Ann. I. H. Poincaré – PR 37, 4 (2001) 481–502

S0246-0203(00)01067-0/FLA

STRICT POSITIVITY OF THE SOLUTION TO A 2-DIMENSIONAL SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT CUTOFF

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Received 23 November 1999, revised 28 April 2000

ABSTRACT. – We consider a 2-dimensional spatially homogeneous Boltzmann equation without cutoff, which we relate to a Poisson driven nonlinear S.D.E. We know from [8] that this S.D.E. admits a solution V_t, and that for each $t > 0$, the law of V_t admits a density $f(t,.)$. This density satisfies the Boltzmann equation. We use here the stochastic calculus of variations for Poisson functionals, in order to prove that f does never vanish. © 2001 Éditions scientifiques et médicales Elsevier SAS

Keywords: Boltzmann equation without cutoff; Poisson measure; Stochastic calculus of variations

1991 MSC: 60H07; 82C40; 35B65

RÉSUMÉ. – Nous considérons une équation de Boltzmann bidimensionnelle, spatialement homogène sans cutoff. Nous associons à cette équation une équation différentielle stochastique poissonnienne non linéaire. Nous savons par [8] que cette E.D.S. admet une solution V_t, et que pour chaque $t > 0$, la loi de V_t admet une densité $f(t,.)$. La fonction $f(t,v)$ obtenue satisfait l'équation de Boltzmann. Nous utilisons ici le calcul des variations stochastiques pour des fonctionnelles de mesures de Poisson, afin de prouver que f ne s'annule jamais. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction and statement of the main result

The 2-dimensional spatially homogeneous Boltzmann equation of Maxwellian molecules deals with the density $f(t,v)$ of particles which have the speed $v \in \mathbb{R}^2$ at the instant $t \geqslant 0$ in a sufficiently dilute (2-dimensional) gas:

$$\frac{\partial f}{\partial t}(t,v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta=-\pi}^{\pi} \left[f(t,v')f(t,v'_*) - f(t,v)f(t,v_*)\right]\beta(\theta)\,d\theta\,dv_*, \tag{1.1}$$

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where, if R_θ is the rotation of angle θ,

$$v' = \frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right); \qquad v'_* = \frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right). \tag{1.2}$$

The new speeds v' and v'_* are the velocities of two molecules which had the speeds v and v_*, after a collision of angle θ. The "cross section" β is an even and positive function on $[-\pi,\pi]\backslash\{0\}$ which explodes at 0 as $1/|\theta|^s$ with $s \in]1,3[$ in the case of interactions in $1/r^\alpha$, with $\alpha > 2$. Thus, the natural assumption (which we will suppose) is

$$\int_0^\pi \theta^2 \beta(\theta)\, d\theta < \infty. \tag{1.3}$$

In this case, Eq. (1.1) is said to be without cutoff. The case with cutoff, namely when $\int_0^\pi \beta(\theta)\, d\theta < \infty$, has been much investigated by the analysts, and they have obtained some existence, regularity and strict positivity results.

In this paper, we prove, by using the stochastic calculus of variations on the Poisson space, a strict lowerbound for the solution f of (1.1) built in [8], in the case where the cross section sufficiently explodes.

To this aim, we use a probabilistic approach to the Boltzmann equations of Maxwellian molecules first introduced by Tanaka [19], and more recently by Desvillettes, Graham and Méléard [7,11] in the one dimensional case, see also [8] for the case of Eq. (1.1). Indeed, we build a non classical Poisson driven S.D.E., of which we denote by V_t the solution. This S.D.E. is related to Eq. (1.1) in the following sense: its probability flow $\mathcal{L}(V_t)$ is a measure solution of (1.1). In [8], the Malliavin calculus is used to prove that for each $t > 0$, $\mathcal{L}(V_t)$ admits a smooth density $f(t,v)$, which satisfies (1.1) in a weak sense.

The strict positivity of f seems to be unknown by the analysts in the case without cutoff, and might be useful to justify computations in which the entropy appears. In the case with cutoff, much more is known: Pulvirenti and Wennberg have proved a Maxwellian lowerbound in [18]. Their method is based on the separation of the gain and loss terms, which typically cannot be used in the present case.

Lowerbounds of the density for Wiener functionals have been worked out by Aida, Kusuoka and Stroock [1], Ben Arous and Léandre [3], see also Bally and Pardoux [2]. In the case of Poisson functionals, the strict positivity of the density in small time has been studied by Léandre [15], Ishikawa [12], and Picard [17].

The first result of strict positivity of the density for Poisson functionnals is due to Léandre [16], who was considering simple jump processes with finite variations. In [10], we have given a sufficient condition for the strict positivity in every time for one-dimensional Poisson-driven S.D.E.s, and this approach does allow to deal almost only with processes with infinite variations. In [9], we have applied this method to the Kac equation without cutoff, which is a caricatural one-dimensional version of the Boltzmann equation.

The strict positivity of the density for general 2-dimensional Poisson driven S.D.E.s seems to be a very difficult problem, but in the case of the S.D.E. related to (1.1), the method works quite easily. The main differences between the one-dimensional caricatural Kac equation and Eq. (1.1) are the following. First, we have to deal with a determinant. We thus have to assume an additional condition on the support of the initial distribution. Furthermore, we have to prove that for each $t > 0$, the support of $f(t,.)$ contains that of the initial distribution. Another technical problem is that one cannot solve explicitely Doléans–Dade equations with values in $\mathcal{M}_{2\times 2}(\mathbb{R})$.

Let us now be precise. First, we define the solutions of (1.1) in the following (weak) sense.

DEFINITION 1.1. – Let P_0 be a probability on $\mathbb{R}^2$ that admits a moment of order 2. A positive function f on $\mathbb{R}^+ \times \mathbb{R}^2$ is a solution of (1.1) with initial data P_0 if for every test function $\phi \in C_b^2(\mathbb{R}^2)$,

$$\begin{aligned}\int_{v\in\mathbb{R}^2} f(t,v)\phi(v)\,dv &= \int_{v\in\mathbb{R}^2} \phi(v)P_0(dv) - \frac{b}{2}\int_0^t \int_{v\in\mathbb{R}^2}\int_{v^*\in\mathbb{R}^2} \langle \phi'(v), v-v^*\rangle\, dv^* dv\, ds \\ &+ \int_0^t \int_{v\in\mathbb{R}^2}\int_{v^*\in\mathbb{R}^2}\int_{-\pi}^{\pi} f(s,v)f(s,v^*)\big[\phi(v')-\phi(v) \\ &- \langle \phi'(v), v'-v\rangle\big]\beta(\theta)\,d\theta\, dv^*\, dv\, ds, \end{aligned} \tag{1.4}$$

where ϕ' denotes the gradient of ϕ, where $\langle .\,,\,.\rangle$ stands for the scalar product in $\mathbb{R}^2$, where v' is defined by (1.2), and where

$$b = \int_{-\pi}^{\pi} (1-\cos\theta)\beta(\theta)\,d\theta. \tag{1.5}$$

In [8], one assumes that

Assumption (H):
1. The initial distribution P_0 admits a moment of order 2, and β satisfies (1.3),
2. $\beta = \beta_0 + \beta_1$, where β_1 is even and positive on $[-\pi,\pi]\backslash\{0\}$, and there exists $k_0 > 0$, $\theta_0 \in]0,\pi[$, and $r \in]1,3[$ such that $\beta_0(\theta) = \frac{k_0}{|\theta|^r} 1_{[-\theta_0,\theta_0]}(\theta)$,
3. P_0 is not a Dirac mass.

Let us also consider the following random elements:

Notation 1.2. – Assume (H)-1. We denote by N a Poisson measure on $[0,\infty[\times[0,1]\times[-\pi,\pi]$, with intensity measure:

$$\nu(d\theta, d\alpha, ds) = \beta(\theta)\,d\theta\,d\alpha\,ds \tag{1.6}$$

and by $\tilde{N}$ the associated compensated measure. We consider a $\mathbb{R}^2$-valued random variable V_0 independent of N, of which the law is P_0. We will consider $[0,1]$ as a probability space, denote by $d\alpha$ the Lebesgue measure on $[0,1]$, and denote by E_α and $\mathcal{L}_\alpha$ the expectation and law on $([0,1],\mathcal{B}([0,1]),d\alpha)$.

If (H)-2 also holds, we suppose that $N = N_0 + N_1$, where N_0 and N_1 are two independent Poisson measures on $[0, \infty[\times [0,1] \times [-\pi, \pi]$, with intensity measures:

$$\nu_0(d\theta, d\alpha, ds) = \beta_0(\theta)\, d\theta\, d\alpha\, ds; \qquad \nu_1(d\theta, d\alpha, ds) = \beta_1(\theta)\, d\theta\, d\alpha\, ds. \tag{1.7}$$

In this case, we also assume that our probability space is the canonical one associated with the independent random elements V_0, N_0, and N_1:

$$\begin{aligned}&(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\\ &\quad = (\Omega', \mathcal{F}', \{\mathcal{F}'\}, P') \otimes \big(\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}, P^0\big) \otimes \big(\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}, P^1\big).\end{aligned} \tag{1.8}$$

The following theorem is proved in [8] (Theorems 2.8 and 2.9).

THEOREM 1.3. – *Assume* (H)-1. *There exists a* $\mathbb{R}^2$*-valued càdlàg adapted process* $\{V_t(\omega)\}$ *on* Ω *and a* $\mathbb{R}^2$*-valued process* $\{W_t(\alpha)\}$ *on* $[0,1]$ *such that, if*

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}, \tag{1.9}$$

then

$$\left.\begin{aligned} V_t(\omega) = {} & V_0(\omega) + \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\big(V_{s-}(\omega) - W_{s-}(\alpha)\big)\tilde{N}(\omega, d\theta\, d\alpha\, ds) \\ & - \frac{b}{2}\int_0^t \int_0^1 \big(V_{s-}(\omega) - W_{s-}(\alpha)\big) d\alpha\, ds, \\ \mathcal{L}_\alpha(W) = {} & \mathcal{L}(V); \qquad E\Big(\sup_{[0,T]} \|V_t\|^2\Big) < \infty. \end{aligned}\right\} \tag{1.10}$$

The obtained law $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ *is unique.*

Finally, the main theorem of [8] (Theorem 3.1) is the following.

THEOREM 1.4. – *Assume* (H). *Let* (V, W) *be a solution of* (1.10). *Then for all* $t > 0$, *the law of* V_t *admits a density* $f(t,.)$ *with respect to the Lebesgue measure on* $\mathbb{R}^2$. *The obtained function* f *is a solution of the Boltzmann equation* (1.1) *in the sense of Definition* 1.1.

It is also proved in [8] (Theorems 3.2 and 3.3) that under an additional assumption, the solution f is regular in the following sense: for each $t > 0$, $f(t,.)$ is in $C^\infty(\mathbb{R}^2)$, and f is continuous on $]0, \infty[\times \mathbb{R}^2$.

Let us now give our assumption, which is more stringent than (H): we need a stronger explosion of the cross section, and the support of the initial distribution has to be large enough.

Assumption (SP):

1. The same as (H)-1,
2. The same as (H)-2, but with $r \in [2, 3[$,

3. For each $X_0 \in \mathbb{R}^2$, there exist $0 < \varepsilon < \eta < \infty$ such that

$$P_0\big(\big\{X \in \mathbb{R}^2 / |X^x - X_0^x| < \varepsilon, |X^y - X_0^y| > \eta\big\}\big) > 0, \tag{1.11}$$

$$P_0\big(\big\{X \in \mathbb{R}^2 / |X^y - X_0^y| < \varepsilon, |X^x - X_0^x| > \eta\big\}\big) > 0. \tag{1.12}$$

Our main result is the following:

THEOREM 1.5. – *Assume* (*SP*), *and consider the solution* f *in the sense of Definition* 1.1 *of Eq.* (1.1) *built in Theorem* 1.4. *There exists a strictly positive function* $g(t,v)$ *on* $]0,+\infty[\times\mathbb{R}^2$, *continuous in* v, *such that for all* $t>0$, *all* $\phi \in C_b^+(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \phi(v) f(t,v)\,dv \geqslant \int_{\mathbb{R}^2} \phi(v) g(t,v)\,dv. \tag{1.13}$$

In particular, if f *is continuous in* v, *then* f *is strictly positive on* $]0,+\infty[\times\mathbb{R}^2$.

Let us say a word about our assumptions. (*SP*)-1 is quite reasonable. Indeed, the analysts almost always assume that P_0 admits a density (see, e.g., Desvillettes, [6]); the assumption $\int \|v\|^2 P_0(dv) < \infty$ means that the energy of the initial system is finite; and (1.3) is physically natural. (*SP*)-2 means that the cross section contains a sufficiently "large" and "regular" part, which will allow us to use the Malliavin calculus. Notice that the fact that $r \geqslant 2$ means that $\int |\theta|\beta(\theta)\,d\theta = \infty$: we really need a strong explosion of the cross section. Finally, (*SP*)-3 is a technical condition. Notice that (*SP*)-3 is satisfied if $\mathrm{supp} P_0$ contains $\{(x,0), x \geqslant 0\} \cup \{(0,y), y \geqslant 0\}$, or even $\{(n,0), n \in \mathbb{N}\} \cup \{(0,n), n \in \mathbb{N}\}$. If the support of P_0 is bounded, then the condition is not satisfied.

Finally, let us notice that in our proof, we check the following lemma:

LEMMA 1.6. – *Assume* (H)-1, *and consider a solution* (V,W) *of* (1.10). *Then for each* $t>0$,

$$\mathrm{supp}\, P_0 \subset \mathrm{supp}\, \mathcal{L}(V_t). \tag{1.14}$$

The present work is organized as follows. In Section 2, we prove Lemma 1.6. In the third section, we state a criterion of strict positivity of the density for Poisson functionals, which we apply to V_t in the next sections.

In the whole work, we will assume at least (H)-1, *use Notation* 1.2, *and consider a solution* (V,W) *of* (1.10). *We will always work on the time interval* $[0,T]$, *for some* $T>0$ *fixed. We will denote by* K *a constant of which the value may change from line to line.*

2. Conservation of the support

This section is dedicated to the proof of Lemma 1.6, which will be useful to prove Theorem 1.5. We fix $X_0 \in \mathrm{supp}\, P_0 = \mathrm{supp}\, P \circ V_0^{-1}$, $\varepsilon > 0$, and $t > 0$. We have to show that

$$P\big(\|V_t - X_0\| \leqslant \varepsilon\big) > 0. \tag{2.1}$$

The main idea of the proof is very simple: since V_0 and N are independent, we can build a subset of Ω, of positive probability, on which V_0 is near X_0 and N is very small. On this subset, V_t will be near V_0, and thus near X_0.

For $p \in \mathbb{N}^*$, we denote by N^p the restriction $N|_{[0,T]\times[0,1]\times\{[-\pi,\pi]/[-1/p,1/p]\}}$, which is a finite Poisson measure. Then, we split V_t into

$$V_t = V_0 + A_t^p + B_t^p, \tag{2.2}$$

where

$$A_t^p = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\big(V_{s-} - W_{s-}(\alpha)\big) N^p(d\theta\, d\alpha\, ds) \tag{2.3}$$

and, if $b_p = \int_{-1/p}^{1/p} (1-\cos\theta)\beta(\theta)\,d\theta$,

$$\begin{aligned} B_t^p = {} & \int_0^t \int_0^1 \int_{-1/p}^{1/p} A(\theta)\big(V_{s-} - W_{s-}(\alpha)\big) \tilde{N}(d\theta\, d\alpha\, ds) \\ & - \frac{b_p}{2} \int_0^t \int_0^1 \big(V_{s-} - W_{s-}(\alpha)\big)\, d\alpha\, ds. \end{aligned} \tag{2.4}$$

We consider the set

$$\Omega_p = \big\{ \|V_0 - X_0\| < \varepsilon/2;\ N^p \equiv 0 \big\} \tag{2.5}$$

of which the probability is strictly positive (for each p), since V_0 and N^p are independent, since $X_0 \in \operatorname{supp} P \circ V_0^{-1}$, and since N^p is a *finite* Poisson measure.

It is clear from (2.2) and (2.5), since Ω_p belongs to $\sigma(V_0, N^p)$, and thanks to the Bienaymé–Tchebichev inequality applied to the conditional probability measure $P(\,.\,|\sigma(V_0, N^p))$, that

$$\begin{aligned} P(\|V_t - X_0\| \leqslant \varepsilon) & \geqslant P\big(\|V_0 - X_0\| \leqslant \varepsilon/2;\ A_t^p = 0;\ \big\|B_t^p\big\| \leqslant \varepsilon/2\big) \\ & \geqslant P\big(\Omega_p;\ \big\|B_t^p\big\| \leqslant \varepsilon/2\big) \\ & \geqslant E\big(1_{\Omega_p} P\big(\big\|B_t^p\big\| \leqslant \varepsilon/2 \mid \sigma\big(V_0, N^p\big)\big)\big) \\ & \geqslant E\bigg(1_{\Omega_p}\bigg\{1 - \frac{4}{\varepsilon^2} E\big(\big\|B_t^p\big\|^2 \mid \sigma\big(V_0, N^p\big)\big)\bigg\}\bigg). \end{aligned} \tag{2.6}$$

Since $N|_{[0,T]\times[0,1]\times[-1/p,1/p]}$ is independent of V_0 and N^p, it clearly is a Poisson measure under the conditional probability measure $P(\,.\,|\sigma(V_0, N^p))$. Thus, using Burkholder's inequality, the facts that $E_\alpha(\sup_{[0,T]} \|W_t\|^2) < \infty$, and $\|A(\theta)\| \leqslant K\theta^2$, we see that

$$\begin{aligned} & E\big(\big\|B_t^p\big\|^2 \mid \sigma\big(V_0, N^p\big)\big) \\ & \quad \leqslant K \int_0^t \int_0^1 \int_{-1/p}^{1/p} \theta^2 \big[E\big(\|V_s\|^2 \mid \sigma\big(V_0, N^p\big)\big) + \|W_s(\alpha)\|^2\big] \beta(\theta)\, d\theta\, d\alpha\, ds \end{aligned}$$

$$+ Kb_p^2 \int_0^t \int_0^1 \left[E\left(\|V_s\|^2 \mid \sigma\left(V_0, N^p\right)\right) + \|W_s(\alpha)\|^2\right] d\alpha\, ds$$

$$\leqslant u_p \left[1 + \int_0^t E\left(\|V_s\|^2 \mid \sigma\left(V_0, N^p\right)\right) ds\right], \tag{2.7}$$

where the sequence u_p decreases to 0 when p goes to infinity. Furthermore, thanks to (2.2) and the definition of Ω_p,

$$1_{\Omega_p} \|V_t\| \leqslant 1_{\Omega_p} \left[\|X_0\| + \varepsilon + \left\|B_t^p\right\|\right] \tag{2.8}$$

from which we deduce the existence of a constant K, not depending on p, such that

$$1_{\Omega_p} E\left(\|V_t\|^2 \mid \sigma\left(V_0, N^p\right)\right) \leqslant 1_{\Omega_p} \left[K + K \int_0^t E\left(\|V_s\|^2 \mid \sigma\left(V_0, N^p\right)\right) ds\right]. \tag{2.9}$$

Gronwall's lemma allows us to conclude that

$$1_{\Omega_p} E\left(\|V_t\|^2 \mid \sigma\left(V_0, N^p\right)\right) \leqslant K 1_{\Omega_p}. \tag{2.10}$$

Finally, using (2.7), we obtain

$$1_{\Omega_p} E\left(\left\|B_t^p\right\|^2 \mid \sigma\left(V_0, N^p\right)\right) \leqslant K u_p 1_{\Omega_p}. \tag{2.11}$$

Using (2.6), we see that

$$P(\|V_t - X_0\| \leqslant \varepsilon) \geqslant E\left[1_{\Omega_p}\left(1 - Ku_p/\varepsilon^2\right)\right] \geqslant \left(1 - Ku_p/\varepsilon^2\right) P(\Omega_p). \tag{2.12}$$

Recalling that for each p, $P(\Omega_p) > 0$, and choosing p large enough, in order that $u_p \leqslant \varepsilon^2/K$, we deduce (2.1), and Lemma 1.6 follows.

3. A criterion of strict positivity

This section contains two parts. We first introduce some general notations and definitions about Bismut's approach of the Malliavin calculus on our Poisson space. Then we adapt the criterion of strict positivity of Bally and Pardoux [2] (which deals with the Wiener functionals) to our probability space.

In the following definition, we precise the perturbations we will use. We have already introduced such a perturbation in [8], but we have to define here all the possible perturbations.

DEFINITION 3.1. – A predictable $\mathbb{R}^2$-valued function $v(\omega, s, \theta, \alpha)$ on $\Omega \times [0, T] \times [-\theta_0, \theta_0] \times [0, 1]$ is said to be a "perturbation" if for all fixed ω, s, α, $v(\omega, s, ., \alpha)$ is C^1

on $[-\theta_0, \theta_0]$, and if there exist some even positive (deterministic) functions η and ρ on $[-\theta_0, \theta_0]$ such that

$$\|v(s,\theta,\alpha)\| \leqslant \eta(\theta); \qquad \|v'(s,\theta,\alpha)\| \leqslant \rho(\theta), \tag{3.1}$$

$$\eta(\theta) \leqslant \frac{|\theta|}{2}; \qquad \eta(-\theta_0) = \eta(\theta_0) = 0, \tag{3.2}$$

$$\text{if } \xi(\theta) = \rho(\theta) + r2^{r+2}\frac{\eta(\theta)}{|\theta|} \quad \text{then } \|\xi\|_\infty \leqslant \frac{1}{2} \text{ and } \xi \in L^1(\beta_0(\theta)\,d\theta). \tag{3.3}$$

Notice that thanks to (3.3), η and ρ are in $L^1 \cap L^\infty(\beta_0(\theta)\,d\theta)$.

Consider now a fixed perturbation v. For $\lambda \in B(0,1)$ (this ball is that of $\mathbb{R}^2$), we set

$$\gamma^\lambda(s,\theta,\alpha) = \theta + \langle \lambda, v(s,\theta,\alpha)\rangle, \tag{3.4}$$

where $\langle , \rangle$ denotes the scalar product of $\mathbb{R}^2$. Thanks to (3.1), (3.2), and (3.3), it is easy to check that for each $\lambda, s, \alpha, \omega$, $\gamma^\lambda(s, .\,, \alpha)$ is an increasing bijection from $[-\theta_0, \theta_0]\backslash\{0\}$ into itself. Then we denote by $N_0^\lambda = \gamma^\lambda(N_0)$ the image measure of N_0 by γ^λ: for any Borel subset A of $[0,T] \times [-\theta_0, \theta_0] \times [0,1]$,

$$N_0^\lambda(A) = \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A\big(s, \gamma^\lambda(s,\theta,\alpha), \alpha\big) N_0(d\theta\,d\alpha\,ds). \tag{3.5}$$

We also define the shift S^λ on Ω by

$$V_0 \circ S^\lambda = V_0; \qquad N_0 \circ S^\lambda = N_0^\lambda; \qquad N_1 \circ S^\lambda = N_1. \tag{3.6}$$

We will need the following predictable function:

$$Y^\lambda(s,\theta,\alpha) = \frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)}\big(1 + \langle \lambda, v'(s,\theta,\alpha)\rangle\big). \tag{3.7}$$

Then it is easy to check that for all λ,

$$\gamma^\lambda\big(Y^\lambda.\nu_0\big) = \nu_0. \tag{3.8}$$

Indeed, for any Borel set $A \subset [0,T] \times [0,1] \times [-\pi, \pi]$,

$$\begin{aligned}
&\gamma^\lambda\big(Y^\lambda.\nu_0\big)(A) \\
&\quad= \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A\big(s,\alpha,\gamma^\lambda(s,\theta,\alpha)\big) Y^\lambda(s,\theta,\alpha)\,\beta_0(\theta)\,d\theta\,d\alpha\,ds \\
&\quad= \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A\big(s,\alpha,\gamma^\lambda(s,\theta,\alpha)\big) \times \frac{\partial}{\partial\theta}\gamma^\lambda(s,\theta,\alpha) \times \beta_0\big(\gamma^\lambda(s,\theta,\alpha)\big)\,d\theta\,d\alpha\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_0^1 \int_{-\pi}^{\pi} 1_A(s,\alpha,\theta')\beta_0(\theta')\,d\theta'\,d\alpha\,ds \\
&= \nu_0(A),
\end{aligned}
\tag{3.9}
$$

where the last inequality comes from the substitution $\theta' = \gamma^\lambda(s,\theta,\alpha)$.

We will also need the following inequality: for all $\lambda, \mu \in B(0,1)$ (recall that ξ is defined in (3.3)),

$$
\left|Y^\lambda(s,\theta,\alpha) - Y^\mu(s,\theta,\alpha)\right| \leqslant \|\lambda-\mu\| \times \xi(\theta) \tag{3.10}
$$

which we now prove, using (3.1), (3.2), and (3.3).

$$
\begin{aligned}
&\left|Y^\lambda(s,\theta,\alpha) - Y^\mu(s,\theta,\alpha)\right| \\
&\quad\leqslant \frac{\beta_0(\gamma^\lambda(s,\theta,\alpha))}{\beta_0(\theta)} \times \|\lambda-\mu\| \times \|v'(s,\theta,\alpha)\| \\
&\qquad + \left|1 + \langle \mu, v'(s,\theta,\alpha)\rangle\right| \times \frac{|\beta_0(\gamma^\mu(s,\theta,\alpha)) - \beta_0(\gamma^\lambda(s,\theta,\alpha))|}{\beta_0(\theta)} \\
&\quad\leqslant \|\lambda-\mu\| \times \rho(\theta) \times \left[1 + \frac{|\gamma^\lambda(s,\theta,\alpha)-\theta| \times \sup_{[\theta,\gamma^\lambda(s,\theta,\alpha)]} |\beta_0'(\phi)|}{\beta_0(\theta)}\right] \\
&\qquad + \frac{3}{2} \times \frac{|\gamma^\lambda(s,\theta,\alpha) - \gamma^\mu(s,\theta,\alpha)| \times \sup_{[\gamma^\mu(s,\theta,\alpha),\gamma^\lambda(s,\theta,\alpha)]} |\beta_0'(\phi)|}{\beta_0(\theta)}
\end{aligned}
$$

(we have used the fact that $\rho \leqslant 1/2$, which is obvious from (3.3)). But for all λ, μ, it is easily checked that

$$
\begin{aligned}
\sup_{[\gamma^\mu(s,\theta,\alpha),\gamma^\lambda(s,\theta,\alpha)]} |\beta_0'(\phi)| &\leqslant \sup_{[|\theta|-\eta(\theta),|\theta|+\eta(\theta)]} |\beta_0'(\phi)| \leqslant k_0 r / \bigl(|\theta| - \eta(\theta)\bigr)^{r+1} \\
&\leqslant 2^{r+1} r k_0 / |\theta|^{r+1},
\end{aligned}
$$

since $\eta(\theta) \leqslant |\theta|/2$. We finally obtain

$$
\begin{aligned}
&\left|Y^\lambda(s,\theta,\alpha) - Y^\mu(s,\theta,\alpha)\right| \\
&\quad\leqslant \|\lambda-\mu\| \times \rho(\theta) \times \left[1 + r2^{r+1}\eta(\theta)/|\theta|\right] + \frac{3}{2} r 2^{r+1} \|\lambda-\mu\| \times \eta(\theta)/|\theta| \\
&\quad\leqslant \|\lambda-\mu\| \times \left[\rho(\theta) + r2^{r+1}\frac{\eta(\theta)}{|\theta|} \times (\rho(\theta) + 3/2)\right] \\
&\quad\leqslant \|\lambda-\mu\| \times \xi(\theta)
\end{aligned}
\tag{3.11}
$$

and (3.10) is proved.

We also consider the following martingale

$$
M_t^\lambda = \int_0^t \int_0^1 \int_{-\pi}^{\pi} \bigl(Y^\lambda(s,\theta,\alpha) - 1\bigr)\,\tilde{N}_0(d\theta\,d\alpha\,ds) \tag{3.12}
$$

and its Doléans–Dade exponential (see Jacod and Shiryaev [14])

$$G_t^\lambda = \mathcal{E}(M^\lambda)_t = \mathrm{e}^{M_t^\lambda} \prod_{0\leqslant s\leqslant t} (1+\Delta M_s^\lambda)\mathrm{e}^{-\Delta M_s^\lambda}. \tag{3.13}$$

Since $|Y^\lambda - 1| \leqslant \xi \leqslant 1/2$, it is clear that G^λ is always strictly positive on $[0, T]$. We now set $P^\lambda = G_T^\lambda.P$. Using Eq. (3.8), and the Girsanov theorem for random measures (see Jacod and Shiryaev [14], p. 157) one can show that $P^\lambda \circ (S^\lambda)^{-1} = P$, i.e. that the law of (V_0, N_0^λ, N_1) under P^λ is the same as the one of (V_0, N_0, N_1) under P.

Finally, it is quite easy, by using the explicit expression (3.13) of G^λ, to check the following lemma.

LEMMA 3.2. – *Let v be a perturbation, and G^λ the associated exponential martingale. Then for all $t > 0$, all $\omega \in \Omega$, the map $\lambda \mapsto G_t^\lambda(\omega)$ is continuous on $B(0, 1)$.*

We now give the criterion of strict positivity we will use.

THEOREM 3.3. – *Let X be a $\mathbb{R}^2$-valued random variable on Ω, and let $X_0 \in \mathbb{R}^2$. Assume that there exists a sequence v_n of perturbations such that, if $X^n(\lambda) = X \circ S_n^\lambda$, then for all n, the map*

$$\lambda \mapsto X^n(\lambda) \tag{3.14}$$

is a.s. twice differentiable on $B(0, 1)$. Assume that there exist $c > 0$, $\delta > 0$, and $k < \infty$, such that for all $r > 0$,

$$\lim_{n\to\infty} P\big(\Lambda^n(r)\big) > 0 \tag{3.15}$$

where

$$\begin{aligned}\Lambda^n(r) = \bigg\{ \|X - X_0\| < r, \left|\det \frac{\partial}{\partial\lambda} X^n(0)\right| \geqslant c,\\ \sup_{\|\lambda\|\leqslant\delta} \left[\left\|\frac{\partial}{\partial\lambda} X^n(\lambda)\right\| + \left\|\frac{\partial^2}{\partial\lambda^2} X^n(\lambda)\right\|\right] \leqslant k\bigg\}.\end{aligned} \tag{3.16}$$

Then there exists a continuous function $\theta_{X_0}(.) : \mathbb{R}^2 \mapsto \mathbb{R}_+$, such that $\theta_{X_0}(X_0) > 0$, and such that for all $\phi \in C_b^+(\mathbb{R}^2)$,

$$E\big(\phi(X)\big) \geqslant \int_{\mathbb{R}^2} \phi(y)\theta_{X_0}(y)\,dy. \tag{3.17}$$

In order to prove this criterion, it suffices to copy the proof of Theorem 3.3 in [10] or Theorem 2.3 in [9]. Let us just recall the 2-dimensional version of the uniform local inverse theorem used in the proof, that can be found in Aida, Kusuoka and Stroock [1]:

LEMMA 3.4. – *Let $c > 0$, $\delta > 0$, and $k < \infty$ be fixed. Consider the following set*:

$$\mathcal{G} = \Big\{ g : \mathbb{R}^2 \mapsto \mathbb{R}^2 / |\det g'(0)| \geqslant c, \sup_{|x|\leqslant\delta} \big[\|g(x)\| + \|g'(x)\| + \|g''(x)\|\big] \leqslant k \Big\}. \tag{3.18}$$

Then there exist $\alpha > 0$ *and* $R > 0$ *such that for every* $g \in \mathcal{G}$*, there exists a neighborhood* $\mathcal{V}_g$ *of* 0 *contained in* $B(0, R)$ *such that* g *is a diffeomorphism from* $\mathcal{V}_g$ *to* $B(g(0), \alpha)$.

We finally state a useful remark, of which the proof can be found in [10], Remark 3.5.

Remark 3.5. – Let X be a $\mathbb{R}^2$-valued random variable on Ω. Assume that for every $X_0 \in \operatorname{supp} P \circ X^{-1}$, the assumptions of Theorem 3.3 are satisfied. Then the law of X is bounded below by a measure admitting a strictly positive continuous density on $\mathbb{R}^2$ with respect to the Lebesgue measure on $\mathbb{R}^2$.

From now on, $T > 0$ *is fixed, and so is* $X_0 \in \mathbb{R}^2$.

In the next section, we will consider a fixed perturbation v_n, and we will compute $V_t^n(\lambda)$ and its derivatives for any $t \in [0, T]$. Section 5 is devoted to the explicit choice of the sequence v_n of perturbations. In Section 6, we will first prove that for some $\beta > 0$, some $\delta > 0$, a.s.,

$$\liminf_{n\to\infty} \left| \det \frac{\partial}{\partial\lambda} V_T^n(0) \right| \geqslant \beta 1_{\{\|V_T - X_0\| \leqslant \delta\}}. \tag{3.19}$$

Then we will check that for some constant K, for all $n \in \mathbb{N}$, all $\lambda \in B(0, 1)$,

$$\left\| \frac{\partial}{\partial\lambda} V_T^n(\lambda) \right\| + \left\| \frac{\partial^2}{\partial\lambda^2} V_T^n(\lambda) \right\| \leqslant K. \tag{3.20}$$

Finally, we will easily conclude.

4. Differentiability of the perturbed process

In this section, we consider a fixed perturbation v_n. We compute $V_t^n(\lambda) = V_t \circ S_n^\lambda$ and its derivatives with respect to λ. The rigorous proof of the differentiability of similar processes can be found in [8] or [9].

In order to compute $V_t^n(\lambda)$, it suffices to replace each ω by $S_n^\lambda(\omega)$, and to use the definition of S_n^λ:

$$\begin{aligned} V_t^n(\lambda) = V_0 &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\big(V_{s-}^n(\lambda) - W_{s-}(\alpha)\big)\tilde{N}(d\theta\, d\alpha\, ds) \\ &- \frac{b}{2} \int_0^t \int_0^1 \big(V_{s-}^n(\lambda) - W_{s-}(\alpha)\big)\, d\alpha\, ds \\ &+ \int_0^t \int_0^1 \int_{-\pi}^{\pi} \big(A\big(\gamma_n^\lambda(s, \theta, \alpha)\big) - A(\theta)\big)\big(V_{s-}^n(\lambda) - W_{s-}(\alpha)\big) N_0(d\theta\, d\alpha\, ds). \end{aligned} \tag{4.1}$$

We now introduce the following semi-martingale, with values in $\mathcal{M}_{2\times 2}(\mathbb{R})$:

$$K_t^n(\lambda) = \int_0^t \int_0^1 \int_{-\pi}^{\pi} A(\theta)\tilde{N}(d\theta\, d\alpha\, ds) - \frac{b}{2} I t$$

$$+\int_0^t\int_0^1\int_{-\pi}^{\pi}\big(A\big(\gamma_n^\lambda(s,\theta,\alpha)\big)-A(\theta)\big)N_0(d\theta\,d\alpha\,ds),\tag{4.2}$$

where I is the unit 2×2 matrix. Differentiating (4.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}V_t^n(\lambda)=\int_0^t dK_s^n(\lambda).\,\frac{\partial}{\partial\lambda}V_{s-}^n(\lambda)+\int_0^t\int_0^1\int_{-\pi}^{\pi}A'\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\\ \times\big(V_{s-}^n(\lambda)-W_{s-}(\alpha)\big)v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds).\end{aligned}\tag{4.3}$$

We have used the notation

$$\begin{pmatrix}a\\ b\end{pmatrix}(x\quad y)=\begin{pmatrix}ax & ay\\ bx & by\end{pmatrix}.$$

The 2×2 matrix $\frac{\partial}{\partial\lambda}V_t^n(\lambda)$ is given by

$$\left(\frac{\partial}{\partial\lambda_x}V_t^n(\lambda)\quad \frac{\partial}{\partial\lambda_y}V_t^n(\lambda)\right).$$

We thus see that $\frac{\partial}{\partial\lambda}V_t^n(\lambda)$ satisfies a linear S.D.E. We thus are able to compute its explicit expression, which we now do.

First consider the Doléans–Dade exponential $\mathcal{E}(K^n(\lambda))$ defined as the solution of:

$$\mathcal{E}\big(K^n(\lambda)\big)_t=I+\int_0^t dK^n(\lambda)_s.\,\mathcal{E}\big(K^n(\lambda)\big)_{s-}.\tag{4.4}$$

Since $I+\Delta K_s^n(\lambda)$ is always invertible (use the explicit expression of $A(\theta)$), we know from Jacod [13], that $\mathcal{E}(K^n(\lambda))$ is a.s. invertible for all $t\in[0,T]$.

Using the main result of Jacod, [13], we deduce that

$$\begin{aligned}\frac{\partial}{\partial\lambda}V_t^n(\lambda)&=\mathcal{E}\big(K^n(\lambda)\big)_t\int_0^t\int_0^1\int_{-\pi}^{\pi}\mathcal{E}\big(K^n(\lambda)\big)_{s-}^{-1}\big(I+\Delta K_s^n(\lambda)\big)^{-1}A'\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\\ &\quad\times\big(V_{s-}^n(\lambda)-W_{s-}(\alpha)\big)v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds)\\ &=\mathcal{E}\big(K^n(\lambda)\big)_t\int_0^t\int_0^1\int_{-\pi}^{\pi}\mathcal{E}\big(K^n(\lambda)\big)_{s-}^{-1}\big(I+A\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\big)^{-1}A'\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\\ &\quad\times\big(V_{s-}^n(\lambda)-W_{s-}(\alpha)\big)v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds).\end{aligned}\tag{4.5}$$

The last equality comes from the fact that N_0 and N_1 are independent, thus they never jump at the same time (a.s.), and hence $I+\Delta K_s^n(\lambda)$ is taken in account in the integral against N_0 only when the jump $\Delta K_s^n(\lambda)$ comes from N_0.

Exactly in the same way, one can compute the second derivative:

$$
\begin{aligned}
\frac{\partial^2}{\partial\lambda^2}V_t^n(\lambda) = \mathcal{E}\big(K^n(\lambda)\big)_t \int_0^t\int_0^1\int_{-\pi}^{\pi} & \mathcal{E}\big(K^n(\lambda)\big)_{s-}^{-1}\big(I + A\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\big)^{-1} \\
& \times \left[2A'\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\frac{\partial}{\partial\lambda}V_{s-}^n(\lambda) + A''\big(\gamma_n^\lambda(s,\theta,\alpha)\big)\right. \\
& \left.\times \big(V_{s-}^n(\lambda) - W_{s-}(\alpha)\big)v_n^T(s,\theta,\alpha)\right]v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds).
\end{aligned}
\tag{4.6}
$$

Here, $\frac{\partial^2}{\partial\lambda^2}V_t^n(\lambda)$ is given by

$$
\left(\frac{\partial}{\partial\lambda_x}\frac{\partial}{\partial\lambda}V_t^n(\lambda) \quad \frac{\partial}{\partial\lambda_y}\frac{\partial}{\partial\lambda}V_t^n(\lambda)\right),
$$

and we have used the notation

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x \quad y) = \begin{pmatrix} ax & bx & ay & by \\ cx & dx & cy & dy \end{pmatrix}.
$$

We will frequently use the following lemma. Recall that if M is a 2×2 matrix, then $\|M\|_{op} = \sup_{\|X\|=1}\|MX\|$.

LEMMA 4.1. – *For all* $0 \leqslant s \leqslant t$,

$$
\big\|\mathcal{E}\big(K^n(\lambda)\big)_t\mathcal{E}\big(K^n(\lambda)\big)_{s-}^{-1}\big\|_{op} \leqslant 1. \tag{4.7}
$$

To prove this lemma, we first solve the Doléans–Dade equation in a very simple case.

LEMMA 4.2. – *Let U be a $\mathcal{M}_{2\times 2}(\mathbb{R})$-valued process that can be written as the finite sum of its jumps*: *for some* $0 \leqslant T_1 < \cdots < T_k \leqslant T$,

$$
U_t = \sum_{i=1}^k \Delta U_{T_i} 1_{\{T_i \leqslant t\}}. \tag{4.8}
$$

Then

$$
\mathcal{E}(U)_t = \prod_{i=1}^k \big(I + \Delta U_{T_i} 1_{\{T_i \leqslant t\}}\big), \tag{4.9}
$$

where $\prod_{i=1}^k A_i = A_k.A_{k-1}\ldots A_1$.

Proof. – It is immediate. Since

$$
\mathcal{E}(U)_t = I + \sum_{i=1}^k 1_{\{T_i \leqslant t\}}\Delta U_{T_i}.\mathcal{E}(U)_{T_i -} \tag{4.10}
$$

it suffices to work recursively on the time intervals $[T_i, T_{i+1}[$. □

Proof of Lemma 4.1. – Let us denote by N^ε, N_0^ε, and N_1^ε the restrictions to $[0,T]\times[0,1]\times\{[-\pi,\pi]/[-\varepsilon,\varepsilon]\}$ of N, N_0, and N_1. We also set $b^\varepsilon = \int_{\{[-\pi,\pi]/[-\varepsilon,\varepsilon]\}}(1 -$

$\cos\theta)\beta(\theta)\,d\theta$. We denote by $K_t^{n,\varepsilon}(\lambda)$ the semi-martingale given by (4.2) with $\tilde{N}^\varepsilon$, N_0^ε, and b^ε instead of $\tilde{N}$, N_0, and b. A standard computation shows that

$$E\Big(\sup_{t\in[0,T]}\big\|\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_t-\mathcal{E}\big(K^n(\lambda)\big)_t\big\|^2\Big)\underset{\varepsilon\to 0}{\longrightarrow}0. \tag{4.11}$$

Furthermore, splitting $\tilde{N}^\varepsilon(d\theta\,d\alpha\,ds)$ into $N^\varepsilon(d\theta\,d\alpha\,ds)-1_{\{|\theta|\in[\varepsilon,\pi]\}}\beta(\theta)\,d\theta\,d\alpha\,ds$, one can check that

$$K_t^{n,\varepsilon}(\lambda)=\int_0^t\int_0^1\int_{-\pi}^{\pi}A\big(\gamma_n^\lambda(s,\theta,\alpha)\big)N_0^\varepsilon(d\theta\,d\alpha\,ds)+\int_0^t\int_0^1\int_{-\pi}^{\pi}A(\theta)N_1^\varepsilon(d\theta\,d\alpha\,ds). \tag{4.12}$$

Thus $K_t^{n,\varepsilon}(\lambda)$ satisfies the assumptions of Lemma 4.2. Thus, if $0\leqslant T_1\leqslant\cdots\leqslant T_k$ denote the successive times of its jumps, we know that

$$\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_t=\prod_{i=1}^k\big(I+\Delta K_{T_i}^{n,\varepsilon}(\lambda)1_{\{T_i\leqslant t\}}\big). \tag{4.13}$$

Hence, if $0\leqslant s\leqslant t$,

$$\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_t\,\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_{s-}^{-1}=\prod_{i=1}^k\big(I+\Delta K_{T_i}^{n,\varepsilon}(\lambda)1_{\{s<T_i\leqslant t\}}\big). \tag{4.14}$$

But every jump of $K^{n,\varepsilon}(\lambda)$ can be written as $A(\phi)$, for some $\phi\in[-\pi,\pi]$. One easily checks that for all ϕ, $\|I+A(\phi)\|_{op}\leqslant 1$. Thus it is clear that for all $\varepsilon>0$, all $0\leqslant s\leqslant t$,

$$\big\|\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_t\,\mathcal{E}\big(K^{n,\varepsilon}(\lambda)\big)_{s-}^{-1}\big\|_{op}\leqslant 1. \tag{4.15}$$

From (4.11), we deduce that there exists a sequence ε_k decreasing to 0 such that a.s.,

$$\sup_{t\in[0,T]}\big\|\mathcal{E}\big(K^{n,\varepsilon_k}(\lambda)\big)_t-\mathcal{E}\big(K^n(\lambda)\big)_t\big\|\underset{k\to\infty}{\longrightarrow}0. \tag{4.16}$$

One easily concludes: a.s., $\mathcal{E}(K^{n,\varepsilon_k}(\lambda))_t$ goes to $\mathcal{E}(K^n(\lambda))_t$ for all $t\in[0,T]$. Thus a.s., for all $0\leqslant s<t$, $\mathcal{E}(K^{n,\varepsilon_k}(\lambda))_t$ and $\mathcal{E}(K^{n,\varepsilon_k}(\lambda))_{s-}^{-1}$ go to $\mathcal{E}(K^n(\lambda))_t$ and $\mathcal{E}(K^n(\lambda))_{s-}^{-1}$ respectively, and hence $\mathcal{E}(K^{n,\varepsilon_k}(\lambda))_t\,\mathcal{E}(K^{n,\varepsilon_k}(\lambda))_{s-}^{-1}$ go to $\mathcal{E}(K^n(\lambda))_t\,\mathcal{E}(K^n(\lambda))_{s-}^{-1}$. □

5. Choice of the sequence of perturbations

Our aim is now to choose a sequence of perturbations such that (3.19) and (3.20) are satisfied. An easy computation, using (4.3), shows that

$$\frac{\partial}{\partial\lambda}V_T^n(0)=-\frac{1}{2}\mathcal{E}(K)_T\int_0^T\int_0^1\int_{-\pi}^{\pi}\mathcal{E}(K)_{s-}^{-1}J_n(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds), \tag{5.1}$$

where $K = K^n(0)$, which obviously does not depend on n, see (4.2) and (3.4), and where the 2×2 matrix $J_n(s, \theta, \alpha)$ is given by

$$\begin{pmatrix} v_n^x(s,\theta,\alpha)\big[\big(V_{s-}^y - W_{s-}^y(\alpha)\big) & v_n^y(s,\theta,\alpha)\big[\big(V_{s-}^y - W_{s-}^y(\alpha)\big) \\ \qquad + f(\theta)\big(V_{s-}^x - W_{s-}^x(\alpha)\big)\big] & \qquad + f(\theta)\big(V_{s-}^x - W_{s-}^x(\alpha)\big)\big] \\ v_n^x(s,\theta,\alpha)\big[-\big(V_{s-}^x - W_{s-}^x(\alpha)\big) & v_n^y(s,\theta,\alpha)\big[-\big(V_{s-}^x - W_{s-}^x(\alpha)\big) \\ \qquad + f(\theta)\big(V_{s-}^y - W_{s-}^y(\alpha)\big)\big] & \qquad + f(\theta)\big(V_{s-}^y - W_{s-}^y(\alpha)\big)\big] \end{pmatrix} \tag{5.2}$$

where $f(\theta) = (1 + \cos\theta)^{-1}\sin\theta$. The main idea for choosing v_n is the following: first, we will get rid of the random terms $\mathcal{E}(K)_T$ and $\mathcal{E}(K)_{s-}^{-1}$, by using a localization procedure at T, and by using the a.s. continuity of $\mathcal{E}(K)$ at T. Then we will compute the determinant of $\frac{\partial}{\partial\lambda} V_T^n(0)$ in the most natural way: we will write it as $ad - bc$. Then we will choose v_n^x and v_n^y in such a way that ad is large but bc is small.

Let us now define rigorously our perturbation. First, we recall the following Lemma, that can be found in [9]. This lemma uses the fact that in (SP)-2, $r \geqslant 2$, i.e. that $\int |\theta|\beta(\theta)\,d\theta = \infty$.

LEMMA 5.1. – *Assume* (SP)-1, 2. *One can build a sequence ϕ_n of positive, even, C^1 functions on $[-\theta_0, \theta_0]$ such that $\phi_n(-\theta_0) = \phi_n(\theta_0) = 0$, such that $\phi_n(\theta) \leqslant k|\theta| \wedge (1/2)$ for some $k \leqslant 1/2$, such that if*

$$\xi_n(\theta) = |\phi_n'(\theta)| + r2^{r+2}\frac{\phi_n(\theta)}{|\theta|}, \tag{5.3}$$

then $\xi_n \in L^1(\beta_0(\theta)\,d\theta)$ and $\xi_n \leqslant 1/2$, and such that there exists a sequence a_n, decreasing to 0 *when n tends to infinity, and satisfying*

$$a_n \int_{-\theta_0}^{\theta_0} \phi_n(\theta)\beta_0(\theta)\,d\theta \longrightarrow \infty, \tag{5.4}$$

$$a_n \int_{-\theta_0}^{\theta_0} |\theta|\phi_n(\theta)\beta_0(\theta)\,d\theta \longrightarrow 0. \tag{5.5}$$

Then we prove a lemma which uses assumption (SP)-3. For some $0 < \varepsilon < \eta < k < \infty$, we set

$$\mathcal{H}_s^x = \big\{\alpha \in [0,1]/\big|W_{s-}^x(\alpha) - X_0^x\big| < \varepsilon, \eta < \big|W_{s-}^y(\alpha) - X_0^y\big| < k\big\}, \tag{5.6}$$

$$\mathcal{H}_s^y = \big\{\alpha \in [0,1]/\big|W_{s-}^y(\alpha) - X_0^y\big| < \varepsilon, \eta < \big|W_{s-}^x(\alpha) - X_0^x\big| < k\big\}. \tag{5.7}$$

LEMMA 5.2. – *Assume* (SP), *and recall that $X_0 \in \mathbb{R}^2$ is fixed. There exist $q > 0$, $0 < \varepsilon < \eta < k$ such that for all $s \in [T/2, T]$,*

$$P_\alpha\big(\mathcal{H}_s^x\big) \geqslant q; \qquad P_\alpha\big(\mathcal{H}_s^y\big) \geqslant q. \tag{5.8}$$

Proof. – First we consider the constants $0<\varepsilon<\eta$ associated with X_0 by assumption (SP)-3. It is clearly possible to choose $k<\infty$ large enough, in such a way that

$$P_\alpha\big(\mathcal{H}_0^x\big)>0; \qquad P_\alpha\big(\mathcal{H}_0^y\big)>0. \tag{5.9}$$

It is thus clear from Lemma 1.6 that for all $s\in[0,T]$,

$$P_\alpha\big(\mathcal{H}_s^x\big)>0; \qquad P_\alpha\big(\mathcal{H}_s^y\big)>0. \tag{5.10}$$

On the other hand, the map $t\mapsto\mathcal{L}(V_t)=\mathcal{L}_\alpha(W_t)$ is weakly continuous, since V satisfies a quite simple S.D.E. We also know from Theorem 1.4 that for all $t>0$ (and thus for all $t\in[T/2,T]$), $\mathcal{L}(V_t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Since $\mathcal{H}_s^x$ (respectively $\mathcal{H}_s^y$) can be written as $\{W_{s-}\in O^x\}$ (respectively $\{W_{s-}\in O^y\}$) for some open subset O^x (respectively O^y) of $\mathbb{R}^2$, we deduce that the maps $s\mapsto P_\alpha(\mathcal{H}_s^x)$ and $s\mapsto P_\alpha(\mathcal{H}_s^y)$ are continuous. Since continuous functions which never vanish on a compact interval are bounded below by a strictly positive constant $q>0$, one easily concludes. □

We now are able to define our perturbation. First consider the processes on $[T-a_n,T]$ (recall that a_n and ϕ_n were defined in Lemma 5.1):

$$Z_t^{n,x}=\int_{T-a_n}^{t}\int_0^1\int_{-\pi}^{\pi}1_{\mathcal{H}_s^x}(\alpha)\phi_n(\theta)N_0(d\theta\,d\alpha\,ds), \tag{5.11}$$

$$Z_t^{n,y}=\int_{T-a_n}^{t}\int_0^1\int_{-\pi}^{\pi}1_{\mathcal{H}_s^y}(\alpha)\phi_n(\theta)N_0(d\theta\,d\alpha\,ds). \tag{5.12}$$

We fix $c>0$ (which will be chosen later), and we set

$$T_n^x=\inf\big\{t>T-a_n/Z_t^{n,x}\geqslant c\big\}, \tag{5.13}$$

$$T_n^y=\inf\big\{t>T-a_n/Z_t^{n,y}\geqslant c\big\}. \tag{5.14}$$

We now denote by $sg(x)$ the sign of x. The constant $\delta>0$ will be chosen later. We set

$$v_n^x(s,\theta,\alpha)=1_{\{\|V_{s-}-X_0\|<\delta\}}1_{[T-a_n,T_n^x\wedge T]}(s)1_{\mathcal{H}_s^x}(\alpha)sg\big(V_{s-}^y-W_{s-}^y(\alpha)\big)\phi_n(\theta), \tag{5.15}$$

$$v_n^y(s,\theta,\alpha)=-1_{\{\|V_{s-}-X_0\|<\delta\}}1_{[T-a_n,T_n^y\wedge T]}(s)1_{\mathcal{H}_s^y}(\alpha)sg\big(V_{s-}^x-W_{s-}^x(\alpha)\big)\phi_n(\theta). \tag{5.16}$$

For each n, v_n is a perturbation (see Definition 3.1), since it is predictable, and since it satisfies (3.1), (3.2), and (3.3) thanks to Lemma 5.1.

The following lemma is the key of the proof.

LEMMA 5.3. – *The following convergence holds*

$$\lim_{n\to\infty}P\big(T_n^x<T;\ T_n^y<T\big)=1. \tag{5.17}$$

Proof. – Let us just check the convergence for T_n^x.

$$
\begin{aligned}
P\left(T_n^x < T\right) &= P\left(Z_T^{n,x} \geqslant c\right)\\
&\geqslant 1 - \mathrm{e}^c E\left(\mathrm{e}^{-Z_T^{n,x}}\right)\\
&\geqslant 1 - \mathrm{e}^c \exp\left\{-\int_{T-a_n}^{T}\int_{\mathcal{H}_s^x}\int_{-\pi}^{\pi}\left(1-\mathrm{e}^{-\phi_n(\theta)}\right)\beta_0(\theta)\,d\theta\,d\alpha\,ds\right\}\\
&\geqslant 1 - \mathrm{e}^c \exp\left\{-a_n \times q \times \frac{1}{2}\int_{-\pi}^{\pi}\phi_n(\theta)\beta_0(\theta)\,d\theta\right\} \qquad (5.18)
\end{aligned}
$$

which goes to 1 thanks to Eq. (5.4). We have used Lemma 5.2 and the fact that since ϕ_n is smaller than $1/2$, $1 - \mathrm{e}^{-\phi_n} \geqslant \phi_n/2$. □

6. Conclusion

We are now able to prove Theorem 1.5. We begin with the following proposition:

PROPOSITION 6.1. – *Recall that $X_0 \in \mathbb{R}^2$ is fixed. There exist some constants $\delta > 0$, $\beta > 0$ such that a.s.,*

$$
\liminf_{n\to\infty}\left|\det \frac{\partial}{\partial\lambda}V_T^n(0)\right| \geqslant \beta 1_{\{\|V_T - X_0\| < \delta\}}. \qquad (6.1)
$$

First recall that

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}V_T^n(0) = \mathcal{E}(K)_T \int_0^T\int_0^1\int_{-\pi}^{\pi} \mathcal{E}(K)_{s-}^{-1}\left(I + A(\theta)\right)^{-1}A'(\theta)\\
\times \left(V_{s-} - W_{s-}(\alpha)\right)v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds), \qquad (6.2)
\end{aligned}
$$

where $K_t = K_t^n(0)$. First, we get rid of the random terms $\mathcal{E}(K)_T$ and $\mathcal{E}(K)_{s-}^{-1}$.

LEMMA 6.2. – *Consider*

$$
D_T^n = \int_0^T\int_0^1\int_{-\pi}^{\pi}\left(I + A(\theta)\right)^{-1}A'(\theta)\left(V_{s-} - W_{s-}(\alpha)\right)v_n^T(s,\theta,\alpha)N_0(d\theta\,d\alpha\,ds). \qquad (6.3)
$$

Then a.s.,

$$
\liminf_{n\to\infty}\left|\det\frac{\partial}{\partial\lambda}V_T^n(0)\right| = \liminf_{n\to\infty}\left|\det D_T^n\right|. \qquad (6.4)
$$

Proof. – We just have to check that a.s., when n goes to infinity,

$$
\left\|\frac{\partial}{\partial\lambda}V_T^n(0) - D_T^n\right\| \longrightarrow 0. \qquad (6.5)
$$

First, it is clear that $\|A'(\theta)\| \leqslant K$. From (5.6), (5.7), and (5.15), (5.16), we deduce that

$$
\left[\|V_{s-}\| + \|W_{s-}(\alpha)\|\right]\|v_n(s,\theta,\alpha)\| \leqslant \left[2\|X_0\| + \delta + k\right]\|v_n(s,\theta,\alpha)\|. \qquad (6.6)
$$

Thus

$$\begin{aligned}
\left\| \frac{\partial}{\partial\lambda} V_T^n(0) - D_T^n \right\| &\leqslant K \sup_{[T-a_n,T]} \left\| \mathcal{E}(K)_T\, \mathcal{E}(K)_{s-}^{-1} - I \right\| \\
&\quad \times \int_{T-a_n}^{T} \int_0^1 \int_{-\pi}^{\pi} \left[\left| v_n^x(s,\theta,\alpha) \right| + \left| v_n^y(s,\theta,\alpha) \right| \right] N_0(d\theta\, d\alpha\, ds) \\
&\leqslant K \sup_{[T-a_n,T]} \left\| \mathcal{E}(K)_T\, \mathcal{E}(K)_{s-}^{-1} - I \right\| \times \left[Z^{n,x}_{T_n^x} + Z^{n,y}_{T_n^y} \right] \\
&\leqslant K(2c+1) \sup_{[T-a_n,T]} \left\| \mathcal{E}(K)_T\, \mathcal{E}(K)_{s-}^{-1} - I \right\| \qquad (6.7)
\end{aligned}$$

thanks to the definitions of v_n, Z^n, and T_n. This term goes to 0, because the map $t \mapsto \mathcal{E}(K)_t$ is a.s. continuous at T. □

Proof of Proposition 6.1. – Thanks to the previous lemma, it suffices to check the proposition with D_T^n instead of $\frac{\partial}{\partial\lambda} V_T^n(0)$. First notice that

$$D_T^n = -\frac{1}{2} \int_0^t \int_0^1 \int_{-\pi}^{\pi} J_n(s,\theta,\alpha) N_0(d\theta\, d\alpha\, ds), \qquad (6.8)$$

where J_n was defined by (5.2). Computing the determinant in the most simple way, we write it of the form

$$\det D_T^n = \frac{1}{4} \times \left[H_T^{n,xx} H_T^{n,yy} - H_T^{n,xy} H_T^{n,yx} \right]. \qquad (6.9)$$

We want to prove that $H_T^{n,xx}$ and $H_T^{n,yy}$ are large, and that $H_T^{n,xy}$ and $H_T^{n,yx}$ are small.

First, we prove a lowerbound for $H_T^{n,xx} = \int_0^T \int_0^1 \int_{-\pi}^{\pi} J_n^{xx}(s,\theta,\alpha) N_0(d\theta\, d\alpha\, ds)$. First, we deduce from (5.6) and (5.15) that

$$\begin{aligned}
J_n^{xx}(s,\theta,\alpha) &\geqslant \left(\left| V_{s-}^y - W_{s-}^y(\alpha) \right| - |f(\theta)| \left| V_{s-}^x - W_{s-}^x(\alpha) \right| \right) \\
&\quad \times 1_{\mathcal{H}_s^x}(\alpha) 1_{\|V_{s-} - X_0\| < \delta} 1_{[T-a_n, T\wedge T_n^x]}(s) \phi_n(\theta) \\
&\geqslant \left((\eta - \delta) - |f(\theta)| (\varepsilon + \delta) \right) 1_{\mathcal{H}_s^x}(\alpha) 1_{\|V_{s-} - X_0\| < \delta} \\
&\quad \times 1_{[T-a_n, T\wedge T_n^x]}(s) \phi_n(\theta). \qquad (6.10)
\end{aligned}$$

Furthermore,

$$|f(\theta)| \leqslant \left| \frac{\sin\theta}{1+\cos\theta} \right| \leqslant \left| \frac{\theta}{1+\cos\theta_0} \right| \leqslant K|\theta|.$$

We thus obtain

$$\begin{aligned}
H_T^{n,xx} \geqslant (\eta - \delta) &\int_{T-a_n}^{T\wedge T_n^x} \int_0^1 \int_{-\pi}^{\pi} 1_{\{\|V_{s-} - X_0\| < \delta\}} 1_{\mathcal{H}_s^x}(\alpha) \phi_n(\theta) N_0(d\theta\, d\alpha\, ds) \\
- K(\varepsilon + \delta) &\int_{T-a_n}^{T} \int_0^1 \int_{-\pi}^{\pi} |\theta| \phi_n(\theta) N_0(d\theta\, d\alpha\, ds)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant (\eta-\delta)\inf_{[T-a_n,T]} 1_{\{\|V_{s-}-X_0\|<\delta\}} \times Z^{n,x}_{T\wedge T^x_n} \\
&\quad - K\int_{T-a_n}^{T\wedge T^x_n}\int_0^1\int_{-\pi}^{\pi} |\theta|\phi_n(\theta) N_0(d\theta\, d\alpha\, ds).
\end{aligned}
\tag{6.11}
$$

Thanks to (5.5), the second term clearly goes to 0 a.s. On the other hand, we know from Lemma 5.3 that

$$
\liminf_{n\to\infty} Z^{n,x}_{T\wedge T^x_n} \geqslant c. \tag{6.12}
$$

Since V is a.s. continuous at T, we deduce that $\inf_{[T-a_n,T]} 1_{\{\|V_{s-}-X_0\|<\delta\}}$ goes to $1_{\{\|V_T-X_0\|<\delta\}}$.

This way, we obtain a.s.,

$$
\liminf H^{n,xx}_T \geqslant (\eta-\delta)\times c\times 1_{\{\|V_T-X_0\|<\delta\}}. \tag{6.13}
$$

The same lowerbound holds for $\liminf_{n\to\infty} H^{n,yy}_T$.

We now compute $H^{n,xy}_T = \int_0^T\int_0^1\int_{-\pi}^{\pi} J^{xy}_n(s,\theta,\alpha) N_0(d\theta\, d\alpha\, ds)$. Thanks to the definition of v^x_n and $\mathcal{H}^x_s$,

$$
\begin{aligned}
J^{xy}_n(s,\theta,\alpha) &\leqslant \big((\varepsilon+\delta)+|f(\theta)|(k+\delta)\big) 1_{\mathcal{H}^x_s}(\alpha) \\
&\quad \times 1_{\|V_{s-}-X_0\|<\delta} 1_{[T-a_n,T\wedge T^x_n]}(s)\phi_n(\theta).
\end{aligned}
\tag{6.14}
$$

Hence,

$$
\begin{aligned}
H^{n,xy}_T &\leqslant \sup_{[T-a_n,T]} 1_{\{\|V_{s-}-X_0\|<\delta\}} \times (\varepsilon+\delta) Z^{n,x}_{T\wedge T^x_n} \\
&\quad + K\int_{T-a_n}^{T}\int_0^1\int_{-\pi}^{\pi} |\theta|\phi_n(\theta) N_0(d\theta\, d\alpha\, ds).
\end{aligned}
\tag{6.15}
$$

The second term goes to 0 a.s., thanks to (5.5). The definitions of $Z^{n,x}$ and T^x_n, and the fact that $\phi_n \leqslant 1/2$ yield that $Z^{n,x}_{T\wedge T^x_n} \leqslant c+1/2$. Finally, using the a.s. continuity of V at T, we deduce that a.s.,

$$
\limsup_{n\to\infty} \big|H^{n,xy}_T\big| \leqslant (\varepsilon+\delta)(c+1/2) 1_{\{\|V_T-X_0\|<\delta\}}. \tag{6.16}
$$

The same upperbound holds for $\limsup_{n\to\infty} |H^{n,yx}_T|$.

We finally deduce from (6.9) that

$$
\liminf_{n\to\infty} \big|\det D^n_T\big| \geqslant \big[c^2(\eta-\delta)^2-(c+1/2)^2(\varepsilon+\delta)^2\big] 1_{\{\|V_T-X_0\|<\delta\}}. \tag{6.17}
$$

Thus Proposition 6.1 will be proved if we exhibit $\delta>0$ and $c>0$ such that $c(\eta-\delta) > (c+1/2)(\varepsilon+\delta)$. Since $0<\varepsilon<\eta$, this is clearly possible: choose

$$
\delta = \frac{\eta-\varepsilon}{3}; \qquad c = \frac{\eta+2\varepsilon}{\eta-\varepsilon}. \tag{6.18}
$$

The first part of our criterion is satisfied. □

We still have to check the following result.

PROPOSITION 6.3. – *There exists a constant $K<\infty$ such that for all n,*

$$P\left(\sup_{\|\lambda\|\leqslant 1}\left\{\left\|\frac{\partial}{\partial\lambda}V_T^n(\lambda)\right\|+\left\|\frac{\partial^2}{\partial\lambda^2}V_T^n(\lambda)\right\|\right\}\leqslant K\right)=1. \tag{6.19}$$

Proof. – First, we prove a Lipschitz property (in λ) for $V_t^n(\lambda)$. Setting $U_t^n(\lambda,\mu)=V_t^n(\lambda)-V_t^n(\mu)$,

$$\begin{aligned}U_t^n(\lambda,\mu)=&\int_0^t dK_s^n(\lambda).\,U_{s-}^n(\lambda,\mu)+\int_0^t\int_0^1\int_{-\pi}^{\pi}\left[A\left(\gamma_n^\lambda(s,\theta,\alpha)\right)-A\left(\gamma_n^\mu(s,\theta,\alpha)\right)\right]\\&\times\left[V_{s-}^n(\mu)-W_{s-}(\alpha)\right]N_0(d\theta\,d\alpha\,ds).\end{aligned} \tag{6.20}$$

Thus, using again the result of Jacod [13],

$$\begin{aligned}U_t^n(\lambda,\mu)=\mathcal{E}\left(K^n(\lambda)\right)_t\int_0^t\int_0^1\int_{-\pi}^{\pi}&\mathcal{E}\left(K^n(\lambda)\right)_{s-}^{-1}\left(I+A\left(\gamma_n^\lambda(s,\theta,\alpha)\right)\right)^{-1}\\&\times\left[A\left(\gamma_n^\lambda(s,\theta,\alpha)\right)-A\left(\gamma_n^\mu(s,\theta,\alpha)\right)\right]\\&\times\left[V_{s-}^n(\mu)-W_{s-}(\alpha)\right]N_0(d\theta\,d\alpha\,ds).\end{aligned} \tag{6.21}$$

But, since $|\gamma_n^\lambda(s,\theta,\alpha)|\leqslant\theta_0<\pi$, it is clear that $\|(I+A(\gamma_n^\lambda(s,\theta,\alpha)))^{-1}\|\leqslant K$. Furthermore, one easily checks that

$$\left\|A\left(\gamma_n^\lambda(s,\theta,\alpha)\right)-A\left(\gamma_n^\mu(s,\theta,\alpha)\right)\right\|\leqslant K\|\lambda-\mu\|\times\|\upsilon_n(s,\theta,\alpha)\|. \tag{6.22}$$

Using also Lemma 4.1, we deduce that

$$\begin{aligned}\left\|U_t^n(\lambda,\mu)\right\|\leqslant K\|\lambda-\mu\|\int_0^t\int_0^1\int_{-\pi}^{\pi}&\left[\left\|V_{s-}^n(\mu)\right\|+\|W_{s-}(\alpha)\|\right]\\&\times\|\upsilon_n(s,\theta,\alpha)\|N_0(d\theta\,d\alpha\,ds).\end{aligned} \tag{6.23}$$

In particular, if $\mu=0$,

$$\|V_t^n(\lambda)\|\leqslant\|V_t\|+KY, \tag{6.24}$$

where

$$\begin{aligned}Y&=\int_0^T\int_0^1\int_{-\pi}^{\pi}\left[\|V_{s-}\|+\|W_{s-}(\alpha)\|\right]\times\|\upsilon_n(s,\theta,\alpha)\|N_0(d\theta\,d\alpha\,ds)\\&\leqslant[2\|X_0\|+\delta+k]\times\left(Z^{n,x}_{T_n^x}+Z^{n,y}_{T_n^y}\right)\\&\leqslant[2\|X_0\|+\delta+k]\times(2c+1)\leqslant K.\end{aligned} \tag{6.25}$$

We have used the definitions of υ_n, Z_n, T_n, and $\mathcal{H}_s$.

Let us now turn to the first derivative. We use expression (4.5). Using the same arguments as above, and inequalities (6.24) and (6.25),

$$\begin{aligned}\left\|\frac{\partial}{\partial\lambda}V_t^n(\lambda)\right\| &\leqslant K\int_0^T\int_0^1\int_{-\pi}^{\pi}\left(\left\|V_{s-}^n(\lambda)\right\|+\|W_{s-}(\alpha)\|\right)\|v_n(s,\theta,\alpha)\|N_0(d\theta\,d\alpha\,ds)\\ &\leqslant K\int_0^T\int_0^1\int_{-\pi}^{\pi}\left(\|V_{s-}\|+Y+\|W_{s-}(\alpha)\|\right)\|v_n(s,\theta,\alpha)\|N_0(d\theta\,d\alpha\,ds)\\ &\leqslant K\times\left(Z^{n,x}_{T_n^x}+Z^{n,y}_{T_n^y}\right)\\ &\leqslant K. \end{aligned} \tag{6.26}$$

Exactly in the same way, one can check that for some constant K, for all λ, t, n,

$$\left\|\frac{\partial^2}{\partial\lambda^2}V_t^n(\lambda)\right\|\leqslant K. \tag{6.27}$$

We thus have proved Proposition 6.3. □

We are now able to conclude.

Proof of Theorem 1.5. – We have fixed $X_0\in\mathbb{R}^2$, and we have found a sequence of perturbations such that, for some $\beta>0$, $\delta>0$, $K<\infty$,

$$\text{a.s.,}\quad \liminf_{n\to\infty}\left|\det\frac{\partial}{\partial\lambda}V_T^n(0)\right|\geqslant\beta 1_{\{\|V_T-X_0\|\leqslant\delta\}}, \tag{6.28}$$

$$\forall n\in\mathbb{N}^*,\quad P\left(\sup_{\|\lambda\|\leqslant 1}\left\{\left\|\frac{\partial}{\partial\lambda}V_T^n(\lambda)\right\|+\left\|\frac{\partial^2}{\partial\lambda^2}V_T^n(\lambda)\right\|\right\}\leqslant K\right)=1, \tag{6.29}$$

from which we easily deduce, for all $r>0$,

$$\begin{aligned}\liminf_n P\Bigg(&\|V_T-X_0\|\leqslant r;\ \left|\det\frac{\partial}{\partial\lambda}V_T^n(0)\right|\geqslant\beta/2;\\ &\sup_{\|\lambda\|\leqslant 1}\left\{\left\|\frac{\partial}{\partial\lambda}V_T^n(\lambda)\right\|+\left\|\frac{\partial^2}{\partial\lambda^2}V_T^n(\lambda)\right\|\right\}\leqslant K\Bigg)\geqslant P(\|V_T-X_0\|\leqslant r\wedge\delta).\end{aligned} \tag{6.30}$$

It is thus clear that every X_0 in the support of $\mathcal{L}(V_T)$ satisfies the assumptions of Theorem 3.3. Applying Remark 3.5 drives immediately to the conclusion. □

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