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A SINGULAR LARGE DEVIATIONS PHENOMENON

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ABSTRACT. – Consider $\{X_t^\varepsilon\colon\ t \geqslant 0\}$ $(\varepsilon > 0)$, the solution starting from 0 of a stochastic differential equation, which is a small Brownian perturbation of the one-dimensional ordinary differential equation $x_t' = \operatorname{sgn}(x_t)|x_t|^\gamma$ $(0 < \gamma < 1)$. Denote by $p_t^\varepsilon(x)$ the density of X_t^ε. We study the exponential decay of the density as $\varepsilon \to 0$. We prove that, for the points (t, x) lying between the extremal solutions of the ordinary differential equation, the rate of the convergence is different from the rate of convergence in large deviations theory (although respected for the points (t, x) which does not lie between the extremals). Proofs are based on probabilistic (large deviations theory) and analytic (viscosity solutions for Hamilton–Jacobi equations) tools. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – On considère $\{X_t^\varepsilon\colon\ t \geqslant 0\}$ $(\varepsilon > 0)$, la solution issue de zéro d'une équation différentielle stochastique définie comme une petite perturbation brownienne de l'équation différentielle ordinaire $x_t' = \operatorname{sgn}(x_t)|x_t|^\gamma$ $(0 < \gamma < 1)$. On note $p_t^\varepsilon(x)$ la densité de X_t^ε. Le but de cet article est d'étudier la décroissance exponentielle de la densité quand $\varepsilon \to 0$. On montre que la vitesse de convergence est différente de celle rencontrée dans la théorie classique des grandes déviations lorsque le point (t, x) est situé entre les trajectoires des solutions extrémales de l'équation différentielle ordinaire. En revanche, pour les points qui ne sont pas situés entre les extrémales, la vitesse est identique à celle de la théorie des grandes déviations. Les preuves reposent sur des arguments probabilistes (théorie des grandes déviations) et analytiques (solutions de viscosité pour des équations de Hamilton–Jacobi). © 2001 Éditions scientifiques et médicales Elsevier SAS

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Introduction

Let $0 < T < \infty$, $\{B_t\colon\ t \geqslant 0\}$ an one-dimensional Brownian motion, and consider the stochastic differential equation on $[0, T]$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon\,\mathrm{d}B_t + \mathrm{b}(X_t^\varepsilon)\,\mathrm{d}t, \\ X_0^\varepsilon = x_0. \end{cases}$$

Let us denote by P_ε the law of the process $X_\cdot^\varepsilon$. It is classical that the family $\{P_\varepsilon\colon\ \varepsilon > 0\}$ is weakly relatively compact and, as ε tends to zero, every cluster value P has its support contained in the set of paths x which are solutions of the dynamical system

$$\begin{cases} x'(t) = \mathrm{b}(x(t)), \\ x(0) = x_0. \end{cases} \tag{1}$$

If (1) has an unique solution (for instance, if b is a Lipschitz function), then by the large deviations theory, it is known that P_ε is exponentially tight and therefore P_ε converges to P exponentially fast, as ε tends to zero.

If (1) has more than one solution, in [1] it is proved that, under suitable conditions, there is just one limit value in law, concentrated on at most two paths: the extremal solutions of (1) (see Fig. 1).

The aim of this paper is to study the precise convergence of P_ε towards P in the following case: take $0 < \gamma < 1$ and let P_ε be the law of the solution of the stochastic differential equation:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon\,\mathrm{d}B_t + \mathrm{sgn}(X_t^\varepsilon)|X_t^\varepsilon|^\gamma\,\mathrm{d}t, \\ X_0^\varepsilon = 0. \end{cases} \tag{2}$$

We can see this equation as a small random perturbation of the dynamical system:

$$\begin{cases} x_t' = \mathrm{sgn}(x_t)|x_t|^\gamma, \\ x_0 = 0. \end{cases} \tag{3}$$

Let us denote by $p_t^\varepsilon(\cdot)$ the density of X_t^ε with respect to the Lebesgue measure. We observe that if $|x| \neq (t(1-\gamma))^{1/1-\gamma}$, i.e. if (t, x) does not belong to the graph of one of the extremal solutions of problem (3), then the density tends to zero, corroborating the results in [1].

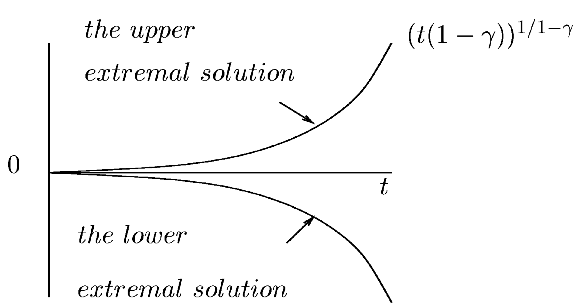

Fig. 1. Solutions of the dynamical system.

Let us describe our main results. According to the position of the point (t, x), we emphasize two kinds of rate.

If the point (t, x) is such that $|x| > \{t(1-\gamma)\}^{1/1-\gamma}$, there exists a positive function k_t such that:

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln p_t^\varepsilon(x) = -k_t(|x|).$$

This means that the density has an exponential decay with rate ε^2, as in large deviations theory. The rate is the same as in the case when the dynamical system has an unique solution. For instance, if the drift b is a Lipschitz function the rate agrees to the rate in Freidlin–Wentzell theorem for random perturbations of the dynamical systems (see, [7, p. 31]).

If the point (t, x) lies in the domain between the two extremals, that is, if $|x| < \{t(1-\gamma)\}^{1/1-\gamma}$, then the density has an exponential decay with a different rate, namely $\varepsilon^{2(1-\gamma)/(1+\gamma)}$. Precisely, we show that, for such points (t, x):

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{2(1-\gamma)}{(1+\gamma)}} \ln p_t^\varepsilon(x) = \lambda_1 \left(\frac{x^{1-\gamma}}{1-\gamma} - t \right).$$

Here λ_1 is the first positive eigenvalue of the Schrödinger operator:

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\gamma}{2|x|^{1-\gamma}} + \frac{|x|^{2\gamma}}{2}.$$

Let us note that, in the particular case $\gamma = 0$, the calculation is explicit (see Proposition 3 below).

The plan of the paper is as follows. In the first section we recall some existence results for stochastic and ordinary equations and also the results of [1], for the drift $\mathrm{b}(x) := \mathrm{sgn}(x)|x|^\gamma$, $0 \leqslant \gamma < 1$. Moreover we give some representations of the density p_t^ε. In particular, we give an expansion in terms of eigenvalues and eigenfunctions of the Schrödinger operator. This was already studied by Kac [10] for continuous potentials and we adapt this result to our situation. Section 2 is devoted to the convergence of the density in logarithmic scale with rate ε^2. We compute the limit for the points (t, x) which does not lie between the extremals (see Theorem 1) and we give an upper bound for the other points. In the last section we treat the convergence of the density in logarithmic scale with the rate $\varepsilon^{2(1-\gamma)/(1+\gamma)}$, for the points (t, x) lying between the extremals. The precise limit is obtained in Theorem 2 by the study, developed in Section 3, of the viscosity solution of a Hamilton–Jacobi equation (see [2] or [8]). Although the ideas are inspired by [2], there are several new difficulties, since, for example, b is not a Lipschitz function.

1. Preliminaries

1.1. Existence results

In this subsection we recall some existence results for the stochastic differential equation (2), for the ordinary differential equation and the convergence result of [1].

PROPOSITION 1. – *There exists a unique strong solution of* (2). *Moreover, for any Borel measurable function* f,

$$\mathbb{E}\left[f(X_t^\varepsilon)\right] = \mathbb{E}\left[f(\varepsilon B_t)\exp\left\{\frac{|B_t|^{\gamma+1}}{(\gamma+1)\varepsilon^{1-\gamma}} - \frac{\gamma}{2\varepsilon^{1-\gamma}}\int_0^t |B_s|^{\gamma-1}\,\mathrm{d}s - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}\,\mathrm{d}s\right\}\right]. \tag{4}$$

Proof. – The existence, weak uniqueness and non explosion results are consequences of Girsanov theorem and Novikov criterion (which is satisfied here since $\gamma < 1$). Pathwise uniqueness is a consequence of Proposition 3.2 in [13, p. 370]. Applying Girsanov theorem, we get

$$\mathbb{E}\left[f\left(\frac{X_t^\varepsilon}{\varepsilon}\right)\right] = \mathbb{E}\left[f(B_t)\exp\left\{\frac{1}{\varepsilon^{1-\gamma}}\int_0^t \operatorname{sgn}(B_s)|B_s|^\gamma\,\mathrm{d}B_s - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}\,\mathrm{d}s\right\}\right],$$

and (4) is a consequence of Itô–Tanaka formula (thanks to convexity) and the occupation time formula. □

We study now the dynamical system (3) and the behaviour of the law P_ε of the process $X_\cdot^\varepsilon$, as $\varepsilon \to 0$:

PROPOSITION 2. – *Eq.* (3) *admits an infinity of solutions*:

$$\left\{c_\gamma(t-\lambda)_+^{1/1-\gamma}, \lambda \geqslant 0; -c_\gamma(t-\lambda)_+^{1/1-\gamma}, \lambda \geqslant 0\right\},$$

where c_γ *is a constant. Let us denote by*

$$\rho_{1,2}(t) = \pm\{(1-\gamma)t\}^{1/1-\gamma}$$

the extremal solutions of the dynamical system. Then P_ε *tends to* $\frac{1}{2}\delta_{\rho_1} + \frac{1}{2}\delta_{\rho_2}$, *as* $\varepsilon \to 0$.

Proof. – The existence result is obvious. By Theorem 5.2 in [1, p. 291]: if P is any cluster value of $\{P_\varepsilon\}$, as $\varepsilon \to 0$, then P is concentrated on the extremal solutions ρ_1 and ρ_2:

$$P = \frac{1}{2}\delta_{\rho_1} + \frac{1}{2}\delta_{\rho_2}. \qquad \square$$

1.2. The particular case: $\gamma = 0$

Let us note that in the case $\gamma = 0$ the calculation is explicit, we compute the density and we show that the diffusion tends towards the extremal solutions (in a generalized sense, namely a.e. differentiable) of the following differential equation:

$$\begin{cases} x_t' = \operatorname{sgn}(x_t), \\ x_0 = 0, \end{cases}$$

which are $\rho_{1,2}(t) = \pm t$. In this particular case, the diffusion $X^{\varepsilon}_{\cdot}$ is solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X^{\varepsilon}_t = \varepsilon\,\mathrm{d}B_t + \mathrm{sgn}(X^{\varepsilon}_t)\,\mathrm{d}t, \\ X^{\varepsilon}_0 = 0, \end{cases} \tag{2'}$$

and we can compute the density $p^{\varepsilon}_t(x)$ of X^{ε}_t with respect to the Lebesgue measure:

PROPOSITION 3. – *Let us denote* $\varphi(x) = \int_x^{\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y$. *Then,*

$$p^{\varepsilon}_t(x) = \frac{1}{\varepsilon\sqrt{2\pi t}} \exp -\left\{ \frac{(|x|-t)^2}{2\varepsilon^2 t} \right\} - \frac{1}{\varepsilon^2\sqrt{2\pi}} \varphi\left(\frac{|x|}{\varepsilon\sqrt{t}} + \frac{\sqrt{t}}{\varepsilon} \right) \exp \frac{2|x|}{\varepsilon^2}. \tag{5}$$

Moreover, as $\varepsilon \to 0$,

$$p^{\varepsilon}_t(x) \sim \frac{1}{\varepsilon\sqrt{2\pi t}} \left(1 - \frac{t}{(|x|+t)} \right) \exp -\frac{(|x|-t)^2}{2\varepsilon^2 t}, \quad \textit{if } x \neq 0,$$

$$p^{\varepsilon}_t(x) \sim \frac{\varepsilon}{\sqrt{2\pi t^3}} \exp -\frac{t}{2\varepsilon^2}, \quad \textit{if } x = 0.$$

In particular, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^*$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln p^{\varepsilon}_t(x) = -\frac{(|x|-t)^2}{2t}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln \mathbb{P}\big(\big||X^{\varepsilon}_t| - t\big| \geqslant \delta \big) = -\frac{\delta^2}{2t}.$$

Proof. – Using Girsanov theorem and the Itô–Tanaka formula we get

$$\mathbb{E}\big[f(X^{\varepsilon}_t)\big] = \mathbb{E}\left[f(\varepsilon|B_t|) \exp\left\{ \frac{|B_t|}{\varepsilon} - \frac{2L_t}{\varepsilon} - \frac{t}{2\varepsilon^2} \right\} \right],$$

where f is a Borel measurable even function (one can consider only even functions since $-X^{\varepsilon}$ is a solution of (2′) too) and L_t is the Brownian local time at level 0.

Moreover, by Levy's theorem, $(|B_t|, 2L_t)$ has the same law as $(S_t - B_t, S_t)$, where $S_t = \sup_{0\leqslant s\leqslant t} B_s$, hence

$$\mathbb{E}\big[f(X^{\varepsilon}_t)\big] = \mathbb{E}\left[f\big(\varepsilon(S_t - B_t)\big) \exp\left\{ -\frac{B_t}{\varepsilon} - \frac{t}{2\varepsilon^2} \right\} \right],$$

where the law of (B_t, S_t) is well known (see, for instance, [11, Proposition 8.1, p. 95]):

$$\mathbb{P}(B_t \in \mathrm{d}a, S_t \in \mathrm{d}b) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left\{ -\frac{(2b-a)^2}{2t} \right\} \mathrm{d}a\,\mathrm{d}b, \quad \text{for } a \leqslant b, b \geqslant 0.$$

Hence

$$\mathbb{E}\left[f(X_t^\varepsilon)\right] = \int_0^\infty \int_{-\infty}^b \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2b-a)^2}{2t} - \frac{a}{\varepsilon} - \frac{t}{2\varepsilon^2}\right\} f\big(\varepsilon(b-a)\big)\,\mathrm{d}a\,\mathrm{d}b.$$

We make the change of variables $x := \varepsilon(2b-a)$ and $y := \varepsilon(b-a)$ and we obtain

$$\begin{aligned}
\mathbb{E}\left[f(X_t^\varepsilon)\right] &= \frac{2}{\varepsilon^3\sqrt{2\pi t^3}} \int_0^\infty \int_y^\infty x \exp\left\{-\frac{x^2}{2\varepsilon^2 t} + \frac{2y}{\varepsilon^2} - \frac{x}{\varepsilon^2} - \frac{t}{2\varepsilon^2}\right\} f(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{2}{\varepsilon\sqrt{2\pi t}} \int_0^\infty f(y) \exp\left\{\frac{2y}{\varepsilon^2} - \frac{(y+t)^2}{2\varepsilon^2 t}\right\} \mathrm{d}y \\
&\quad - \frac{2}{\varepsilon^2\sqrt{2\pi}} \int_0^\infty \left(\int_{(y+t)/(\varepsilon\sqrt{t})}^\infty \exp -\frac{v^2}{2}\,\mathrm{d}v\right) f(y) \exp\frac{2y}{\varepsilon^2}\,\mathrm{d}y.
\end{aligned}$$

From this equality we get the expression of the density (5). Moreover, using the Laplace method we obtain the equivalents in the statement of the proposition. □

1.3. Some representations of the density

In this subsection we shall describe some useful representations of the density of X_t^ε, solution of Eq. (2), for arbitrary $0 < \gamma < 1$.

PROPOSITION 4. – *For $t > 0$, $\varepsilon > 0$, $x \in \mathbb{R}$:*

$$\begin{aligned}
p_t^\varepsilon(x) &= \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\} \\
&\quad \times \mathbb{E}\left[\exp\left\{-\frac{\gamma t}{2}\int_0^1 \left|xs + \varepsilon\sqrt{t}b_s\right|^{\gamma-1}\mathrm{d}s - \frac{t}{2\varepsilon^2}\int_0^1 \left|xs + \varepsilon\sqrt{t}b_s\right|^{2\gamma}\mathrm{d}s\right\}\right],
\end{aligned} \tag{6}$$

where $\{b_t\colon t \in [0,1]\}$ is the standard Brownian bridge.

Proof. – By (4) in Proposition 1 and by the scaling property of the Brownian motion, we obtain

$$\begin{aligned}
\mathbb{E}\left[f(X_t^\varepsilon)\right] = \mathbb{E}\Bigg[f(\varepsilon\sqrt{t}B_1) \exp\Bigg\{&\frac{t^{(\gamma+1)/2}}{(\gamma+1)\varepsilon^{1-\gamma}}|B_1|^{\gamma+1} \\
&- \frac{\gamma t^{(\gamma+1)/2}}{2\varepsilon^{1-\gamma}}\int_0^1 |B_s|^{\gamma-1}\mathrm{d}s - \frac{t^{\gamma+1}}{2\varepsilon^{2-2\gamma}}\int_0^1 |B_s|^{2\gamma}\mathrm{d}s\Bigg\}\Bigg].
\end{aligned}$$

Let us decompose the Brownian motion as follows:

$$B_t = gt + b_t,$$

where g is a standard Gaussian random variable independent of the Brownian bridge b. Therefore,

$$\mathbb{E}\left[f(X_t^\varepsilon)\right] = \int_{\mathbb{R}} \frac{f(\varepsilon\sqrt{t}y)}{\sqrt{2\pi}} \exp\left\{\frac{t^{(\gamma+1)/2}}{(\gamma+1)\varepsilon^{1-\gamma}}|y|^{\gamma+1} - \frac{y^2}{2}\right\} \mathrm{d}y$$
$$\times \mathbb{E}\left[\exp\left\{-\frac{\gamma t^{(\gamma+1)/2}}{2\varepsilon^{1-\gamma}} \int_0^1 |ys+b_s|^{\gamma-1}\,\mathrm{d}s - \frac{t^{\gamma+1}}{2\varepsilon^{2-2\gamma}} \int_0^1 |ys+b_s|^{2\gamma}\,\mathrm{d}s\right\}\right].$$

By the change of variable $x = \varepsilon\sqrt{t}y$, the above formula becomes

$$\mathbb{E}\left[f(X_t^\varepsilon)\right] = \int_{\mathbb{R}} \frac{f(x)}{\varepsilon\sqrt{2\pi t}} \exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\}$$
$$\times \mathbb{E}\left[\exp\left\{-\frac{\gamma t}{2} \int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{\gamma-1}\,\mathrm{d}s - \frac{t}{2\varepsilon^2} \int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{2\gamma}\,\mathrm{d}s\right\}\right] \mathrm{d}x$$

and we obtain the expression of the density (6). □

Another useful expression of a density is contained in the following:

COROLLARY 1. – *For $t > 0$, $\varepsilon > 0$ and $x \in \mathbb{R}$,*

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\}$$
$$\times \mathbb{E}_{\frac{x}{\varepsilon(s(\varepsilon))^{1/2}}}\left[\exp\left\{-\int_0^{t/s(\varepsilon)} \frac{V(B_s)}{2}\,\mathrm{d}s\right\} \Bigg| B_{\frac{t}{s(\varepsilon)}} = 0\right], \tag{7}$$

where we denoted $s(\varepsilon) := \varepsilon^{(2(1-\gamma))/(1+\gamma)}$ and the potential V is given by:

$$V(x) := \frac{\gamma}{|x|^{1-\gamma}} + |x|^{2\gamma}. \tag{8}$$

Proof. – By conditioning with respect to $\{B_t = x\}$ in (4) we obtain

$$\mathbb{E}\left[f(X_t^\varepsilon)\right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} f(\varepsilon x) \exp\left\{\frac{1}{(\gamma+1)\varepsilon^{1-\gamma}}|x|^{\gamma+1} - \frac{x^2}{2t}\right\} \mathrm{d}x$$
$$\times \mathbb{E}_0\left[\exp -\left\{\frac{\gamma}{2\varepsilon^{1-\gamma}} \int_0^t |B_s|^{\gamma-1}\,\mathrm{d}s - \frac{1}{2\varepsilon^{2-2\gamma}} \int_0^t |B_s|^{2\gamma}\,\mathrm{d}s\right\} \Bigg| B_t = x\right].$$

The functional of the Brownian motion which appears in the integral on the right hand side of the previous equality is time reversal invariant. Therefore we obtain

$$\mathbb{E}_0\left[\exp -\left\{\frac{\gamma}{2\varepsilon^{1-\gamma}} \int_0^t |B_s|^{\gamma-1}\,\mathrm{d}s - \frac{1}{2\varepsilon^{2-2\gamma}} \int_0^t |B_s|^{2\gamma}\,\mathrm{d}s\right\} \Bigg| B_t = x\right]$$

$$= \mathbb{E}_x\left[\exp-\left\{\frac{\gamma}{2\varepsilon^{1-\gamma}}\int_0^t |B_s|^{\gamma-1}\,\mathrm{d}s - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}\,\mathrm{d}s\right\}\middle| B_t = 0\right].$$

By scaling we get (7). □

The following result contains an expansion of the density of X_t^ε in terms of the eigenvalues and the eigenfunctions of a Schrödinger operator. This type of expression was already considered in [10, p. 194] for continuous potentials.

PROPOSITION 5. – *For $t > 0$, $\varepsilon > 0$ and $x \in \mathbb{R}$:*

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}}\exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2}\right\}\sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t/s(\varepsilon)}\psi_j(0)\psi_j\left(\frac{|x|}{\varepsilon s(\varepsilon)^{1/2}}\right), \tag{9}$$

where λ_j and ψ_j are the eigenvalues and the normalized eigenfunctions of the operator on $L^2(\mathbb{R})$:

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}V(x),$$

where V is given by (8). *Moreover the series is uniform convergent for fixed t and x belonging to a compact set of $\mathbb{R}$.*

Proof. – Let us consider the following one-parameter semi-group:

$$(T_t f)(x) := \mathbb{E}_x\left[f(B_t)\exp-\frac{1}{2}\int_0^t V(B_s)\,\mathrm{d}s\right],$$

and we shall denote by $a_t(x, y)$ the density of the semi-group with respect to the Lebesgue measure:

$$a_t(x, y) := \frac{\mathrm{e}^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\mathbb{E}_x\left[\exp-\frac{1}{2}\int_0^t V(B_s)\,\mathrm{d}s\middle| B_t = y\right].$$

Therefore, by (7) we can write

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}}\exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2}\right\}a_{\frac{t}{s(\varepsilon)}}\left(0, \frac{x}{\varepsilon s(\varepsilon)^{1/2}}\right).$$

Let us note that, by the definition, the semi-group T_t preserve the positivity. Moreover, the generator of $T_t = \mathrm{e}^{-Ht}$ is $-H$, with H a positive self-adjoint operator. Indeed, this second property is true for self-adjoint contraction semi-groups (see, [5, Theorem 4.6, p. 99]) and we can prove that

$$\|T_t\|_{L^2(\mathbb{R}),L^2(\mathbb{R})} \leqslant \sup_{x\in\mathbb{R}}\mathbb{E}_x\left[\exp-\frac{1}{2}\int_0^t V(B_s)\,\mathrm{d}s\right] \leqslant 1,$$

since $V(\cdot) \geqslant 0$ (see, for instance, [3, p. 271]).

It can be shown (see [6, Lemma 2.1, p. 339]) that the density of a semi-group satisfying the previous properties and which is a trace class operator, can be developed as

$$a_t(x, y) = \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda_j t} \psi_j(x)\psi_j(y).$$

Here the λ_j and ψ_j are the eigenvalues and the normalized eigenfunctions of the discrete spectrum of the equation

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}V(x)\psi(x) = \lambda\psi(x).$$

Moreover the convergence of the series is uniform over all compact sets of $\mathbb{R} \times \mathbb{R}$.

To obtain the result (9) we shall prove that T_t is a trace class operator. Clearly,

$$a_t(x, x) \leqslant \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left[\exp -\frac{1}{2}\int_0^t \left|x + b_s^{0,t}\right|^{2\gamma} \mathrm{d}s\right] =: \tilde{a}_t(x, x), \tag{10}$$

where $b_s^{0,t}$ is the Brownian bridge from 0 to 0 over $[0, t]$ (thus the standard Brownian bridge is $b_s = b_s^{0,1}$) and $\tilde{a}_t(x, y)$ is the density of the semi-group generated by the Schrödinger operator

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\widetilde{V}(x) \quad \text{with } \widetilde{V}(x) := |x|^{2\gamma}.$$

Since $\widetilde{V} \in L^2_{\mathrm{loc}}(\mathbb{R})$ and $\lim_{x\to\infty} \widetilde{V}(x)/|x|^\gamma = +\infty$ we can deduce that this operator is a class trace operator (see also [4, Theorem 3.2, p. 488]). By Mercer's theorem (see, for instance, [12, p. 65]), we get

$$\int_{\mathbb{R}} \tilde{a}_t(x, x)\,\mathrm{d}x < \infty,$$

and then, by (10),

$$\int_{\mathbb{R}} a_t(x, x)\,\mathrm{d}x < \infty.$$

Again by Mercer's theorem, we deduce that T_t is a trace class operator. □

Remark 2. – In the particular case $\gamma = 1/2$ we can find, by straightforward calculation, an equivalent of $\psi_j(x)$:

$$\psi_j(x) \sim \exp\left\{-\frac{2}{3}x^{3/2} + 2\sqrt{x}\lambda_j\right\}, \quad \text{as } x \to \infty.$$

This enables to think that $\varepsilon^{2/3} \ln p_t^\varepsilon(x)$ tends to $\lambda_1(2\sqrt{|x|} - t)$, if (t, x) lies between the extremal solutions $\rho_{1,2}(t) = \pm t^2/4$ (here $s(\varepsilon) = \varepsilon^{2/3}$).

The second part of this remark can be proved in the following simple case $x = \varepsilon s(\varepsilon)^{1/2}$ but for any $0 < \gamma < 1$:

COROLLARY 2. – *For $t > 0$, $0 < \gamma < 1$,*

$$\lim_{\varepsilon\to 0} s(\varepsilon) \ln p_t^\varepsilon\big(\varepsilon s(\varepsilon)^{1/2}\big) = -\lambda_1 t. \tag{11}$$

Moreover, the convergence is uniform on any compact subset of $\mathbb{R}_+^*$.

Proof. – By (9) we get

$$p_t^\varepsilon\big(\varepsilon s(\varepsilon)^{1/2}\big) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}} \exp\left\{\frac{1}{(\gamma+1)}\right\} \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t/s(\varepsilon)} \psi_j(0)\psi_j(1).$$

Since V is bounded from below, there exists a constant $K > 0$ such that

$$\text{for all } j \geqslant 1, \quad \|\psi_j\|_\infty \leqslant K \|\psi_j\|_2$$

(see [4, Lemma 3.1, p. 488]). Therefore, by classical convergence theorems,

$$p_t^\varepsilon\big(\varepsilon s(\varepsilon)^{1/2}\big) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}} \exp\left\{\frac{1}{(\gamma+1)} - \frac{\lambda_1 t}{s(\varepsilon)}\right\} \big(\psi_1(0)\psi_1(1) + \mathrm{o}(s(\varepsilon))\big)$$

and we obtain the announced result. It is not difficult to modify this proof to obtain the uniform convergence. □

2. Convergence of $\varepsilon^2 \ln p_t^\varepsilon(x)$

The purpose of this section is to study the behaviour of $\varepsilon^2 \ln p_t^\varepsilon(x)$. The result will be sharp if (t, x) does not lie between the two extremal solutions of (3).

THEOREM 1. – *If $|x| > \{t(1-\gamma)\}^{1/1-\gamma}$ then there exists a positive function k_t such that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) = -k_t(|x|). \tag{12}$$

Remark 3. – We also prove that if $|x| \leqslant \{t(1-\gamma)\}^{1/1-\gamma}$ then

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \leqslant 0, \tag{13}$$

but this result will be improved in Section 3.

Proof of Theorem 1. – Clearly, by (6) we can write

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\} \mathbb{E}\left[\exp\left(-F(\varepsilon b) - \frac{G(\varepsilon b)}{\varepsilon^2}\right)\right],$$

where

$$F(\varepsilon b) = \frac{\gamma t}{2} \int_0^1 \left| xu - \sqrt{t}\varepsilon b_u \right|^{\gamma - 1} \mathrm{d}u$$

and

$$G(\varepsilon b) = \frac{t}{2} \int_0^1 \left| xu - \sqrt{t}\varepsilon b_u \right|^{2\gamma} \mathrm{d}u.$$

(i) (*An upper bound for* $\limsup_{\varepsilon \to 0} \varepsilon^2 \ln p_t^\varepsilon(x)$.)
We have

$$p_t^\varepsilon(x) \leqslant \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{ \frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t} \right\} \mathbb{E}\left[\exp - \frac{G(\varepsilon b)}{\varepsilon^2} \right] =: r_t^\varepsilon(x).$$

G is a continuous lower bounded functional of the Brownian bridge. Therefore, to study $r_t^\varepsilon(x)$ we use the Varadhan principle (see, for instance [7, p. 43]). Hence, applying the logarithm we obtain

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \leqslant \frac{|x|^{\gamma+1}}{\gamma+1} - \frac{x^2}{2t} - \frac{1}{2} \inf_{\phi \in H_0^1} A(\phi),$$

where, for $\phi \in H_0^1$,

$$A(\phi) := t \int_0^1 \left| xu - \sqrt{t}\phi(u) \right|^{2\gamma} \mathrm{d}u + \int_0^1 \phi'^2(u)\,\mathrm{d}u. \tag{14}$$

Here

$$H_0^1 := \left\{ \phi(t) = \int_0^t f(s)\,\mathrm{d}s\colon\ f \in L^2([0,1]) \text{ and } \phi(1) = 0 \right\},$$

endowed with the norm

$$\|\phi\|_{H_0^1} := \left(\int_0^1 |\phi'(s)|^2\,\mathrm{d}s \right)^{1/2}.$$

We compute the infimum of the functional A in the following:

PROPOSITION 6. – *There exists a positive function k_t such that*

$$\inf_{\phi \in H_0^1} A(\phi) = \begin{cases} \frac{2|x|^{1+\gamma}}{\gamma+1} - \frac{x^2}{t} & \text{if } |x| \leqslant \{t(1-\gamma)\}^{1/(1-\gamma)}, \\ \frac{2|x|^{1+\gamma}}{\gamma+1} - \frac{x^2}{t} + 2k_t(|x|) & \text{otherwise.} \end{cases} \tag{15}$$

We can finish the proof of the theorem and we postpone the proof of Proposition 6. Using (15) we deduce (13) and

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \leqslant -k_t(|x|).$$

(ii) (*A lower bound for* $\liminf_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x)$.)

Let us just note that F explodes when (t,x) lies between the extremals. In the following we assume that $x > \{t(1-\gamma)\}^{1/1-\gamma}$. Let us denote $\kappa := \frac{1}{2}(x - \sqrt{t}\phi_0'(0)) > 0$. Here ϕ_0 is the function which minimizes the functional A (see the proof of Proposition 6 below). It results from the proof of Proposition 6 below that ϕ_0 belongs to the following open set

$$\mathcal{U} := \left\{\phi \in \mathcal{C}([0,1]):\ xu - \sqrt{t}\phi(u) > \kappa u,\ \forall u \in [0,1]\right\}.$$

Moreover, there exists $\eta > 0$ such that

$$\max_{\phi\in\mathcal{U}} F(\phi) \leqslant \eta.$$

Take $\delta > 0$ and let $\mathcal{V}$ be a neighbourhood of ϕ_0 such that

$$\max_{\phi\in\mathcal{V}} G(\phi) \leqslant G(\phi_0) + \delta.$$

Let us denote $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$. Then we can write

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon^2 \ln \mathbb{E}\left[\exp -\left(F(\varepsilon b) + \frac{G(\varepsilon b)}{\varepsilon^2}\right)\right] \\
&\quad \geqslant \liminf_{\varepsilon\to 0} \varepsilon^2 \ln \mathbb{E}\left[\exp -\left(F(\varepsilon b) + \frac{G(\varepsilon b)}{\varepsilon^2}\right)\mathbb{1}_{\{\varepsilon b\in\mathcal{W}\}}\right] \\
&\quad \geqslant \liminf_{\varepsilon\to 0} \varepsilon^2 \ln \mathbb{P}(\varepsilon b \in \mathcal{W}) - \lim_{\varepsilon\to 0} \varepsilon^2\eta - \max_{\phi\in\mathcal{W}} G(\phi).
\end{aligned}$$

By Schilder's theorem (see for instance [7, p. 18]), we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon^2 \ln \mathbb{E}\left[\exp -\left(F(\varepsilon b) + \frac{G(\varepsilon b)}{\varepsilon^2}\right)\right] \\
&\quad \geqslant -\inf_{\phi\in\mathcal{W}\cap H_0^1} \frac{1}{2}\int_0^1 |\phi'(u)|^2\,\mathrm{d}u - G(\phi_0) - \delta \\
&\quad \geqslant -\frac{1}{2}A(\phi_0) - \delta.
\end{aligned}$$

Letting $\delta \to 0$ we get

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \geqslant \frac{|x|^{\gamma+1}}{\gamma+1} - \frac{x^2}{2t} - \frac{1}{2}\inf_{\phi\in H_0^1} A(\phi).$$

By (15) we obtain the limit (13). This ends the proof of Theorem 1 except for the proof of Proposition 6. □

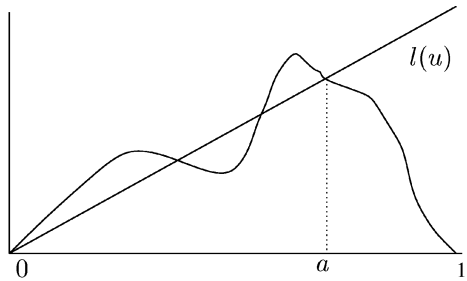

Fig. 2. Description of a.

Proof of Proposition 6. – First, we can assume that $x \geqslant 0$. Indeed, if $x \leqslant 0$ it suffices to replace in (6) b_u by $-b_u$ which are identical in law, to obtain the result.

(i) Let $\phi \in H_0^1$ and let us denote (see also Fig. 2)

$$a = \sup\left\{0 \leqslant u \leqslant 1\colon\ \phi(u) = \frac{xu}{\sqrt{t}}\right\}.$$

It is obvious that on [0,1], the straight line $l(s) := xs/\sqrt{t}$ minimizes the functional

$$\phi \mapsto \int_0^1 \left|xs - \sqrt{t}\phi(s)\right|^{2\gamma} \mathrm{d}s.$$

Moreover

$$\int_0^a \phi'^2(u)\,\mathrm{d}u \geqslant \frac{ax^2}{t} = \int_0^a \left(l'(u)\right)^2 \mathrm{d}u, \quad \forall \phi \in H_0^1.$$

Indeed

$$\frac{ax}{\sqrt{t}} = \phi(a) = \left|\int_0^a \phi'(u)\,\mathrm{d}u\right| \leqslant \sqrt{a}\left(\int_0^a \phi'^2(u)\,\mathrm{d}u\right)^{1/2}.$$

(ii) We show that there exists $\phi_0 \in H_0^1$ such that $A(\phi_0) = \inf A(\phi)$. Take a minimizing sequence ϕ_n of A. Since this sequence is bounded in H_0^1 there exists a subsequence, still denoted by ϕ_n, weakly convergent to some ϕ_0. This implies pointwise convergence of ϕ_n to ϕ_0, and by Lebesgue theorem, convergence of the first part of $A(\phi_n)$ to the first part of $A(\phi_0)$. As a byproduct one gets convergence of the L^2 norm of ϕ_n' to the one of ϕ_0' and combined with weak convergence it yields strong convergence. Hence $A(\phi_n)$ goes to $A(\phi_0)$ which, in turn, realizes the infimum. By (i) we see that on $[0,a]$, $\phi_0 = l$. Let us notice also that

$$\phi_0(u) < l(u) \quad \text{for all } u \in \,]a,1]. \tag{16}$$

For any $h \in H_0^1$ compactly supported in $]a,1]$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} A(\phi_0 + \lambda h)\bigg|_{\lambda=0} = 0$$

(this differentiation is allowed since, for $u \in]a,1]$, $xu - \sqrt{t}\phi_0(u) > 0$). By (14) and (16) we obtain

$$\int_0^1 \gamma t^{3/2} \left|xu - \sqrt{t}\phi_0(u)\right|^{2\gamma-1} h(u)\,\mathrm{d}u - \int_0^1 \phi_0'(u)h'(u)\,\mathrm{d}u = 0. \tag{17}$$

Let us denote $y(u) := xu - \sqrt{t}\phi_0(u) > 0$. Then from (17) we obtain that y verifies the differential equation: $y''(u) = \gamma t^2 y^{2\gamma-1}(u)$ in a weak sense on $]a,1]$, with $y(a) = 0$ and $y(1) = x$ (thanks to continuity of y). We deduce that y verifies in a weak sense

$$\frac{\mathrm{d}(y')^2}{\mathrm{d}u} = 2y'y'' = 2y'\left(\gamma t^2 y^{2\gamma-1}\right) = 2\gamma t^2 y^{2\gamma-1} y'.$$

Therefore, for all $\varepsilon > 0$,

$$\begin{aligned}\left(y'(u)\right)^2 &= \left(y'(a+\varepsilon)\right)^2 + 2\gamma t^2 \int_{a+\varepsilon}^{u} y(x)^{2\gamma-1} y'(x)\,\mathrm{d}x \\ &= \left(y'(a+\varepsilon)\right)^2 + t^2 y(u)^{2\gamma} - t^2 y(a+\varepsilon)^{2\gamma}. \end{aligned}\tag{18}$$

This equality implies that y' can be extended as a continuous function on the whole $[a,1]$.

(iii) We shall prove that, for $a > 0$, y satisfies:

$$u = a + \frac{y(u)^{1-\gamma}}{t(1-\gamma)} \quad \text{for all } u \in [a,1]. \tag{19}$$

We need to compute $y'(a+)$. Let us suppose that $y'(a+) > 0$ then $|y(u)|^{2\gamma-1}$ is integrable in a neighborhood of a and formula (17) extends to any h. Now, this implies that the second derivative of y is a function. Since $y'(a-) = 0$, this contradicts $y'(a+) > 0$. Hence $y'(a+) = 0$ and we obtain by (18)

$$y'(u)^2 = t^2 y(u)^{2\gamma}$$

or,

$$u = a + \int_{y(a)}^{y(u)} \frac{\mathrm{d}x}{tx^{\gamma}} = a + \frac{y(u)^{1-\gamma}}{t(1-\gamma)}.$$

Finally take $u = 1$, since $y(1) = x$, we get

$$a = 1 - \frac{x^{1-\gamma}}{t(1-\gamma)}, \tag{20}$$

and the condition $a > 0$ can be written as

$$x < \{t(1-\gamma)\}^{1/(1-\gamma)},$$

namely (t,x) lies between the two extremals.

(iv) We need to compute the minimum of A

$$\begin{aligned}\inf A(\phi) &= A(\phi_0) = A\bigl(y(\cdot)/\sqrt{t} - l(\cdot)\bigr)\\ &= t\int_0^1 \bigl|xu - \sqrt{t}\phi_0(u)\bigr|^{2\gamma}\,\mathrm{d}u + \int_0^1 \phi_0'^2(u)\,\mathrm{d}u\\ &= \frac{x^2a}{t} + t\int_a^1 \bigl|xu - \sqrt{t}\phi_0(u)\bigr|^{2\gamma}\,\mathrm{d}u + \int_a^1 \phi_0'^2(u)\,\mathrm{d}u,\end{aligned}$$

since $\phi_0(u) = l(u)$ on $[0,a]$, or

$$\begin{aligned}A(\phi_0) &= \frac{x^2a}{t} + t\int_a^1 y(u)^{2\gamma}\,\mathrm{d}u + \int_a^1 \frac{(x-y'(u))^2}{t}\,\mathrm{d}u\\ &= \frac{x^2a}{t} + t\int_a^1 y(u)^{2\gamma}\,\mathrm{d}u + \frac{x^2(1-a)}{t} - \frac{2x}{t}\int_a^1 y'(u)\,\mathrm{d}u + \frac{1}{t}\int_a^1 y'^2(u)\,\mathrm{d}u.\end{aligned}$$

By (19) we obtain:

$$A(\phi_0) = \frac{x^2}{t} + 2t\int_a^1 \bigl\{t(1-\gamma)(u-a)\bigr\}^{\frac{2\gamma}{1-\gamma}}\,\mathrm{d}u - 2x\int_a^1 \bigl\{t(1-\gamma)(u-a)\bigr\}^{\frac{\gamma}{1-\gamma}}\,\mathrm{d}u,$$

which can be written, by change of variable $v = t(1-\gamma)(u-a)$ and by (20), as

$$A(\phi_0) = \frac{x^2}{t} + \frac{2}{1-\gamma}\int_0^{x^{1-\gamma}} v^{\frac{2\gamma}{1-\gamma}}\,\mathrm{d}v - \frac{2x}{t(1-\gamma)}\int_0^{x^{1-\gamma}} v^{\frac{\gamma}{1-\gamma}}\,\mathrm{d}v.$$

Then we get the first part of (15) by straightforward calculation.

(v) Assume now $a = 0$ which means, by (iii), that:

$$x \geqslant \bigl(t(1-\gamma)\bigr)^{1/(1-\gamma)}.$$

As in (iii), the solution of the problem (18) satisfies

$$y'(u)^2 = t^2 y(u)^{2\gamma} + y'(0)^2. \tag{21}$$

However in this case $(a = 0)$ we have not the explicit value of $y'(0)$, as in (iii).

(vi) We need to compute the minimum of A

$$\inf A(\phi) = A(\phi_0) = t\int_0^1 y(u)^{2\gamma}\,\mathrm{d}u + \int_0^1 \frac{(x-y'(u))^2}{t}\,\mathrm{d}u$$

$$= \frac{2}{t}\int_0^1 y'(u)^2\,\mathrm{d}u - \frac{y'(0)^2}{t} - \frac{x^2}{t}.$$

Since y is positive on $]0,1]$, y' does not vanish thanks to the differential equation (21) thus is positive. Therefore it is allowed to apply the following change of variable

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{\sqrt{y'(0)^2 + t^2 y^{2\gamma}}}, \tag{22}$$

and we get

$$A(\phi_0) = \frac{2}{t}\int_0^x \sqrt{y'(0)^2 + t^2 y^{2\gamma}}\,\mathrm{d}y - \frac{y'(0)^2}{t} - \frac{x^2}{t}.$$

By straightforward calculation we obtain

$$A(\phi_0) = \frac{2x\sqrt{y'(0)^2 + t^2 x^{2\gamma}}}{(1+\gamma)t} + \frac{(\gamma-1)y'(0)^2}{(1+\gamma)t} - \frac{x^2}{t} = \frac{2x^{\gamma+1}}{1+\gamma} - \frac{x^2}{t} + 2k_t(x),$$

where

$$t(1+\gamma)k_t(x) := x\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - tx^{1+\gamma} + \frac{\gamma-1}{2}y'(0)^2.$$

Let us prove that $k_t(x) > 0$. By the change of variable (22), we get

$$1 = \int_0^x \frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y = \int_0^x \frac{\mathrm{d}y}{\sqrt{y'(0)^2 + t^2 y^{2\gamma}}}.$$

Therefore as a function of x, $y'(0)$ is continuous, strictly increasing and differentiable for $x \geqslant \{t(1-\gamma)\}^{1/1-\gamma}$. Moreover the derivative with respect to x of $y'(0)$ is equal to

$$\frac{\gamma y'(0)}{x + (\gamma-1)\sqrt{y'(0)^2 + t^2 x^{2\gamma}}} \geqslant 0.$$

Therefore we can compute $k_t'(x)$ for $x > \{t(1-\gamma)\}^{1/1-\gamma}$:

$$k_t'(x) = \frac{1}{t}\Big(\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - tx^{\gamma}\Big) > 0, \quad \text{since } y'(0) > 0.$$

Observe that $k_t(\{t(1-\gamma)\}^{1/1-\gamma}) = 0$ by (iv) and that $k_t'(x)$ is positive for $x > \{t(1-\gamma)\}^{1/1-\gamma}$, so $k_t(x) > 0$. This ends the proof of the second part of (15). □

Remark 4. – Using a probabilistic method (see [9]), we can obtain an upper bound in the particular case $\gamma = 1/2$. Precisely we can prove that, for $|x| \leqslant t^2/4$,

$$\limsup_{\varepsilon\to 0} \varepsilon^{2/3}\ln p_t^{\varepsilon}(x) \leqslant a_1'\big(t/2 - \sqrt{|x|}\big),$$

where a_1' is the greater negative zero of the derivative of the Airy function Ai. In the proof of this upper bound we use the following result concerning a functional of the standard Brownian bridge $\{b_u, u \in [0,1]\}$, which can be interesting in itself. For $0 \leqslant a < 1$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2/3} \ln \mathbb{E}\left[\exp -\frac{1}{\varepsilon}\int_0^a |b_u|\,\mathrm{d}u\right] = \frac{a_1' a}{2^{1/3}}$$

(see also [16], [14]). The improvement of the upper bound in the general case will be presented in the following section.

3. Viscosity solution of a Hamilton–Jacobi equation

In Theorem 1 we obtained the behaviour of $p_t^\varepsilon(x)$, if (t,x) does not lie between the extremals. The aim of this section is to study the behaviour for (t,x) lying between the extremals, namely we study $s(\varepsilon)\ln p_t^\varepsilon(x)$, with $s(\varepsilon) = \varepsilon^{(2(1-\gamma))/(1+\gamma)}$.

THEOREM 2. – *If (t,x) belongs to the domain contained between the extremal solutions of* (3), *then*

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^\varepsilon(x) = -\lambda_1\left(t - \frac{|x|^{1-\gamma}}{1-\gamma}\right). \tag{23}$$

Here λ_1 is the first positive eigenvalue of the Schrödinger operator:

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\gamma}{2|x|^{1-\gamma}} + \frac{|x|^{2\gamma}}{2}.$$

Our study is based on a particular tool: the viscosity solutions of parabolic partial differential equations. For a study of these solutions the reader may consult the book of Barles [2] or the one of Fleming [8].

First we shall introduce some domains of the first quadrant plane:

$$\Omega := \left\{(t,x)\colon\ 0 < t < T,\ 0 < x < \{(1-\gamma)t\}^{1/1-\gamma}\right\},$$

$$\widetilde{\Omega} := \left\{(t,x)\colon\ 0 < t \leqslant T,\ 0 < x < \{(1-\gamma)t\}^{1/1-\gamma}\right\},$$

$$\widehat{\Omega} := \left\{(t,x)\colon\ 0 < t \leqslant T,\ 0 \leqslant x < \{(1-\gamma)t\}^{1/1-\gamma}\right\},$$

$$\Omega^\varepsilon := \left\{(t,x)\colon\ (1-\gamma)\varepsilon^{4/(1+\gamma)} < t < T,\ \varepsilon s(\varepsilon)^{1/2} < x < \{(1-\gamma)t\}^{1/1-\gamma}\right\},$$

$$\widetilde{\Omega}^\varepsilon := \left\{(t,x)\colon\ (1-\gamma)\varepsilon^{4/(1+\gamma)} < t \leqslant T,\ \varepsilon s(\varepsilon)^{1/2} < x < \{(1-\gamma)t\}^{1/1-\gamma}\right\},$$

$$\widehat{\Omega}^\varepsilon := \left\{(t,x)\colon\ (1-\gamma)\varepsilon^{4/(1+\gamma)} < t \leqslant T,\ \varepsilon s(\varepsilon)^{1/2} \leqslant x < \{(1-\gamma)t\}^{1/1-\gamma}\right\}.$$

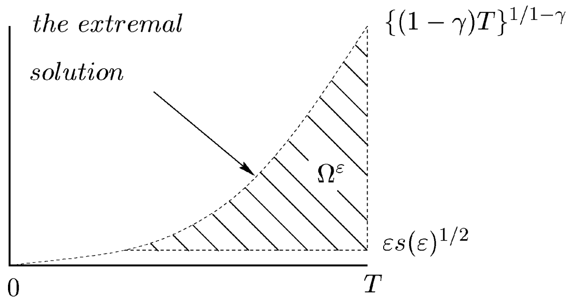

Fig. 3. The domain Ω^ε.

Let us consider the following parabolic partial differential equation in $U \subset \mathbb{R}^2$ (we shall precise U below):

$$\frac{\partial u}{\partial t} + H\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{24}$$

where H is a real Hamiltonian defined on $\mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}$. We assume that H is elliptic in the following sense:

$$H(t, x, u, p, q_1) \leqslant H(t, x, u, p, q_2), \quad \text{if } q_2 \leqslant q_1.$$

We recall the notion of viscosity solution for (24) and we need a slightly different definition than the one in [2] (see Definition 2.1, p. 11 or Definition 4.1, p. 80), since the domains which we consider are not open nor closed.

Definition 1. – Let u be a bounded upper semi-continuous (u.s.c.) (respectively lower semi-continuous (l.s.c.)) function on a connected set U with connected boundary. u is a viscosity sub-solution (respectively super-solution) of (24) on U, if for all $\varphi \in \mathcal{C}^2(U)$, whenever $(t_0, x_0) \in U$ is a point of local maximum (local minimum) of $u - \varphi$, then

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H\left(t_0, x_0, u(t_0, x_0), \frac{\partial \varphi}{\partial x}(t_0, x_0), \frac{\partial^2 \varphi}{\partial x^2}(t_0, x_0)\right) \leqslant 0 \text{ (respectively } \geqslant 0). \tag{25}$$

PROPOSITION 7. – *Let us define*

$$u^\varepsilon(t, x) := -s(\varepsilon) \ln\left(p_t^\varepsilon(x) + \mathrm{e}^{-D/s(\varepsilon)}\right), \tag{26}$$

where $D > 0$. Then u^ε is a viscosity solution of

$$\frac{\partial u^\varepsilon}{\partial t} + H_\varepsilon\left(t, x, u^\varepsilon, \frac{\partial u^\varepsilon}{\partial x}, \frac{\partial^2 u^\varepsilon}{\partial x^2}\right) = 0 \quad in\ \widetilde{\Omega}^\varepsilon, \tag{27}$$

corresponding to the Hamiltonian:

$$H_\varepsilon(t, x, u, p, q) := -\frac{\varepsilon^2}{2} q + \frac{\varepsilon^{\frac{4\gamma}{1+\gamma}}}{2} p^2 + x^\gamma p - \gamma x^{\gamma-1} s(\varepsilon)\left(1 - \exp\frac{u-D}{s(\varepsilon)}\right). \tag{28}$$

Remark 5. – The reason to introduce the exponential term, with $D > 0$, in the definition of u^ε is that this last function is bounded. Clearly, by choosing D large enough, this term in the logarithm scale will not change the limit as $\varepsilon \to 0$.

Proof of Proposition 7. –

(a) First, we shall prove that Eq. (27) is verified on Ω^ε in classical sense.

Since V, the potential given by (8) of the Schrödinger operator in the statement of Theorem 3, is uniformly Hölder continuous on a neighbourhood of any $x \neq 0$ (see [15, Definition 2 p. 122]), by Theorem 1 p. 127 in [15] we deduce that the function

$$(t,x) \mapsto \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{E}\left[\exp -\frac{1}{2}\int_0^t V(B_s)\,\mathrm{d}s \,\middle|\, B_t = x\right]$$

is a classical solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2} V u \quad \text{on }]0,T] \times \mathbb{R}^*.$$

Thus, by similar arguments, using (7) we obtain that $p^\varepsilon \in \mathcal{C}^{1,2}(\Omega^\varepsilon)$. By logarithmic transform, we get that u^ε is a classical solution of

$$\frac{\partial u}{\partial t} + H_\varepsilon\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0 \quad \text{on } \Omega^\varepsilon,$$

where H_ε is given by (28).

(b) Moreover all classical solutions are viscosity solutions, hence u^ε is a viscosity solution on Ω^ε. It suffices to verify that u^ε is a viscosity solution on $\widetilde{\Omega}^\varepsilon \setminus \Omega^\varepsilon$. Take now $\varphi \in \mathcal{C}^2(\widetilde{\Omega}^\varepsilon)$ such that $(T, x_0) \in \widetilde{\Omega}^\varepsilon$ is a local maximum of $u^\varepsilon - \varphi$. Replacing φ by $\varphi + (x - x_0)^4 + (t - T)^2$ the first and the second derivative at (T, x_0) do not change, and so we can assume that (T, x_0) is a point of local strict maximum. The idea is to adapt the reasoning for the points of Ω^ε to the point (T, x_0). To do this, we need the following:

LEMMA 1. – *Let $(u_\eta)_\eta$ be a sequence of u.s.c. functions which converges towards u, uniformly over all compact subsets of a bounded set U. We suppose that u can be extended to an u.s.c. function on $\bar{U}$. If (τ, ξ) is a local strict maximum of u then there exists $(\tau_\eta, \xi_\eta) \in \bar{U}$ which is a point of local maximum of u_η such that* $\lim_{\eta \to 0}(\tau_\eta, \xi_\eta) = (\tau, \xi)$.

We can finish the proof of Proposition 8. Let us consider the function:

$$\Xi_\eta(t,x) := u^\varepsilon(t,x) - \varphi(t,x) - \frac{\eta}{T-t}, \quad \eta > 0.$$

By Lemma 1 applied on Ω^ε, there exists a sequence $(t_\eta, x_\eta) \in \widetilde{\Omega}^\varepsilon$ of local maxima of Ξ_η which converges to (T, x_0), as $\eta \to 0$. Clearly, $\lim_{t \to T} \Xi_\eta(t,x) = -\infty$. Hence $t_\eta < T$

and for η small enough $(t_\eta, x_\eta) \in \Omega^\varepsilon$. Since u^ε is a viscosity sub-solution on Ω^ε we get:

$$\frac{\eta}{(T-t_\eta)^2} + \frac{\partial\varphi}{\partial t}(t_\eta, x_\eta) + H_\varepsilon\left(t_\eta, x_\eta, u^\varepsilon(t_\eta, x_\eta), \frac{\partial\varphi}{\partial x}(t_\eta, x_\eta), \frac{\partial^2\varphi}{\partial x^2}(t_\eta, x_\eta)\right) \leqslant 0.$$

By the continuity of u^ε, letting $\eta \to 0$ we obtain

$$\frac{\partial\varphi}{\partial t}(T, x_0) + H_\varepsilon\left(T, x_0, u^\varepsilon(T, x_0), \frac{\partial\varphi}{\partial x}(T, x_0), \frac{\partial^2\varphi}{\partial x^2}(T, x_0)\right) \leqslant 0.$$

The same argument can be used to prove that u^ε is a super-solution. This ends the proof of Proposition 8 except for the proof of Lemma 1. □

Proof of Lemma 1. – The result is clear for $(\tau, \xi) \in U$ (see [2, Lemma 4.2 p. 88]). Let us suppose that $(\tau, \xi) \in \partial U$. Take $r > 0$ and we define the compact set $K^r = \overline{B((\tau, \xi), r)} \cap \bar{U}$, where B is an Euclidean ball centred in (τ, ξ) with radius r such that (τ, ξ) is a global strict maximum on K^r. The u.s.c. function u_η reaches its maximum on the compact set K^r at (τ_η, ξ_η). We extract a sub-sequence, denoted for simplicity again by (τ_η, ξ_η), which converges to $(\bar{\tau}, \bar{\xi})$, as $\eta \to 0$. Assume that $(\bar{\tau}, \bar{\xi}) \notin \partial U$. Since u is u.s.c. and since (τ, ξ) is a strict maximum, there exists $(t, y) \in K^r$ such that

$$u(\tau, \xi) > u(t, y) > u(\bar{\tau}, \bar{\xi}).$$

This inequality can not be true! Indeed, $u_\eta(\tau_\eta, \xi_\eta)$ tends to $u(\bar{\tau}, \bar{\xi})$ and $u_\eta(t, y)$ tends to $u(t, y)$, these two convergences being uniform. Hence, $(\bar{\tau}, \bar{\xi}) \in \partial U$. Moreover, we know that $\|(\bar{\tau}, \bar{\xi}) - (\tau, \xi)\| \leqslant r$. We can choose a sequence $(\bar{\tau}^r, \bar{\xi}^r)$ which tends to (τ, ξ), as $r \to 0$. By diagonalization, we can find a sequence $(\tau_\eta, \xi_\eta) \in \bar{U}$ which converges to (τ, ξ), as $\eta \to 0$. □

Our aim is to take the limit as $\varepsilon \to 0$ in the Hamilton–Jacobi equation (27). We prove the following stability result:

PROPOSITION 8. – *Let us denote*

$$\bar{u}(t, x) := \limsup_{\varepsilon \to 0, s \to t, y \to x, (s,y) \in \widehat{\Omega}^\varepsilon} u^\varepsilon(s, y), \quad \textit{for all } (t, x) \in \widehat{\Omega}. \tag{29}$$

Then $\bar{u}$ is a viscosity sub-solution of the equation

$$\frac{\partial u}{\partial t} + H_0\left(x, \frac{\partial u}{\partial x}\right) = 0 \quad \textit{on } \widetilde{\Omega}, \tag{30}$$

with the Hamiltonian

$$H_0(x, p) := x^\gamma p. \tag{31}$$

If we denote $\underline{u} = \liminf u^\varepsilon$, with a limit taken as previously, then $\underline{u}$ is a viscosity super-solution of (30).

The proof of this result is similar to that of Theorem 4.1 p. 85 in [2], except for the fact that the stability result is stated on a closed set. Here we only need the following:

LEMMA 2. – *Let* $(v_\varepsilon)_\varepsilon$ *be a sequence of u.s.c. functions having a local uniform bound on* $\widehat{\Omega}^\varepsilon$. *Let us denote by* $\bar{v} = \limsup v_\varepsilon$ *as in* (29). *We assume that* $\bar{v}$ *has a local strict maximum on* $\widehat{\Omega}$. *Then, there exists a sub-sequence* $(v_{\varepsilon'})_{\varepsilon'}$ *of* $(v_\varepsilon)_\varepsilon$ *and a sequence* $(r_{\varepsilon'}, z_{\varepsilon'})_{\varepsilon'} \in \widehat{\Omega}^\varepsilon$ *such that*: *for all* $\varepsilon' > 0$, $v_{\varepsilon'}$ *reaches a local maximum on* $\widehat{\Omega}^\varepsilon$ *at* $(r_{\varepsilon'}, z_{\varepsilon'})$ *and*

$$\lim_{\varepsilon'\to 0}(r_{\varepsilon'}, z_{\varepsilon'}) = (r,z), \qquad \lim_{\varepsilon'\to 0} v_{\varepsilon'}(r_{\varepsilon'}, z_{\varepsilon'}) = \bar{v}(r,z).$$

The proof of this lemma is similar to the proof of Lemma 4.2, p. 88 in [2].

We also prove an uniqueness result contained in the following:

PROPOSITION 9. – *For all* $(t,x) \in \widehat{\Omega}$,

$$\bar{u}(t,x) = \underline{u}(t,x). \tag{32}$$

Proof of Proposition 9. – (i) First, we prove (32) for $(t,x) \in]0,T] \times \{0\}$. Take $\varphi \in \mathcal{C}^2(\widehat{\Omega})$ such that $(t_0, 0)$ is a local maximum of $\bar{u} - \varphi$. By Lemma 2, there exists a sequence of points $(t_\varepsilon, x_\varepsilon)$ of local maxima of u^ε on $\widehat{\Omega^\varepsilon}$ such that:

$$\lim_{\varepsilon\to 0}(t_\varepsilon, x_\varepsilon) = (t_0, 0).$$

We take a sub-sequence if necessary and we study then two different situations:

(a) either $(t_\varepsilon, x_\varepsilon) \in \widetilde{\Omega}^\varepsilon$ and taking the limit as $\varepsilon \to 0$ in Eq. (27) we get

$$\frac{\partial\varphi}{\partial t} - 1 \leqslant 0; \tag{33}$$

(b) or $(t_\varepsilon, x_\varepsilon) \in \widehat{\Omega^\varepsilon} \setminus \widetilde{\Omega}^\varepsilon$. Since, for D large enough,

$$u^\varepsilon(t_\varepsilon, x_\varepsilon) = -s(\varepsilon)\ln\bigl(p^\varepsilon_{t_\varepsilon}\bigl(\varepsilon s(\varepsilon)^{1/2}\bigr) + \mathrm{e}^{-D/s(\varepsilon)}\bigr)$$

tends to $\lambda_1 t_0$ as $\varepsilon \to 0$ (see Corollary 2), we get $\bar{u}(t_0, 0) = \lambda_1 t_0$.

Take a particular function φ_0 which does not verify (33):

$$\bar{u}(t,x) - \varphi_0(t,x) := \bar{u}(t,x) - \frac{x}{\eta} - \eta^2\cosh\left(\frac{t-t_0}{\eta^2}\right) - \frac{t-t_0}{\eta}. \tag{34}$$

Denote by (t_η, x_η) the point of maximum of $\bar{u} - \varphi_0$ on $\widehat{\Omega}$ and we shall prove that $(t_\eta, x_\eta) \notin \widetilde{\Omega}$. Clearly

$$(\bar{u} - \varphi_0)(t_\eta, x_\eta) \geqslant (\bar{u} - \varphi_0)(t_0, 0) = \bar{u}(t_0, 0) - \eta^2.$$

Since $\bar{u}$ is bounded, we obtain, by (34) $\lim_{\eta\to 0} x_\eta = 0$, $\lim_{\eta\to 0} t_\eta = t_0$ and $\lim_{\eta\to 0}\frac{t_\eta - t_0}{\eta} = 0$. Moreover since $\bar{u}$ is a viscosity sub-solution, if $(t_\eta, x_\eta) \in \widetilde{\Omega}$, we get

$$\frac{\partial\varphi_0}{\partial t}(t_\eta, x_\eta) + x_\eta^\gamma\frac{\partial\varphi_0}{\partial x}(t_\eta, x_\eta) \leqslant 0. \tag{35}$$

Clearly, by (34),

$$\frac{\partial \varphi_0}{\partial t}(t_\eta, x_\eta) = \sinh\left(\frac{t_\eta - t_0}{\eta^2}\right) + \frac{1}{\eta} \quad \text{and} \quad \frac{\partial \varphi_0}{\partial x}(t_\eta, x_\eta) = \frac{1}{\eta}.$$

It is obvious that neither Eq. (33) nor Eq. (35) can be verified by φ_0 with η small enough. Hence $(t_\eta, x_\eta) \notin \widetilde{\Omega}$ and so $(t_\eta, x_\eta) \in]0, T] \times \{0\}$. Moreover, since we are in case (b), $\bar{u}(t_\eta, 0) = \lambda_1 t_\eta$. We deduce that

$$\bar{u}(t_0, 0) \leqslant \lambda_1 t_\eta + \eta - \varphi_0(t_\eta, 0).$$

As $\eta \to 0$ we get

$$\bar{u}(t_0, 0) \leqslant \lambda_1 t_0.$$

Using the same reasoning for $\underline{u}$ we obtain that $\bar{u} = \underline{u}$ on $]0, T] \times \{0\}$.

(ii) Second, we prove (32) for $(t, x) \in \widetilde{\Omega}$. It suffices to verify (32) on the compact set

$$K_\delta = \left\{(t, x)\colon\ x \leqslant \left((t-\delta)(1-\gamma)\right)^{1/1-\gamma}\right\} \cap \widetilde{\Omega}$$

for any $\delta > 0$. Let us note that the inequation

$$\frac{\partial \varphi}{\partial t}(t, x) + x^\gamma \frac{\partial \varphi}{\partial x}(t, x) \leqslant 0$$

is verified on the boundary $\{(t, x)\colon\ x = \{(t-\delta)(1-\gamma)\}^{1/1-\gamma}\} \cap \widetilde{\Omega}$. To show this fact we proceed as in the proof of Proposition 7(b) by taking $\varphi \in \mathcal{C}^2(K_\delta)$ and the sequence of functions

$$\Xi_\eta(t, x) := \bar{u}(t, x) - \varphi(t, x) + \frac{\eta}{x^{1-\gamma} - (t-\delta)(1-\gamma)}.$$

We shall compute

$$M := \sup_{K_\delta}(\bar{u} - \underline{u}). \tag{36}$$

Let us assume that $M > 0$ and, as we have already seen, this maximum cannot be reached for $x = 0$. Take $\alpha > 0$. The function

$$\bar{u}_\alpha(t, x) := \bar{u}(t, x) - \alpha t$$

is a sub-solution of the equation

$$\frac{\partial u}{\partial t} + H_0\left(x, \frac{\partial u}{\partial x}\right) + \alpha = 0.$$

Let us denote

$$\Psi_\eta(t, s, x, y) := \bar{u}_\alpha(t, x) - \underline{u}(s, y) - \frac{(x-y)^2}{\eta^2} - \frac{(t-s)^2}{\eta^2}, \tag{37}$$

and let $(t_\eta, s_\eta, x_\eta, y_\eta)$ be a point where Ψ_η reaches a local maximum. Then $\bar{u}_\alpha - \chi_1$ reaches a local maximum at (t_η, x_η), where χ_1 denotes the function

$$\chi_1(t, x) := \underline{u}(s_\eta, y_\eta) + \frac{(x - y_\eta)^2}{\eta^2} + \frac{(t - s_\eta)^2}{\eta^2}.$$

By the same argument, $\chi_2 - \underline{u}$ reaches a local maximum at $(s_{\eta'}, y_{\eta'})$ where χ_2 denotes the function

$$\chi_2(s, y) := \bar{u}_\alpha(t_\eta, x_\eta) - \frac{(x_\eta - y)^2}{\eta^2} - \frac{(t_\eta - s)^2}{\eta^2}.$$

To finish the proof we need the following:

LEMMA 3. – *There exist $\rho > 0$ and $(t_\eta, s_\eta, x_\eta, y_\eta)$, a sequence of maxima of the function Ψ_η given by* (37), *such that*

$$\lim_{\eta\to 0}(x_\eta - y_\eta)^2/\eta^2 = 0, \tag{38}$$

$$x_\eta > \rho \quad \textit{and} \quad y_\eta > \rho \quad \textit{for } \eta \textit{ small enough.} \tag{39}$$

We return to the proof of Proposition 9. By Lemma 3, there exists a sub-sequence $(t_{\eta'}, s_{\eta'}, x_{\eta'}, y_{\eta'})$ of $(t_\eta, s_\eta, x_\eta, y_\eta)$, such that $x_{\eta'} > 0$. Since $\bar{u}_\alpha$ is a viscosity sub-solution, we get

$$\frac{\partial \chi_1}{\partial t}(t_{\eta'}, x_{\eta'}) + H_0\left(x_{\eta'}, \frac{\partial \chi_1}{\partial x}(t_{\eta'}, x_{\eta'})\right) + \alpha \leqslant 0,$$

hence

$$\frac{2(t_{\eta'} - s_{\eta'})}{\eta'^2} + H_0\left(x_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}\right) + \alpha \leqslant 0. \tag{40}$$

By the same argument, since $\underline{u}$ is a viscosity super-solution we get

$$\frac{2(t_{\eta'} - s_{\eta'})}{\eta'^2} + H_0\left(y_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}\right) \geqslant 0. \tag{41}$$

Subtracting (41) from (40) we obtain

$$-\frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}(x_{\eta'}^\gamma - y_{\eta'}^\gamma) = H_0\left(x_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}\right) - H_0\left(y_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}\right) \leqslant -\alpha.$$

Taking the limit as $\eta \to 0$, and using (38) and (39), we get $0 \leqslant -\alpha$. This is in contradiction with the assumption $\alpha > 0$. The proof is complete except for the proof of Lemma 3. □

Remark 6. – Obviously,

$$u(t, x) = \lambda_1\left(t - \frac{x^{1-\gamma}}{1-\gamma}\right)$$

is a classical solution of (30), which verify $u(t,0)=\lambda_1 t$. Hence, by the proof of Proposition 9, we deduce an uniqueness result and we get that $\bar{u}=\underline{u}=u$.

Proof of Lemma 3. – Put $M_\eta=\sup_{K_\delta}\Psi_\eta=\Psi_\eta(t_\eta,s_\eta,x_\eta,y_\eta)$. Then, by (37), for any (t,x) and (s,y) belonging to K_δ,

$$\bar{u}_\alpha(t,x)-\underline{u}(s,y)-\frac{(x-y)^2}{\eta^2}-\frac{(t-s)^2}{\eta^2}\leqslant M_\eta.$$

Taking $(t,x)=(s,y)$ we get

$$(\bar{u}_\alpha-\underline{u})(t,x)\leqslant M_\eta,$$

hence

$$M^\alpha:=\sup_{K_\delta}(\bar{u}_\alpha-\underline{u})\leqslant M_\eta.$$

Since

$$M^\alpha\leqslant\bar{u}_\alpha(t_\eta,x_\eta)-\underline{u}(s_\eta,y_\eta)-\frac{(x_\eta-y_\eta)^2}{\eta^2}-\frac{(t_\eta-s_\eta)^2}{\eta^2},$$

and by the fact that M^α, $\bar{u}_\alpha$ and $\underline{u}$ are bounded, there exists $k>0$, such that

$$\frac{(x_\eta-y_\eta)^2}{\eta^2}+\frac{(t_\eta-s_\eta)^2}{\eta^2}\leqslant k,\quad \text{for all } \eta>0.$$

We can extract a sub-sequence $(t_{\eta'},s_{\eta'},x_{\eta'},y_{\eta'})$ which converges to $(t,s,x,y)\in K_\delta$ and such that $\{(x_{\eta'}-y_{\eta'})^2/\eta'^2\colon\ \eta'\geqslant 0\}$ converges. Since $x_{\eta'}-y_{\eta'}\to 0$ and $t_{\eta'}-s_{\eta'}\to 0$, as $\eta'\to 0$, we deduce that $t=s$ and $x=y$.

Furthermore,

$$\begin{aligned}M^\alpha&\leqslant\liminf M_{\eta'}\leqslant\limsup M_{\eta'}\\&\leqslant\bar{u}_\alpha(t,x)-\underline{u}(t,x)-\lim_{\eta'\to 0}\frac{(x_{\eta'}-y_{\eta'})^2}{\eta'^2}-\liminf_{\eta'\to 0}\frac{(t_{\eta'}-s_{\eta'})^2}{\eta'^2}\leqslant M^\alpha.\end{aligned}$$

Hence

$$\lim_{\eta'\to 0}M_{\eta'}=M^\alpha\quad\text{and}\quad\lim_{\eta'\to 0}\frac{(x_{\eta'}-y_{\eta'})^2}{\eta'^2}=0.$$

Assume that $(t,x)\in\,]0,T]\times\{0\}$. The preceeding inequality yields

$$M^\alpha\leqslant\bar{u}_\alpha(t,x)-\underline{u}(t,x)=-\alpha t$$

since $\bar{u}=\underline{u}$ on $]0,T]\times\{0\}$. For α small enough M^α is positive which contradicts our last inequality and (39) is justified. □

Finally, by the symmetry of $p_t^\varepsilon(\cdot)$, the result of Theorem 2 is an easy consequence of Remark 6 and of the following:

PROPOSITION 10. – *For* $(t,x)\in\widetilde{\Omega}$,

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^{\varepsilon}(x)=\lambda_1\left(\frac{x^{1-\gamma}}{1-\gamma}-t\right), \tag{42}$$

and the convergence is uniform on each compact subset of $\widetilde{\Omega}$. *Moreover,*

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^{\varepsilon}(0)=-\lambda_1 t,\quad \forall t>0.$$

Proof of the Proposition 10. – This proof is an adaptation of the proof of a result in [2] (see Lemma 4.1, p. 86).

Let K be a compact subset of $\widetilde{\Omega}$. First, we show that $\lim_{\varepsilon\to 0} u^{\varepsilon}=u$, uniformly on K. By Proposition 9, $\bar{u}=\underline{u}=u$ on K. This means that u is a continuous function (since $\bar{u}$ is u.s.c. and $\underline{u}$ is l.s.c.). Hence, by (26) $u^{\varepsilon}-u$ is also a continuous function and $M_{\varepsilon}:=\sup_K(u^{\varepsilon}-u)$ is reached at $(t_{\varepsilon},x_{\varepsilon})\in K$.

Since u^{ε} is bounded, we can extract a sub-sequence $(t_{\varepsilon'},x_{\varepsilon'})$, such that $(t_{\varepsilon'},x_{\varepsilon'})\to(t,x)\in K$ and $M_{\varepsilon'}\to(\limsup_{\varepsilon}M_{\varepsilon})$, as $\varepsilon'\to 0$. By (29) we get

$$\limsup_{\varepsilon'\to 0} u^{\varepsilon'}(t_{\varepsilon'},x_{\varepsilon'})\leqslant\bar{u}(t,x).$$

Hence

$$\limsup_{\varepsilon\to 0}\sup_K\big(u^{\varepsilon}-u\big)=\limsup_{\varepsilon'\to 0}\big(u^{\varepsilon'}(t_{\varepsilon'},x_{\varepsilon'})-u(t_{\varepsilon'},x_{\varepsilon'})\big)\leqslant\bar{u}(t,x)-u(t,x)=0.$$

By similar arguments, we obtain:

$$\limsup_{\varepsilon\to 0}\sup_K\big(u-u^{\varepsilon}\big)\leqslant 0.$$

Therefore,

$$\limsup_{\varepsilon\to 0}\sup_K\big(u-u^{\varepsilon}\big)=0.$$

On the other hand,

$$\limsup_{\varepsilon\to 0} u^{\varepsilon}(t,x)=\min\Big\{\limsup_{\varepsilon\to 0}\big(-s(\varepsilon)\ln p_t^{\varepsilon}(x)\big),D\Big\},$$
$$\liminf_{\varepsilon\to 0} u^{\varepsilon}(t,x)=\min\Big\{\liminf_{\varepsilon\to 0}\big(-s(\varepsilon)\ln p_t^{\varepsilon}(x)\big),D\Big\}.$$

We deduce that, for D large enough, the term $\exp(-D/s(\varepsilon))$ will not change the limits as ε tends to zero, since $u(t,x)=\lambda_1(t-\frac{x^{1-\gamma}}{1-\gamma})$ is bounded. Hence, $s(\varepsilon)\ln p_t^{\varepsilon}(x)$ converges uniformly on each compact set of $\widetilde{\Omega}$ to $\lambda_1(\frac{x^{1-\gamma}}{1-\gamma}-t)$.

Finally, for $x=0$, we use formula (9) which we proved in Proposition 5:

$$p_t^{\varepsilon}(0)=\frac{1}{\varepsilon s(\varepsilon)^{1/2}}\sum_{j=1}^{\infty}\mathrm{e}^{-\lambda_j t/s(\varepsilon)}\psi_j^2(0).$$

By a similar reasoning as in the proof of Corollary 2, we can show that:

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^{\varepsilon}(0) = -\lambda_1 t. \qquad \square$$

Note

This work is the starting point in proving a large deviations principle in a more general context, subject which will be treated elsewhere (see [9]).

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