\mathcal{F} = \left\{ u \in H^1(E_a \cup E_b)\colon \begin{array}{ll} \gamma_a^- u \in H^1(\Gamma) & \text{if } \kappa_a > 0, \\ \gamma_a^- u \in H^{1/2}(\Gamma) & \text{if } \kappa_a = 0, \\ \gamma_b^+ u \in H^1(\Gamma) & \text{if } \kappa_b > 0, \\ \gamma_b^+ u \in H^{1/2}(\Gamma) & \text{if } \kappa_b = 0 \end{array} \right\}.$$

(iv) Let $m(\mathrm{d}\xi) = \delta_a(\mathrm{d}\xi)$ and $\rho(\mathrm{d}\xi) = \kappa_a \delta_a(\mathrm{d}\xi)$, where $-\infty < a < \infty$ and κ_a is a nonnegative constant. Then the corresponding Dirichlet space is given by

$$\mathcal{E}(u,u) = \frac{\kappa_a}{2}\int_\Gamma \sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a)\big)^2\,\mathrm{d}x' + \frac{1}{4}\iint_{\Gamma\times\Gamma}\big\{u(x',a)-u(y',a)\big\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y', \tag{5.52}$$

$$\mathcal{F} = \begin{cases} H^1(\Gamma_a), & \text{if } \kappa_a > 0,\\ H^{1/2}(\Gamma_a), & \text{if } \kappa_a = 0,\end{cases} \tag{5.53}$$

where $\Gamma_a = \Gamma \times \{a\}$ and U_0 is given by (5.49). The Dirichlet space corresponding to the case $\kappa_a = 0$ is obtained in [7, Example 6.2.2]. The coefficient $1/4$ in the right hand side of (5.52) is twice of $1/8$ in (6.2.33) of [7], because (5.52) is derived from the Brownian motion on the whole space $\mathbb{R}^d$ and (6.2.33) in [7] is derived from the reflecting Brownian motion on the half space.

(v) Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=0}^\infty$ be increasing sequences such that $-\infty < b_0 < a_1 < b_1 < \cdots < a_n < b_n < a_{n+1} < b_{n+1} < \cdots$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a_* < \infty$. Let

$$m(\mathrm{d}\xi) = \sum_{n=1}^\infty m_n I_{(b_{n-1},a_n)}(\xi)\,\mathrm{d}\xi + m_*\delta_{a_*}(\mathrm{d}\xi),$$

where $m_n > 0$, $m_* \geqslant 0$ and $\sum_{n=1}^\infty m_n(a_n - b_{n-1}) + m_* < \infty$. Hence m is a finite measure on $\mathbb{R}$. In this case $S_m = \bigcup_{n=1}^\infty [b_{n-1}, a_n] \cup \{a_*\}$ and $\mathbb{R}\setminus S_m = (-\infty, b_0) \cup (\bigcup_{n=1}^\infty (a_n, b_n)) \cup (a_*, \infty)$. Let

$$\rho(\mathrm{d}\xi) = \sum_{n=1}^\infty \kappa_{a_n}\delta_{a_n}(\mathrm{d}\xi) + \sum_{n=0}^\infty \kappa_{b_n}\delta_{b_n}(\mathrm{d}\xi) + \kappa_*\delta_{a_*}(\mathrm{d}\xi),$$

where $\kappa_{a_n} \geqslant 0$, $\kappa_{b_n} \geqslant 0$, $\kappa_* \geqslant 0$, and $\sum_{n=1}^\infty \kappa_{a_n} + \sum_{n=0}^\infty \kappa_{b_n} + \kappa_* < \infty$. Hence ρ is a finite measure on $\mathbb{R}$ and $\operatorname{supp}[\rho] \subset S_m$. Then (5.6) is reduced to

$$\begin{aligned}
\mathcal{E}(u,u) = {} & \sum_{n=1}^\infty \frac{1}{2}\int_{E_n}\sum_{i=1}^{d}\big(\partial_{x^i}u(x)\big)^2\,\mathrm{d}x + \sum_{n=1}^\infty \frac{\kappa_{a_n}}{2}\int_\Gamma \sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a_n)\big)^2\,\mathrm{d}x' \\
& + \sum_{n=0}^\infty \frac{\kappa_{b_n}}{2}\int_\Gamma \sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',b_n)\big)^2\,\mathrm{d}x' + \frac{\kappa_*}{2}\int_\Gamma \sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a_*)\big)^2\,\mathrm{d}x' \\
& + \frac{1}{8}\sum_{n=1}^\infty \iint_{\Gamma\times\Gamma}\big\{u(x',a_n)-u(y',a_n)\big\}^2 U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{8}\sum_{n=0}^\infty \iint_{\Gamma\times\Gamma}\big\{u(x',b_n)-u(y',b_n)\big\}^2 U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{8}\iint_{\Gamma\times\Gamma}\big\{u(x',a_*)-u(y',a_*)\big\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y'
\end{aligned}$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\left\{u(x',a_n)-u(y',b_n)\right\}^2 V_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y',$$

for $u \in C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$, where $E_n = \Gamma \times (b_{n-1}, a_n)$, $n = 1, 2, \ldots$, U_0 is defined by (5.49), $U_n \equiv U_{a_n b_n}$ is defined by (5.50) with $a = a_n,\ b = b_n$, and $V_n \equiv V_{a_n b_n}$ is defined by (5.51) with $a = a_n,\ b = b_n$ for each $n = 1, 2, \ldots$.

The domain $\mathcal{F}$ is the completion of $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ with respect to the norm $\mathcal{E}_1(\cdot,\cdot)^{1/2}$, where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)_{L^2(\Omega_\mu,\mu)}$. Therefore every $u \in \mathcal{F}$ possesses at least the following properties.

(v.1) If $m_* > 0$, then $u(\cdot, a_*) \in H^1(\Gamma)$ or $u(\cdot, a_*) \in H^{1/2}(\Gamma)$ according to $\kappa_* > 0$ or $\kappa_* = 0$.

(v.2) $u \in H^1(E_n)$ for $n = 1, 2, \ldots$ and

$$\|u\|_{(C)}^2 := \sum_{n=1}^{\infty}\left\{\int_{E_n}\sum_{i=1}^{d}(\partial_{x^i}u)^2\,\mathrm{d}x + m_n\int_{E_n}u^2\,\mathrm{d}x\right\} < \infty.$$

(v.3) Both of $\gamma_{a_n}^- u$ and $\gamma_{b_{n-1}}^+ u$ belong to $H^{1/2}(\Gamma)$ for $n = 1, 2, \ldots$, and

$$\begin{aligned}\|u\|_{(J)}^2 :=& \sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\left\{\gamma_{a_n}^-u(x')-\gamma_{a_n}^-u(y')\right\}^2 U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &+\sum_{n=0}^{\infty}\iint_{\Gamma\times\Gamma}\left\{\gamma_{b_n}^+u(x')-\gamma_{b_n}^+u(y')\right\}^2 U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &+\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\left\{\gamma_{a_n}^-u(x')-\gamma_{b_n}^+u(y')\right\}^2 V_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y' < \infty.\end{aligned}$$

(v.4) For each $n = 1, 2, \ldots$, $\gamma_{a_n}^- u \in H^1(\Gamma)$ if $\kappa_{a_n} > 0$. For each $n = 0, 1, 2, \ldots$, $\gamma_{b_n}^+ u \in H^1(\Gamma)$ if $\kappa_{b_n} > 0$. Further

$$\sum_{n=1}^{\infty}\kappa_{a_n}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_{a_n}^-u)^2\,\mathrm{d}x' + \sum_{n=0}^{\infty}\kappa_{b_n}\int_{\Gamma}\sum_{i=1}^{d-1}(\partial_{x^i}\gamma_{b_n}^+u)^2\,\mathrm{d}x' < \infty.$$ [5]

Here $\gamma_{a_n}^-$ and $\gamma_{b_{n-1}}^+$ are the trace operators on $\Gamma_{a_n} := \Gamma \times \{a_n\}$ and $\Gamma_{b_{n-1}} := \Gamma \times \{b_{n-1}\}$ from the domain E_n, respectively.

By means of above properties, it is easy to see that every $u \in \mathcal{F}$ satisfies the following identity and finiteness condition.

$$\gamma_{a_n}^-u(x') - \gamma_{b_{n-1}}^+u(x') = \int_{b_{n-1}}^{a_n}\partial_{x^d}u(x',x^d)\,\mathrm{d}x^d, \quad \text{a.e. } x' \in \Gamma. \tag{5.54}$$

[5] We use the convention $0 \cdot \infty = 0$ throughout this paper.

$$\sum_{n=1}^{\infty}\int_{b_{n-1}}^{a_n}\sum_{i=1}^{d}\left(\partial_{x^i}u\left(x',x^d\right)\right)^2\mathrm{d}x^d$$
$$+\sum_{n=1}^{\infty}\int_{\Gamma}\left\{\gamma_{a_n}^{-}u(x')-\gamma_{a_n}^{-}u(y')\right\}^2U_n(x',y')\,\mathrm{d}y'$$
$$+\sum_{n=0}^{\infty}\int_{\Gamma}\left\{\gamma_{b_n}^{+}u(x')-\gamma_{b_n}^{+}u(y')\right\}^2U_n(x',y')\,\mathrm{d}y'$$
$$+\sum_{n=1}^{\infty}\int_{\Gamma}\left\{\gamma_{a_n}^{-}u(x')-\gamma_{b_n}^{+}u(y')\right\}^2V_n(x',y')\,\mathrm{d}y'<\infty,\quad \text{a.e. } x'\in\Gamma. \tag{5.55}$$

Let us remark that, for every $u\in\mathcal{F}$, there exists the limit

$$\gamma_*u(x'):=\lim_{n\to\infty}\gamma_{\varepsilon_n}u(x'),\quad \text{a.e. } x'\in\Gamma, \tag{5.56}$$

where $\varepsilon_n=a_n$ or b_n, $\gamma_{\varepsilon_n}=\gamma_{a_n}^{-}$ if $\varepsilon_n=a_n$, $=\gamma_{b_n}^{+}$ if $\varepsilon_n=b_n$, for each $n\in\mathbb{N}$, and further it holds that

$$\gamma_*u(x')=u(x',a_*),\quad \text{a.e. } x'\in\Gamma,\quad \text{if } m_*>0, \tag{5.57}$$
$$\|\gamma_*u\|_{L^2(\Gamma)}\leqslant c_1\left\{\|u\|_{(C)}+\|u\|_{(J)}+\|\gamma_{b_0}^{+}u\|_{L^2(\Gamma)}\right\}, \tag{5.58}$$

where c_1 is a positive constant independent of $u\in\mathcal{F}$. In order to prove above properties (5.56)–(5.58), it is enough to show that there is a positive constant c_2 such that, for $u\in\mathcal{F},\ m,\ n=0,1,2,\dots\ (m<n)$, and for a.e. $x'\in\Gamma$,

$$\left|\gamma_{\varepsilon_n}u(x')-\gamma_{\varepsilon_m}u(x')\right|^2\leqslant c_2\left\{\sum_{k=m}^{n-1}\int_{b_k}^{a_{k+1}}\left(\partial_{x^d}u\left(x',x^d\right)\right)^2\mathrm{d}x^d\right.$$
$$+\sum_{k=m}^{n}\int_{\Gamma}\left|\gamma_{a_k}^{-}u(x')-\gamma_{a_k}^{-}u(y')\right|^2U_k(x',y')\,\mathrm{d}y'$$
$$\left.+\sum_{k=m}^{n}\int_{\Gamma}\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|^2V_k(x',y')\,\mathrm{d}y'\right\}. \tag{5.59}$$

We give a proof of (5.59) in the case where $\varepsilon_n=a_n,\ \varepsilon_m=b_m$. Since

$$\gamma_{a_n}^{-}u(x')-\gamma_{b_m}^{+}u(x')=\sum_{k=m}^{n-1}\left\{\gamma_{a_{k+1}}^{-}u(x')-\gamma_{b_k}^{+}u(x')\right\}+\sum_{k=m+1}^{n-1}\left\{\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y'_k)\right\}$$
$$+\sum_{k=m+1}^{n-1}\left\{\gamma_{a_k}^{-}u(y'_k)-\gamma_{a_k}^{-}u(x')\right\},$$

for $y'_k\in\Gamma$ $(k=m+1,\dots,n-1)$, by virtue of (5.54), we find that

$$\left|\gamma_{a_n}^{-}u(x')-\gamma_{b_m}^{+}u(x')\right|^2\leqslant 3\left\{\sum_{k=m}^{n-1}\int_{b_k}^{a_{k+1}}\left(\partial_{x^d}u\left(x',x^d\right)\right)^2\mathrm{d}x^d\cdot\sum_{k=m}^{n-1}(a_{k+1}-b_k)\right.$$

$$
+\left(\sum_{k=m+1}^{n-1}\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y_k')\right|\right)^2
+\left(\sum_{k=m+1}^{n-1}\left|\gamma_{a_k}^{-}u(y_k')-\gamma_{a_k}^{-}u(x')\right|\right)^2\Bigg\}. \tag{5.60}
$$

Put $A_k = \int_\Gamma V_k(x', y')\,\mathrm{d}y'$, which is independent of x' because $V_k(x', y')$ is a function of $|x'-y'|$ by means of (5.51). By virtue of Lemma 5.2, $c_3 := \inf_{k\in\mathbb{N},\, x'\in\Gamma}(b_k - a_k)A_k > 0$. Let $\nu_k(x', \mathrm{d}y') = A_k^{-1}V_k(x', y')\,\mathrm{d}y'$. Integrating the second term of the right hand side of (5.60) by the product measure $\prod_{j=m+1}^{n-1}\nu_j(x', \mathrm{d}y_j')$ over the set Γ^{n-m-1}, we find that

$$
\begin{aligned}
&\int_{\Gamma^{n-m-1}}\left(\sum_{k=m+1}^{n-1}\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y_k')\right|\right)^2\prod_{j=m+1}^{n-1}\nu_j(x',\mathrm{d}y_j')\\
&=\sum_{k=m+1}^{n-1}\int_\Gamma\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|^2\nu_k(x',\mathrm{d}y')\\
&\quad+\sum_{k\neq l}\int_\Gamma\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|\nu_k(x',\mathrm{d}y')\\
&\quad\times\int_\Gamma\left|\gamma_{b_l}^{+}u(x')-\gamma_{a_l}^{-}u(z')\right|\nu_l(x',\mathrm{d}z')\\
&\leqslant\left\{\sum_{k=m+1}^{n-1}\left(\int_\Gamma\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|^2\nu_k(x',\mathrm{d}y')\right)^{1/2}\right\}^2\\
&\leqslant\sum_{k=m+1}^{n-1}\int_\Gamma\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|^2V_k(x',y')\,\mathrm{d}y'\times\sum_{k=m+1}^{n-1}\frac{1}{A_k}\\
&\leqslant\frac{a_*-b_0}{c_3}\sum_{k=m+1}^{n-1}\int_\Gamma\left|\gamma_{b_k}^{+}u(x')-\gamma_{a_k}^{-}u(y')\right|^2V_k(x',y')\,\mathrm{d}y'.
\end{aligned}
$$

Noting that $V_k(x', y') \leqslant U_k(x', y')$, and integrating both sides of (5.60) by the product measure $\prod_{j=m+1}^{n-1}\nu_j(x', \mathrm{d}y_j')$ over the set Γ^{n-m-1}, we obtain (5.59).

From above observation, it follows that the Dirichlet space $(\mathcal{E}, \mathcal{F})$ is given as follows:

$$
\begin{aligned}
\mathcal{E}(u,u)=&\sum_{n=1}^{\infty}\frac{1}{2}\int_{E_n}\sum_{i=1}^{d}\left(\partial_{x^i}u(x)\right)^2\mathrm{d}x+\sum_{n=1}^{\infty}\frac{\kappa_{a_n}}{2}\int_\Gamma\sum_{i=1}^{d-1}\left(\partial_{x^i}\gamma_{a_n}^{-}u(x')\right)^2\mathrm{d}x'\\
&+\sum_{n=0}^{\infty}\frac{\kappa_{b_n}}{2}\int_\Gamma\sum_{i=1}^{d-1}\left(\partial_{x^i}\gamma_{b_n}^{+}u(x')\right)^2\mathrm{d}x'+\frac{\kappa_*}{2}\int_\Gamma\sum_{i=1}^{d-1}\left(\partial_{x^i}\gamma_*u(x')\right)^2\mathrm{d}x'\\
&+\frac{1}{8}\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\left\{\gamma_{a_n}^{-}u(x')-\gamma_{a_n}^{-}u(y')\right\}^2U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y'
\end{aligned}
$$

$$+\frac{1}{8}\sum_{n=0}^{\infty}\iint_{\Gamma\times\Gamma}\{\gamma_{b_n}^{+}u(x')-\gamma_{b_n}^{+}u(y')\}^2 U_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_{*}u(x')-\gamma_{*}u(y')\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{4}\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\{\gamma_{a_n}^{-}u(x')-\gamma_{b_n}^{+}u(y')\}^2 V_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y',$$

$$\mathcal{F}=\left\{u\colon \begin{array}{l} u \text{ satisfies (v.2), (v.3), (v.4), and further } \gamma_* u\in H^1(\Gamma) \\ \text{or } \gamma_* u\in H^{1/2}(\Gamma) \text{ according to } \kappa_*>0 \text{ or } \kappa_*=0 \end{array}\right\}.$$

(vi) Let us consider the case where $E_n=\emptyset$ for every $n\in\mathbb{N}$ in (v). Namely, let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence with $\lim_{n\to\infty}a_n=a_*\in\mathbb{R}$. Let

$$m(\mathrm{d}\xi)=\sum_{n=1}^{\infty}m_n\delta_{a_n}(\mathrm{d}\xi)+m_*\delta_{a_*}(\mathrm{d}\xi),$$

where $m_n>0,\ m_*\geqslant 0,\ \sum_{n=1}^{\infty}m_n+m_*<\infty$. Hence m is a finite measure on $\mathbb{R}$. In this case $S_m=\bigcup_{n=1}^{\infty}\{a_n\}\cup\{a_*\}$ and $\mathbb{R}\setminus S_m=(-\infty,a_1)\cup(\bigcup_{n=1}^{\infty}(a_n,a_{n+1}))\cup(a_*,\infty)$. Let

$$\rho(\mathrm{d}\xi)=\sum_{n=1}^{\infty}\kappa_n\delta_{a_n}(\mathrm{d}\xi)+\kappa_*\delta_{a_*}(\mathrm{d}\xi),$$

where $\kappa_n\geqslant 0,\ \kappa_*\geqslant 0,\ \sum_{n=1}^{\infty}\kappa_n+\kappa_*<\infty$. Hence ρ is a finite measure on $\mathbb{R}$ and $\mathrm{supp}[\rho]\subset S_m$. Then (5.6) is reduced to

$$\mathcal{E}(u,u)=\sum_{n=1}^{\infty}\frac{\kappa_n}{2}\int_{\Gamma}\sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a_n)\big)^2\,\mathrm{d}x'+\frac{\kappa_*}{2}\int_{\Gamma}\sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a_*)\big)^2\,\mathrm{d}x'$$
$$+\frac{1}{8}\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\{u(x',a_n)-u(y',a_n)\}^2\big(U_{n-1}(x',y')+U_n(x',y')\big)\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{u(x',a_*)-u(y',a_*)\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$+\frac{1}{4}\sum_{n=1}^{\infty}\iint_{\Gamma\times\Gamma}\{u(x',a_n)-u(y',a_{n+1})\}^2 V_n(x',y')\,\mathrm{d}x'\,\mathrm{d}y',$$

for $u\in C_0^{\infty}(\mathbb{R}^d)|_{\Omega_\mu}$, where U_0 is defined by (5.49), $U_n\equiv U_{a_na_{n+1}}$ is defined by (5.50) with $a=a_n,\ b=a_{n+1}$, and $V_n\equiv V_{a_na_{n+1}}$ is defined by (5.51) with $a=a_n,\ b=a_{n+1}$ for each $n=1,2,\ldots.$

Since the domain $\mathcal{F}$ is the completion of $C_0^{\infty}(\mathbb{R}^d)|_{\Omega_\mu}$ with respect to the norm $\mathcal{E}_1(\cdot,\cdot)^{1/2}$, every $u\in\mathcal{F}$ has the following properties.

(vi.1) If $m_* > 0$, then $u(\cdot, a_*) \in H^1(\Gamma)$ or $u(\cdot, a_*) \in H^{1/2}(\Gamma)$ according to $\kappa_* > 0$ or $\kappa_* = 0$.

(vi.2) $u(\cdot, a_n) \in H^{1/2}(\Gamma)$ for $n = 1, 2, \ldots$, and

$$\|u\|_{(J)}^2 := \sum_{n=1}^{\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_n) - u(y', a_n)\}^2 \big(U_{n-1}(x', y') + U_n(x', y')\big)\, dx'\, dy' + \sum_{n=1}^{\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_n) - u(y', a_{n+1})\}^2 V_n(x', y')\, dx'\, dy' < \infty.$$

Further, by the same argument as in (v), we find that, for every $u \in \mathcal{F}$, there exists the limit

$$\gamma_* u(x') := \lim_{n\to\infty} u(x', a_n), \quad \text{a.e. } x' \in \Gamma,$$

and

$$\gamma_* u(x') = u(x', a_*), \quad \text{a.e. } x' \in \Gamma, \quad \text{if } m_* > 0,$$
$$\|\gamma_* u\|_{L^2(\Gamma)} \leqslant c_1 \big\{ \|u\|_{(J)} + \|u(\cdot, a_1)\|_{L^2(\Gamma)} \big\},$$

where c_1 is a positive constant independent of $u \in \mathcal{F}$. Thus the Dirichlet space $(\mathcal{E}, \mathcal{F})$ is given as follows.

$$\begin{aligned}
\mathcal{E}(u, u) = {} & \sum_{n=1}^{\infty} \frac{\kappa_n}{2} \int_{\Gamma} \sum_{i=1}^{d-1} \big(\partial_{x^i} u(x', a_n)\big)^2\, dx' + \frac{\kappa_*}{2} \int_{\Gamma} \sum_{i=1}^{d-1} \big(\partial_{x^i} \gamma_* u(x')\big)^2\, dx' \\
& + \frac{1}{8} \sum_{n=1}^{\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_n) - u(y', a_n)\}^2 \big(U_{n-1}(x', y') + U_n(x', y')\big)\, dx'\, dy' \\
& + \frac{1}{8} \iint_{\Gamma\times\Gamma} \{\gamma_* u(x') - \gamma_* u(y')\}^2 U_0(x', y')\, dx'\, dy' \\
& + \frac{1}{4} \sum_{n=1}^{\infty} \iint_{\Gamma\times\Gamma} \{u(x', a_n) - u(y', a_{n+1})\}^2 V_n(x', y')\, dx'\, dy',
\end{aligned}$$

$$\mathcal{F} = \left\{ u\colon \begin{array}{l} u \text{ satisfies (vi.2), and further } \gamma_* u \in H^1(\Gamma) \text{ or} \\ \gamma_* u \in H^{1/2}(\Gamma) \text{ according to } \kappa_* > 0 \text{ or } \kappa_* = 0 \end{array} \right\}.$$

(vii) Let E be the triadic Cantor set, that is, $E = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{e\in\{0,2\}^{n-1}} (a_e, b_e)$, where $a_e = \sum_{i=1}^{n-1} \varepsilon_i 3^{-i} + 3^{-n}$ and $b_e = a_e + 3^{-n}$, for $e = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \{0, 2\}^{n-1}$. Let m be the Cantor measure, that is, a finite measure corresponding to the Cantor function. Then $S_m = E$. Let

$$\rho(d\xi) = \sum_{n=1}^{\infty} \sum_{e\in\{0,2\}^{n-1}} \big\{ \kappa_{a_e} \delta_{a_e}(d\xi) + \kappa_{b_e} \delta_{b_e}(d\xi) \big\},$$

where κ_{a_e} and κ_{b_e} are nonnegative constants and $\sum_{n=1}^{\infty} \sum_{e\in\{0,2\}^{n-1}} \{\kappa_{a_e} + \kappa_{b_e}\} < \infty$. Then ρ is a finite measure and $\mathrm{supp}[\rho] \subset S_m$. Therefore (5.6) is reduced to

$$
\begin{aligned}
\mathcal{E}(u,u) = {} & \frac{1}{2}\sum_{n=1}^{\infty}\sum_{e\in\{0,2\}^{n-1}}\kappa_{a_e}\int_{\Gamma}\sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',a_e)\big)^2\,\mathrm{d}x' \\
& + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{e\in\{0,2\}^{n-1}}\kappa_{b_e}\int_{\Gamma}\sum_{i=1}^{d-1}\big(\partial_{x^i}u(x',b_e)\big)^2\,\mathrm{d}x' \\
& + \frac{1}{8}\iint_{\Gamma\times\Gamma}\{u(x',0)-u(y',0)\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{8}\iint_{\Gamma\times\Gamma}\{u(x',1)-u(y',1)\}^2 U_0(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{8}\sum_{n=1}^{\infty}\sum_{e\in\{0,2\}^{n-1}}\iint_{\Gamma\times\Gamma}\{u(x',a_e)-u(y',a_e)\}^2 U_e(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{8}\sum_{n=1}^{\infty}\sum_{e\in\{0,2\}^{n-1}}\iint_{\Gamma\times\Gamma}\{u(x',b_e)-u(y',b_e)\}^2 U_e(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
& + \frac{1}{4}\sum_{n=1}^{\infty}\sum_{e\in\{0,2\}^{n-1}}\iint_{\Gamma\times\Gamma}\{u(x',a_e)-u(y',b_e)\}^2 V_e(x',y')\,\mathrm{d}x'\,\mathrm{d}y',
\end{aligned}
$$

for $u\in C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$, where U_0 is defined by (5.49), U_e is defined by (5.50) with $a=a_e,\ b=b_e$, and V_e is defined by (5.51) with $a=a_e,\ b=b_e$. The domain $\mathcal{F}$ is the completion of $C_0^\infty(\mathbb{R}^d)|_{\Omega_\mu}$ with respect to the norm $\{\mathcal{E}(\cdot,\cdot)+(\cdot,\cdot)_{L^2(\Omega_\mu,\mu)}\}^{1/2}$. Although we cannot determine the domain $\mathcal{F}$, we may say that

$$
\mathcal{F}\subset\left\{u\in L^2(\Omega_\mu,\mu)\colon \sup_{N\in\mathbb{N}}\int_{\Gamma\times E\times\Gamma\times E}\left|u\left(x',x^d\right)-u\left(y',y^d\right)\right|^2\nu_N\left(\mathrm{d}x'\,\mathrm{d}x^d\,\mathrm{d}y'\,\mathrm{d}y^d\right)<\infty\right\},
$$

where ν_N's are Radon measures on $\Gamma\times E\times\Gamma\times E$ defined by

$$
\begin{aligned}
& \nu_N\left(\mathrm{d}x'\mathrm{d}x^d\mathrm{d}y'\mathrm{d}y^d\right) \\
& \quad = U_0(x',y')\left\{\mathrm{d}x'\delta_0\left(\mathrm{d}x^d\right)\mathrm{d}y'\delta_0\left(\mathrm{d}y^d\right)+\mathrm{d}x'\delta_1\left(\mathrm{d}x^d\right)\mathrm{d}y'\delta_1\left(\mathrm{d}y^d\right)\right\} \\
& \qquad + \sum_{n=1}^{N}\sum_{e\in\{0,2\}^{n-1}}\left\{U_e(x',y')\left\{\mathrm{d}x'\delta_{a_e}\left(\mathrm{d}x^d\right)\mathrm{d}y'\delta_{a_e}\left(\mathrm{d}y^d\right)\right.\right. \\
& \qquad \left.\left. + \mathrm{d}x'\delta_{b_e}\left(\mathrm{d}x^d\right)\mathrm{d}y'\delta_{b_e}\left(\mathrm{d}y^d\right)\right\}+V_e(x',y')\,\mathrm{d}x'\delta_{a_e}\left(\mathrm{d}x^d\right)\mathrm{d}y'\delta_{b_e}\left(\mathrm{d}y^d\right)\right\}.
\end{aligned}
$$

6. The partial differential equation for the limit process

Let $\{a_k\}_{k=-\infty}^{\infty}$ and $\{b_k\}_{k=-\infty}^{\infty}$ be sequences such that $a_k<b_k<a_{k+1}<b_{k+1},\ k=0,\ \pm1,\ \pm2,\dots$, and $\lim_{k\to-\infty}a_k=\lim_{k\to-\infty}b_k=-\infty,\ \lim_{k\to\infty}a_k=\lim_{k\to\infty}b_k=\infty$. Let $m(\mathrm{d}\xi)=\sum_{k=-\infty}^{\infty}m_k I_{(b_k,a_{k+1})}(\xi)\,\mathrm{d}\xi$ and $\rho(\mathrm{d}\xi)=\sum_{k=-\infty}^{\infty}\{\kappa_{a_k}\delta_{a_k}(\mathrm{d}\xi)\ +\kappa_{b_k}\delta_{b_k}(\mathrm{d}\xi)\}$, where m_k's are positive constants, and κ_{a_k}'s and κ_{b_k}'s are nonnegative constants.

m and ρ are Radon measures on $\mathbb{R}$, $\operatorname{supp}[\rho] \subset S_m = \bigcup_{k=-\infty}^{\infty} [b_k, a_{k+1}]$, and $\mathbb{R} \setminus S_m = \bigcup_{k=-\infty}^{\infty} (a_k, b_k)$. In view of Theorem 5.1, the corresponding Dirichlet form is given by

$$
\begin{aligned}
\mathcal{E}(u,u) = {} & \sum_{k=-\infty}^{\infty} \frac{1}{2} \int_{E_k} \left\{ \sum_{i,j=1}^{d-1} a^{ij}(x') \partial_{x^i} u(x) \partial_{x^j} u(x) + a^{dd}\left(x^d\right) \partial_{x^d} u(x) \partial_{x^d} u(x) \right\} \mathrm{d}x \\
& + \sum_{k=-\infty}^{\infty} \frac{\kappa_{a_k}}{2} \int_{\Gamma} \sum_{i,j=1}^{d-1} a^{ij}(x') \partial_{x^i} \gamma_{a_k}^{-} u(x') \partial_{x^j} \gamma_{a_k}^{-} u(x') \,\mathrm{d}x' \\
& + \sum_{k=-\infty}^{\infty} \frac{\kappa_{b_k}}{2} \int_{\Gamma} \sum_{i,j=1}^{d-1} a^{ij}(x') \partial_{x^i} \gamma_{b_k}^{+} u(x') \partial_{x^j} \gamma_{b_k}^{+} u(x') \,\mathrm{d}x' \\
& + \frac{1}{8} \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{a_k}^{-} u(x') - \gamma_{a_k}^{-} u(y') \right\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' \\
& + \frac{1}{8} \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{b_k}^{+} u(x') - \gamma_{b_k}^{+} u(y') \right\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' \\
& + \frac{1}{4} \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{a_k}^{-} u(x') - \gamma_{b_k}^{+} u(y') \right\}^2 V_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y',
\end{aligned}
$$

where $E_k = \Gamma \times (b_k, a_{k+1})$, $\Gamma_{a_k} = \Gamma \times \{a_k\}$, $\Gamma_{b_k} = \Gamma \times \{b_k\}$, $\gamma_{a_k^-}$ $[\gamma_{b_k}^{+}]$ is the trace operator on Γ_{a_k} $[\Gamma_{b_k}]$ from E_{k-1} $[E_k]$, and U_k, V_k are given by (5.13) and (5.17). The domain $\mathcal{F}$ is the space of all measurable functions u defined on $E := \bigcup_{k=-\infty}^{\infty} E_k$ satisfying the following properties.

($\mathcal{F}$.1) $u \in H^1(E_k)$ for $k = 0, \pm 1, \pm 2, \ldots$, and

$$
\sum_{k=-\infty}^{\infty} \left\{ \int_{E_k} \sum_{i=1}^{d} (\partial_{x^i} u)^2 \,\mathrm{d}x + m_k \int_{E_k} u^2 \,\mathrm{d}x \right\} < \infty.
$$

($\mathcal{F}$.2) For each $k = 0, \pm 1, \pm 2, \ldots$, $\gamma_{a_k}^{-} u \in H^1(\Gamma)$ or $H^{1/2}(\Gamma)$ according to $\kappa_{a_k} > 0$ or $\kappa_{a_k} = 0$, and $\gamma_{b_k}^{+} u \in H^1(\Gamma)$ or $H^{1/2}(\Gamma)$ according to $\kappa_{b_k} > 0$ or $\kappa_{b_k} = 0$. Further

$$
\begin{aligned}
& \sum_{k=-\infty}^{\infty} \kappa_{a_k} \int_{\Gamma} \sum_{i=1}^{d-1} (\partial_{x^i} \gamma_{a_k}^{-} u)^2 \,\mathrm{d}x' + \sum_{k=-\infty}^{\infty} \kappa_{b_k} \int_{\Gamma} \sum_{i=1}^{d-1} (\partial_{x^i} \gamma_{b_k}^{+} u)^2 \,\mathrm{d}x' \\
& \quad + \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{a_k}^{-} u(x') - \gamma_{a_k}^{-} u(y') \right\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' \\
& \quad + \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{b_k}^{+} u(x') - \gamma_{b_k}^{+} u(y') \right\}^2 U_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' \\
& \quad + \sum_{k=-\infty}^{\infty} \iint_{\Gamma \times \Gamma} \left\{ \gamma_{a_k}^{-} u(x') - \gamma_{b_k}^{+} u(y') \right\}^2 V_k(x', y') \,\mathrm{d}x' \,\mathrm{d}y' < \infty.
\end{aligned}
$$

Put $\mathcal{E}_\lambda(u,v)=\mathcal{E}(u,v)+\lambda(u,v)_{L^2(E,\mu)}$ for $\lambda>0$. By virtue of the general theory, for each $f\in L^2(E,\mu)$ and $\lambda>0$, there exists a unique element u of $\mathcal{F}$ denoted by $G_\lambda f$ such that

$$\mathcal{E}_\lambda(u,v)=(f,v)_{L^2(E,\mu)},\quad v\in\mathcal{F}. \tag{6.1}$$

It is known that $\{G_\lambda,\lambda>0\}$ is Markovian in the sense that $0\leqslant\lambda G_\lambda f\leqslant 1$ whenever $0\leqslant f\leqslant 1$ ([7]). Hence it is defined as a bounded linear operator on $C(\overline{E})$, where $\overline{E}=\bigcup_{k=-\infty}^{\infty}\overline{E}_k$ with $\overline{E}_k=E_k\cup\Gamma_{a_{k+1}}\cup\Gamma_{b_k}$. We pose the following assumption on the sequences $\{a_k\}_{k=-\infty}^{\infty}$ and $\{b_k\}_{k=-\infty}^{\infty}$.

$$\sum_{0\leqslant k<\infty} m_k(a_{k+1}-b_k)^2=\infty, \tag{6.2}$$

$$\sum_{-\infty<k<0} m_k(a_{k+1}-b_k)^2=\infty, \tag{6.3}$$

which imply that (4.5) and (4.6) are satisfied. Therefore we can derive the following assertion from Theorem 4.3.

THEOREM 6.1. – *Under the conditions* (6.2) *and* (6.3) *it holds that*

$$G_\lambda\big(\widehat{C}(\overline{E})\big)\subset\widehat{C}(\overline{E}),\quad \lambda>0.$$

We are going to observe the assertion of Theorem 6.1 from a point of view of partial differential equation theory. In order to proceed our argument, we need the following assumption.

$$a^{ij}\in C^{1,\alpha}(\Gamma),\quad i,j=1,\ldots,d-1, \tag{6.4}$$

for an $\alpha\in(0,1]$. Here $C^{1,\alpha}(\Gamma)$ is the space of all continuously differentiable functions on Γ with bounded derivatives and with the finite norm

$$\|f\|_{1,\alpha}=\sum_{|r|=0,1}\|\partial^r f\|+\sum_{|r|=1}\big[\partial^r f\big]_\alpha,$$

where

$$\|f\|=\sup_{x\in\Gamma}|f(x)|,\quad [f]_\alpha=\sup_{x,y\in\Gamma}\frac{|f(x)-f(y)|}{|x-y|^\alpha},$$
$$\partial^r=(\partial_{x^1})^{r_1}(\partial_{x^2})^{r_2}\cdots(\partial_{x^d})^{r_d},\qquad r=(r_1,r_2,\ldots,r_d),$$
$$|r|=r_1+r_2+\cdots+r_d.$$

In the following, we fix a nonnegative element $\chi\in C_0^\infty(\Gamma)$ such that $\chi(y')=1$ for $|y'|\leqslant 1/2$, $\chi(y')=0$ for $|y'|\geqslant 1$, and $\chi(-y')=\chi(y')$.

PROPOSITION 6.2. – *Under the assumption* (6.4)*, there exists the limit*

$$b_k^i(x')=\lim_{\varepsilon\downarrow 0}\int_{\Gamma\setminus\Delta_\varepsilon^{x'}}\big(y^i-x^i\big)\chi(y'-x')U_k(x',y')\,\mathrm{d}y',\quad x'\in\Gamma, \tag{6.5}$$

for each $i=1,2,\ldots,d-1$, *and* $k=0,\pm1,\pm2,\ldots$, *where* $\Delta_\varepsilon^{x'}=\{y'\in\Gamma\colon |x'-y'|<\varepsilon\}$.

In order to prove Proposition 6.2, we need some estimates of $p'(t,x',y')$. We summarize them first of all. Following the parametric method, we find that $p'(t,x',y')$ is represented as

$$p'(t,x',y') = \overline{q}^{y'}(t,x',y') + \widetilde{q}(t,x',y'), \tag{6.6}$$

where

$$\overline{q}^{z'}(t,x',y') = \left(\det a(z')\right)^{1/2}(2\pi t)^{-(d-1)/2}\exp\left\{-\frac{1}{2t}(x'-y')a(z')\,{}^t(x'-y')\right\}, \tag{6.7}$$

$a(z') = (a_{ij}(z')) = (a^{ij}(z'))^{-1}$, and $\widetilde{q}(t,x',y')$ is the remainder term.

LEMMA 6.3. – *There are positive constants C_{13} and C_{14} such that*

$$|\widetilde{q}(t,x',y')| \leqslant C_{13}t^{-(d-1-\alpha)/2}\mathrm{e}^{-C_{14}|x'-y'|^2/t}, \quad 0<t<\infty,\ x',y'\in\Gamma. \tag{6.8}$$

Proof. – In the following, c_1, c_2, etc. stand for positive constants independent of variables. Let x', y', $z'\in\Gamma$. By means of (6.7),

$$\left|\overline{q}^{z'}(t,x',y')\right| \leqslant c_1 t^{-(d-1)/2}\mathrm{e}^{-c_2|x'-y'|^2/t}, \quad 0<t<\infty. \tag{6.9}$$

Combining this with (4.31), we get

$$|\widetilde{q}(t,x',y')| \leqslant c_3 t^{-(d-1)/2}\mathrm{e}^{-c_4|x'-y'|^2/t}, \quad 0<t<\infty. \tag{6.10}$$

In [13, p. 378, (13.5)], the following estimate is already obtained.

$$|\widetilde{q}(t,x',y')| \leqslant c_5 t^{-(d-1-\alpha)/2}\mathrm{e}^{-c_6|x'-y'|^2/t}, \quad 0<t\leqslant T,$$

where $T>0$ is fixed arbitrarily. This estimate coupled with (6.10) implies (6.8). □

Proof of Proposition 6.2. – We put

$$W_k(y';z') = \left(\det a(z')\right)^{1/2}\int_0^\infty (2\pi t)^{-(d-1)/2}\exp\left\{-\frac{1}{2t}y'a(z')\,{}^t y'\right\}\alpha_k(t)\,\mathrm{d}t, \tag{6.11}$$

for $y',z'\in\Gamma$. We note that there is a positive constant C_{15} such that

$$\left|W_k(y';x') - W_k(y';z')\right| \leqslant C_{15}|x'-z'||y'|^{-d}, \tag{6.12}$$

for every $x',y',z'\in\Gamma$, $k=0,\pm1,\pm2,\ldots$. Indeed, noting (6.4), we see that

$$\begin{aligned}\left|W_k(y';x') - W_k(y';z')\right| &= \left|\int_0^1 \partial_s W_k\big(y';x'+s(z'-x')\big)\,\mathrm{d}s\right|\\ &\leqslant |x'-z'|\int_0^1 |\nabla_{\eta'}W_k(y';\eta')||_{\eta'=x'+s(z'-x')}\,\mathrm{d}s\end{aligned}$$

$$\leqslant c_1|x'-z'|\left\{\int_0^\infty t^{-(d-1)/2}\mathrm{e}^{-c_2|y'|^2/t}\alpha_k(t)\,\mathrm{d}t + |y'|^2\int_0^\infty t^{-(d+1)/2}\mathrm{e}^{-c_2|y'|^2/t}\alpha_k(t)\,\mathrm{d}t\right\},$$

for some positive constants c_1 and c_2. Combining this with (5.24) and (5.25), we get (6.12).

By virtue of (6.6),

$$U_k(x',y') = \int_0^\infty \overline{q}^{y'}(t,x',y')\alpha_k(t)\,\mathrm{d}t + \int_0^\infty \widetilde{q}(t,x',y')\alpha_k(t)\,\mathrm{d}t \equiv U_{k,1}(x',y') + U_{k,2}(x',y').$$

From (6.11),

$$U_{k,1}(x',y') = W_k(y'-x';y').$$

Noting that $\int_{\Gamma\setminus\Delta_\varepsilon^{x'}}(y^i-x^i)\chi(y'-x')W_k(y'-x';x')\,\mathrm{d}y'=0$, we have

$$\int_{\Gamma\setminus\Delta_\varepsilon^{x'}}(y^i-x^i)\chi(y'-x')U_{k,1}(x',y')\,\mathrm{d}y' = \int_{\Gamma\setminus\Delta_\varepsilon^{x'}}(y^i-x^i)\chi(y'-x')\bigl\{W_k(y'-x';y')-W_k(y'-x';x')\bigr\}\,\mathrm{d}y'.$$

By virtue of (6.12),

$$\bigl|W_k(y'-x';y')-W_k(y'-x';x')\bigr| \leqslant C_{15}|x'-y'|^{-d+1}.$$

Therefore there exists the following limit:

$$b_{k,1}^i(x') := \lim_{\varepsilon\downarrow 0}\int_{\Gamma\setminus\Delta_\varepsilon^{x'}}(y^i-x^i)\chi(y'-x')U_{k,1}(x',y')\,\mathrm{d}y' = \int_\Gamma (y^i-x^i)\chi(y'-x')\bigl\{W_k(y'-x';y')-W_k(y'-x';x')\bigr\}\,\mathrm{d}y' = \int_\Gamma y^i\chi(y')\bigl\{W_k(y';x'+y')-W_k(y';x')\bigr\}\,\mathrm{d}y'.$$

By means of (6.8), (5.24) and (5.25),

$$|U_{k,2}(x',y')| \leqslant c_1\int_0^\infty t^{-(d+2-\alpha)/2}\mathrm{e}^{-c_2|x'-y'|^2/t}\,\mathrm{d}t \leqslant c_3|x'-y'|^{-(d-\alpha)},$$

for some positive constants c_i $(i=1,2,3)$. Therefore there exists the following limit:

$$b^i_{k,2}(x') := \lim_{\varepsilon\downarrow 0} \int\limits_{\Gamma\setminus\Delta^{x'}_\varepsilon} \left(y^i - x^i\right)\chi(y'-x')U_{k,2}(x',y')\,\mathrm{d}y'$$
$$= \int\limits_{\Gamma} \left(y^i - x^i\right)\chi(y'-x')U_{k,2}(x',y')\,\mathrm{d}y'.$$

Thus there exists the limit b^i_k defined by (6.5). □

We define the following operators.

$$\mathcal{A}u(x) = \frac{1}{2}\left\{\sum_{i,j=1}^{d-1}\partial_{x^i}\left(a^{ij}(x')\partial_{x^j}u(x)\right) + \partial_{x^d}\left(a^{dd}\left(x^d\right)\partial_{x^d}u(x)\right)\right\},$$

$$\partial_{n_a^-}u(x') = \frac{1}{2}a^{dd}(a)\partial_{x^d}u(x',a), \qquad \partial_{n_b^+}u(x') = -\frac{1}{2}a^{dd}(b)\partial_{x^d}u(x',b),$$

$$\widetilde{\mathcal{A}}^{\pm}_{\xi}u(x') = \frac{\kappa_\xi}{2}\sum_{i,j=1}^{d-1}\partial_{x^i}\left(a^{ij}(x')\partial_{x^j}\gamma^{\pm}_{\xi}u(x')\right),$$

$$\widetilde{\mathcal{B}}^{\pm}_{\xi}u(x') = \frac{1}{4}\int\limits_{\Gamma}\left\{\gamma^{\pm}_{\xi}u(x'+y') - \gamma^{\pm}_{\xi}u(x') - \nabla_{x'}\gamma^{\pm}_{\xi}u(x')\cdot y'\chi(y')\right\}$$
$$\times U_k(x',x'+y')\,\mathrm{d}y' + \frac{1}{4}\sum_{i=1}^{d-1}b^i_k(x')\partial_{x^i}\gamma^{\pm}_{\xi}u(x'),$$

$$\widetilde{\mathcal{C}}^{+-}_{a_k}u(x') = \frac{1}{4}\int\limits_{\Gamma}\left\{\gamma^{+}_{b_k}u(x'+y') - \gamma^{-}_{a_k}u(x')\right\}V_k(x',x'+y')\,\mathrm{d}y',$$

$$\widetilde{\mathcal{C}}^{-+}_{b_k}u(x') = \frac{1}{4}\int\limits_{\Gamma}\left\{\gamma^{-}_{a_k}u(x'+y') - \gamma^{+}_{b_k}u(x')\right\}V_k(x',x'+y')\,\mathrm{d}y',$$

where $\xi = a_k$ or b_k.

THEOREM 6.4. – *Assume* (6.2), (6.3), *and* (6.4). *Let* $f \in \widehat{C}(\overline{E})$ *and* $\lambda > 0$. *Then* $u = G_\lambda f$ *is the unique element in* $\widehat{C}(\overline{E}) \cap \mathcal{F}$ *satisfying the following equations in weak sense*: *for every* $k = 0, \pm 1, \pm 2, \ldots,$

$$(\lambda m_k - \mathcal{A})u = m_k f \qquad \text{in } E_k, \tag{6.13}$$

$$\partial_{n^+_{b_k}}u = \widetilde{\mathcal{A}}^{+}_{b_k}u + \widetilde{\mathcal{B}}^{+}_{b_k}u + \widetilde{\mathcal{C}}^{-+}_{b_k}u \qquad \text{in } \Gamma_{b_k}, \tag{6.14}$$

$$\partial_{n^-_{a_{k+1}}}u = \widetilde{\mathcal{A}}^{-}_{a_{k+1}}u + \widetilde{\mathcal{B}}^{-}_{a_{k+1}}u + \widetilde{\mathcal{C}}^{+-}_{a_{k+1}}u \quad \text{in } \Gamma_{a_{k+1}}. \tag{6.15}$$

Remark 6.5. – We say that an element $u \in \mathcal{F}$ satisfies Eqs. (6.13)–(6.15) in the weak sense if the equations

$$\int\limits_{E_k}(\lambda m_k - \mathcal{A})u(x)v(x)\,\mathrm{d}x = \int\limits_{E_k} m_k f(x)v(x)\,\mathrm{d}x \qquad \text{for } v \in H^1_0(E_k), \tag{6.16}$$

$$\int_\Gamma \partial_{n^+_{b_k}} u(x')\phi(x')\,\mathrm{d}x' = \int_\Gamma \bigl(\widetilde{\mathcal{A}}^+_{b_k}u + \widetilde{\mathcal{B}}^+_{b_k}u + \widetilde{\mathcal{C}}^{-+}_{b_k}u\bigr)(x')\phi(x')\,\mathrm{d}x' \tag{6.17}$$

$$\text{for } \phi \in H^1(\Gamma) \text{ or } H^{1/2}(\Gamma) \text{ according as } \kappa_{b_k} > 0 \text{ or } \kappa_{b_k} = 0$$

hold for (6.13) and (6.14) and a similar equation for (6.15). More precisely, the integrals in the above are understood as follows. Firstly,

$$\int_{E_k} \mathcal{A}u(x)v(x)\,\mathrm{d}x$$
$$= -\frac{1}{2}\int_{E_k}\left\{\sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}u(x)\partial_{x^j}v(x) + a^{dd}\bigl(x^d\bigr)\partial_{x^d}u(x)\partial_{x^d}v(x)\right\}\mathrm{d}x$$

for $v \in H^1_0(E_k)$. Secondly, let $v = H_{\Gamma\times\{b_k\}}\phi$ and take a sequence $\{\rho_n\} \subset C^\infty_0(\mathbb{R})$ such that $\rho_n(\xi) = 1$ for $|\xi - b_k| < 1/n$, $= 0$ for $|\xi - b_k| > 2/n$, and $0 \leqslant \rho_n \leqslant 1$. Then

$$\int_\Gamma \partial_{n^+_{b_k}} u(x')\phi(x')\,\mathrm{d}x' = \lim_{n\to\infty}\frac{1}{2}\int_{E_k} a^{dd}\bigl(x^d\bigr)\partial_{x^d}u(x)\partial_{x^d}\bigl(v(x)\rho_n\bigl(x^d\bigr)\bigr)\,\mathrm{d}x, \tag{6.18}$$

for each $\phi \in H^1(\Gamma)$ or $H^{1/2}(\Gamma)$. Note that the convergence is secured in the following proof of Theorem 6.4. Further, $v = H_{\Gamma\times\{b_k\}}\phi(x)$ can be replaced by any other $v \in H^{3/2}(E_k)$ or $H^1(E_k)$ with $\gamma^+_{b_k}v = \phi$. Thirdly,

$$\int_\Gamma \widetilde{\mathcal{A}}^+_{b_k}u(x')\phi(x')\,\mathrm{d}x' = -\frac{\kappa_{b_k}}{2}\int_\Gamma \sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}\gamma^+_{b_k}u(x')\partial_{x^j}\phi(x')\,\mathrm{d}x',$$

for $\phi \in H^1(\Gamma)$ if $\kappa_{b_k} > 0$, and $\int_\Gamma \widetilde{\mathcal{A}}^+_{b_k}u(x')\phi(x')\,\mathrm{d}x' = 0$ for all $\phi \in H^{1/2}(\Gamma)$ otherwise. Finally,

$$\int_\Gamma \widetilde{\mathcal{B}}_{b_k}u(x')\phi(x')\,\mathrm{d}x' \tag{6.19}$$
$$= -\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_{b_k}u(x') - \gamma_{b_k}u(y')\}\{\phi(x') - \phi(y')\}U_k(x', y')\,\mathrm{d}x'\,\mathrm{d}y',$$
$$\int_\Gamma \widetilde{\mathcal{C}}^{-+}_{b_k}u(x')\,\phi(x')\,\mathrm{d}x' = -\frac{1}{4}\int_\Gamma\int_\Gamma\{\gamma^-_{a_k}u(y') - \gamma^+_{b_k}u(x')\}V_k(x', y')\,\mathrm{d}y'\phi(x')\,\mathrm{d}x',$$

for all $\phi \in H^1(\Gamma)$ or $H^{1/2}(\Gamma)$.

Proof of Theorem 6.4. – We will only prove (6.16) and (6.17). Taking $v \in H^1_0(E_k)$ in (6.1), we obtain (6.16). Take next $v(x)\rho_n(x^d)$ for v in (6.1), where $v = H_{\Gamma\times\{b_k\}}\phi$ and ρ_n given in Remark 6.5. We then have, for sufficiently large n,

$$\lambda m_k\int_{E_k} u(x)v(x)\rho_n\bigl(x^d\bigr)\,\mathrm{d}x + \frac{1}{2}\int_{E_k}\sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}u(x)\partial_{x^j}v(x)\rho_n\bigl(x^d\bigr)\,\mathrm{d}x$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{E_k} a^{dd}\big(x^d\big)\partial_{x^d}u(x)\partial_{x^d}\big(v(x)\rho_n\big(x^d\big)\big)\,\mathrm{d}x\\
&+\frac{\kappa_{b_k}}{2}\int_{\Gamma}\sum_{i,j=1}^{d-1} a^{ij}(x')\partial_{x^i}\gamma_{b_k}^{+}u(x')\partial_{x^j}\phi(x')\,\mathrm{d}x'\\
&+\frac{1}{8}\iint_{\Gamma\times\Gamma}\{\gamma_\xi u(x')-\gamma_\xi u(y')\}\{\phi(x')-\phi(y')\}U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y',\\
&+\frac{1}{4}\int_{\Gamma}\int_{\Gamma}\{\gamma_{a_k}^{-}u(y')-\gamma_{b_k}^{+}u(x')\}V_k(x',y')\,\mathrm{d}y'\phi(x')\,\mathrm{d}x'\\
=&\int_{E_k} f(x)v(x)\rho_n\big(x^d\big)\,\mathrm{d}x.
\end{aligned}
$$

The first two terms and the last term go to 0 as $n\to\infty$. Hence the limit in (6.18) exists and (6.17) follows in the sense of Remark 6.5. □

Finally we give intuitive validity of $\widetilde{\mathcal{B}}_{b_k}^{\pm}$. The expression for $\widetilde{\mathcal{B}}_{a_k}^{\pm}$ is similar.

PROPOSITION 6.6. – *For each* $u\in C_0^\infty(\mathbb{R}^d)|_{\overline{E_k}}$ *and* $\phi\in C_0^\infty(\Gamma)$, (6.19) *holds in the strict sense.*

Proof. – Let $\phi,\ \varphi\in C_0^\infty(\Gamma)$. Then

$$
\begin{aligned}
&\iint_{\Gamma\times\Gamma}\{\phi(x')-\phi(y')\}\{\varphi(x')-\varphi(y')\}U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\
&\quad=\lim_{\varepsilon\downarrow 0}\iint_{\Gamma\times\Gamma\setminus\Delta_\varepsilon}\{\phi(x')-\phi(y')\}\{\varphi(x')-\varphi(y')\}U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\
&\quad\equiv\lim_{\varepsilon\downarrow 0}\Phi_\varepsilon,
\end{aligned}
$$

where $\Delta_\varepsilon=\{(x',y')\in\Gamma\times\Gamma\colon\ |x'-y'|<\varepsilon\}$. Noting that $U_k(x',y')=U_k(y',x')$, we have

$$
\begin{aligned}
\Phi_\varepsilon=&-\iint_{\Gamma\times\Gamma\setminus\Delta_\varepsilon}\phi(x')\{\varphi(y')-\varphi(x')-\nabla_{x'}\varphi(x')\cdot(y'-x')\chi(y'-x')\}\\
&\times U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\
&-\iint_{\Gamma\times\Gamma\setminus\Delta_\varepsilon}\phi(y')\{\varphi(x')-\varphi(y')-\nabla_{y'}\varphi(y')\cdot(x'-y')\chi(x'-y')\}\\
&\times U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\
&-\iint_{\Gamma\times\Gamma\setminus\Delta_\varepsilon}\phi(x')\nabla_{x'}\varphi(x')\cdot(y'-x')\chi(y'-x')U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\\
&-\iint_{\Gamma\times\Gamma\setminus\Delta_\varepsilon}\phi(y')\nabla_{y'}\varphi(y')\cdot(x'-y')\chi(x'-y')U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y'
\end{aligned}
$$

$$
\begin{aligned}
&= -2 \iint\limits_{\Gamma\times\Gamma\setminus\Delta_\varepsilon} \phi(x')\{\varphi(y') - \varphi(x') - \nabla_{x'}\varphi(x')\cdot(y'-x')\chi(y'-x')\} \\
&\quad \times U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad - 2 \iint\limits_{\Gamma\times\Gamma\setminus\Delta_\varepsilon} \phi(x')\nabla_{x'}\varphi(x')\cdot(y'-x')\chi(y'-x')U_k(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\equiv -2\Phi_{1,\varepsilon} - 2\Phi_{2,\varepsilon}.
\end{aligned}
$$

By means of Lemma 5.2 and Proposition 6.2, there exist the limits of $\Phi_{i,\varepsilon},\ i = 1,2$, as $\varepsilon \downarrow 0$ and

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0} \Phi_{1,\varepsilon} &= \int\limits_\Gamma \phi(x')\,\mathrm{d}x' \int\limits_\Gamma \{\varphi(x'+y') - \varphi(x') - \nabla_{x'}\varphi(x')\cdot y'\chi(y')\} \\
&\quad \times U_k(x',x'+y')\,\mathrm{d}y', \\
\Phi_{2,\varepsilon} &= \int\limits_\Gamma \phi(x')\,\mathrm{d}x' \int\limits_{\Gamma\setminus\Delta_\varepsilon^{x'}} \nabla_{x'}\varphi(x')\cdot(y'-x')\chi(y'-x')U_k(x',y')\,\mathrm{d}y' \\
&= \int\limits_\Gamma \phi(x') \sum_{i=1}^{d-1} \partial_{x^i}\varphi(x')\,\mathrm{d}x' \int\limits_{\Gamma\setminus\Delta_\varepsilon^{x'}} (y^i - x^i)\chi(y'-x')U_k(x',y')\,\mathrm{d}y' \\
&\longrightarrow \int\limits_\Gamma \phi(x') \sum_{i=1}^{d-1} \partial_{x^i}\varphi(x') b_k^i(x')\,\mathrm{d}x' \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}
$$

Therefore we obtain (6.19). □

Acknowledgement

The authors would like to thank the anonymous referee for the valuable comments.

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