Ann. I. H. Poincaré – PR 38, 1 (2002) 91–108

S0246-0203(01)01085-8/FLA

THE SPECTRAL GAP FOR A GLAUBER-TYPE DYNAMICS IN A CONTINUOUS GAS ☆

Lorenzo BERTINI [a], **Nicoletta CANCRINI** [b,d], **Filippo CESI** [c,d]

[a] *Dipartimento di Matematica, Università "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*
[b] *Dipartimento di Energetica, Università dell'Aquila, Italy*
[c] *Dipartimento di Fisica, Università "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*
[d] *INFM Unità di Roma "La Sapienza", Roma, Italy*

Received 30 May 2000, revised 5 February 2001

ABSTRACT. – We consider a continuous gas in a d-dimensional rectangular box with a finite range, positive pair potential, and we construct a Markov process in which particles appear and disappear with appropriate rates so that the process is reversible w.r.t. the Gibbs measure. If the thermodynamical paramenters are such that the Gibbs specification satisfies a certain mixing condition, then the spectral gap of the generator is strictly positive uniformly in the volume and boundary condition. The required mixing condition holds if, for instance, there is a convergent cluster expansion. © 2002 Éditions scientifiques et médicales Elsevier SAS

AMS classification: 82C21; 60K35; 82C22; 60J75

Keywords: Spectral gap; Gibbs measures; Continuous systems; Birth and death processes

RÉSUMÉ. – Dans une boîte rectangulaire de dimension d, on considère un gaz continu avec un potentiel à portée finie, pair et positif, et on construit un processus de Markov dans lequel les particules apparaissent et disparaissent avec un taux tel que ce processus soit réversible par rapport à la mesure de Gibbs associée. Si les paramètres thermodynamiques assurent une certaine condition de mélange pour la mesure de Gibbs, nous concluons que le trou spectral associé au générateur est strictement positif, uniformément par rapport au volume de la boîte et aux conditions aux bords. La condition de mélange requise a lieu par exemple lorsqu'il y a convergence du développement viriel. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

We consider a continuous gas in a bounded volume $\Lambda \subset \mathbb{R}^d$, distributed according the Gibbs probability measure associated to a finite range pair potential φ. The Gibbs measure in a volume Λ is given by (see Section 2 for more details)

☆ This work was partially supported by GNAFA and by "Cofinanziamento Murst".
E-mail addresses: lorenzo@carpenter.mat.uniroma1.it (L. Bertini), nicoletta.cancrini@roma1.infn.it (N. Cancrini), filippo.cesi@roma1.infn.it (F. Cesi).

$$\mu_\Lambda^\eta(\mathrm{d}\omega) := \mathrm{e}^{|\Lambda|}\left(Z_\Lambda^\eta\right)^{-1} z^{N_\Lambda(\omega)} \exp\left[-\beta H_\Lambda^\eta(\omega)\right](Q_\Lambda \times \delta_{\Lambda^c,\eta})(\mathrm{d}\omega),$$

where β is the inverse temperature, z the activity, H the Hamiltonian, N_Λ the number of particles, η the boundary condition, Q_Λ the Poisson point process in Λ with intensity 1, and Z_Λ^η is the normalization. Then we introduce a Markov process in which particles may appear and disappear everywhere in Λ with rates such that the process is reversible with respect to the Gibbs measure.

These kind of processes, also called *spatial birth and death processes*, have been constructed by Preston in [12] as a particular case of jump processes, in a more general framework than ours, i.e. without assuming reversibility w.r.t. a Gibbs measure. In [12], and, more recently, in [3], many of their general properties are studied.

In this paper we are interested in the approach to the invariant measure in the L^2 sense, and, in particular, we show that for a positive, finite range, pair potential, if z, β are such that there is a convergent cluster expansion (see condition (CE) in Section 2), then the generator of the process L_Λ^η has a spectral gap which is strictly positive uniformly in the volume and the boundary condition. Convergence of cluster expansion is not actually necessary for our results and it will be only used to prove a mixing condition for the Gibbs measures (Corollary 2.5 below) which could be assumed as a more general hypothesis.

Uniform positivity of the spectral gap has been discussed in several papers for *lattice* spin systems, for either discrete/compact spin spaces [15,16,9,10,7] and unbounded spin spaces [18,17,4]. Within that context the general idea is that the following notions are equivalent

(1) The spectral gap of the generator is strictly positive uniformly in the volume and boundary condition.
(2) The logarithmic Sobolev constant is bounded uniformly in the volume and boundary condition.
(3) The covariance w.r.t. the Gibbs measure of two local functions decays exponentially fast in the distance of the "supports" of the functions, uniformly in the volume and boundary condition.

We observe that for the system we consider in this paper, there is no hope of proving (in general) a logarithmic Sobolev inequality (LSI). Even worse such an inequality fails even for a fixed finite volume. Consider, indeed, the trivial case of $H = 0$. Then the distribution of the number of particles in a volume Λ is Poissonian with mean $z|\Lambda|$. It is easy to verify (see [5], Section 5.1) that the Poissonian distribution does not satisfy a LSI. It is still possible, though, that under stronger conditions on the potential which do not include the case $H = 0$ (e.g. superstability, see [13]) a LSI is indeed satisfied.

Our results are presented in Section 2, while most proofs are postponed to Section 3. Section 4 contains a partial converse of our main result, i.e. that the uniform positivity of the spectral gap implies the exponential decay of the covariance of two local functions. Finally, Section 5 is a brief discussion on the possibility of having a logarithmic Sobolev inequality.

2. Notation and results

The Gibbs measures. Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$; we denote by $\mathcal{B}_b(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ the collection of all bounded Borel sets. For $A \in \mathcal{B}_b(\mathbb{R}^d)$, $|A|$ indicates the Lebesgue measure of A. Let $\mathcal{R}^d \subset \mathcal{B}_b(\mathbb{R}^d)$ be the set of all *rectangles* (cartesian products of finite closed intervals). We consider, as *configuration space*, the set Ω of all locally finite subsets of $\mathbb{R}^d$, i.e.

$$\Omega := \{\omega \subset \mathbb{R}^d\colon \operatorname{card}(\omega \cap A) < \infty \text{ for all bounded subsets } A \text{ of } \mathbb{R}^d\},$$

where $\operatorname{card}(A)$ stands for the cardinality of the set A. We endow Ω with the σ-algebra $\mathcal{F}$ generated by the counting variables $N_A : \omega \to \operatorname{card}(\omega \cap A)$, where $A \in \mathcal{B}_b(\mathbb{R}^d)$. Given $\omega, \eta \in \Omega$ we let $\omega \triangle \eta$ be the symmetric difference of ω and η, i.e. $\omega \triangle \eta := (\omega \cup \eta)\backslash(\omega \cap \eta)$. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, we consider also the finite volume configuration space

$$\Omega_\Lambda := \{\omega \subset \Lambda\colon \omega \text{ is finite}\}$$

with σ-algebra $\mathcal{F}_\Lambda$ generated by the functions N_A, such that A is a Borel subset of Λ. We write $f \in \mathcal{F}_A$ to indicate that the f is $\mathcal{F}_A$-measurable. The function f is said to be *local* if there exists $A \in \mathcal{B}_b(\mathbb{R}^d)$ such that $f \in \mathcal{F}_A$.

For $x, y \in \mathbb{R}^d$ we denote by $d(x, y)$ the Euclidean distance, while $|x|$ stands for $d(x, 0)$. Let $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ be a measurable even function; φ is called a *pair potential*. We assume that φ has *finite range* r, i.e. that $\varphi(x) = 0$ if $|x| > r$. Given $A \subset \mathbb{R}^d$ we let

$$\bar{A}^r := \{x \in \mathbb{R}^d\colon d(x, A) \leqslant r\}.$$

The Hamiltonian $H_\Lambda : \Omega \mapsto \mathbb{R}$, is given by

$$H_\Lambda(\omega) := \sum_{\substack{\{x,y\} \subset \omega \\ \{x,y\} \cap \Lambda \neq \emptyset}} \varphi(x - y).$$

For $\omega, \eta \in \Omega$ we also let $H_\Lambda^\eta(\omega) := H_\Lambda(\omega_\Lambda \eta_{\Lambda^c})$, where $\omega_\Lambda \eta_{\Lambda^c} := (\omega \cap \Lambda) \cup (\eta \cap \Lambda^c)$, Λ^c stands for the complement of Λ, and η is called the *boundary condition*. We denote with Q_Λ the Poisson point process on Λ with intensity 1, and we define $Q_\Lambda^\eta := Q_\Lambda \times \delta_{\Lambda^c,\eta}$, where $\delta_{\Lambda^c,\eta}$ is the probability measure on $(\Omega_{\Lambda^c}, \mathcal{F}_{\Lambda^c})$ which gives mass 1 to the configuration η. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the finite volume *Gibbs measure* in Λ at inverse temperature β, activity z and boundary condition η is given by

$$\mu_\Lambda^\eta(\mathrm{d}\omega) := \mathrm{e}^{|\Lambda|}\left(Z_\Lambda^\eta\right)^{-1} z^{N_\Lambda(\omega)} \exp\left[-\beta H_\Lambda^\eta(\omega)\right] Q_\Lambda^\eta(\mathrm{d}\omega), \tag{2.1}$$

where Z_Λ^η is the appropriate normalization factor (we omit for simplicity the dependence of these quantities on z and β). We denote with $\mu_\Lambda^\eta(f)$ the expectation of f with respect to μ_Λ^η, while $\mu_\Lambda(f)$ denotes the function $\omega \to \mu_\Lambda^\omega(f)$. Explicitly, for all measurable functions f on Ω_Λ, we have

$$\mu_\Lambda^\eta(f) = \left(Z_\Lambda^\eta\right)^{-1} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int\limits_{\Lambda^n} \mathrm{e}^{-\beta H_\Lambda^\eta(x)} f(x)\, \mathrm{d}x,$$

where we have identified the functions on Ω_Λ with the symmetric functions on $\bigcup_{n=0}^{\infty} \Lambda^n$. For a set $X \in \mathcal{F}$ we set $\mu_\Lambda(X) := \mu_\Lambda(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function on X. We write $\mu(f, g)$ to denote the covariance (with respect to μ) of f and g. The family of measures (2.1) satisfies the DLR compatibility conditions

$$\mu_\Lambda\big(\mu_V(X)\big) = \mu_\Lambda(X) \quad \forall X \in \mathcal{F} \quad \forall V, \Lambda \in \mathcal{B}_b\big(\mathbb{R}^d\big), V \subset \Lambda. \tag{2.2}$$

The dynamics. For a given function f on Ω we let

$$D_x^- f(\omega) := f(\omega \backslash \{x\}) - f(\omega) \quad D_x^+ f(\omega) := f(\omega \cup \{x\}) - f(\omega) \quad \omega \in \Omega,\ x \in \mathbb{R}^d, \tag{2.3}$$

where it is understood that $D_x^- f(\emptyset) = 0$. For simplicity we use the notation

$$\big(D_\Lambda^- f \cdot D_\Lambda^- g\big)(\omega) := \sum_{x \in \omega \cap \Lambda} D_x^- f(\omega) D_x^- g(\omega).$$

The stochastic dynamics we want to study is determined by the generators L_Λ, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, defined by

$$(L_\Lambda f)(\omega) := \sum_{x \in \omega \cap \Lambda} D_x^- f(\omega) + z \int\limits_{\Lambda} \mathrm{e}^{-\beta D_x^+ H_\Lambda(\omega)} D_x^+ f(\omega)\, \mathrm{d}x \quad \omega \in \Omega. \tag{2.4}$$

L_Λ^η stands for L_Λ acting on $L^2(\mu_\Lambda^\eta)$ with domain $\mathcal{D}_0(L_\Lambda^\eta)$ given by

$$\mathcal{D}_0(L_\Lambda^\eta) = \big\{ f \in L^2\big(\mu_\Lambda^\eta\big) \colon \exists M \in \mathbb{N}, |f| \leqslant M \text{ and } f(\omega) = 0 \text{ when } N_\Lambda(\omega) > M \big\}. \tag{2.5}$$

The Dirichlet form associated with L_Λ^η is given by

$$\mathcal{E}_\Lambda^\eta(f, g) := \big\langle \big(-L_\Lambda^\eta\big) f, g \big\rangle_{L^2(\mu_\Lambda^\eta)} \quad f, g \in \mathcal{D}_0\big(L_\Lambda^\eta\big)$$

and we let $\mathcal{E}_\Lambda^\eta(f) := \mathcal{E}_\Lambda^\eta(f, f)$. The construction of the corresponding Markov semigroups in the spaces $L^p(\mu_\Lambda^\eta)$, $p \in [1, \infty]$, is more or less standard, and it is summarized below

PROPOSITION 2.1. –
(1) *$\mathcal{D}_0(L_\Lambda^\eta)$ is dense in $L^2(\mu_\Lambda^\eta)$.*
(2) *$\mathcal{E}_\Lambda^\eta(f, g) = \mu_\Lambda^\eta(D_\Lambda^- f \cdot D_\Lambda^- g) = z \int_\Lambda \mathrm{d}x\, \mu_\Lambda^\eta(\mathrm{e}^{-\beta D_x^+ H_\Lambda^\eta} D_x^+ f D_x^+ g)$, for all $f, g \in \mathcal{D}_0(L_\Lambda^\eta)$.*
(3) *L_Λ^η is symmetric on $\mathcal{D}_0(L_\Lambda^\eta)$.*
(4) *$\mathcal{E}_\Lambda^\eta$ is closable and its closure is associated with a self-adjoint extension of L_Λ^η. We maintain the same symbols L_Λ^η and $\mathcal{E}_\Lambda^\eta$ to denote these extensions, and denote by $\mathcal{D}(L_\Lambda^\eta)$, $\mathcal{D}(\mathcal{E}_\Lambda^\eta)$ the respective domains.*

(5) $P_t^{\Lambda,\eta} := e^{tL_\Lambda^\eta}$ *is a positive preserving contraction semigroup* (*thus a Markov semigroup*) *on* $L^p(\mu_\Lambda^\eta)$ *for all* $p \in [1,\infty]$.

Remark. – A more explicit construction for these processes can be found in [12] where sufficient conditions are found which guarantee the uniqueness of the solution of the Kolmogorov's backward equations. Since we are mainly interested in L^p properties, our approach is more direct for our purposes.

Proof. – For statement (1), let $f \in L^2(\mu_\Lambda^\eta)$, and define $f_n \in \mathcal{D}_0(L_\Lambda^\eta)$ by

$$f_n(\omega) := \big[(f(\omega) \wedge n) \vee (-n)\big] \cdot \mathbb{1}_{N_\Lambda(\omega) \leqslant n},$$

where $a \vee b$ ($a \wedge b$) stands for the maximum (minimum) between a and b. The dominated convergence theorem implies that $\|f - f_n\|_{L^2(\mu_\Lambda^\eta)}$ goes to 0. Statement (2) is a simple computation, while (3) is trivially implied by (2). Statement (4) is the well known construction of the Friedrichs extension of a non-positive symmetric operator. In order to prove (5) it is sufficient (and actually necessary) to show that (see Theorems 1.3.2 and 1.3.3 of [2])

(a) If $f \in \mathcal{D}(\mathcal{E}_\Lambda^\eta)$ then $|f| \in \mathcal{D}(\mathcal{E}_\Lambda^\eta)$ and $\mathcal{E}_\Lambda^\eta(|f|) \leqslant \mathcal{E}_\Lambda^\eta(f)$.

(b) If $0 \leqslant f \in \mathcal{D}(\mathcal{E}_\Lambda^\eta)$ then $f \wedge 1 \in \mathcal{D}(\mathcal{E}_\Lambda^\eta)$ and $\mathcal{E}_\Lambda^\eta(f \wedge 1) \leqslant \mathcal{E}_\Lambda^\eta(f)$.

Properties (a) and (b) are obviously true on $\mathcal{D}_0(L_\Lambda^\eta)$ thanks to the expression for the Dirichlet form given in (2), and they can be directly extended to $\mathcal{D}(\mathcal{E}_\Lambda^\eta)$ since $\mathcal{D}(\mathcal{E}_\Lambda^\eta)$ is the closure of $\mathcal{D}_0(L_\Lambda^\eta)$ w.r.t. the norm $[\|f\|^2_{L^2(\mu_\Lambda^\eta)} + \mathcal{E}_\Lambda^\eta(f)]^{1/2}$ □

The spectral gap of L_Λ^η is defined as as

$$\operatorname{gap}(L_\Lambda^\eta) := \inf \operatorname{spec}\big(-L_\Lambda^\eta \restriction \mathbb{1}^\perp\big),$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\mu_\Lambda^\eta)$ orthogonal to the constant functions.

In order to prove our main result we need some kind of mixing property of the Gibbs measure, which we can prove under the hypothesis of a convergent cluster expansion. An explicit condition which guarantees this convergence is the following: let $\xi(\beta) := e\int_{\mathbb{R}^d}(1 - e^{-\beta\varphi(x)})\,dx$. Then we assume

$$z\xi(\beta)/\big(1 - 2z\xi(\beta)\big) < 1. \tag{CE}$$

Our main result is then the following:

THEOREM 2.2. – *Let* $\varphi \geqslant 0$ *be a pair potential with finite range* r. *If* (CE) *holds there exists* $G = G(r, z, \beta)$ *finite such that for all* $\eta \in \Omega$, $\Lambda \in \mathcal{R}^d$,

$$\mu_\Lambda^\eta(f, f) \leqslant G\mathcal{E}_\Lambda^\eta(f), \quad \textit{for all } f \in \mathcal{D}\big(\mathcal{E}_\Lambda^\eta\big). \tag{2.6}$$

Remark. – Poincaré inequality (2.6) is equivalent to any of the following statements:

(1) $\operatorname{gap}(L_\Lambda^\eta)^{-1} \leqslant G$.

(2) $\|P_t^{\Lambda,\eta} f - \mu_\Lambda^\eta f\|_{L^2(\mu_\Lambda^\eta)} \leqslant (\mu_\Lambda^\eta(f,f))^{1/2} e^{-t/G}$, for all $f \in L^2(\mu_\Lambda^\eta)$.

In order to prove the theorem we need to

(1) prove a mixing condition for the Gibbs measures (Corollary 2.5 below),

(2) show that the spectral gap is strictly positive for all rectangles contained in some fixed cube Λ_0 whose size depends on z, β and r (in Proposition 2.6 below we actually show that the spectral gap is strictly positive for any bounded volume).

Given (1) and (2), there are several standard arguments (see the papers cited in the introduction) which produce Theorem 2.2 for *lattice spin systems*. The easiest approach is perhaps the one given in Theorem 4.5 in [8]. We will adapt the same strategy to our system. The proof will follow the scheme

$$\text{Lemma 2.3} + \text{cluster expansion} \Rightarrow \text{Corollary 2.4} \Rightarrow \text{Corollary 2.5},$$
$$\text{Corollary 2.5} + \text{Proposition 2.6} \Rightarrow \text{Theorem 2.2}.$$

Our first result is a general upper bound for the covariance of two local functions.

LEMMA 2.3. – *Let $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and let Λ_f, Λ_g be two Borel subsets of Λ such that $\Lambda_f \cap \Lambda_g = \emptyset$. For all $z > 0, \beta \geqslant 0, \eta \in \Omega$ and all pairs of local functions f, g with $f \in \mathcal{F}_{\Lambda_f}$ and $g \in \mathcal{F}_{\Lambda_g}$, we have*

$$\left|\mu_\Lambda^\eta(f, g)\right| \leqslant \mu_\Lambda^\eta(|f|)\mu_\Lambda^\eta(|g|) \sup_{\eta \in \Omega}\left[\frac{Z^\eta_{\Lambda\setminus(\Lambda_f\cup\Lambda_g)} Z^\eta_\Lambda}{Z^\eta_{\Lambda\setminus\Lambda_f} Z^\eta_{\Lambda\setminus\Lambda_g}} - 1\right]. \tag{2.7}$$

Remark. – One may wonder how we can bound the covariance of two functions in terms of their L^1 (rather than L^2) norm. This is possible because f, g have disjoint "supports", i.e. $\Lambda_f \cap \Lambda_g = \emptyset$.

Using standard cluster expansion, one can estimate the logarithm of the ratio of the partition functions appearing in (2.7) (see Lemma 4 of [14]) and obtain

COROLLARY 2.4. – *Assume $\varphi \geqslant 0$ and let z, β be such that* (CE) *holds. Then there exist $\alpha = \alpha(r, z, \beta)$ and $m = m(r, z, \beta)$ such that for all $\Lambda, \Lambda_f, \Lambda_g \in \mathcal{B}_b(\mathbb{R}^d)$ such that $\Lambda_f \subset \Lambda$, $\Lambda_g \subset \Lambda$, $d(\Lambda_f, \Lambda_g) > 2r$, and $|\bar{\Lambda}^r_f| \wedge |\bar{\Lambda}^r_g| \leqslant \exp(md(\Lambda_f, \Lambda_g))$, we have*

$$\left|\mu_\Lambda^\eta(f, g)\right| \leqslant \alpha\mu_\Lambda^\eta(|f|)\mu_\Lambda^\eta(|g|)\mathrm{e}^{-md(\Lambda_f,\Lambda_g)} \quad \forall f \in \mathcal{F}_{\Lambda_f},\ g \in \mathcal{F}_{\Lambda_g},\ \forall \eta \in \Omega. \tag{2.8}$$

This result has an immediate consequence, which will be useful for our purposes.

COROLLARY 2.5. – *If $\varphi \geqslant 0$ and* (CE) *holds, there exist $\alpha = \alpha(r, z, \beta)$ and $m = m(r, z, \beta)$ such that for all $\Lambda, \Lambda_f \in \mathcal{B}_b(\mathbb{R}^d)$, $\Lambda_f \subset \Lambda$,*

$$\left|\mu_\Lambda^\eta(f) - \mu_\Lambda^\omega(f)\right| \leqslant \alpha\mu_\Lambda^\eta(|f|)\mathrm{e}^{-md(\Lambda_f,\eta\,\triangle\,\omega)} \tag{2.9}$$

for all $\omega, \eta \in \Omega$, for all $f \in \mathcal{F}_{\Lambda_f}$ such that $d(\Lambda_f, \eta \triangle \omega) > 3r$, and $|\bar{\Lambda}^r_f| \leqslant \exp[m(d(\Lambda_f, \eta \triangle \omega) - r)]$.

Remark. – As we said before, Corollary 2.5 is the only ingredient we need (together with the positivity of the spectral gap in a given finite volume) in order to prove Theorem 2.2. We observe here that inequality (2.9) is very strong, because of the factor $\mu_\Lambda^\eta(|f|)$ in the RHS. Let for instance $\eta = \emptyset$ (free boundary condition). Then (2.9) implies that the difference $|\mu_\Lambda^\emptyset(f) - \mu_\Lambda^\omega(f)|$ is bounded *uniformly* in ω. Thus one cannot hope that (2.9) holds for a large class of interactions. On the other side (2.9) is stronger than we actually need. To be more precise what we really need is inequality (3.11) in the next section.

Proof. – Let $h_\Lambda^{\omega,\eta}$ be the density of μ_Λ^ω w.r.t. μ_Λ^η. Then

$$h_\Lambda^{\omega,\eta} = \frac{\exp[-\beta(H_\Lambda^\omega - H_\Lambda^\eta)]}{\mu_\Lambda^\eta(\exp[-\beta(H_\Lambda^\omega - H_\Lambda^\eta)])}$$

so $h_\Lambda^{\omega,\eta}$ is measurable w.r.t. $\mathcal{F}_A$, where $A := \{x \in \Lambda\colon\ d(x, \omega \triangle \eta) \leqslant r\}$. Therefore from Corollary 2.4 it follows that

$$\begin{aligned}\left|\mu_\Lambda^\eta(f) - \mu_\Lambda^\omega(f)\right| &= \left|\mu_\Lambda^\eta[f(1 - h_\Lambda^{\omega,\eta})]\right| = \left|\mu_\Lambda^\eta(f, h_\Lambda^{\omega,\eta})\right| \\ &\leqslant \alpha\mu_\Lambda^\eta(|f|)\mu_\Lambda^\eta\big(h_\Lambda^{\omega,\eta}\big)\mathrm{e}^{-md(\Lambda_f, A)} \leqslant \alpha\mu_\Lambda^\eta(|f|)\mathrm{e}^{-m[d(\Lambda_f, \omega\triangle\eta) - r]}\end{aligned}$$

and we get the result, after redefining α. □

Finally we will show that the spectral gap is strictly positive in any bounded volume.

PROPOSITION 2.6. – *If $\varphi \geqslant 0$, then*

$$\mu_\Lambda^\eta(f, f) \leqslant 2\big(\mathrm{e}^{z|\Lambda|} - 1\big)\mathcal{E}_\Lambda^\eta(f) \quad \forall f \in \mathcal{D}\big(\mathcal{E}_\Lambda^\eta\big)$$

for all $z > 0$, $\beta \geqslant 0$, $\eta \in \Omega$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

3. Proofs

3.1. Proof of Lemma 2.3

If A is a Borel subset of Λ and h is an $\mathcal{F}_A$ measurable function on Ω, we have (see (2.1))

$$\mu_\Lambda^\eta(h) = Q_A^\eta(\rho_{\Lambda,A}h) \tag{3.1}$$

where

$$\rho_{\Lambda,A}(\omega) := \mathrm{e}^{|A|}z^{N_A(\omega)}\exp\big[-\beta H_A\big(\omega \cap \big(A \cup \Lambda^c\big)\big)\big]\frac{Z_{\Lambda\setminus A}^\omega}{Z_\Lambda^\omega}.$$

Notice that the Hamiltonian does not include the interactions between A and $\Lambda\setminus A$, since those terms are included in the partition function $Z_{\Lambda\setminus A}$. From (3.1), if we let

$$R_{\Lambda,f,g}(\omega) := \frac{\rho_{\Lambda,\Lambda_f\cup\Lambda_g}(\omega)}{\rho_{\Lambda,\Lambda_f}(\omega)\rho_{\Lambda,\Lambda_g}(\omega)} = \frac{Z_{\Lambda\setminus(\Lambda_f\cup\Lambda_g)}^\omega Z_\Lambda^\omega}{Z_{\Lambda\setminus\Lambda_f}^\omega Z_{\Lambda\setminus\Lambda_g}^\omega} \tag{3.2}$$

we obtain

$$\begin{aligned}\mu_\Lambda^\eta(f, g) &= \mu_\Lambda^\eta(fg) - \mu_\Lambda^\eta(f)\mu_\Lambda^\eta(g) \\ &= Q_{\Lambda_f\cup\Lambda_g}^\eta\big[fg\rho_{\Lambda,\Lambda_f}\rho_{\Lambda,\Lambda_g}(R_{\Lambda,f,g} - 1)\big].\end{aligned}$$

Hence

$$\begin{aligned}\left|\mu_\Lambda^\eta(f,g)\right| &\leqslant \sup|R_{\Lambda,f,g}-1|\,Q^\eta_{\Lambda_f\cup\Lambda_g}\big[|fg|\rho_{\Lambda,\Lambda_f}\rho_{\Lambda,\Lambda_g}\big]\\ &= \sup|R_{\Lambda,f,g}-1|\mu_\Lambda^\eta(|f|)\mu_\Lambda^\eta(|f|).\end{aligned}$$

3.2. Proof of Proposition 2.6

By writing the covariance $\mu_\Lambda^\eta(f,f)$ in the product coupling, we get

$$\begin{aligned}\mu_\Lambda^\eta(f,f) &= \frac{1}{2}\int \mu_\Lambda^\eta(\mathrm{d}\omega)\mu_\Lambda^\eta(\mathrm{d}\tilde\omega)\big[f(\omega)-f(\tilde\omega)\big]^2\\ &= \frac{1}{2}\big(Z_\Lambda^\eta\big)^{-2}\sum_{\substack{n,m=0\\(n,m)\neq(0,0)}}^{\infty}\frac{z^n}{n!}\frac{z^m}{m!}\int\limits_{\Lambda^n\times\Lambda^m}\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\beta[H_\Lambda^\eta(x)+H_\Lambda^\eta(y)]}\big[f(x)-f(y)\big]^2.\end{aligned}\tag{3.3}$$

Let $D_i^- f(x_1,\dots,x_n) := f(x_1,\dots,\hat{x}_i,\dots,x_n)-f(x_1,\dots,x_n)$, where $\hat{x}_i$ denotes that the variable x_i is omitted. By telescopic sums we have

$$f(x_1,\dots,x_n)-f(y_1,\dots,y_m) = -\sum_{k=1}^{n} D_k^- f(x_1,\dots,x_k)+\sum_{h=1}^{m} D_h^- f(y_1,\dots,y_h)$$

whence, by Schwarz inequality,

$$\begin{aligned}&\frac{1}{2}\big[f(x_1,\dots,x_n)-f(y_1,\dots,y_m)\big]^2\\ &\quad\leqslant n\sum_{k=1}^{n}\big[D_k^- f(x_1,\dots,x_k)\big]^2+m\sum_{h=1}^{m}\big[D_h^- f(y_1,\dots,y_h)\big]^2\end{aligned}$$

which, plugged into (3.3), yields

$$\begin{aligned}\mu_\Lambda^\eta(f,f) \;&\leqslant\; 2\big(Z_\Lambda^\eta\big)^{-1}\big(1-\mathrm{e}^{-z|\Lambda|}\big)\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{z^n}{(n-1)!}\\ &\qquad\times\int\limits_{\Lambda^n}\mathrm{d}x\,\mathrm{e}^{-\beta H_\Lambda^\eta(x)}\big[D_k^- f(x_1,\dots,x_k)\big]^2,\end{aligned}\tag{3.4}$$

where we used

$$\big(Z_\Lambda^\eta\big)^{-1}\sum_{m=1}^{\infty}\frac{z^m}{m!}\int\limits_{\Lambda^m}\mathrm{d}y\,\mathrm{e}^{-\beta H_\Lambda^\eta(y)} = 1-\big(Z_\Lambda^\eta\big)^{-1}\leqslant 1-\mathrm{e}^{-z|\Lambda|}.$$

Last inequality holds because $\varphi\geqslant 0$ so that $\mathrm{e}^{-\beta H_\Lambda^\eta(y)}\leqslant 1$. By the same reason, for $k\leqslant n$ we have $H_\Lambda^\eta(x_1,\dots,x_n)\geqslant H_\Lambda^\eta(x_1,\dots,x_k)$; therefore, by using (3.4),

$$\begin{aligned}\mu_\Lambda^\eta(f,f) &\leqslant 2\big(Z_\Lambda^\eta\big)^{-1}\big(1-\mathrm{e}^{-z|\Lambda|}\big)\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{z^n|\Lambda|^{n-k}}{(n-1)!}\int\limits_{\Lambda^k}\mathrm{d}x\,\mathrm{e}^{-\beta H_\Lambda^\eta(x)}\big[D_k^- f(x)\big]^2\\ &= 2\big(Z_\Lambda^\eta\big)^{-1}\big(1-\mathrm{e}^{-z|\Lambda|}\big)\sum_{k=1}^{\infty}\frac{z^k}{(k-1)!}\int\limits_{\Lambda^k}\mathrm{d}x\,\mathrm{e}^{-\beta H_\Lambda^\eta(x)}\big[D_k^- f(x)\big]^2\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{n=k}^{\infty} \frac{z^{n-k}|\Lambda|^{n-k}}{(n-k)!} \frac{(n-k)!(k-1)!}{(n-1)!} \\
&\leqslant 2\mathrm{e}^{z|\Lambda|}\big(1-\mathrm{e}^{-z|\Lambda|}\big)\big(Z_\Lambda^\eta\big)^{-1} \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \int_{\Lambda^k} \mathrm{d}x\, \mathrm{e}^{-\beta H_\Lambda^\eta(x)} \big[D_k^- f(x)\big]^2 \\
&= 2\mathrm{e}^{z|\Lambda|}\big(1-\mathrm{e}^{-z|\Lambda|}\big)\big(Z_\Lambda^\eta\big)^{-1} \sum_{k=1}^{\infty} \frac{z^k}{k!} \int_{\Lambda^k} \mathrm{d}x\, \mathrm{e}^{-\beta H_\Lambda^\eta(x)} \sum_{j=1}^{k} \big[D_j^- f(x)\big]^2 \\
&= 2\big(\mathrm{e}^{z|\Lambda|}-1\big)\mathcal{E}_\Lambda^\eta(f),
\end{aligned}
$$

where the last identity follows from $\mathcal{E}_\Lambda^\eta(f) = \mu_\Lambda^\eta(|D_\Lambda^- f|^2)$ (see (2) of Proposition 2.1). □

3.3. Proof of Theorem 2.2

Notation. Throughout this proof we let, for brevity,

$$
\|f\|_p := \|f\|_{L^p(\mu_\Lambda^\eta)}, \quad \langle f, g\rangle := \langle f, g\rangle_{L^2(\mu_\Lambda^\eta)}, \quad p \in [1, \infty]. \tag{3.5}
$$

As we said earlier Theorem 2.2 is a consequence of Corollary 2.5 and Proposition 2.6. We proceed more or less as in [8], Theorem 4.5.

Basically what we want to show is that the gap stays (almost) the same if we double the volume. Let $\Lambda \subset \mathcal{R}^d$ and assume that the longest side of Λ has length $\sim L$ and it corresponds to the direction e_d in $\mathbb{R}^d$. We write $\Lambda = A \cup B$, where A and B are two rectangles of roughly the same size with a small overlap in the direction e_d. The overlap is order $\sqrt{L}$. We then claim that Corollary 2.5 implies the existence of $c_1 = c_1(r, z, \beta)$, $c_2 = c_2(r, z, \beta)$ and $L_0 = L_0(r, z, \beta)$ such that

$$
\mu_\Lambda^\eta(f, f) \leqslant \big(1 + c_1 \mathrm{e}^{-c_2 m\sqrt{L}}\big)\mu_\Lambda^\eta\big(\mu_A(f, f) + \mu_B(f, f)\big) \quad \forall L \geqslant L_0. \tag{3.6}
$$

We observe that this inequality holds with $c_1 = 0$ if A and B are disjoint and non–interacting, so that $\mu_\Lambda^\eta = \mu_A^\eta \times \mu_B^\eta$. The factor $c_1 \exp(-c_2 m\sqrt{L})$ measures in a certain sense the weak interaction between A and B. The proof of inequality (3.6) relies on the following lemma:

LEMMA 3.1. – *Let $\Lambda, A, B \in \mathcal{B}_b(\mathbb{R}^d)$, with $\Lambda = A \cup B$. Assume that for some $\eta \in \Omega$, $\varepsilon \in [0, \sqrt{2} - 1)$, $p \in [1, \infty]$, we have*

$$
\begin{aligned}
&\big\|\mu_B g - \mu_\Lambda^\eta g\big\|_p \leqslant \varepsilon \|g\|_p \quad \forall g \in L^p\big(\Omega, \mathcal{F}_{A^c}, \mu_\Lambda^\eta\big), \\
&\big\|\mu_A g - \mu_\Lambda^\eta g\big\|_p \leqslant \varepsilon \|g\|_p \quad \forall g \in L^p\big(\Omega, \mathcal{F}_{B^c}, \mu_\Lambda^\eta\big).
\end{aligned} \tag{3.7}
$$

Then

$$
\mu_\Lambda^\eta(f, f) \leqslant \big(1 - 2\varepsilon - \varepsilon^2\big)^{-1} \mu_\Lambda^\eta\big(\mu_A(f, f) + \mu_B(f, f)\big) \quad f \in L^2\big(\mu_\Lambda^\eta\big). \tag{3.8}
$$

Remark. – A similar result was obtained in Proposition 4.4 in [1] (see also Proposition 3.5 in [8]). The proof given there is somehow more complicated and it is based on the explicit expression for the semigroup of the "2-block dynamics", given by $e^{t(\mu_A+\mu_B-2)}$. We present below a shorter and more direct approach.

Proof. –
Step 1. Reduction to the case $p=2$. Let $p\in[1,\infty]$, let $q^{-1}:=1-p^{-1}$, and, for $r\in[1,\infty]$, consider the linear operators

$$T_{B,r}: L^r(\Omega,\mathcal{F}_{A^c},\mu_\Lambda^\eta)\ni f\mapsto \mu_B(f)-\mu_\Lambda^\eta(f)\in L^r(\Omega,\mathcal{F}_{B^c},\mu_\Lambda^\eta),$$
$$T_{A,r}: L^r(\Omega,\mathcal{F}_{B^c},\mu_\Lambda^\eta)\ni f\mapsto \mu_A(f)-\mu_\Lambda^\eta(f)\in L^r(\Omega,\mathcal{F}_{A^c},\mu_\Lambda^\eta).$$

Inequality (3.7) says that $\|T_{B,p}\|\leqslant\varepsilon$ and $\|T_{A,p}\|\leqslant\varepsilon$. Using (repeatedly) the DLR conditions (2.2), we have, for all $f\in L^p(\Omega,\mathcal{F}_{A^c},\mu_\Lambda^\eta)$, $g\in L^q(\Omega,\mathcal{F}_{B^c},\mu_\Lambda^\eta)$,

$$\begin{aligned}\langle\mu_B f,g\rangle&=\mu_\Lambda^\eta\big((\mu_B f)g\big)=\mu_\Lambda^\eta\big(\mu_B(fg)\big)=\mu_\Lambda^\eta(fg)\\&=\mu_\Lambda^\eta\big(\mu_A(fg)\big)=\mu_\Lambda^\eta\big(f(\mu_A g)\big)=\langle f,\mu_A g\rangle,\end{aligned}$$

thus

$$\langle T_{B,p}f,g\rangle=\langle f,T_{A,q}g\rangle\quad\forall f\in L^p(\Omega,\mathcal{F}_{A^c},\mu_\Lambda^\eta),\ \forall g\in L^q(\Omega,\mathcal{F}_{B^c},\mu_\Lambda^\eta).$$

This shows that, if $p<\infty$, identifying the dual of L^p with L^q, $T_{A,q}$ is the adjoint of $T_{B,p}$, while if $p=\infty$ then $T_{B,p}$ is the adjoint of $T_{A,q}$. In both cases $\|T_{A,q}\|=\|T_{B,p}\|\leqslant\varepsilon$. By the Riesz–Thorin interpolation theorem $\|T_{A,2}\|\leqslant\varepsilon$. Interchanging A and B we also find $\|T_{B,2}\|\leqslant\varepsilon$.

Step 2. Conclusion. Let $f\in L^2(\mu_\Lambda^\eta)$ and assume (without losing generality) $\mu_\Lambda^\eta f=0$. Then, recalling (3.5),

$$\mu_\Lambda^\eta(f,f)=\mu_\Lambda^\eta(f^2)-\mu_\Lambda^\eta(f\mu_A f)+\mu_\Lambda^\eta(f\mu_A f)=\mu_\Lambda^\eta\big(\mu_A(f,f)\big)+\mu_\Lambda^\eta(f\mu_A f).\tag{3.9}$$

The second term can be written as

$$\begin{aligned}\mu_\Lambda^\eta(f\mu_A f)&=\big\langle(f-\mu_B f),\mu_A f\big\rangle+\langle\mu_B f,\mu_A f\rangle\\&=\big\langle(f-\mu_B f),\mu_A f\big\rangle+\langle f,\mu_B\mu_A f\rangle\\&\leqslant\|f-\mu_B f\|_2\|\mu_A f\|_2+\|f\|_2\|\mu_B\mu_A f\|_2\\&\leqslant\big[\|f-\mu_B f\|_2+\varepsilon\|f\|_2\big]\|\mu_A f\|_2\\&=\big[\|f-\mu_B f\|_2+\varepsilon\|f\|_2\big]\mu_\Lambda^\eta(f\mu_A f)^{1/2},\end{aligned}\tag{3.10}$$

where the second and the last equalities follow from the DLR conditions (2.2), while in the second inequality we have used (3.7) with $p=2$. From (3.10) we get

$$\begin{aligned}\mu_\Lambda^\eta(f\mu_A f)&\leqslant\|f-\mu_B f\|_2^2+\varepsilon^2\|f\|_2^2+2\varepsilon\|f\|_2\|f-\mu_B f\|_2\\&\leqslant\mu_\Lambda^\eta\big(\mu_B(f,f)\big)+\|f\|_2^2(2\varepsilon+\varepsilon^2)\end{aligned}$$

which, together with (3.9), implies (3.8) □

In order to proceed with the proof of Theorem 2.2 we go back to the geometry of A and B described before (3.6) and we want to show that Corollary 2.5 implies that inequality (3.7) holds with, say, $p=\infty$, for all boundary conditions. In fact, if $g \in \mathcal{F}_{A^c}$, and L is large enough so that Corollary 2.5 can be applied, we have, by (2.2)

$$\begin{aligned}\left\|\mu_B g - \mu_\Lambda^\eta g\right\|_\infty &= \left\|\mu_B g - \mu_\Lambda^\eta \mu_B g\right\|_\infty \leqslant \sup_{\omega,\tau\in\Omega:\omega_{\Lambda^c}=\tau_{\Lambda^c}} \left|\mu_B^\omega g - \mu_B^\tau g\right| \\ &\leqslant \alpha\|g\|_1 \mathrm{e}^{-md(\Lambda\setminus A,\Lambda\setminus B)} \leqslant \alpha\|g\|_\infty \mathrm{e}^{-md(\Lambda\setminus A,\Lambda\setminus B)}.\end{aligned} \tag{3.11}$$

The same bound applies to the quantity $\|\mu_A g - \mu_\Lambda^\eta g\|_\infty$. Therefore (3.6) follows from Lemma 3.1 and from our choice of the geometry of the sets A and B.

The next step is to bound the quantity $\mu_\Lambda^\eta(\mu_A(f,f) + \mu_B(f,f))$ in terms of the Dirichlet form $\mathcal{E}_\Lambda^\eta(f)$. Given $V \in \mathcal{R}^d$, let

$$G_V := G_V(r,z,\beta) := \sup_{\eta\in\Omega}\bigl(\mathrm{gap}\bigl(L_V^\eta\bigr)\bigr)^{-1}.$$

We have, then

$$\begin{aligned}\mu_\Lambda^\eta\bigl(\mu_A(f,f)+\mu_B(f,f)\bigr) &\leqslant (G_A\vee G_B)\mu_\Lambda^\eta\bigl[\mu_A\bigl(|D_A^-(f)|^2\bigr)+\mu_B\bigl(|D_B^-(f)|^2\bigr)\bigr] \\ &\leqslant (G_A\vee G_B)\mu_\Lambda^\eta\bigl[|D_{A\cup B}^-(f)|^2+|D_{A\cap B}^-(f)|^2\bigr] \\ &= (G_A\vee G_B)\bigl[\mathcal{E}_\Lambda^\eta(f)+\mu_\Lambda^\eta\bigl(|D_{A\cap B}^-(f)|^2\bigr)\bigr].\end{aligned} \tag{3.12}$$

From (3.6) and (3.12) we get

$$\mu_\Lambda^\eta(f,f) \leqslant \bigl(1+c_1\mathrm{e}^{-c_2 m\sqrt{L}}\bigr)(G_A\vee G_B)\bigl[\mathcal{E}_\Lambda^\eta(f)+\mu_\Lambda^\eta\bigl(|D_{A\cap B}^-(f)|^2\bigr)\bigr]. \tag{3.13}$$

At this point one may be tempted to discourage, because if we bound the term $\mu_\Lambda^\eta(|D_{A\cap B}^-(f)|^2)$ with $\mathcal{E}_\Lambda^\eta$ then we get

$$G_\Lambda \leqslant \bigl(2+c_1\mathrm{e}^{-c_2 m\sqrt{L}}\bigr)(G_A\vee G_B)$$

which implies that if we (roughly) double the volume, the inverse G_Λ of the spectral gap also (roughly) doubles. But, as observed in [8], one can *average over the location of the overlap*. Consider in fact a sequence of pairs $\{A_i, B_i\}_{i=1}^s$, where, for instance, $s := \lfloor L^{1/3}\rfloor$, where $\lfloor\cdot\rfloor$ is the integral part. By averaging (3.13) over i we obtain

$$\mu_\Lambda^\eta(f,f) \leqslant \bigl(1+c_1\mathrm{e}^{-c_2 m\sqrt{L}}\bigr)\sup_i(G_{A_i}\vee G_{B_i})\left[\mathcal{E}_\Lambda^\eta(f)+\frac{1}{s}\mu_\Lambda^\eta\left(\sum_{i=1}^s |D_{A_i\cap B_i}^-(f)|^2\right)\right]. \tag{3.14}$$

If the sets A_i, B_i are chosen in such a way that $A_i\cap B_i\cap A_j\cap B_j=\emptyset$ for all $i\neq j$ then there exists $L_1 = L_1(r,z,\beta)\geqslant L_0$ such that for all $L\geqslant L_1$

$$G_\Lambda \leqslant \bigl(1+c_1\mathrm{e}^{-c_2 m\sqrt{L}}\bigr)\left(1+\frac{1}{s}\right)\sup_i(G_{A_i}\vee G_{B_i}) \leqslant \left(1+\frac{2}{\lfloor L^{1/3}\rfloor}\right)\sup_i(G_{A_i}\vee G_{B_i}). \tag{3.15}$$

In order to conclude the proof of Theorem 2.2 all is left to do is to organize the geometric iterative construction. Let $l_k := (3/2)^{k/d}$, and let $\mathcal{R}_k^d$ be the set of all rectangles in $\mathcal{R}^d$ which, modulo translations and permutations of the coordinates, are contained in

$$[0, l_{k+1}] \times [0, l_{k+2}] \times \cdots \times [0, l_{k+d}].$$

Let also $G_k := \sup_{V \in \mathcal{R}_k^d} G_V$. The idea behind this construction is that each rectangle in $\mathcal{R}_k^d \backslash \mathcal{R}_{k-1}^d$ can be obtained as a "slightly overlapping union" of two rectangles in $\mathcal{R}_{k-1}^d$. More precisely we have:

PROPOSITION 3.2. – *For all $\Lambda \in \mathcal{R}_k^d \backslash \mathcal{R}_{k-1}^d$ there exists a finite sequence $\{A_i, B_i\}_{i=1}^{s_k}$, where $s_k := \lfloor l_k^{1/3} \rfloor$, such that*
(1) *$\Lambda = A_i \cup B_i$ and $A_i, B_i \in \mathcal{R}_{k-1}^d$, for all $i = 1, \ldots, s_k$,*
(2) *$d(\Lambda \backslash A_i, \Lambda \backslash B_i) \geqslant \frac{1}{8}\sqrt{l_k}$, for all $i = 1, \ldots, s_k$,*
(3) *$A_i \cap B_i \cap A_j \cap B_j = \emptyset$ if $i \neq j$.*

Proof. – Let $\Lambda := [a_1, b_1] \times \cdots \times [a_d, b_d] \in \mathcal{R}_k^d \backslash \mathcal{R}_{k-1}^d$, We can assume $a_n = 0$ and $b_n \leqslant l_{k+n}$, for $n = 1, \ldots, d$. Then necessarily $b_d > l_k$, since, otherwise, $\Lambda \in \mathcal{R}_{k-1}^d$. Define

$$A_i := [0, b_1] \times \cdots \times [0, b_{d-1}] \times \left[0, \frac{b_d}{2} + \frac{2i}{8}\sqrt{l_k}\right],$$
$$B_i := [0, b_1] \times \cdots \times [0, b_{d-1}] \times \left[\frac{b_d}{2} + \frac{2i-1}{8}\sqrt{l_k}, b_d\right].$$

We have $d(\Lambda \backslash A_i, \Lambda \backslash B_i) = \frac{1}{8}\sqrt{l_k}$. Furthermore

$$\frac{b_d}{2} + \frac{2s_k}{8}\sqrt{l_k} \leqslant \frac{l_{k+d}}{2} + \frac{1}{4} l_k^{5/6} \leqslant \frac{3l_k}{4} + \frac{1}{4} l_k^{5/6} \leqslant l_k$$

which, together with the fact that $l_k < b_d$, implies that A_i and B_i are both subsets of Λ. Moreover, since, for all $i = 1, \ldots, s_k$

$$\frac{b_d}{2} + \frac{2i}{8}\sqrt{l_k} \leqslant l_k, \quad b_1 \leqslant l_{k+1}, \ \ldots, \ b_{d-1} \leqslant l_{k-1+d}$$

we find that A_i belongs to $\mathcal{R}_{k-1}^d$. The sets B_i's also belong to $\mathcal{R}_{k-1}^d$, since they are smaller than the A_i's. □

Let then k_0 be the smallest integer such that $l_k \geqslant L_1$. From (3.15) and Proposition 3.2 we obtain that for all $k > k_0$

$$G_k \leqslant \left(1 + 2(3/2)^{-k/(3d)}\right) G_{k-1} \leqslant G_{k_0} \prod_{k=k_0+1}^{\infty} \left(1 + 2(3/2)^{-k/(3d)}\right)$$
$$\leqslant G_{k_0} \exp\left[2\left(1 - (2/3)^{1/(3d)}\right)^{-1}\right]$$

which, together with Proposition 2.6, yields Theorem 2.2. □

4. Spectral gap ⇒ decay of correlations

In this section we prove a partial converse to Theorem 2.2. More precisely, assuming finite range positive pair potential, we get the exponential decay of correlation in a volume Λ with boundary condition η provided our Glauber-type dynamics satisfies a Poincaré inequality in that volume with that boundary condition. Unfortunately, we are not able to prove the exponential decay of correlation as stated in Corollary 2.4 but only with the $L^1(\mathrm{d}\mu_\Lambda^\eta)$ norm which appears on the RHS of (2.8) replaced by a much stronger norm; therefore we do not have equivalence in Theorem 2.2.

The argument leading to the result previously outlined is well-known in the context of lattice (bounded or unbounded) spin systems, see e.g. [8–10,15–18]. Below we stress the main differences in the continuous case we are dealing with.

Recalling the operator D_x^+, defined in (2.3), we introduce the following semi-norm. For $f \in \mathcal{F}_\Lambda$ we set

$$|\!|\!| f |\!|\!| := \int_\Lambda \mathrm{d}x \, \| D_x^+ f \|_{L^\infty(\mathrm{d}\mu_\Lambda^\eta)} \tag{4.1}$$

which is the continuous analogous of Liggett's triple norm defined in Ch.1 of [6]. We show next that the mapping $x \to \| D_x^+ f \|_{L^\infty(\mathrm{d}\mu_\Lambda^\eta)}$ is indeed measurable. We first notice that $(x,\omega) \to D_x^+ f(\omega)$ is measurable w.r.t. the product σ-algebra $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$. For this purpose it is enough to show that $t : \mathbb{R}^d \times \Omega \mapsto \Omega$, defined by $t(x,\omega) := \omega \cup x$ is measurable. Since $\mathcal{F}$ is generated by the functions $(N_A)_{A \in \mathcal{B}_b(\mathbb{R}^d)}$, if we show that $(t^{-1} \circ N_A^{-1})\{k\} \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$ for all nonnegative integers k, the measurability of t follows. But

$$\begin{aligned}\left(t^{-1} \circ N_A^{-1}\right)\{k\} &= \left\{(x,\omega)\colon N_A(\omega \cup x) = k\right\} \\ &= \left\{(x,\omega)\colon N_A(\omega) = k, x \notin A \backslash \omega\right\} \cup \left\{(x,\omega)\colon N_A(\omega) = k-1, x \in A \backslash \omega\right\}\end{aligned}$$

thus the only problem is to show that the set $M := \{(x,\omega)\colon x \in \omega\} \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$. But M can be written as $\{(x,\omega)\colon 0 \in \vartheta_{-x}\omega\}$, where ϑ_x is the translation by x, and, since the mapping $(x,\omega) \to \vartheta_x \omega$ is measurable (see, for instance [11]), we have that M is measurable. We have thus shown that $(x,\omega) \to D_x^+ f(\omega)$ is measurable. By consequence $x \to \| D_x^+ f \|_{L^p(\mathrm{d}\mu_\Lambda^\eta)}$ is measurable for all $p \in [1,\infty)$. Finally, $\| D_x^+ f \|_{L^\infty(\mathrm{d}\mu_\Lambda^\eta)} = \lim_{p \to \infty} \| D_x^+ f \|_{L^p(\mathrm{d}\mu_\Lambda^\eta)}$, so $x \to \| D_x^+ f \|_{L^\infty(\mathrm{d}\mu_\Lambda^\eta)}$ is also measurable.

The main result in this section is:

THEOREM 4.1. – *Let $\varphi \geqslant 0$ be of finite range r. If there exists $G < \infty$ such that*

$$\mu_\Lambda^\eta(f,f) \leqslant G \mathcal{E}_\Lambda^\eta(f) \quad \forall f \in \mathcal{D}\left(\mathcal{E}_\Lambda^\eta\right),$$

then there are $m = m(G,r,z) > 0$ and $\alpha = \alpha(G,r,z) < \infty$ such that the following holds. For any $\Lambda_f, \Lambda_g \in \mathcal{B}_b(\mathbb{R}^d)$ such that $\Lambda_f \subset \Lambda$, $\Lambda_g \subset \Lambda$, $\Lambda_f \cap \Lambda_g = \emptyset$, and $|\bar{\Lambda}_f^r| \wedge |\bar{\Lambda}_g^r| \leqslant \exp(m d(\Lambda_f,\Lambda_g))$, we have

$$\left|\mu_\Lambda^\eta(f,g)\right| \leqslant \alpha\left(\|f\|_{L^2(\mathrm{d}\mu_\Lambda^\eta)} \|g\|_{L^2(\mathrm{d}\mu_\Lambda^\eta)} + z |\!|\!| f |\!|\!| \, |\!|\!| g |\!|\!|\right) \mathrm{e}^{-m d(\Lambda_f,\Lambda_g)} \quad \forall f \in \mathcal{F}_{\Lambda_f},\ g \in \mathcal{F}_{\Lambda_g}.$$

The key ingredient in proving the above Theorem is the following *finite speed of propagation* lemma.

LEMMA 4.2. – *Let $\varphi \geqslant 0$ be of finite range r. Then there are $\delta = \delta(r,z) > 0$ and $M = M(r,z) < \infty$ such that the following holds. For any $\Lambda_f, \Lambda_g \in \mathcal{B}_b(\mathbb{R}^d)$ such that $\Lambda_f \subset \Lambda$, $\Lambda_g \subset \Lambda$, $\Lambda_f \cap \Lambda_g = \emptyset$, and $|\bar{\Lambda}^r_f| \wedge |\bar{\Lambda}^r_g| \leqslant \exp(\delta d(\Lambda_f, \Lambda_g))$, we have*

$$\left|\mu^\eta_\Lambda\left(P^{\Lambda,\eta}_t(fg) - P^{\Lambda,\eta}_t f P^{\Lambda,\eta}_t g\right)\right| \leqslant Mz\mathrm{e}^{Mt-\delta d(\Lambda_f,\Lambda_g)} |||f||| \, |||g||| \tag{4.2}$$

for any $f \in \mathcal{F}_{\Lambda_f} \cap L^2(\mathrm{d}\mu^\eta_\Lambda)$ and $g \in \mathcal{F}_{\Lambda_g} \cap L^2(\mathrm{d}\mu^\eta_\Lambda)$.

Remark. – For compact or (suitable) unbounded spin systems, one can prove a bound analogous to (4.2) with its LHS replaced by $\|P_t(fg) - P_t f P_t g\|_{L^\infty(\mathrm{d}\mu^\eta_\Lambda)}$; see e.g. Proposition 4.18, Chapter I in [6] and [17] respectively. In the continuous case we do not get such a stronger bound; however Lemma 4.2 as stated is precisely what we need to prove Theorem 4.1.

Proof of Theorem 4.1. – We can assume $\mu^\eta_\Lambda f = \mu^\eta_\Lambda g = 0$. Since μ^η_Λ is the invariant measure for $P^{\Lambda,\eta}_t$ we have

$$\begin{aligned}\left|\mu^\eta_\Lambda(f,g)\right| &= \left|\mu^\eta_\Lambda\left(P^{\Lambda,\eta}_t(fg)\right)\right| = \left|\mu^\eta_\Lambda\left(P^{\Lambda,\eta}_t f P^{\Lambda,\eta}_t g\right) + \mu^\eta_\Lambda\left(P^{\Lambda,\eta}_t(fg) - P^{\Lambda,\eta}_t f P^{\Lambda,\eta}_t g\right)\right| \\ &\leqslant \left\|P^{\Lambda,\eta}_t f\right\|_{L^2(\mathrm{d}\mu^\eta_\Lambda)}\left\|P^{\Lambda,\eta}_t g\right\|_{L^2(\mathrm{d}\mu^\eta_\Lambda)} + \left|\mu^\eta_\Lambda\left(P^{\Lambda,\eta}_t(fg) - P^{\Lambda,\eta}_t f P^{\Lambda,\eta}_t g\right)\right|,\end{aligned}$$

where we used Schwarz inequality. For δ and M as in Lemma 4.2, choose $t = \delta(2M)^{-1} d(\Lambda_f, \Lambda_g)$ and apply $\|P^{\Lambda,\eta}_t f\|_{L^2(\mathrm{d}\mu^\eta_\Lambda)} \leqslant \mathrm{e}^{-t/G}\|f\|_{L^2(\mathrm{d}\mu^\eta_\Lambda)}$ together with the bound (4.2) to get the result. □

Proof of Lemma 4.2. – Since the volume Λ and the boundary condition η are kept fixed we drop them from the notation. We claim for each $f, g \in \mathcal{D}(\mathcal{E})$

$$\mu\left(P_t(fg) - P_t f P_t g\right) = 2\int_0^t \mathrm{d}s\, \mathcal{E}(P_s f, P_s g) \tag{4.3}$$

which is a general identity for self-adjoint Markov semigroups. In order to verify it, let us first consider $f, g \in L^\infty(\mathrm{d}\mu) \cap \mathcal{D}(\mathcal{E})$ and approximate the generator L by the bounded (in $L^2(\mathrm{d}\mu)$) operator L_k defined by $L_k f := -\int_0^k \lambda\, \mathrm{d}E_\lambda(f)$ where $\{E_\lambda, \lambda \in [0,\infty)\}$ is the family of spectral projectors associated to $-L$. We also let $P^k_t := \exp(L_k t)$. Since L_k is bounded, a straightforward computation shows

$$P^k_t(fg) - P^k_t f P^k_t g = \int_0^t \mathrm{d}s\, P^k_{t-s}\left[L_k\left(P^k_s f P^k_s g\right) - P^k_s f L_k P^k_s g - P^k_s g L_k P^k_s f\right].$$

Taking expectation w.r.t. μ and using self–adjointness of L_k we get (4.3) for the approximating semigroup. For $f, g \in L^\infty(\mathrm{d}\mu) \cap \mathcal{D}(\mathcal{E})$ we now take the limit $k \to \infty$ which gives (4.3); finally we extend it to any $f, g \in \mathcal{D}(\mathcal{E})$ by density.

We now fix $f, g \in \mathcal{D}_0(L)$ ($\mathcal{D}_0(L)$ was defined in (2.5)) and prove the bound (4.2) for such functions. The lemma follows then by density (see the proof of Proposition 2.1

for an analogous argument). Given f as above we define $F_t \in L^\infty(\Lambda, \mathrm{d}x)$ as $F_t(x) := \|D_x^+ P_t f\|_{L^\infty(\mathrm{d}\mu)}$. We also let G_t be defined in the same way with f replaced by g. Then, recalling (2) in Proposition 2.1, the identity (4.3) implies

$$\begin{aligned} \left|\mu(P_t(fg) - P_t f P_t g)\right| &= 2z \left| \int_0^t \mathrm{d}s \int_\Lambda \mathrm{d}x\, \mu\left(\mathrm{e}^{-\beta D_x^+ H_\Lambda} D_x^+ P_s f\, D_x^+ P_s g\right)\right| \\ &\leqslant 2z \int_0^t \mathrm{d}s \int_\Lambda \mathrm{d}x\, F_s(x) G_s(x). \end{aligned} \tag{4.4}$$

We claim there are $\delta = \delta(z, r) > 0$ and $M = M(z, r) < \infty$ such that

$$F_t(x) \leqslant M \mathrm{e}^{Mt} \int_\Lambda \mathrm{d}y\, \mathrm{e}^{-\delta d(x,y)} F_0(y). \tag{4.5}$$

Postponing its proof, let us first conclude the Lemma. Since $F_0(y) = 0$ if $y \notin \Lambda_f$, from (4.4) and (4.5) we get

$$\begin{aligned} \left|\mu(P_t(fg) - P_t f P_t g)\right| \leqslant{}& 2zM^2 \mathrm{e}^{2Mt} \int_\Lambda \mathrm{d}y\, F_0(y) \int_\Lambda \mathrm{d}y'\, G_0(y') \\ & \times \sup_{\substack{y \in \Lambda_f \\ y' \in \Lambda_g}} \int_\Lambda \mathrm{d}x\, \mathrm{e}^{-\delta[d(x,y) + d(x,y')]} \\ \leqslant{}& zCM^2 \mathrm{e}^{2Mt} |\Lambda_f| \wedge |\Lambda_g| \, |||f||| \, |||g||| \, \mathrm{e}^{-\delta d(\Lambda_f, \Lambda_g)/2} \end{aligned}$$

for some constant $C = C(\delta)$. Redefining δ and M, the bound (4.2) follows.

It remains to prove (4.5). We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} D_x^+ P_t f &= L D_x^+ P_t f + [D_x^+, L] P_t f \\ &= L D_x^+ P_t f - D_x^+ P_t f + z \int_\Lambda \mathrm{d}y \left(D_x^+ \mathrm{e}^{-\beta D_y^+ H_\Lambda}\right) T_x D_y^+ P_t f, \end{aligned} \tag{4.6}$$

where $[D_x^+, L]$ denotes the commutator and $(T_x f)(\omega) := f(\omega \cup \{x\})$. The second identity in (4.6) follows by a direct computation from (2.4). By integrating (4.6) we get

$$D_x^+ P_t f = P_t D_x^+ f + \int_0^t \mathrm{d}s\, P_{t-s} \left\{ -D_x^+ P_s f + z \int_\Lambda \mathrm{d}y \left(D_x^+ \mathrm{e}^{-\beta D_y^+ H_\Lambda}\right) T_x D_y^+ P_s f \right\}. \tag{4.7}$$

Of course we need to justify the steps leading to (4.7); as in the case of (4.3) it is better first to approximate L by a bounded operator (in $L^\infty(\mathrm{d}\mu)$) so that (4.6)–(4.7) hold trivially. Noticing that $\|T_x f\|_{L^\infty(\mathrm{d}\mu)} \leqslant \|f\|_{L^\infty(\mathrm{d}\mu)}$ and (since $\varphi \geqslant 0$) $\|D_x^+ \mathrm{e}^{-\beta D_y^+ H_\Lambda}\|_{L^\infty(\mathrm{d}\mu)} \leqslant 1$ we can then remove the truncation in (4.7); we omit the details.

Since P_t is a contraction in $L^\infty(\mathrm{d}\mu)$ from (4.7) we get the bound

$$F_t(x) \leqslant F_0(x) + \int_0^t \mathrm{d}s \left\{ F_s(x) + z \int_\Lambda \mathrm{d}y\, \mathbb{1}_{d(y,x)\leqslant r} F_s(y) \right\}, \tag{4.8}$$

where we used again that T_x is a contraction in $L^\infty(\mathrm{d}\mu)$ and $\|D_x^+ \mathrm{e}^{-\beta D_y^+ H_\Lambda}\|_{L^\infty(\mathrm{d}\mu)} \leqslant \mathbb{1}_{d(y,x)\leqslant r}$ which follows from the finite range assumption.

Let γ be the integral operator on $L^1(\Lambda, \mathrm{d}x)$ with kernel $\gamma(x,y) := z\mathbb{1}_{d(y,x)\leqslant r}$; by iterating (4.8) we get

$$F_t(x) \leqslant \mathrm{e}^t \sum_{k=0}^\infty \frac{t^k}{k!} (\gamma^k F_0)(x).$$

It is now easy to show, by induction on k, the operator γ^k has an integral kernel $\gamma^k(x,y)$ which can be estimated as follows

$$0 \leqslant \gamma^k(x,y) \leqslant z^k \left|\{y' \colon d(0,y') \leqslant r\}\right|^{k-1} \mathbb{1}_{d(x,y)\leqslant kr}$$

by a straightforward computation we then get (4.5). □

5. Logarithmic Sobolev inequalities?

One may wonder whether the Markov processes constructed in Section 2 satisfy a logarithmic Sobolev inequality (LSI), i.e. if there exists $c_s < \infty$ such that

$$\mu_\Lambda^\eta\left(f^2 \log f^2/\|f\|_2^2\right) \leqslant c_s \mathcal{E}_\Lambda^\eta(f). \tag{5.1}$$

The answer is negative as it can be easily shown. We remark that if one could prove that (5.1) holds for a (large enough) fixed bounded volume Λ_0 uniformly in the boundary condition, then a uniform LSI both in the volume and the boundary condition would follow, under a mixing assumption like (2.9), by adapting to the continuous case more or less standard lattice spin techniques. The problem is thus the failure of (5.1) for a given volume. To see this, we make the minimal assumption that the Hamiltonian is *stable*, i.e. that there exists $B > 0$ such that $H_\Lambda^\emptyset(\omega) \geqslant -BN_\Lambda(\omega)$. By consequence, if $\eta \in \Omega$ is an arbitrary boundary condition,

$$H_\Lambda^\eta(\omega) \geqslant -BN_\Lambda(\omega) + \sum_{x\in\omega, y\in\eta} \varphi(x-y) \geqslant \left(-B + \inf\varphi N_{\bar\Lambda^r}(\eta)\right) N_\Lambda(\omega) =: -A(\eta) N_\Lambda(\omega).$$

Let $\rho := \mu_\Lambda^\eta \circ N_\Lambda^{-1}$ be the distribution of the number of particles in Λ. For a function f which can be written as $f = g \circ N_\Lambda$ (i.e. f depends only on the number of particles in Λ), we have

$$\mathcal{E}_\Lambda^\eta(f) = \sum_{k=1}^\infty \rho(k) k \left[g(k-1) - g(k)\right]^2.$$

If we let $g := \mathbb{1}_{[n,\infty)}$, then $g^2 \log(g^2) = 0$, and (5.1) becomes

$$\rho[n,\infty) \log \rho[n,\infty)^{-1} \leqslant c_s \rho(n) n. \tag{5.2}$$

Since $x \log(x^{-1})$ is increasing in $(0, \mathrm{e}^{-1})$, if n is large enough such that $\rho[n,\infty) < \mathrm{e}^{-1}$, we have

$$\rho[n,\infty) \log \rho[n,\infty)^{-1} \geqslant \rho(n) \log \rho(n)^{-1}. \tag{5.3}$$

On the other side, using the stability condition and $Z_\Lambda^\eta \geqslant 1$, we have

$$\rho(n) = \left(Z_\Lambda^\eta\right)^{-1} \frac{z^n}{n!} \int\limits_{\Lambda^n} \mathrm{e}^{-\beta H_\Lambda^\eta(x)}\, \mathrm{d}x \leqslant \frac{z^n}{n!} \mathrm{e}^{A(\eta)n} \Lambda^n$$

which, together with (5.3), shows that (5.2) fails.

The only possibity is then to modify the transition rates. Our choice (2.4) can be considered (since $\varphi \geqslant 0$) as the continuous equivalent of the "Metropolis" algorithm used in finite systems, where the transition rate $i \to j$ is equal to $1 \wedge \mathrm{e}^{-\beta(H(j)-H(i))}$. Another possible choice for the generator is, for instance,

$$(\hat{L}_\Lambda f)(\omega) := \sum_{x \in \omega \cap \Lambda} \mathrm{e}^{-\beta D_x^- H_\Lambda(\omega)} D_x^- f(\omega) + z \int\limits_\Lambda D_x^+ f(\omega)\, \mathrm{d}x \quad \omega \in \Omega, \tag{5.4}$$

which corresponds to a process where particle appear with rate z, and disappear with rate $\mathrm{e}^{-\beta D_x^- H}$. The associated Dirchlet form is

$$\hat{\mathcal{E}}_\Lambda^\eta(f) = z \int\limits_\Lambda \mathrm{d}x\, \mu_\Lambda^\eta\left((D_x^+ f)^2\right)$$

and, since $\hat{\mathcal{E}}_\Lambda^\eta \geqslant \mathcal{E}_\Lambda^\eta$, the generator $\hat{L}_\Lambda^\eta$ has a spectral gap greater than or equal to the spectral gap of L_Λ^η. Inequality (5.2) becomes

$$\rho[n,\infty) \log \rho[n,\infty)^{-1} \leqslant c_s z |\Lambda| \rho(n-1). \tag{5.5}$$

While (5.5) is equivalent to (5.2) when $\varphi = 0$, if for φ in some appropriate class of potentials one knew that the particle distribution ρ behaves like e^{-cn^2} for large n, then (5.5) would hold and a LSI cannot be ruled out. For superstable potentials (see [13]) it is known that $\rho(n) \leqslant \mathrm{e}^{-c_1 n^2 + c_2 n}$, so it could be interesting, in this case, to investigate the possibility of having a LSI for $\hat{L}_\Lambda^\eta$.

Acknowledgements

We thank Fabio Martinelli for many helpful insights and for his "semigroupless" proof of the "2-blocks" Lemma 3.1.

REFERENCES

[1] Cesi F., Maes C., Martinelli F., Relaxation of disordered magnets in the Griffiths regime, Comm. Math. Phys. 188 (1997) 135–173.

[2] Davies E.B., Heat Kernels and Spectral Theory, Cambridge Univ. Press, 1989.

[3] Fernández R., Ferrari P.A., Garcia N.L., Perfect simulation for interacting point processes, loss networks and Ising models, Preprint, 1999.

[4] Ledoux M., Logarithmic Sobolev inequalities for unbounded spin systems revisited, in: Séminaire de Probabilités, XXXV (Berlin), Springer, Berlin, 2001, pp. 167–194.

[5] Ledoux M., Concentration of measure and logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXIII, Lecture Notes in Mathematics, Vol. 1709, Springer, Berlin, 1999, pp. 120–216.

[6] Liggett T.M., Interacting Particle Systems, Springer, Berlin, 1985.

[7] Lu S.L., Yau H.T., Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics, Comm. Math. Phys. 156 (1993) 399–433.

[8] Martinelli F., Lectures on Glauber dynamics for discrete spin models, in: Lectures on Probability Theory and Statistics (Saint-Flour, 1997), Lecture Notes in Mathematics, Vol. 1717, Springer, Berlin, 1999, pp. 93–191.

[9] Martinelli F., Olivieri E., Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case, Comm. Math. Phys. 161 (1994) 447.

[10] Martinelli F., Olivieri E., Approach to equilibrium of Glauber dynamics in the one phase region II: The general case, Comm. Math. Phys. 161 (1994) 487.

[11] Matthes K., Kerstan J., Mecke J., Infinitely Divisible Point Processes, Wiley, Chichester, 1978.

[12] Preston C., Spatial birth-and-death processes, Proceedings of the 40th Session of the International Statistical Institute (Warsaw, 1975), Bull. Inst. Internat. Statist. 46 (1975) 371–391.

[13] Ruelle D., Superstable interactions in classical statistical mechanics, Comm. Math. Phys. 18 (1970) 127–159.

[14] Spohn H., Equilibrium fluctuations for interacting Brownian particles, Comm. Math. Phys. 103 (1986) 1–33.

[15] Stroock D.W., Zegarlinski B., The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition, Comm. Math. Phys. 144 (1992) 303–323.

[16] Stroock D.W., Zegarlinski B., The logarithmic Sobolev inequality for discrete spin on a lattice, Comm. Math. Phys. 149 (1992) 175.

[17] Yoshida N., The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on the lattice, Ann. Inst. H. Poincaré Probab. Statist. 37 (2) (2001) 223–243.

[18] Zegarlinski B., The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice, Comm. Math. Phys. 175 (1996) 401–432.