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PERTURBED AND NON-PERTURBED BROWNIAN TABOO PROCESSES

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ABSTRACT. – In this paper we study the Brownian taboo process, which is a version of Brownian motion conditioned to stay within a finite interval, and the α-perturbed Brownian taboo process, which is an analogous version of an α-perturbed Brownian motion.We are particularly interested in the asymptotic behaviour of the supremum of the taboo process, and our main results give integral tests for upper and lower functions of the supremum as $t \to \infty$. In the Brownian case these include extensions of recent results in Lambert [4], but are proved in a quite different way. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Dans cet article, nous étudions le processus Brownien tabou qui est une version du mouvement Brownien, conditionné à rester dans un intervalle fini, et le processus Brownien tabou α-perturbé qui est une version semblable du mouvement Brownien α-perturbé. Nous sommes particulièrement intéressés par le comportement asymptotique du supremum du processus tabou et nos principaux résultats fournissent des intégrales tests pour des fonctions majorantes et minorantes du supremum lorsque $t \to \infty$. Dans le cas Brownien, ces résultats incluent des extensions de résultats récents de Lambert [4], mais ceux-ci sont prouvés de maníère différente. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The Brownian taboo process, a version of Brownian motion conditioned to stay within a finite interval, was first introduced by Knight in [3]. In a recent paper Lambert [4] has introduced an analogous version of a spectrally negative Lévy process, and proved some results which are new even for the Brownian case. In particular he studied the asymptotic behaviour of the maximum of the taboo process, and in the Brownian case his results are as follows. Let $\mathbb{P}_x$ denote the measure under which the coordinate process $\{X_t, t \geqslant 0\}$ is a Brownian taboo process on [0,a) starting at x, and write $S_t = \sup_{s \leqslant t}\{X_s\}$.

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THEOREM 1.1 (Lambert). –
(i) *For any fixed* $x \in [0, a)$ *and any decreasing non-negative function* f,

$$\mathbb{P}_x\{a - S_t < f(t) \text{ i.o. as } t \to \infty\} = 0 \quad or \quad 1$$

according as $I := \int_1^\infty f(t)\,\mathrm{d}t$ *is finite or infinite.*
(ii) *For any fixed* $x \in [0, a)$

$$\mathbb{P}_x\left\{\limsup_{t\to\infty} \frac{t(a - S_t\}}{\log\log t} = \frac{a^3}{\pi^2}\right\} = 1. \tag{1.1}$$

These results, and their extensions to the spectrally negative Lévy process case, were established in [4] by exploiting the fact that the excursions of the taboo process away from a fixed point form a Poisson point process. An alternative approach is to rephrase these results as statements about the behaviour of the first passage time process $\{T_y, x \leqslant y < a\}$, where $T_y = \inf\{t\colon\ X_t > y\}$. This process has independent increments and an explicit formula for $\mathbb{E}_x\{\mathrm{e}^{-\lambda T_y}\}$ is available. From this, it is easy to see that under $\mathbb{P}_0$ we can write

$$T_y \overset{d}{=} V_y + U_y, \tag{1.2}$$

where, for fixed y, V_y and U_y are independent, non-negative random variables with V_y having an exponential distribution and U_y having a distribution whose tail decays at an exponential rate which is faster than that of V_y. Moreover the parameter $\Theta(y)$ of V_y has the asymptotic behaviour

$$\Theta(a - \varepsilon) \backsim \frac{\varepsilon\pi^2}{a^3} \quad \text{as } \varepsilon \downarrow 0,$$

which explains the appearance of the quantity a^3/π^2 in (1.1). We show that it is possible to exploit (1.2) to get sufficiently good bounds on the tail of the distribution of $T_{a-\varepsilon}$ as $\varepsilon \downarrow 0$ to establish the following improvement of (1.1).

THEOREM 1.2. – *For any fixed* $x \in [0, a)$ *and any increasing non-negative function* g *such that* $f(t) = t^{-1}g(t)$ *decreases,*

$$\mathbb{P}_x\{a - S_t > f(t) \text{ i.o. as } t \to \infty\} = 0 \quad or \quad 1$$

according as $J := \int_1^\infty t^{-1}\mathrm{e}^{-\beta g(t)}\,\mathrm{d}t$ *is finite or infinite, where* $\beta = \pi^2/a^3$.

It is also the case that a similar technique can be used to give an alternative proof of the first statement in Theorem 1.1. Moreover it is clear that if we consider an α-perturbed Brownian taboo process, by which we mean the process we get by taking a suitable harmonic transforn of an α-perturbed Brownian motion, (see Chapters 8 and 9 of [7] for background on this), then we can no longer use Lambert's technique to study the asymptotic behaviour of the maximum. This is because the excursions away from a fixed point of this perturbed taboo process do not form a Poisson point process. However, even though this process is no longer Markovian, its first passage process is a time-inhomogeneous Markov process, and indeed has independent increments. There is also an analogue of (1.2), with the exponentially distributed random variable being replaced by one having a Gamma distribution. Although the technical problems are somewhat more onerous, in section 3 we state and sketch the proofs of results which extend both theorems 1.1 and 1.2 to this perturbed situation.

2. The Brownian case

As previously remarked, the distribution of the first passage process under $\mathbb{P}_x$ is determined by the fact that it has independent increments and satisfies, with $\gamma = \pi/a$,

$$\mathbb{E}_x\{e^{-\lambda T_y}\} = \begin{cases} \frac{\sin y\gamma \sin x\sqrt{\gamma^2-2\lambda}}{\sin x\gamma \sin y\sqrt{\gamma^2-2\lambda}} & \text{if } 0 < x < y < a,\ \lambda < \frac{\gamma^2}{2}, \\ \frac{\sqrt{\gamma^2-2\lambda}\sin y\gamma}{\gamma \sin y\sqrt{\gamma^2-2\lambda}} & \text{if } 0 = x < y < a,\ \lambda < \frac{\gamma^2}{2}. \end{cases} \tag{2.1}$$

The first statement here is a special case of Proposition 3.2 of [4], but can easily be derived from the fact that the Taboo process is a space-time h-transform of Brownian motion killed on exiting $(0, a)$, with $h(x, t) = \sin\gamma x \exp\frac{1}{2}t\gamma^2$. Since $\mathbb{P}_0$ is $\lim_{x\downarrow 0}\mathbb{P}_x$, the second statement also follows.

Introduce the notation $\Theta(y) = \frac{\pi^2}{2a^2}\{(\frac{a}{y})^2 - 1\}$, and for any $0 < b < c \leqslant \infty$ write $D(b, c)$ for the distribution of a non-negative random variable which is zero with probability b/c, and conditioned on being positive, has an Exp(b) distribution. Then $D(b,\infty)$ coincides with the Exp(b) distribution, and a random variable D has the $D(b, c)$ distribution with $c < \infty$ if and only if we can write

$$Y_1 = Y_2 + D,$$

where Y_2 and D are independent, Y_1 has an Exp(b) distribution, and Y_2 has an Exp(c) distribution. We then have

LEMMA 2.1. – *For any $0 \leqslant x < y < a$ we have under $\mathbb{P}_x$*

$$T_y \overset{d}{=} V_y + U_y, \tag{2.2}$$

where the non-negative random variables V_y and U_y are independent, V_y has the $D(\Theta(y), \Theta(x))$ distribution, and

$$\mathbb{P}_x\{U_y > t\} \leqslant c_1 e^{-t\pi^2/a^2} \quad \textit{for all } t \geqslant 0, \tag{2.3}$$

where c_1 is a constant, which depends only on a.

Proof. – Writing $\phi_x(y,\lambda) = \mathbb{E}_x\{e^{-\lambda T_y}\}$ and $\phi(y,\lambda)$ for $\phi_0(y,\lambda)$ we see from (2.1) that $\phi_x(y,\lambda) = \phi(y,\lambda)/\phi(x,\lambda)$ for $x > 0$. Also, if we write $\Theta_k(y) = \frac{\pi^2}{2a^2}\{(\frac{ka}{y})^2 - 1\}$ for $k \geqslant 1$, so that $\Theta_1(y) = \Theta(y)$, we see from the infinite product representation of the sine function that

$$\phi_x(y,\lambda) = \prod_1^\infty \frac{\Theta_k(y)\{\lambda + \Theta_k(x)\}}{\Theta_k(x)\{\lambda + \Theta_k(y)\}} = \prod_1^\infty \varphi_x(y,\lambda,k) \quad \text{say}. \tag{2.4}$$

Since $\varphi_x(y,\lambda,k)$ is the Laplace transform of the $D(\Theta_k(y),\Theta_k(x))$ distribution, the first statement follows. Noting that $\Theta_k(y)\geqslant\Theta_2(y)\geqslant 3\pi^2/(2a^2)$ for $k\geqslant 2$ this formula also shows that for $0\leqslant\theta\leqslant\pi^2/a^2$ we have

$$\mathbb{E}_x\left\{\mathrm{e}^{\theta U_y}\right\}\leqslant\mathbb{E}_0\left\{\mathrm{e}^{\theta U_a}\right\}\leqslant\mathbb{E}_0\left\{\mathrm{e}^{\frac{\pi^2}{a^2}U_a}\right\}:=c_1, \tag{2.5}$$

and the second result follows from Chebychev's inequality. □

The main estimate we need in the proof of Theorem 1.2 is as follows.

LEMMA 2.2. – *Put $\beta=\pi^2/a^3$; then for any fixed $0\leqslant x<a$,*

$$\mathbb{P}_x\{T_{a-\varepsilon}>t\}\backsim\mathrm{e}^{-\beta t\varepsilon}\quad\text{as } t\varepsilon\to\infty \text{ and } t\varepsilon^2\to 0.$$

Proof. – Note first that if $\tilde\varepsilon=\Theta(a-\varepsilon)$ then $t\tilde\varepsilon=t\beta\varepsilon+\mathrm{O}(t\varepsilon^2)$ as $\varepsilon\downarrow 0$. Using the decomposition (2.2) and the bound (2.3) gives

$$\begin{aligned}\mathbb{P}_x\{T_{a-\varepsilon}>t\}&=\int_0^t\mathbb{P}_x\{V_{a-\varepsilon}>t-s\}\mathbb{P}_x\{U_{a-\varepsilon}\in\mathrm{d}s\}+\mathbb{P}_x\{U_{a-\varepsilon}>t\}\\&=\mathrm{e}^{-t\tilde\varepsilon}\left\{1-\frac{\tilde\varepsilon}{\Theta(x)}\right\}\int_0^t\mathrm{e}^{\tilde\varepsilon s}\mathbb{P}_x\{U_{a-\varepsilon}\in\mathrm{d}s\}+\mathrm{O}\left(\mathrm{e}^{-\frac{\pi^2 t}{a^2}}\right),\end{aligned}$$

and the result follows since the first inequality in (2.5) gives $\mathbb{E}_x\{\mathrm{e}^{\tilde\varepsilon U_{a-\varepsilon}}\}\to 1$. □

Proof of Theorem 1.2. – It is well-known (see Csáki [1] for a rigorous argument in a similar situation) that we can restrict attention to the "critical" case, so henceforth we assume that for $t\geqslant t_0$

$$\frac{1}{2\beta}\log\log t\leqslant g(t)\leqslant\frac{3}{2\beta}\log\log t. \tag{2.6}$$

Let $A_n=\{a-S_{t_n}>f(t_n)\}=\{T_{a-f(t_n)}>t_n\}$, where $t_n=\mathrm{e}^n$, $n\geqslant 1$. A simple calculation shows that $J<\infty$ is equivalent to the convergence of $\sum_1^\infty\mathrm{e}^{-\beta h_n}$, where $h_n=g(t_n)$. Plainly (2.6) implies that $\sqrt{t_n}f(t_n)\to 0$ and $t_nf(t_n)\to\infty$ so we can apply Lemma 2.2 to get

$$\mathbb{P}_x\{A_n\}\backsim\exp\bigl(-\beta t_nf(t_n)\bigr)=\exp(-\beta h_n).$$

Then the Borel–Cantelli lemma establishes the result when $J<\infty$.

Now assume that $J=\infty$, so that $\sum_1^\infty\mathbb{P}_x\{A_n\}=\infty$. We want to use the Kochen–Stone modification of the Borel–Cantelli lemma to deduce from this that $\mathbb{P}_x\{A_n \text{ i.o.}\}=1$. Note that for $j>i$ with $r_n=a-f(t_n)$ we have

$$\begin{aligned}\mathbb{P}_x\{A_i\cap A_j\}&=\int_0^{r_i}\mathbb{P}_x\{A_i,X_{t_i}\in\mathrm{d}y\}\mathbb{P}_y\{T_{r_j}>t_j-t_i\}\\&\leqslant\mathbb{P}\{T_{r_j}>t_j-t_i\}\int_0^{r_i}\mathbb{P}_x\{A_i,X_{t_i}\in\mathrm{d}y\}=\mathbb{P}\{T_{r_j}>t_j-t_i\}\mathbb{P}_x\{A_i\}.\end{aligned}$$

Since $(t_j - t_i)f(t_j) \to \infty$ as $i \to \infty$ we can apply Lemma 2.2 to get

$$\mathbb{P}\{T_{r_j} > t_j - t_i\} \backsim \exp -\beta(t_j - t_i)f(t_j) = \exp -\beta h_j\bigl(1 - \mathrm{e}^{i-j}\bigr), \tag{2.7}$$

provided that $(t_j - t_i)\{f(t_j)\}^2 \to 0$, and this is immediate from (2.6). Now given an arbitrary $\delta > 0$ we put $m_i = \min(n \geqslant 1\colon\ h_{i+k} \leqslant \delta \mathrm{e}^k$ for all $k \geqslant n)$, $i = 1, 2, \ldots$. It is easy to see from (2.6) that for all large enough i

$$m_i \leqslant 1 + \log h_{2i} \leqslant 1 + \frac{3}{2b}\log 2i.$$

Thus there exists N_δ such that, for all large enough n,

$$\begin{aligned}\sum_{i=N_\delta}^{n}\sum_{j=i+1}^{i+m_i}\mathbb{P}_x\{A_i \cap A_j\} &\leqslant (1+\delta)\sum_{i=N_\delta}^{n}\sum_{j=i+1}^{i+m_i}\mathbb{P}_x\{A_i\}\exp -\beta h_j\bigl(1-\mathrm{e}^{i-j}\bigr)\\ &\leqslant (1+\delta)\sum_{i=N_\delta}^{n} m_i\mathbb{P}_x\{A_i\}\exp -\beta h_i\bigl(1-\mathrm{e}^{-1}\bigr)\\ &\leqslant (1+\delta)\sum_{i=N_\delta}^{n} m_i\mathbb{P}_x\{A_i\} i^{-\frac{1}{2}(1-\mathrm{e}^{-1})} \leqslant c_2\sum_{i=1}^{n}\mathbb{P}_x\{A_i\}.\end{aligned}$$

But also, since $h_j(1-\mathrm{e}^{i-j}) \geqslant h_j - \delta$ when $j > i + m_i$,

$$\begin{aligned}\sum_{i=N_\delta}^{n}\sum_{j>i+m_i}^{n}\mathbb{P}_x\{A_i \cap A_j\} &\leqslant (1+\delta)\sum_{i=N_\delta}^{n}\sum_{j>i+m_i}^{n}\mathbb{P}_x\{A_i\}\mathrm{e}^{\beta\delta}\mathrm{e}^{-\beta h_j}\\ &\leqslant (1+2\delta)\mathrm{e}^{\beta\delta}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{P}_x\{A_i\}\mathbb{P}_x\{A_j\},\end{aligned}$$

and since δ is arbitrary, it follows that

$$\limsup_{n\to\infty}\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}_x\{A_i \cap A_j\}}{(\sum_{i=1}^{n}\mathbb{P}_x\{A_i\})^2} \leqslant 1,$$

and the result follows. □

3. The perturbed case

If B is a standard Brownian motion starting from zero, $\alpha < 1$ is a constant, and $S_t^B = \sup_{0\leqslant s\leqslant t} B_s$, then the process Y defined by

$$Y_t = B_t + \frac{\alpha}{1-\alpha}S_t^B, \quad t \geqslant 0,$$

is called an α-perturbed Brownian motion. It is immediate that $S_t^Y = \sup_{0\leqslant s\leqslant t} Y_s$ is given by

$$S_t^Y = \frac{1}{1-\alpha}S_t^B,$$

and it follows that Y is the pathwise unique solution of the functional equation

$$Y_t = B_t + \alpha S_t^Y, \quad t \geqslant 0.$$

(For more information about this process see Chapters 8 and 9 of [7] and the references given there.)

It is not difficult to construct an h-transform of the bivariate Markov process consisting of an α-perturbed Brownian motion killed when it exits $(0, a)$ and its supremum process, which corresponds to conditioning the α-perturbed Brownian motion to remain within this interval. We will refer to [6] for the details of this calculation, and merely record that the required function is

$$h(x, s, t) = \frac{\sin \gamma x}{\{\sin \gamma s\}^\alpha} \exp \frac{1}{2} t\gamma^2,$$

where again $\gamma = \pi/a$, and as previously noted, the perturbation parameter satisfies $\alpha < 1$. We call this an α-perturbed taboo process, and in this section $\mathbb{P}_x^{(\alpha)}$ will denote the measure under which the coordinate process is a version of this process starting from x. The result corresponding to Theorem 1.2 is as follows.

THEOREM 3.1. – *For any fixed $x \in [0, a)$ and any increasing non-negative function g such that $f(t) = t^{-1} g(t)$ is decreasing,*

$$\mathbb{P}_x^{(\alpha)}\{a - S_t > f(t) \text{ i.o. as } t \to \infty\} = 0 \quad \text{or} \quad 1$$

according as $K := \int_1^\infty t^{-1} g(t)^{-\alpha} \mathrm{e}^{-\beta g(t)}\, \mathrm{d}t$ is finite or infinite, where $\beta = \pi^2/a^3$.

Remark 1. – A consequence of this result is that, with $\log_k(\cdot)$ denoting the kth iterate of $\log(\cdot)$, and $\bar{\alpha} = 1 - \alpha$,

$$\mathbb{P}_x^{(\alpha)}\left\{\limsup_{t\to\infty} \frac{t(a - S_t) - \beta^{-1} \log_2 t}{\log_3 t} = \frac{\bar{\alpha}}{\beta}\right\} = 1,$$

so that the effect of the perturbation is only felt on the $\log_3 t$ scale.

The result corresponding to the first part of Theorem 1.1 is:

THEOREM 3.2. – *For any fixed $x \in [0, a)$ and any non-negative function f such that $g(t) = 1/(tf(t))$ increases to ∞,*

$$\mathbb{P}_x\{a - S_t < f(t) \text{ i.o. as } t \to \infty\} = 0 \quad \text{or} \quad 1$$

according as

$$L := \int_1^\infty \frac{\mathrm{d}t}{tg(t)^{\bar{\alpha}}}$$

is finite or infinite.

The key to our analysis is

LEMMA 3.3. – *Under* $\mathbb{P}_x^{(\alpha)}$ *the first passage process* $\{T_y, x \leqslant y < a\}$ *has independent increments and*

$$\mathbb{E}_x^{(\alpha)}\{e^{-\lambda T_y}\} = \left(\mathbb{E}_x\{e^{-\lambda T_y}\}\right)^{\bar{\alpha}},$$

where the righthand side is given explicitly in (2.1).

Proof. – The first statement follows from the fact $\{(X_t, S_t),\ t \geqslant 0\}$ is a Markov process under $\mathbb{P}_x^{(\alpha)}$. Also the Laplace transform of the time at which an α-perturbed Brownian motion first exits a finite interval is known, (see, e.g., [2]), and the second result follows by a simple calculation. □

Next, we introduce, for any $0 < b < c \leqslant \infty$ the distribution $D^{(\bar{\alpha})}(b, c)$ of a non-negative random variable with Laplace transform $\{b(\lambda + c)/(c(\lambda + b))\}^{\bar{\alpha}}$ if $c < \infty$, and Laplace transform $\{b/(\lambda + b)\}^{\bar{\alpha}}$ if $c = \infty$. Then $D^{(\bar{\alpha})}(b, \infty)$ coincides with the $\Gamma(\bar{\alpha}, b)$ distribution, and a random variable D has the $D^{(\bar{\alpha})}(b, c)$ distribution with $c < \infty$ if and only if we can write

$$Y_1 = Y_2 + D, \tag{3.1}$$

where Y_1 and Y_2 have $\Gamma(\bar{\alpha}, b)$ and $\Gamma(\bar{\alpha}, c)$ distributions and Y_2 and D are independent. In the case $\alpha = 0$ the tail behaviour of this distribution is obvious, but now a little work is required.

LEMMA 3.4. – *If* D *has a* $D^{(\bar{\alpha})}(b, c)$ *distribution with* $c \leqslant \infty$ *fixed,* $bt \to \infty$*, and* $b^2 t \to 0$ *then*

$$\Gamma(\bar{\alpha}) P(D > t) \backsim (bt)^{-\alpha} e^{-bt}. \tag{3.2}$$

Proof. – If $c = \infty$ we know that bD has a $\Gamma(\bar{\alpha}, 1)$ distribution and the result is immediate. When $c < \infty$ we have $\Gamma(\bar{\alpha}) P(D > t) \leqslant \Gamma(\bar{\alpha}) P(Y_1 > t) \backsim (bt)^{-\alpha} e^{-bt}$, so we only need a corresponding lower bound. For this we write $\eta = 2b/c$ and use (3.1) to get

$$\begin{aligned}\Gamma(\bar{\alpha}) P(Y_2 \leqslant \eta t) P(D > t) &\geqslant \Gamma(\bar{\alpha}) P\big(Y_1 > t(1 + \eta)\big) - \Gamma(\bar{\alpha}) P(Y_2 > \eta t) \\ &\backsim (bt)^{-\alpha} e^{-bt} e^{-2b^2 t/c} + O\big((\eta t)^{-\alpha} e^{-c\eta t}\big) \backsim (bt)^{-\alpha} e^{-bt}.\end{aligned}$$

Since $P(Y_2 \leqslant \eta t) \to 1$, the result follows. □

The analogue of Lemma 2.1 is straightforward:

LEMMA 3.5. – *For any* $0 \leqslant x < y < a$ *we have under* $\mathbb{P}_x^{(\alpha)}$

$$T_y \overset{d}{=} V_y + U_y, \tag{3.3}$$

where the non-negative random variables V_y *and* U_y *are independent,* V_y *has the* $D^{(\bar{\alpha})}(\Theta(y), \Theta(x))$ *distribution, and*

$$\mathbb{P}_x^{(\alpha)}\{U_y > t\} \leqslant c_2 e^{-t\pi^2/a^2} \quad \textit{for all } t \geqslant 0, \tag{3.4}$$

where c_2 *is a constant, which depends only on* a *and* α.

Proof. – The proof is the same as that of Lemma 2.1. □

The result corresponding to Lemma 2.2 now follows.

LEMMA 3.6. – *Put* $\beta = \pi^2/a^3$; *then for any fixed* $0 \leqslant x < a$,

$$\Gamma(\bar{\alpha})\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} > t\} \backsim (\beta t\varepsilon)^{-\alpha} \mathrm{e}^{-\beta t\varepsilon} \quad \textit{as } t\varepsilon \to \infty \textit{ and } t\varepsilon^2 \to 0.$$

Proof. – It is immediate from (3.3), Lemma 3.4, and the fact that $\tilde{\varepsilon} = \Theta(a - \varepsilon) = \beta\varepsilon + \mathrm{O}(\varepsilon^2)$ as $\varepsilon \downarrow 0$ that

$$\Gamma(\bar{\alpha})\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} > t\} \geqslant \Gamma(\bar{\alpha})\mathbb{P}_x^{(\alpha)}\{V_{a-\varepsilon} > t\} \backsim (t\tilde{\varepsilon})^{-\alpha}\mathrm{e}^{-t\tilde{\varepsilon}} \backsim (\beta t\varepsilon)^{-\alpha}\mathrm{e}^{-\beta t\varepsilon}.$$

But with $\tilde{\eta} = 2a^2\tilde{\varepsilon}/\pi^2$

$$\begin{aligned}\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} > t\} &\leqslant \mathbb{P}_x^{(\alpha)}\{V_{a-\varepsilon} > t(1-\tilde{\eta})\}\mathbb{P}_x^{(\alpha)}\{U_{a-\varepsilon} \leqslant \tilde{\eta}t\} + \mathbb{P}_x^{(\alpha)}\{U_{a-\varepsilon} > \tilde{\eta}t\} \\ &\backsim (t\tilde{\varepsilon})^{-\alpha}\mathrm{e}^{-\tilde{t}\varepsilon}/\Gamma(\bar{\alpha}) + \mathrm{O}\big(\mathrm{e}^{-\tilde{\eta}t\pi^2/a^2}\big) \backsim (\beta t\varepsilon)^{-\alpha}\mathrm{e}^{-\beta t\varepsilon}/\Gamma(\bar{\alpha}).\end{aligned}$$ □

Proof of Theorem 3.1. – This follows the same lines as the proof of Theorem 1.2, so we omit some of the details. As before, we will assume (2.6) is in force, and again put $A_n = \{a - S_{t_n} > f(t_n)\} = \{T_{a-f(t_n)} > t_n\}$, where $t_n = \mathrm{e}^n, n \geqslant 1$. A simple calculation shows that $K < \infty$ is equivalent to the convergence of $\sum_1^\infty (h_n)^{-\alpha}\mathrm{e}^{-\beta h_n}$, where $h_n = g(t_n)$. Then Lemma 3.6 gives

$$\Gamma(\bar{\alpha})\mathbb{P}_x^{(\alpha)}\{A_n\} \backsim (\beta h_n)^{-\alpha}\exp(-\beta h_n),$$

and the Borel–Cantelli lemma establishes the result when $K < \infty$.

Now assume that $K = \infty$, so that $\sum_1^\infty \mathbb{P}_x^{(\alpha)}\{A_n\} = \infty$. As before we need to estimate $\mathbb{P}_x^{(\alpha)}\{A_i \cap A_j\}$, and here the fact that $\{X_t, t \geqslant 0\}$ is not Markov under $\mathbb{P}_x^{(\alpha)}$ introduces a complication. Note that for $j > i$ with $r_n = a - f(t_n)$ we have

$$\mathbb{P}_x^{(\alpha)}\{A_i \cap A_j\} = \int_0^{r_i}\int_y^{r_i} \mathbb{P}_x^{(\alpha)}\{A_i, X_{t_i} \in dy, S_{t_i} \in \mathrm{d}z\}\mathbb{P}_{y,z}^{(\alpha)}\{T_{r_j} > t_j - t_i\}, \tag{3.5}$$

where $\mathbb{P}_{y,z}^{(\alpha)}$ stands for the measure under which the coordinate process is an α-perturbed taboo process satisfying the initial conditions $(X_0, S_0) = (y, z)$. Under this measure we have the decomposition

$$T_{r_j} = T^{(1)} + T^{(2)}, \tag{3.6}$$

where $T^{(1)}$ and $T^{(2)}$ are independent, $T^{(1)}$ has the distribution of T_z under the *unperturbed* measure $\mathbb{P}_y$, and $T^{(2)}$ has the distribution of T_{r_j} under the *perturbed* measure $\mathbb{P}_z^{(\alpha)}$. Now if $\alpha > 0$ it is clear that

$$\mathbb{P}_y(T_z > t) \leqslant \mathbb{P}_y^{(\alpha)}(T_z > t) \leqslant \mathbb{P}^{(\alpha)}(T_z > t),$$

where $\mathbb{P}^{(\alpha)} = \mathbb{P}_0^{(\alpha)}$, and hence, from (3.6) we get $\mathbb{P}_{y,z}^{(\alpha)}\{T_{r_j} > t\} \leqslant \mathbb{P}^{(\alpha)}\{T_{r_j} > t\}$. Using this in (3.5) and appealing to Lemma 3.6 we see that, when $\alpha > 0$, we have

$$\mathbb{P}_x^{(\alpha)}\{A_j \mid A_i\} \leqslant \mathbb{P}^{(\alpha)}\{T_{r_j} > t_j - t_i\} \backsim \frac{\exp -\beta h_j(1-\mathrm{e}^{i-j})}{\Gamma(\bar{\alpha})\{\beta h_j(1-\mathrm{e}^{i-j})\}^{\alpha}}. \tag{3.7}$$

It is now easy to conclude the proof in this case, as the final part of the proof of Theorem 1.2 requires only minor modifications.

In the case $\alpha < 0$ we use the fact that, in (3.6), $T^{(1)}$ and $T^{(2)}$ are stochastically dominated by independent random variables $W^{(1)}$ and $W^{(2)}$ which have the distribution of T_{r_i} under the measure $\mathbb{P}$, and the distribution of T_{r_j} under the measure $\mathbb{P}^{(\alpha)}$ to see that, for any $\theta \in (0,1)$,

$$\begin{aligned}\mathbb{P}_x^{(\alpha)}\{A_j \mid A_i\} &\leqslant P\{W^{(1)} + W^{(2)} > t_j - t_i\} \\ &\leqslant P\{W^{(1)} > \theta(t_j - t_i)\} + P\{W^{(2)} > (1-\theta)(t_j - t_i)\}.\end{aligned}$$

With the choice of $\theta = f(t_j)/f(t_i)$ the requirements of Lemma 2.2 are satisfied and

$$\begin{aligned}P\{W^{(1)} > \theta(t_j - t_i)\} &\backsim \exp -\beta\theta(t_j - t_i)f(t_i) \\ &= \exp -\beta(t_j - t_i)f(t_j) = \mathrm{o}\{\mathbb{P}^{(\alpha)}\{T_{r_j} > t_j - t_i\}\},\end{aligned}$$

because $\alpha < 0$, and it is easy to see that this term is asymptotically neglible. We can also apply Lemma 3.6 to get

$$\begin{aligned}P\{W^{(2)} > (1-\theta)(t_j - t_i)\} &\backsim \frac{\exp -\beta h_j(1-\mathrm{e}^{i-j})}{\Gamma(\bar{\alpha})\{\beta(1-\theta)h_j(1-\mathrm{e}^{i-j})\}^{\alpha}} \exp \theta\beta h_j(1-\mathrm{e}^{i-j}) \\ &\leqslant \frac{\exp -\beta h_j(1-\mathrm{e}^{i-j})}{\Gamma(\bar{\alpha})\{\beta h_j(1-\mathrm{e}^{i-j})\}^{\alpha}} \exp \beta \mathrm{e}^{j-i} h_j^2/h_i.\end{aligned}$$

Since it follows from (2.6) that, for a suitable c_3

$$\lim_{i\to\infty} \sup_{j \geqslant i + c_3} \left(\frac{\mathrm{e}^{j-i} h_j^2}{h_i}\right) = 0,$$

it is not difficult to modify the argument used in the final part of the proof of Theorem 1.2 to get the required conclusion. □

Clearly the proof of Theorem 3.2 will involve the behaviour of $\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} \leqslant t\}$, and this is given in the following.

LEMMA 3.7. – (i) *Suppose that $t \to \infty$ and $\varepsilon t \downarrow 0$. Then for any fixed $x \in [0, a)$*

$$\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} \leqslant t\} \backsim \frac{(\beta\varepsilon t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)}. \tag{3.8}$$

(ii) *Given arbitrary $\delta > 0$ there exists $K_\delta < \infty$ such that for all ε_1 sufficiently small, $t\varepsilon_1$ sufficiently large and all $\varepsilon_2 \in (0, \varepsilon_1)$*

$$\mathbb{P}_{a-\varepsilon_1}^{(\alpha)}\{T_{a-\varepsilon_2} \leqslant t\} \leqslant K_\delta\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{\overline{\alpha}} + \frac{(1+\delta)(\beta\varepsilon_2 t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)}. \tag{3.9}$$

Proof. – First note that, for any $\eta \in [0, t]$,

$$\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} \leqslant t\} \geqslant \mathbb{P}^{(\alpha)}\{T_{a-\varepsilon} \leqslant t\} \geqslant \mathbb{P}^{(\alpha)}\{V_{a-\varepsilon} \leqslant t-\eta\}\mathbb{P}^{(\alpha)}\{U_{a-\varepsilon} \leqslant \eta\}.$$

Under $\mathbb{P}^{(\alpha)}$ $V_{a-\varepsilon}$ has a $\Gamma(\bar{\alpha}, \tilde{\varepsilon})$ distribution, so choosing $\eta = \sqrt{t}$, so that $\eta/t \to 0$ we have

$$\mathbb{P}^{(\alpha)}\{V_{a-\varepsilon} \leqslant t-\eta\} \backsim \frac{(\tilde{\varepsilon} t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)} \backsim \frac{(\beta \varepsilon t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)}.$$

But since $\eta \to \infty$ we see from (2.3) that $\mathbb{P}^{(\alpha)}\{U_{a-\varepsilon} \leqslant \eta\} \to 1$, and this proves one half of (3.8).

To get the other half, we note that

$$\mathbb{P}_x^{(\alpha)}\{T_{a-\varepsilon} \leqslant t\} \leqslant \mathbb{P}_x^{(\alpha)}\{V_{a-\varepsilon} \leqslant t\}.$$

Assuming that $\bar{\alpha}$ is not a positive integer (the contrary case is easier to deal with) and writing $\Delta = \tilde{\varepsilon}/\Theta(x)$, Lemma 3.3 gives

$$\mathbb{E}_x^{(\alpha)}\{\mathrm{e}^{-\lambda\tilde{\varepsilon}V_{a-\varepsilon}}\} = \left\{\Delta + \frac{1-\Delta}{1+\lambda}\right\}^{\bar{\alpha}} = \Delta^{\bar{\alpha}} \sum_0^\infty \binom{\bar{\alpha}}{k}\left(\frac{1-\Delta}{\Delta(1+\lambda)}\right)^k.$$

Inverting the Laplace transform, we see that $\mathbb{P}_x^{(\alpha)}\{\tilde{\varepsilon}V_{a-\varepsilon} = 0\} = \Delta^{\bar{\alpha}}$ and that $\tilde{\varepsilon}V_{a-\varepsilon}$ has, under $\mathbb{P}_x^{(\alpha)}$, a density on $(0, \infty)$ given by

$$\Delta^{\bar{\alpha}}\mathrm{e}^{-y}\sum_1^\infty \binom{\bar{\alpha}}{k}\left(1-\Delta^{-1}\right)^k \frac{y^{k-1}}{(k-1)!} \leqslant \Delta^{\bar{\alpha}}\sum_1^\infty \binom{\overline{\alpha}}{k}\left(1-\Delta^{-1}\right)^k \frac{y^{k-1}}{(k-1)!}. \tag{3.10}$$

It follows that with $y > 0$ and $z = y(1-\Delta^{-1})$,

$$\begin{aligned}\mathbb{P}_x^{(\alpha)}\{\tilde{\varepsilon}V_{a-\varepsilon} \leqslant y\} &\leqslant \Delta^{\bar{\alpha}}\sum_0^\infty \binom{\bar{\alpha}}{k}\left(1-\Delta^{-1}\right)^k \frac{y^k}{k!} \\ &= \Delta^{\bar{\alpha}}\sum_0^\infty (-1)^k \frac{\Gamma(\alpha+k-1)}{\Gamma(\alpha-1)k!}\left(1-\Delta^{-1}\right)^k \frac{y^k}{k!} \\ &= \Delta^{\bar{\alpha}}F(\alpha-1; 1; -z) = \Delta^{\bar{\alpha}}\mathrm{e}^{-z}F(\bar{\alpha}+1; 1; z),\end{aligned} \tag{3.11}$$

where $F(b; c; \cdot)$ denotes the confluent hypergeometric function and we have used a standard transformation result. (See [5, p. 267].) Now if x is fixed putting $y = \tilde{\varepsilon}t$ we see that $z \backsim \tilde{\varepsilon}t/\Delta \to \infty$ so we can use the known asymptotic behaviour of the hypergeometric function (see [5, p. 289]) to conclude that

$$\Delta^{\bar{\alpha}}\mathrm{e}^{-z}F(\bar{\alpha}+1; 1; z) \backsim \frac{\Delta^{\bar{\alpha}}z^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)} \backsim \frac{(\tilde{\varepsilon}t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)} \backsim \frac{(\beta\varepsilon t)^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+1)},$$

which finishes the proof of (i). For (ii) we note that the same asymptotic result shows that there exists z_δ with

$$\sup_{z\geqslant z_\delta} z^{-\bar\alpha}\mathrm{e}^{-z}F(\bar\alpha+1;1;z)\leqslant\frac{1+\delta}{\Gamma(\bar\alpha+1)}.$$

Now apply (3.11) with $x=a-\varepsilon_1$, $\varepsilon=\varepsilon_2$ and $y=t\varepsilon_2$ so that $\Delta=\tilde\varepsilon_2/\tilde\varepsilon_1\backsim\varepsilon_2/\varepsilon_1$, to see that (3.9) holds if we take $K_\delta=2F(\bar\alpha+1;1;z_\delta)$. □

Proof of Theorem 3.2. – Let $B_n=\{a-S_{t_n}<f(t_n)\}=\{T_{a-f(t_n)}\leqslant t_n\}$, where $t_n=\mathrm{e}^n$, $n\geqslant 1$. A simple calculation shows that $L<\infty$ is equivalent to the convergence of $\sum_1^\infty\{h(n)\}^{-\bar\alpha}$, where $h_n=g(t_n)$. Since x is fixed we can apply (i) of Lemma 3.7 to get

$$\Gamma(\bar\alpha+1)\mathbb{P}_x\{B_n\}\backsim\left\{\beta t_n f(t_n)\right\}^{\bar\alpha}=\{\beta h_n\}^{-\bar\alpha}.$$

Then the Borel–Cantelli lemma establishes the result when $L<\infty$.

Now assume that $L=\infty$, so that $\sum_1^\infty\mathbb{P}_x\{B_n\}=\infty$. Note that for $j>i$ with $r_n=a-f(t_n)$ we have

$$\begin{aligned}\mathbb{P}_x^{(\alpha)}\{B_i\cap B_j\}&=\int_0^{t_i}\mathbb{P}_x^{(\alpha)}\{T_{r_i}\in\mathrm{d}s\}\mathbb{P}_{r_i}^{(\alpha)}\{T_{r_j}\leqslant t_j-s\}\leqslant\mathbb{P}_{r_i}^{(\alpha)}\{T_{r_j}\leqslant t_j\}\int_0^{t_i}\mathbb{P}_x^{(\alpha)}\{T_{r_i}\in\mathrm{d}s\}\\&=\mathbb{P}_{r_i}^{(\alpha)}\{B_j\}\mathbb{P}_x^{(\alpha)}\{B_i\}.\end{aligned}$$

It follows from (ii) of Lemma 3.7 that for arbitrary $\delta>0$,

$$\begin{aligned}\mathbb{P}_{r_i}^{(\alpha)}\{B_j\}&\leqslant K_\delta\left(\frac{f(t_j)}{f(t_i)}\right)^{\bar\alpha}+\frac{(1+\delta)(\beta t_j f(t_j))^{\bar\alpha}}{\Gamma(\bar\alpha+1)}\\&=K_\delta\left(\frac{t_ih_i}{t_jhj}\right)^{\bar\alpha}+\frac{(1+\delta)(\beta h_j)^{\bar\alpha}}{\Gamma(\bar\alpha+1)}\\&\leqslant K_\delta\mathrm{e}^{-\bar\alpha(j-i)}+\frac{(1+\delta)(\beta h_j)^{\bar\alpha}}{\Gamma(\bar\alpha+1)}.\end{aligned}$$

From this, since δ is arbitrary, it is immediate that

$$\limsup_{n\to\infty}\frac{\sum_{i=1}^n\sum_{j=1}^n\mathbb{P}_x\{B_i\cap B_j\}}{(\sum_{i=1}^n\mathbb{P}_x\{B_i\})^2}\leqslant 1,\tag{3.12}$$

and the key step in the proof is finished. □

Remark 2. – An interesting question is whether or not the tail sigma-field of the first passage-time process is trivial under $\mathbb{P}_x^{(\alpha)}$. In the case $\alpha=0$, the triviality can be seen as a consequence of the ergodicity of the (Markovian) taboo process, which was established in [4]; we have not been able to resolve this question when $\alpha\neq 0$. If this sigma-field is trivial when $\alpha\neq 0$, some of our proofs would be shorter, since it would only be necessary to show, for example, that the lim sup in (3.12) is finite.

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