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INTEGRATED DENSITY OF STATES OF SELF-SIMILAR STURM–LIOUVILLE OPERATORS AND HOLOMORPHIC DYNAMICS IN HIGHER DIMENSION

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ABSTRACT. – We investigate the integrated density of states of a Sturm–Liouville operator $\frac{d}{dm}\frac{d}{dx}$ when the measure m is constructed from a self-similar measure on the interval $[0, 1]$. We show that this involves the dynamics of a rational map on the complex projective plane $\mathbb{P}^2$, and we give an explicit formula for the integrated density of states in terms of the Green function of this map. This allows to deduce several results on the structure of the integrated density of states by a study of the dynamics of this map. This operator is a particular case of the so-called diffusions on self-similar sets and is relevant in this context. Indeed it is the first example, except for the sets of the Sierpinski gasket type (usually called decimable), where a connection is established between the spectrum of the operator and the dynamics of the iterates of a certain rational map. Therefore it is a new step toward a generalization of the initial work of Rammal and Toulouse (1983) and Rammal (1984). © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous nous intéresserons à la densité d'états intégrée d'un opérator de Sturm–Liouville $\frac{d}{dm}\frac{d}{dx}$ quand m est construite à partir d'une mesure auto-similaire sur $[0, 1]$. Ceci implique la dynamique d'une application définie sur le plan projectif complexe $\mathbb{P}^2$. Nous donnons une formule explicite reliant la densité d'états à la fonction de Green de cette application. Une étude de la dynamique de cette application permet de donner plusieurs résultats sur la structure de la densité d'états intégrée. Cette étude est surtout significative dans le cadre des diffusions sur des ensembles auto-similaires car elle présente le premier exemple, hors les ensembles du type du Sierpinski gasket (appelés décimables), où une relation est établie entre le spectre de l'opérateur et la dynamique d'une certaine application rationnelle et donc une nouvelle étape vers une généralisation des travaux de Rammal–Toulouse (1983) et Rammal (1984). © 2001 Éditions scientifiques et médicales Elsevier SAS

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In this text we will be interested in a Sturm–Liouville operator $\frac{d}{dm}\frac{d}{dx}$ when m is a self-similar measure on the interval $[0,1]$. This operator is a particular case of a class of operators nicknamed "self-similar laplacians on self-similar sets" or "diffusions on fractals" (cf, for example, [1]). These operators have received a certain attention these last two decades, and of particular importance is the understanding of the structure of their spectrum. A very striking and interesting aspect of these operators is the relations between their spectral properties and the iteration of certain rational maps.

These relations have been discovered and initially investigated by Rammal and Toulouse [23] and Rammal [22] in the case of the Sierpinski graph which is an infinite graph based on the Sierpinski gasket. This graph is constructed as an increasing sequence of finite graphs and they exhibited a polynomial map $z \to z(5-z)$ that relates the spectrum of the difference operators defined on 2 successive graphs: precisely they showed that if λ is an eigenvalue at step $n+1$ then $\lambda(5-\lambda)$ is an eigenvalue at step n. This law was usally called the spectral decimation of the Sierpinski gasket. Thanks to this law and to a functional equation relating some functions of the spectrum on successive steps, Rammal gave a beautiful description of the spectrum of the discrete operator defined on the Sierpinski graph. Using similar ideas, Fukushima and Shima investigated the spectrum of the continuous operator on the Sierpinski gasket itself [11].

The Sierpinski gasket is a particular example of the class of finitely ramified self-similar sets (or p.c.f. self-similar sets, cf [14]): on such sets one can define some self-similar operators which play the role of a Laplace operator. At the exception of some particular cases like the Sierpinski gasket and the class of decimable sets introduced in [12], the spectrum of these operators do not satisfy a law of decimation, i.e., one cannot find a rational map relating the spectrum of the operator on successive scales, and the legitimate question of how to generalize the initial work of Rammal is not answered. Working on this question we were able to identify several new objects that seem important to understand the general situation. We present them on a particular example where we are able to relate explicitely the spectrum of the operator to the dynamics of a certain renormalization map defined on the complex projective plane $\mathbb{P}^2$.

Let us now describe our model and our results. Consider the unit interval $I=[0,1]$ and a real α, $0<\alpha<1$. The 2 homotheties $\Psi_1(x)=\alpha x$ and $\Psi_2(x)=1-(1-\alpha)(1-x)$ give the structure of a self-similar set to the interval $I=[0,1]$, i.e., we have $I=\Psi_1(I)\cup\Psi_2(I)$ (the interval I is even finitely ramified or a p.c.f. self-similar set in the setting of [14]). For a real $0<b<1$ we consider the unique probability measure on $[0,1]$ such that:

$$\int f\,dm = b\int f\circ\Psi_1\,dm + (1-b)\int f\circ\Psi_2\,dm, \tag{0.1}$$

for all continuous function $f\in C([0,1])$. For $b\neq\alpha$ the measure m is singular with respect to the Lebesgue measure (for $\alpha=b$ it is the Lebesgue measure). We will be interested in the Sturm–Liouville operator $L=\frac{d}{dm}\frac{d}{dx}$ defined by $Lf=g$ if and only if $f(x)=ax+b+\int_0^x\int_0^y g(u)\,dm(u)\,dy$, for some reals a and b. We choose either Neuman or Dirichlet boundary conditions on $[0,1]$. It is easy to see that the associated Dirichlet form $a(f,g)=\int_0^1 Lfg\,dm=\int_0^1 f'g'\,dx$ satisfies the following self-similarity relation:

$$a(f)=\alpha^{-1}a(f\circ\Psi_1)+(1-\alpha)^{-1}a(f\circ\Psi_2), \tag{0.2}$$

N.B.: we will always write $a(f)$ for $a(f, f)$ when a is a quadratic form.

The relations (0.1) and (0.2) are the 2 relations which characterise a "self-similar Dirichlet space", the operator L is therefore a self-similar Laplacian as defined for example in [1,14,25].

In all the text we will restrict our study to a particular choice for b: $b = 1 - \alpha$. This ensures that the scaling in time $\gamma_1 = (\alpha b)^{-1}$ in the left sub-interval $\Psi_1(I)$ is equal to the scaling in time $\gamma_2 = ((1 - \alpha)(1 - b))^{-1}$ in the right sub-interval $\Psi_2(I)$. We set $\gamma = \gamma_1 = \gamma_2 = (\alpha(1 - \alpha))^{-1}$. In Section 1.2.2 we justify this choice in connexion with the assumptions usually considered in the case of random Schrödinger operators.

The measure m is extended to the interval $I_{\langle n\rangle} = \Psi_1^{-n}(I)$ by the measure $m_{\langle n\rangle}$ defined by:

$$\int_0^{\alpha^{-n}} f\, dm_{\langle n\rangle} = b^{-n} \int_0^1 f \circ \Psi_1^{-n}\, dm. \tag{0.3}$$

We denote by $\nu^+_{\langle n\rangle}$ (resp. $\nu^-_{\langle n\rangle}$) the counting measures of the solutions of the eigenvalue problem $\frac{d}{dm_{\langle n\rangle}} \frac{d}{dx} f = \lambda f$ with Neuman (resp. Dirichlet) boundary conditions on $I_{\langle n\rangle}$. It is clear by construction that the eigenvalues on $I_{\langle n\rangle}$ are the image by a scaling of ratio γ^{-n} of the eigenvalue on I. We call the integrated density of states the weak limit (when it exists) of the sequence

$$\lim_{n\to\infty} \frac{1}{2^n} \nu^{\pm}_{\langle n\rangle}, \tag{0.4}$$

and we denote it by μ (here we follow the maybe confusing terminology of [3] and [21], in which the integrated density of states is the measure that counts the number of eigenvalues per unit volume at some energy level).

Let us now introduce the renormalization map. Denote by $F = \{0, 1\}$ the set of boundary points of I and set $F^{(1)} = \{0, \alpha, 1\} = \Psi_1(F) \cup \Psi_2(F)$. With a quadratic form Q on $\mathbb{R}^F$ we associate a quadratic form $Q^{(1)}$ on $\mathbb{R}^{F^{(1)}}$ by the following formula:

$$Q^{(1)}(f) = \alpha^{-1} Q(f \circ \Psi_1) + (1 - \alpha)^{-1} Q(f \circ \Psi_2), \quad \forall f \in \mathbb{R}^{F^{(1)}}. \tag{0.5}$$

If Q is a positive quadratic form, then so is $Q^{(1)}$, and we can define the "trace"on the subset F of the quadratic form $Q^{(1)}$ as the quadratic form $Q^{(1)}_F$ on $\mathbb{R}^F$ given by:

$$Q^{(1)}_F(f) = \inf_{g|_F = f} Q^{(1)}(g) = Q^{(1)}(H_Q f), \quad \forall f \in \mathbb{R}^F, \tag{0.6}$$

where $H_Q f$ is the harmonic continuation of f with respect to the positive quadratic form $Q^{(1)}$. We define the renormalization operator T by:

$$TQ = Q^{(1)}_F. \tag{0.7}$$

Using the representation by a symmetric matrix of the form

$$q_1 \begin{pmatrix} q_1 & q \\ q & q_2 \end{pmatrix} \tag{0.8}$$

a quadratic form Q on $\mathbb{R}^F$ can be represented by the 3-tuple (q_1, q_2, q). With these coordinates the map T is given by:

$$T\big((q_1,q_2,q)\big)=\frac{\alpha^{-1}}{q_1+\delta^{-1}q_2}\big(q_1\big(q_1+\delta^{-1}q_2\big)-\delta^{-1}q^2,\delta q_2\big(q_1+\delta^{-1}q_2\big)-\delta q^2,-q^2\big),\tag{0.9}$$

where we set $\delta=\frac{\alpha}{1-\alpha}$. The map T can be extended to $\mathbb{C}^3\setminus\{q_1+\delta^{-1}q_2=0\}$. Consider the map $R:\mathbb{C}^3\to\mathbb{C}^3$ obtained from T by removing the singularities, i.e., we set: $R((q_1,q_2,q))=\alpha(q_1+\delta^{-1}q_2)T((q_1,q_2,q))$. The maps T and R are respectively 1 and 2-homogeneous and induce the same map f defined on the complex projective plane $\mathbb{P}^2$ (in fact f can only be defined on $\mathbb{P}^2$ minus one point called a point of indeterminacy, cf Section 3.1) and given by the following formula:

$$f([x,y,z])=\big[x\big(x+\delta^{-1}y\big)-\delta^{-1}z^2,\delta y\big(x+\delta^{-1}y\big)-\delta z^2,-z^2\big]\tag{0.10}$$

($[x,y,z]$ denotes the image in $\mathbb{P}^2$ of the point $(x,y,z)\in\mathbb{C}^3$, following the usual notations, cf [27]).

A lot of information on the dynamics of this map is contained in the Green function: this function $G:\mathbb{C}^3\to\mathbb{R}\cup\{-\infty\}$ is defined as the limit:

$$G(x)=\lim_{n\to\infty}\frac{1}{2^n}\log\big\|R^n(x)\big\|,\quad x\in\mathbb{C}^3.\tag{0.11}$$

The function G has the important property of being plurisubharmonic (this means that it is subharmonic when restricted to complex lines, and satisfies some smoothness conditions).

We now introduce the last ingredient necessary to state our result. For $\lambda\geqslant 0$ we set $a_\lambda(f)=a(f)+\lambda\int f^2\,dm$. The quadratic form a_λ defines a regular Dirichlet form on I and we can consider its trace (in the sense of [9, Section 6]) on the subset $F=\{0,1\}$ defined as the Dirichlet form $A_{(\lambda)}$ on $\mathbb{R}^F$ given by:

$$A_{(\lambda)}(f)=\inf\{a_\lambda(g),g_{|F}=f\}=a_\lambda(H_\lambda f),\quad f\in\mathbb{R}^F,\tag{0.12}$$

where $H_\lambda f$ denotes the harmonic continuation of f with respect to the Dirichlet form a_λ. The function $\lambda\to A_{(\lambda)}$ can be extended into a meromorphic function on $\mathbb{C}$ with poles included in the Dirichlet spectrum of a. The choice we made for the parameter b implies that the curve $A_{(\lambda)}$ is invariant by T, more precisely we have:

$$T(A_{(\lambda)})=A_{(\gamma\lambda)}.\tag{0.13}$$

The map R introduced previously leaves the hyperplan $\{q=-1\}$ invariant and if we denote by $\phi(\lambda)$ the projection of $A_{(\lambda)}$ on the hyperplan $\{q=-1\}$ then $\lambda\to\phi(\lambda)$ defines a holomorphic curve invariant by the map R, i.e., we have:

$$R\big(\phi(\lambda)\big)=\phi(\gamma\lambda).\tag{0.14}$$

We are now ready to state the main result of this text given in Theorem 3.1. We prove that the integrated density of states exists and has the following expression:

$$\mu=\frac{1}{2\pi}\Delta(G\circ\phi).\tag{0.15}$$

It is well-known that in the case of 1-dimensional Schrödinger operators the Lyapunov exponent and the integrated density of states are related by the so-called Thouless formula. This is also true here and in fact we define a certain Lyapunov exponent $\zeta(\lambda)$ (cf formula (3.51)) associated with the differential equation $\frac{d}{dm}\frac{d}{dx} = \lambda f$ on $\mathbb{R}_+$ and we show that:

$$\zeta(\lambda) = G \circ \phi(\lambda). \tag{0.16}$$

From formula (0.15) and an explicit analysis of the dynamics of f we deduce several results on the structure of the measure μ: we prove that it charges no point and that for $\alpha \neq \frac{1}{2}$ it is supported by a Cantor subset of $\mathbb{R}_-$ (for $\alpha = \frac{1}{2}$, the operator is the usual Laplacian, and we recover the classical results). Using a result of [27] on the Hölder regularity of the Green function, we are also able to prove the Hölder regularity of the integrated density of states and of the Lyapunov exponent for some values of the parameter δ. We don't know wether this restriction on the values of the parameter comes from our technique or from a deeper phenomenon.

Let us also mention that we simultaneously treat the case of the natural underlying discrete operator defined on the so called pre-fractal. We get the same expression for the integrated density of states when the complex curve $\phi(\lambda)$ is replaced by a complex line $\tilde{\phi}(\lambda)$. Finally we would like to point out that the technics we use are quite different from the one developped for 1-dimensional random Schrödinger operators. In particular we make no use of the propagator of the associated differential equation in the investigation of the integrated density of states and we prove separately formula (0.15) and formula (0.16) (we do not deduce (0.15) from (0.16) as in the Thouless formula). This comes from the fact that we developped these technics in our attempt to understand the general situation of finitely ramified fractals where most of the 1-dimensional technics break down.

In the course of the text we introduce several new objects, most of them are general to finitely ramified self-similar sets and we would like to emphasize and clarify the role of some of them.

The good renormalization map to be considered is the map on the projective space associated with T. In particular it is defined on the projective space associated with the set of quadratic forms on $\mathbb{R}^F$ invariant by the "natural" group of symmetries of the problem (in general it is natural to associate with our self-similar set a group of isometries, enventually empty, which leave invariant the structure; this group is then considered as the natural group of symmetries of the set, cf [25]). For the Sierpinski gasket, due to the symmetries of the problem, the set of quadratic forms is of dimension 2 and the map is then defined on the 1-dimensional projective space. The map R we introduced is the natural lift on $\mathbb{C}^3$ of the map f (cf [8] or [27]). The relation between the maps T and R is in general difficult to analyse. It also appears that the theory of iteration of rational maps of $\mathbb{P}^k$, as it has been recently developped, in particular by Fornaess and Sibony (cf [6–8,27]), is crucial in our problem. This promises a rich and complicated general picture since the iteration of rational maps of higher dimension contains many new phenomenons compared with the one-dimensional situation.

The map $\phi(\lambda)$ makes the link between the spectral problem which is something 1-dimensional, and the renormalization map which is intrisically defined on a space of higher dimension. In the case of the Sierpinski gasket the renormalization map was expressed directly on the spectral parameter λ; this was possible since the parameter

λ could in itself parametrize the projective space of quadratic forms. In general the renormalization map does not act on the parameter λ but on a bigger space, which has a priori nothing to do with the spectral problem.

In the course of the proof of formula (0.15) we introduce a sequence of plurisubharmonic functions $H^{\pm}_{\langle n\rangle}$. Most of the information on the density of states is contained in the limit of $\frac{1}{2^n}H^{\pm}_{\langle n\rangle}$. These functions satisfy a functional equation involving the renormalization map T. This functional equation already appeared in the case of the Sierpinski gasket in the work of Rammal [22] and is the tool to relate the limit of $\frac{1}{2^n}H^{\pm}_{\langle n\rangle}$ to the dynamics of the renormalization map. In general it is not easy to analyse this functional equation and the limit of these functions. In this example most of the difficulties are overcome by explicit computations (cf the proof of formula (2.45)).

Let us now describe the organization of the paper. In Section 1 we settle the notations, definitions and basic properties. In Section 2 we introduce the curve $A_{(\lambda)}$, the renormalization map T and the sequence of plurisubharmonic functions $H^{\pm}_{\langle n\rangle}$. The two main results of this section are Proposition 2.1 where we relate the counting measures $\nu^{\pm}_{\langle n\rangle}$ to the sequence of functions $H^{\pm}_{\langle n\rangle}$ and to the curve $A_{(\lambda)}$ and Proposition 2.3 where we establish the functional equation and deduce an expression of $H^{\pm}_{\langle n\rangle}$ in terms of the iterates of the map R. In Section 3 we first describe the dynamics of f and introduce the Green function and the holomorphic curve $\phi(\lambda)$. We prove the formula on the integrated density of states in Theorem 3.1; the main step is to prove that the sequence $\frac{1}{2^n}H^{\pm}_{\langle n\rangle}$ converges to the Green function. Thanks to the expression we got in Section 2 this reduces to prove that the current of integration on the preimages of a certain rational curve, suitly renormalized, converges to the Green current. This is a classical problem (cf [27]) but here cannot be deduced from general results. In Section 3.4 we introduce the Lyapunov exponent $\zeta(\lambda)$ and give its expression in terms of the Green function. In Section 3.5 we give a result on the Hölder regularity of the integrated density of states and of the Lyapunov exponent. In Section 4 we make some remarks and conjectures about the problem on general finitely ramified fractals.

1. Self-similar diffusions on the interval [0, 1]

1.1. Notations

Let $I=[0,1]$ be the unit interval and α be a real such that $0<\alpha<1$. We set $\delta=\frac{\alpha}{1-\alpha}$. We define the two $\mathbb{R}$-similitudes Ψ_1,Ψ_2 by:

$$\Psi_1(x)=\alpha x,\qquad \Psi_2(x)=1-(1-\alpha)(1-x).$$

So, $\Psi_1(I)=[0,\alpha]$, $\Psi_2(I)=[\alpha,1]$ and the interval I is self-similar with respect to (Ψ_1,Ψ_2).

Let b be a real such that $0<b<1$. It is classical that there exists a unique probability measure m on I such that

$$\int_0^1 f\,dm=b\int_0^1 f\circ\Psi_1\,dm+(1-b)\int_0^1 f\circ\Psi_2\,dm,\quad \forall f\in C(I). \tag{1.1}$$

N.B.: This measure is the image by $\pi : \{0,1\}^{\mathbb{N}} \to I$ defined by $\pi((\varepsilon_0, \ldots)) = \sum \varepsilon_i \alpha^i$ of the product of Bernoulli measures on $\{0,1\}$ with parameter $1-b$. For $\alpha \neq b$, the measure m is singular with respect to the Lebesgue measure, for $\alpha = b$ it is the Lebesgue measure.

We denote by L^+ the operator $\frac{d}{dm}\frac{d}{dx}$ with Neuman boundary condition on I, i.e., it is the operator defined on the domain:

$$\left\{ f \in L^2(I,m), \exists g \in L^2(I,m),\ f(x) = ax + b + \int_0^x \int_0^y g(z)\,dm(z)\,dy,\right.$$

$$\left. f'(0) = f'(1) = 0 \right\}, \quad \text{by } L^+ f = g. \tag{1.2}$$

We denote by L^- the corresponding operator with Dirichlet boundary conditions on I.

In this text we will often take the point of view of Dirichlet forms since the self similarity of the process can be read very easily on the measure and on the Dirichlet form (it appears clearly now that Dirichlet forms are the most tractable objects when considering self-similar operators on self-similar sets, cf for example [1]). The operator L^+ is the infinitesimal generator of the regular Dirichlet space $(a, \mathcal{D})$ defined by:

$$\mathcal{D} = \left\{ f \in L^2(I,m),\ \ f \text{ is absolutely continuous and } f' \in L^2(I,dx) \right\}, \tag{1.3}$$

$$a(f,g) = \int_0^1 f'g'\,dx, \quad \forall f,g \in \mathcal{D} \tag{1.4}$$

(cf [9, Example 1.2.2]). The operator L^- is associated with a restricted to the domain

$$\mathcal{D}^- = \{ f \in \mathcal{D},\ f(0) = f(1) = 0 \}. \tag{1.5}$$

In fact, the Markov process canonically associated with $(a, \mathcal{D})$ on $L^2(I,m)$ is a time changed process of the usual Brownian motion. More precisely, let (B_t, P_x) be the usual reflected Brownian motion on I and $\mathcal{A}_t$ be the additive functional defined by $\mathcal{A}_t = \int_0^1 L_t^x\,dm(x)$, where L_t^x denotes the local time at point x, then the process (B_{τ_t}, P_x), where $\tau_t = \inf\{s, \mathcal{A}_s \geqslant t\}$, is the Markov process associated with $(a, \mathcal{D})$ on $L^2(I,m)$. We denote this process by (X_t, P_x).

By a change of variables we see that a satisfies:

$$a(f) = \alpha^{-1} a(f \circ \Psi_1) + (1-\alpha)^{-1} a(f \circ \Psi_2), \quad \forall f,g \in \mathcal{D}. \tag{1.6}$$

N.B.: Here and in the sequel we simply write $a(f)$ for $a(f,f)$ when a is a symmetric bilinear form.

Let $\gamma_1 = (\alpha b)^{-1}$, $\gamma_2 = (1-\alpha)^{-1}(1-b)^{-1}$ and a_λ be the Dirichlet form defined by:

$$a_\lambda(f,g) = a(f,g) + \lambda \int fg\,dm, \quad \forall f,g \in \mathcal{D}. \tag{1.7}$$

We can combine (1.1) and (1.6) in

$$a_\lambda(f) = \alpha^{-1} a_{\gamma_1^{-1}\lambda}(f \circ \Psi_1) + (1-\alpha)^{-1} a_{\gamma_2^{-1}\lambda}(f \circ \Psi_2), \quad \forall f \in \mathcal{D}. \tag{1.8}$$

These properties can be translated into scaling relations on the process. Precisely, let $T = \inf\{s, X_s \in \{0,1\}\}$ and $T_i = \inf\{s, X_s \in \Psi_i(\{0,1\})\}$, $i = 1, 2$, then the following equality between Markov processes holds:

$$\big(\Psi_i(X_{t\wedge T}), P_x\big) = (X_{\gamma_i^{-1}t\wedge T_i}, P_{\Psi_i(x)}). \tag{1.9}$$

Remark 1.1. – These are the scaling relations of the process, but we must note that they involve a discrete range of scales and not the real line like for the usual Brownian motion. This, very roughly, can predict the phenomenon of oscillation in the asymptotic distribution of eigenvalues.

From now on we make the following choice for the measure m:

(H) We choose $b = 1-\alpha$, so that $\gamma_1 = \gamma_2 = \alpha^{-1}(1-\alpha)^{-1}$ (that we denote γ).

This choice implies that the process spends the same time in each of the intervals $\Psi_1(I)$ and $\Psi_2(I)$.

1.2. Extension of the states space. Definition of the integrated density of states

1.2.1. The continuous case

Here we extend the states space I, the Dirichlet form a and the measure m by a natural scaling. Let $I_{\langle n\rangle} = \Psi_1^{-n}(I) = [0, \alpha^{-n}]$. The set $I_{\langle n\rangle}$ is the union of 2^n intervals "identical" to I. Indeed, for $(i_1, \dots, i_n) \in \{1,2\}^n$ we set $\Psi_{i_1,\dots,i_n} = \Psi_{i_n} \circ \cdots \circ \Psi_{i_1}$ and $I_{i_1,\dots,i_n} = \Psi_{i_1,\dots,i_n}(I_{\langle n\rangle})$, so that $I_{1,\dots,1} = I$ and

$$I_{\langle n\rangle} = \bigcup_{i_1,\dots,i_n} I_{i_1,\dots,i_n}. \tag{1.10}$$

We extend the measure m to the states space $I_{\langle n\rangle}$: we define $m_{\langle n\rangle}$ by

$$\int_{I_{\langle n\rangle}} f\, dm_{\langle n\rangle} = (1-\alpha)^{-n} \int_I f \circ \Psi_1^{-n}\, dm, \quad f \in C(I_{\langle n\rangle}). \tag{1.11}$$

We denote by $L^+_{\langle n\rangle}$ (resp. $L^-_{\langle n\rangle}$) the Sturm–Liouville operator $\frac{d}{dm_{\langle n\rangle}}\frac{d}{dx}$ with Neuman (resp. Dirichlet) boundary conditions. Of course $L^+_{\langle n\rangle}$ is the infinitesimal generator of the Dirichlet space $(a_{\langle n\rangle}, \mathcal{D}_{\langle n\rangle})$ given by:

$$\mathcal{D}_{\langle n\rangle} = \big\{f \in L^2(I_{\langle n\rangle}, m_{\langle n\rangle}),\ f' \text{ exists and } f' \in L^2(I_{\langle n\rangle}, dx)\big\}, \tag{1.12}$$

$$a_{\langle n\rangle}(f) = \int_0^{\alpha^{-n}} (f')^2\, dx = \alpha^n a\big(f \circ \Psi_1^{-n}\big), \quad \forall f \in \mathcal{D}_{\langle n\rangle}. \tag{1.13}$$

The operator $L^-_{\langle n\rangle}$ is associated with $a_{\langle n\rangle}$ when restricted to the corresponding Dirichlet domain $\mathcal{D}^-_{\langle n\rangle}$.

These formulas define continuations of a and m since if $f \in \mathcal{D}_{\langle n\rangle}$ is such that $\operatorname{supp}(f) \subset [0,1]$ then $a_{\langle n\rangle}(f) = a(f)$ and $\int f\, dm_{\langle n\rangle} = \int f\, dm$.

The sequence of measures $m_{\langle n\rangle}$ induces a measure $m_{\langle\infty\rangle}$ on $I_{\langle\infty\rangle} = \mathbb{R}_+$. We denote by $L^+_{\langle\infty\rangle}$ and $(a_{\langle\infty\rangle}, \mathcal{D}_{\langle\infty\rangle})$ (resp. $L^-_{\langle\infty\rangle}$ and $(a_{\langle n\rangle}, \mathcal{D}^-_{\langle\infty\rangle})$) the corresponding Sturm–Liouville operator and Dirichlet space with Neuman (resp. Dirichlet) boundary condition at point 0.

Let $0=\lambda^+_{\langle n\rangle,0}>\lambda^+_{\langle n\rangle,1}\geqslant\cdots\geqslant\lambda^+_{\langle n\rangle,k}\geqslant\cdots$ be the list of eigenvalues of the operators $L^+_{\langle n\rangle}$ (i.e., the solutions of $\frac{d}{dm_{\langle n\rangle}}\frac{d}{dx}=\lambda f$ with Neuman boundary conditions on $I_{\langle n\rangle}$). We consider the counting measure:

$$\nu^+_{\langle n\rangle}=\sum_{k=0}^{\infty}\delta_{\lambda^+_{\langle n\rangle,k}}. \tag{1.14}$$

N.B.: δ_x denotes the Dirac mass at the point x.

Let $0>\lambda^-_{\langle n\rangle,1}\geqslant\lambda^-_{\langle n\rangle,2}\geqslant\cdots\geqslant\lambda^-_{\langle n\rangle,k}\geqslant\cdots$ be the list of eigenvalues of $L^-_{\langle n\rangle}$ (i.e., the solutions of $\frac{d}{dm_{\langle n\rangle}}\frac{d}{dx}=\lambda f$ with Dirichlet boundary conditions on $I_{\langle n\rangle}$). We consider the counting measure:

$$\nu^-_{\langle n\rangle}=\sum_{k=1}^{\infty}\delta_{\lambda^-_{\langle n\rangle,k}}. \tag{1.15}$$

DEFINITION-PROPOSITION 1.1. – *If $\frac{1}{2^n}\nu^\pm_{\langle n\rangle}$ converges weakly to the* (*same*) *measure μ, we say that the integrated density of states of the operator $\frac{d}{dm_{\langle\infty\rangle}}\frac{d}{dx}$ on $\mathbb{R}_+$ exists and is μ. Its repartition function, denoted by $F(\lambda)=\int_0^\lambda d\mu$, $\lambda\leqslant 0$, satisfies*:

$$F(\gamma\lambda)=2F(\lambda). \tag{1.16}$$

Remark 1.2. – We adopt here the terminology of [3] and [21] even if it is a bit misleading. The integrated density of states is the measure μ, not its repartition function.

Remark 1.3. – The existence of the integrated density of states (and the fact that the weak limit is the same for $\nu^+_{\langle n\rangle}$ and $\nu^-_{\langle n\rangle}$) can be proved directely using the technics of [10] but our aim is to investigate some of its fine properties.

Remark 1.4. – The formula (1.16) also means that the repartition function can be written $F(\lambda)=\lambda^\rho g(\lambda)$ where $\rho=\log\gamma/\log 2$ and $g(\lambda)$ is a positive function satisfying $g(\gamma\lambda)=g(\lambda)$. The function $F(\lambda)$ also represents the asymptotic repartition of eigenvalues of $(a,\mathcal{D})$ on $L^2(I,m)$, since, using the scaling relation we easily see that the counting function $\mathcal{N}^\pm(\lambda)=\#\{k,|\lambda^\pm_k|\leqslant\lambda\}$ is equivalent to $F(-\lambda)$ when λ tends to infinity (when the integrated density of states exists). In [16], Lapidus and Kigami studied in general (i.e., for general finitely ramified fractals and different scaling ratios γ_i) the behaviour of the function $\mathcal{N}^\pm(\lambda)$. In our context, if γ_1 and γ_2 are not equal, their result states as follows: let ρ be the real such that ${\gamma_1}^{-\rho}+{\gamma_2}^{-\rho}=1$ then if $\log\gamma_1$ and $\log\gamma_2$ are not rationally linked then $\mathcal{N}^\pm(\lambda)$ is equivalent to $C\lambda^\rho$ for a real $C>0$, and if the additive group generated by $\log\gamma_1$ and $\log\gamma_2$ is $(\log p)\mathbb{Z}$ $(p>1)$, then $\mathcal{N}^\pm(\lambda)$ is equivalent to $g(\lambda)\lambda^\rho$ where $g(\lambda)$ is a positive function satisfying $g(p\lambda)=g(\lambda)$. These two cases are called respectively the non-arithmetic and the arithmetic cases. In the arithmetic case, some natural questions are: is the function g constant, is it continuous? For $\gamma_1=\gamma_2$ we will answer to these questions by giving an explicit formula for $F(\lambda)$.

Proof. – Let $F^\pm_{\langle n\rangle}(\lambda)=\int_0^\lambda d\nu^\pm_{\langle n\rangle}$ for $\lambda\leqslant 0$. From relations (1.11) and (1.13) we deduce that

$$F^{\pm}_{\langle n+1\rangle}(\lambda)=F^{\pm}_{\langle n\rangle}(\gamma\lambda) \tag{1.17}$$

and this implies relation (1.16) when $\frac{1}{2^n}F_{\langle n\rangle}$ converges. □

Finally we give the counterpart on $a_{\langle n\rangle}$ and $m_{\langle n\rangle}$ of the scaling relations (1.1) and (1.6). To simplify the notations we adopt the convention that $a(f_{|I_{i_1,\dots,i_n}})$ and $\int f_{|I_{i_1,\dots,i_n}}\,dm$ stand for $a(f\circ\Psi_{i_1,\dots,i_n}\circ\Psi_1^{-n})$ and $\int f\circ\Psi_{i_1,\dots,i_n}\circ\Psi_1^{-n}\,dm$ for a function $f\in\mathcal{D}_{\langle n\rangle}$ (i.e., we transport the form a and the measure m to the interval $I_{i_1,\dots,i_n}$ in a natural way). We have:

$$\int f\,dm_{\langle n\rangle}=\sum_{i_1,\dots,i_n}\alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}\int f_{|I_{i_1,\dots,i_n}}\,dm,\quad \forall f\in\mathcal{D}_{\langle n\rangle}, \tag{1.18}$$

$$a_{\langle n\rangle}(f)=\sum_{i_1,\dots,i_n}\alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}a(f_{|I_{i_1,\dots,i_n}}),\quad \forall f\in\mathcal{D}_{\langle n\rangle}, \tag{1.19}$$

where $\alpha_1=\alpha$ and $\alpha_2=1-\alpha$. On $a_{\langle n\rangle,\lambda}(f)=a_{\langle n\rangle}(f)+\lambda\int f^2\,dm_{\langle n\rangle}$ this is of course translated in:

$$a_{\langle n\rangle,\lambda}(f)=\sum_{i_1,\dots,i_n}\alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}a_\lambda(f_{|I_{i_1,\dots,i_n}}),\quad \forall f\in\mathcal{D}_{\langle n\rangle}. \tag{1.20}$$

1.2.2. Justification of the choice $\gamma_1=\gamma_2$

In this very section we do not assume hypothesis (H), so that γ_1 and γ_2 do no need to be equal. An extra factor comes in formula (1.20) which reads:

$$a_{\langle n\rangle,\lambda}(f)=\sum_{i_1,\dots,i_n}\alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}a_{\gamma_1^n(\gamma_{i_1}\cdots\gamma_{i_n})^{-1}\lambda}(f_{|I_{i_1,\dots,i_n}}),\quad \forall f\in\mathcal{D}_{\langle n\rangle}. \tag{1.21}$$

This extra factor implies that the operator is not the same in each of the cells $I_{i_1,\dots,i_n}$. Indeed, consider a function f in the domain of $L^+_{\langle\infty\rangle}$ such that $\mathrm{supp}(f)\subset I_{i_1,\dots,i_n}$. The infinitesimal generator $\frac{d}{dm_{\langle\infty\rangle}}\frac{d}{dx}$ applied to f gives:

$$\frac{d}{dm_{\langle\infty\rangle}}\frac{d}{dx}f=\gamma_1^{-n}(\gamma_{i_1}\cdots\gamma_{i_n})\frac{d}{dm}\frac{d}{dx}\big(f\circ\Psi_{i_1,\dots,i_n}\circ\Psi_1^{-n}\big). \tag{1.22}$$

Therefore, we can say that the operator satisfies a local invariance by translation if and only if $\gamma_1=\gamma_2$. Of course, even if $\gamma_1=\gamma_2$ the operator is not globally invariant by translation but we believe that this local invariance is the good counterpart of the ergodicity assumed on the law of the potential in the case of random or almost periodic Schrödinger operators. This invariance by translation is known to be essential in the construction of the integrated density of states and in its relation to the spectrum of the operator. In our case, using the results of Kigami and Lapidus [16] on the asymptotic repartition of eigenvalues we can show that the sequence $\frac{1}{2^n}\nu^{\pm}_{\langle n\rangle}$ does not converge to a good measure when $\gamma_1\neq\gamma_2$. Indeed, the result of [16] states as follows: let ρ be the unique positive real such that $\gamma_1^{-\rho}+\gamma_2^{-\rho}=1$, the number of eigenvalues smaller than λ, $\mathcal{N}^{\pm}(\lambda)=\#\{k,\ |\lambda_k^{\pm}|\leqslant\lambda\}$, behaves for large values of λ like $g(\lambda)\lambda^\rho$ where g is a positive constant if $\ln(\gamma_1)$ and $\ln(\gamma_2)$ are not rationally linked, and satisfies $g(p\lambda)=g(\lambda)$ if $\ln(p)\mathbb{Z}$ is the group generated by $\ln(\gamma_1)$ and $\ln(\gamma_2)$. Since by scaling $F^{\pm}_{\langle n\rangle}(\lambda)=\int_0^\lambda\nu^{\pm}_{\langle n\rangle}=F^{\pm}_{\langle 0\rangle}(\gamma_1^n\lambda)=\mathcal{N}^{\pm}(-\gamma_1^n\lambda)$ we see that:

$$\frac{1}{2^n}F^{\pm}_{\langle n\rangle}(\lambda)\sim\frac{1}{2^n}\big\|\gamma_1^n\lambda\big\|^\rho g\big(|\gamma_1^n\lambda|\big), \tag{1.23}$$

for large values of n. But $\gamma_1^\rho > 2$ if $\gamma_1 > \gamma_2$ and $\gamma_1^\rho < 2$ if $\gamma_1 < \gamma_2$. Therefore, $\frac{1}{2^n} F^\pm_{\langle n\rangle}(\lambda)$ converges to $+\infty$ if $\gamma_1 > \gamma_2$ and 0 if $\gamma_1 < \gamma_2$. This can be easily understood from the fact that when we look at the process on large scale (i.e., on $I_{\langle n\rangle}$ for n large) then the coefficients $\gamma_1^{-n}(\gamma_{i_1}\cdots\gamma_{i_n})$ are for most of them either very small if $\gamma_1 > \gamma_2$ or very large if $\gamma_1 < \gamma_2$. So the process on large scales has a tendancy to move very slowly if $\gamma_1 > \gamma_2$ or very fast if $\gamma_1 < \gamma_2$, therefore creating either many small eigenvalues or very few.

Of course, one can define a measure by replacing the renormalizing factor 2^n by $(\gamma_1^\rho)^n$ but it is not clear wether this measure is a interesting object in connection with the operator on the unbounded space when no local invariance by translation is satisfied.

1.2.3. The discrete underlying problem

There is a natural discrete model associated with the continuous one. We also investigate the integrated density of states in this model since there is a nice similarity between the two expressions for the continuous case and the discrete case.

We define $F_{\langle n\rangle}$ as the union of the boundaries of the 2^n intervals $I_{i_1,\dots,i_n}$, i.e., we set $F = F_{\langle 0\rangle} = \partial I = \{0, 1\}$,

$$F_{i_1,\dots,i_n} = \partial I_{i_1,\dots,i_n} = \Psi_{i_1,\dots,i_n} \circ \Psi_1^{-n}(F) \tag{1.24}$$

and $F_{\langle n\rangle} = \bigcup_{i_1,\dots,i_n} F_{i_1,\dots,i_n}$. We also denote $E_{\langle n\rangle} = \mathbb{R}^{F_{\langle n\rangle}}$ and simply $E = \mathbb{R}^F$ (in the terminology of finitely ramified fractals, the set $F_{\langle n\rangle}$ is often called the pre-fractal). We fix a strictly positive probability measure ω on F, i.e., $\omega = c\delta_0 + (1-c)\delta_1$ for a real $0 < c < 1$. Let A be the quadratic form on $\mathbb{R}^F$ defined by:

$$A(f) = \big(f(0) - f(1)\big)^2. \tag{1.25}$$

Up to a constant, A is the unique irreducible conservative Dirichlet form on F. Following the formulas (1.18) and (1.19) we define the measures $\omega_{\langle n\rangle}$ and the quadratic form $A_{\langle n\rangle}$ on $\mathbb{R}^{F_{\langle n\rangle}}$ by:

$$A_{\langle n\rangle}(f) = \sum_{i_1,\dots,i_n} \alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1} A(f_{|F_{i_1,\dots,i_n}}), \quad \forall f \in \mathbb{R}^{F_{\langle n\rangle}}, \tag{1.26}$$

$$\int f\, d\omega_{\langle n\rangle} = \sum_{i_1,\dots,i_n} \alpha^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1} \int f_{|F_{i_1,\dots,i_n}}\, d\omega, \quad \forall f \in \mathbb{R}^{F_{\langle n\rangle}}. \tag{1.27}$$

Naturally, we can extend $\omega_{\langle n\rangle}$ to $F_{\langle\infty\rangle} \simeq \mathbb{N}$ in $\omega_{\langle\infty\rangle}$ and A in $A_{\langle\infty\rangle}$ on the domain $L^2(\mathbb{N}, \omega_{\langle\infty\rangle})$ (since $\sup A_{\langle n\rangle}(f) < \infty$ if $f \in L^2(\mathbb{N}, \omega_{\langle\infty\rangle})$). So $A_{\langle\infty\rangle}, \omega_{\langle\infty\rangle}$ define a discrete Markov process on $\mathbb{N}$ (its transition probabilities are a bit difficult to describe). Denote by $\tilde\nu^+_{\langle n\rangle}$ the counting measure of the spectrum of the infinitesimal generator of $A_{\langle n\rangle}$ on $L^2(F_{\langle n\rangle}, \omega_{\langle n\rangle})$ (and by $\tilde\nu^-_{\langle n\rangle}$ the counting measure of the infinitesimal generator of $A_{\langle n\rangle}$ with Dirichlet condition on $\{0, \alpha^{-n}\}$).

DEFINITION 1.2. – *If $\frac{1}{2^n}\tilde\nu^\pm$ converges to $\tilde\mu$ we say that the integrated density of states of the infinitesimal generator of $A_{\langle\infty\rangle}$ on $L^2(\mathbb{N}, \omega_{\langle\infty\rangle})$ exists and is $\tilde\mu$.*

Remark 1.5. – A priori $\tilde\mu$ depends on the choice of the measure ω. At the opposite to the continuous case, $\tilde\mu$ has no invariance by scaling and in fact is supported by a compact.

2. The sequence of plurisubharmonic functions

2.1. Preliminary results

Let X be a locally compact denumbrable metric space and m be a finite positive Radon measure on X such that $\operatorname{supp}(m) = X$.

Let $(a, \mathcal{D})$ be a regular Dirichlet form on $L^2(X, m)$ such that:

(i) a is irreducible (i.e., $a(f) = 0$ implies that f is constant).

(ii) $(a, \mathcal{D})$ has a compact resolvent.

(iii) There exists $c > 0$ such that $\operatorname{cap}_1(\{x\}) \geqslant c$ for all $x \in X$.

N.B.: $\operatorname{cap}_1(\{x\})$ stands for the 1-capacity of the point $\{x\}$ (cf [9, Section 2]).

The assumption (iii) implies in particular that the functions of the domain have a continuous modification, so that the value at one point can be defined (cf [9, Theorem 2.1.3]).

A second implication of assumption (iii) is that the resolvent R_λ is trace-class. Indeed, it is proved in [9, Example 2.1.2], that $g_1(x, y)$ the kernel of R_1 satisfies $\operatorname{cap}_1(\{x\}) = 1/g_1(x, x)$ so we get:

$$\operatorname{Trace}(R_1) = \int_X g_1(x, x)\, dm \leqslant \frac{1}{c} m(X) < \infty. \tag{2.1}$$

Let F be a finite subset of X. The regularity of the form and assumption (iii) imply that for any $f \in \mathbb{R}^F$ there exists $g \in \mathcal{D}$ such that $g_{|F} = f$.

We define the trace of $(a, \mathcal{D})$ on the subset F as the bilinear form on $\mathbb{R}^F$ defined by:

$$a_F(f) = \inf\{a(g),\ g \in \mathcal{D},\ g_{|F} = f\}, \quad \forall f \in \mathbb{R}^F. \tag{2.2}$$

The irreducibility of $(a, \mathcal{D})$ implies that the infimum in (2.2) is reached on a unique point that we will call the harmonic continuation of f with respect to a.

If F is endowed with a positive measure ω with full support then $(a_F, \mathbb{R}^F)$ is a regular, irreducible Dirichlet form on $L^2(F, \omega)$ (the process associated with a_F and ω on states space F can be represented by a time changed of the initial process associated with $(a, \mathcal{D})$ on $L^2(X, m)$ (cf [9, Theorem 6.2.1], and also [19])).

Remark 2.1. – If F' is a subset of the finite set F, then considering Definition 2.2 we see that $(a_F)_{F'} = a_{F'}$ (where in the first term the trace on F' is applied to the Dirichlet form a_F with domain $\mathbb{R}^F$).

For $\lambda \geqslant 0$ let $a_\lambda(f) = a(f) + \lambda \int_X f^2\, dm$ for $f \in \mathcal{D}$. The bilinear form a_λ is a regular irreducible Dirichlet form satisfying (i), (ii) and (iii). We denote by $A_{(\lambda)} = (a_\lambda)_F$ its trace on the subset F and by $H_\lambda f$ the harmonic continuation of $f \in \mathbb{R}^F$ with respect to a_λ, so that $A_{(\lambda)}(f) = a_\lambda(H_\lambda f)$.

Remark 2.2. – In order to clarify the definition we point out that in the case of Section 1, where $X = [0, 1]$, $a(f) = \int_0^1 (f')^2 dx$, and m is a positive Radon measure with full support we have the following: if $g \in \mathbb{R}^F$ and $f = H_\lambda g$ then f is a solution of the differential equation $\frac{d}{dm}\frac{d}{dx} f = \lambda f$ and when $A_{(\lambda)}$ is considered as a 2×2 matrix:

$$A_{(\lambda)} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} = \begin{pmatrix} -f'(0) \\ f'(1) \end{pmatrix}. \tag{2.3}$$

Set $\mathcal{D}^- = \{f \in \mathcal{D},\ \ f_{|F} = 0\}$ (N.B.: $\mathcal{D}^-$ is the domain with Dirichlet boundary conditions on F; $(a, \mathcal{D}^-)$ is a regular Dirichlet form on $L^2(X \setminus F, m)$).

We denote by $0 > \lambda_1^+ \geqslant \cdots \geqslant \lambda_k^+ \geqslant \cdots$ the negative eigenvalues of the infinitesimal generator associated with $(a, \mathcal{D})$ and by σ_0 the multiplicity of the eingenvalue 0 (which can be 0 or 1, indeed $\sigma_0 = 1$ if $1 \in \mathcal{D}$ and 0 otherwise).

We also denote by $0 > \lambda_1^- \geqslant \cdots \geqslant \lambda_k^- \geqslant \cdots$ the eigenvalues of the infinitesimal generator of $(a, \mathcal{D}^-)$ (in this case 0 is not eigenvalue because of the boundary condition and assumption (i)). Let $f_1^-, \ldots, f_k^-, \ldots$ be an orthonormal basis of eigenfunctions associated with the preceding eigenvalues. The first result gives an expression of $A_{(\lambda)}$:

LEMMA 2.1. – *For any $f \in \mathbb{R}^F$, $\lambda \geqslant 0$:*

$$A_{(\lambda)}(f) = A_{(0)}(f) + \lambda \int (H_0 f)^2\, dm - \lambda^2 \sum_{k=1}^{\infty} \frac{(\int H_0 f f_k^-\, dm)^2}{\lambda - \lambda_k^-}. \tag{2.4}$$

In particular $A_{(\lambda)}$ is meromorphic on $\mathbb{C}$ with at worst simple poles at the points $\{\lambda_1^-, \ldots, \lambda_k^-, \ldots\}$.

Remark 2.3. – This formula is in fact classical on the form $H_0 f - H_\lambda f = \lambda R_\lambda^- H_0 f$ where R_λ^- is the λ-resolvent of $(a, \mathcal{D}^-)$ (cf, for example, [19]).

Proof. – Since $H_\lambda f - H_0 f$ is in $\mathcal{D}^-$ we have that $H_\lambda f = H_0 f + \sum c_k f_k^-$. Using that $a_\lambda(H_\lambda f, f_k^-) = 0$ for all k we get $c_k = -\frac{\lambda}{\lambda - \lambda_k^-} \int H_0 f f_k^-\, dm$. But

$$\begin{aligned} A_{(\lambda)}(f) &= a_\lambda(H_\lambda f) = a_\lambda\Big(H_0 f, H_0 f + \sum c_k f_k^-\Big) \\ &= a(H_0 f) + \lambda \int (H_0 f)^2\, dm + \lambda \sum c_k \int H_0 f f_k^-\, dm, \end{aligned}$$

and the result follows. □

The second result of this section gives a relation between the Neuman spectrum, the Dirichlet spectrum and the trace $A_{(\lambda)}$. We define some kind of infinite dimensional determinants by the following formula: for $\lambda \in \mathbb{C}$ we set

$$d^+(\lambda) = \lambda^{\sigma_0} \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k^+}\right), \tag{2.5}$$

$$d^-(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k^-}\right). \tag{2.6}$$

The existence of these functions comes from the fact that the resolvents of $(a, \mathcal{D})$ and $(a, \mathcal{D}^-)$ are trace class.

In the sequel we will adopt the following convention. If Q is a quadratic form on $\mathbb{R}^F$ then we will denote by $\det(Q)$ the determinant of the symmetric matrix associated with Q when $\mathbb{R}^F$ is endowed with the usual scalar product. Then we have:

LEMMA 2.2. – *There exists a constant $C > 0$ such that for all $\lambda \in \mathbb{C} \setminus \{\lambda_1^-, \ldots, \lambda_k^-, \ldots\}$:*

$$\det(A_{(\lambda)}) = C\frac{d^+(\lambda)}{d^-(\lambda)}. \tag{2.7}$$

Remark 2.4. – We recently discovered that this formula has already been proved in a more general context (cf [4,5]). We nevertheless give our proof for the sake of completeness.

Before giving the proof we state a result a bit more general when X is finite. We denote by $\mathcal{Q}$ (resp. $\mathcal{Q}^+$) the set of quadratic forms on $\mathbb{R}^X$ (resp. positive quadratic forms). When $Q \in \mathcal{Q}^+$ we define the trace of the quadratic form Q by

$$Q_F(f) = \inf\{Q(g), g_{|F} = f\}, \quad f \in \mathbb{R}^F \tag{2.8}$$

(of course this definition coincides with formula (2.2) when Q is moreover a Dirichlet form). It is easy to see that the coefficients of Q_F are rational in the coefficients of Q and that the trace can be extended to $\mathcal{Q}$ minus a finite union of hypersurfaces (where the denominator is null). We denote by Q^- the restriction of Q to the subspace $\{f \in \mathbb{R}^X,\ f_{|F} = 0\}$ (identified with $\mathbb{R}^{X\setminus F}$). With the previous convention for the determinant we have:

LEMMA 2.3. – *For all $Q \in \mathcal{Q}$, the following equality is true* (*when all terms are defined*):

$$\det(Q_F) = \frac{\det Q}{\det Q^-}. \tag{2.9}$$

We first prove Lemma 2.3.

Proof. – We first set some notations. We denote by Δ, Δ_F, Δ^- the symmetric matrices associated with Q, Q_F, Q^- thanks to the canonical scalar products on $\mathbb{R}^X$, $\mathbb{R}^F$ and $\mathbb{R}^{X\setminus F}$. If $Q \in \mathcal{Q}^+$ then the harmonic continuation exists and is denoted by H, i.e., we have $Q_F(f) = Q(Hf)$ for $f \in \mathbb{R}^F$.

Let $P : \mathbb{R}^X \to \mathbb{R}^F \times \mathbb{R}^{X\setminus F}$ be defined by:

$$P(f) = \bigl(f_{|F}, f - H(f_{|F})\bigr), \tag{2.10}$$

then easily:

$$\Delta = {}^t P \begin{pmatrix} \Delta_F & 0 \\ 0 & \Delta^- \end{pmatrix} P \tag{2.11}$$

and since P is expressed by:

$$P = \begin{pmatrix} \mathrm{Id} & 0 \\ \star & \mathrm{Id} \end{pmatrix}. \tag{2.12}$$

We see that $\det(\Delta) = \det(\Delta_F)\det(\Delta^-)$ which is the desired result. □

Proof of Lemma 2.2. – we approximate the Dirichlet forms a_λ to reduce to a finite dimensional space where the proof of Lemma 2.3 can be reproduced. Let $f_0^+, f_1^+, \ldots$ be a basis of eigenvalues associated with $\lambda_0^+ = 0 > \lambda_1^+ \geqslant \cdots$ (we start at λ_1^+ if 0 is not eigenvalue, i.e., if $\sigma_0 = 0$). We denote by $\mathcal{D}^n$ the vector space generated by $\{f_0^+, \ldots, f_n^+\}$. We set:

$$\mathcal{D}^{n,-} = \{ f \in \mathcal{D}^n,\ f_{|F} = 0 \}. \tag{2.13}$$

We denote by a_λ^n and $a_\lambda^{n,-}$ the restriction of a_λ respectively to the subspace $\mathcal{D}^n$ and $\mathcal{D}^{n,-}$.

There exists $N > 0$ such that for any $f \in \mathbb{R}^F$ we can find $g \in \mathcal{D}^N$ with $g_{|F} = f$. For $n \geqslant N$ we can define the trace of a_λ^n on the subset F by:

$$A^n_{(\lambda)}(f) = (a_\lambda^n)_F(f) = \inf\{a_\lambda^n(g),\ g \in \mathcal{D}^n,\ g_{|F} = f\}, \quad \forall f \in \mathbb{R}^F. \tag{2.14}$$

We first prove that $A^n_{(\lambda)}$ converges to $A_{(\lambda)}$.

For $f \in \mathbb{R}^F$ we denote by $H_\lambda f$ the harmonic prolongation of f with respect to a_λ. We denote by Π^n the orthogonal projection from $\mathcal{D}$ to the subspace $\mathcal{D}^n$. It is clear that:

$$A^n_{(\lambda)}\big((\Pi^n(H_\lambda f))_{|F}\big) \leqslant A_{(\lambda)}(f) \leqslant A^n_{(\lambda)}(f). \tag{2.15}$$

Since $A^n_{(\lambda)}$ is a decreasing sequence we have for $n \geqslant N$:

$$A^n_{(\lambda)}(f) \leqslant A^n_{(\lambda)}\big((\Pi^n(H_\lambda f))_{|F}\big) + A^N_{(\lambda)}\big(f - (\Pi^n(H_\lambda f))_{|F}\big), \tag{2.16}$$

so it is enough to prove that $f - (\Pi^n(H_\lambda f))_{|F}$ tends to 0 on F to prove the convergence of $A^n_{(\lambda)}$. If $H_\lambda f = \sum_k c_k^+ f_k^+$ is the decomposition of $H_\lambda f$ on the basis of eigenvectors, then it is equivalent to prove that $(\sum_{k=n}^\infty c_k^+ f_k^+)_{|F}$ converges to 0 but this comes from the fact that the value at a point x of a function $\phi \in \mathcal{D}$ can be defined by:

$$\phi(x) = a_\lambda\big(g_\lambda(x,\cdot), \phi\big), \tag{2.17}$$

where $g_\lambda(x, y)$ is the kernel of the λ-resolvent (and $g_\lambda(x,\cdot) \in \mathcal{D}$ for all $x \in X$) and the application $\phi \to \phi(x)$ is continuous on $\mathcal{D}$ with the L^2 norm. Finally, we proved that $A^n_{(\lambda)}$ converges to $A_{(\lambda)}$.

Let $d^{n,+}(\lambda)$ and $d^{n,-}(\lambda)$ be the functions associated with the spectrum of a_λ^n and $a_\lambda^{n,-}$ (on $\mathcal{D}^n$ and $\mathcal{D}^{n,-}$) then by definition we have:

$$d^{n,+}(\lambda) = \lambda^{\sigma_0} \prod_{k=1}^n \left(1 - \frac{\lambda}{\lambda_k^+}\right), \tag{2.18}$$

and $d^{n,+}$ converges to d^+. A similar relation is not true for $d^{n,-}$ since $\mathcal{D}^{n,-}$ is not the space generated by the first $(n - \#F)$ Dirichlet eigenvectors. Nevertheless, $d^{n,-}$ converges to d^-. Indeed, the infinitesimal generator of $a^{n,-}$ (extended to $\mathcal{D}^-$ by 0) converges to a^- in the strong resolvent sense (cf [3], Definition I.1.6 and Proposition I.1.18) and this implies that the non-zero eigenvalues of $a^{n,-}$ converge to the eigenvalues of a^-.

Mimicking the proof of Lemma 2.3 we can prove that:

$$\det\big(A^n_{(\lambda)}\big) = c_n \frac{d^{n,+}(\lambda)}{d^{n,-}(\lambda)}, \tag{2.19}$$

for a constant $c_n > 0$. Indeed for $n \geqslant N$, $(\mathcal{D}^{n,-})^\perp$ (as a subset of $\mathcal{D}^n$) can be identified with $\mathbb{R}^F$ and

$$P_\lambda^n : \mathcal{D}^n = (\mathcal{D}^{n,-})^\perp \oplus \mathcal{D}^{n,-} \to \mathbb{R}^F \times \mathcal{D}^{n,-} \tag{2.20}$$

$$f \to \big(f_{|F}, f - H_\lambda^n f\big), \tag{2.21}$$

can be viewed as an endomorphism of $\mathcal{D}^n$ which has the form (2.12). Denoting by Δ_λ^n and $\Delta_\lambda^{n,-}$ the symmetric matrices associated with a_λ^n and $a_\lambda^{n,-}$ when $\mathcal{D}^n$ is endowed by the scalar product induced by the measure m (i.e., the restriction of the scalar product on $\mathcal{D}$), we get:

$$\det(\Delta_\lambda^n) = \tilde{c}_n \det(A_{(\lambda)}^n) \det(\Delta_\lambda^{n,-}), \tag{2.22}$$

where $\tilde{c}_n > 0$ (this extra constant comes from the fact that the scalar product induced on $\mathbb{R}^F$ by the identification with $(\mathcal{D}^{n,-})^\perp$ is not the usual scalar product on $\mathbb{R}^F$).

It follows from formula (2.19) that c_n must converge to a constant $c > 0$ and that relation (2.7) is trueqed

2.2. The sequence of plurisubharmonic functions $H_{\langle n\rangle}^{\pm}$

We come back to the situation and notations of Section 1 and we recall that:

$$F = \{0, 1\}, \tag{2.23}$$

$$F_{i_1,\dots,i_n} = \Psi_{i_1,\dots,i_n} \circ \Psi_1^{-n}(F), \quad (i_1, \dots, i_n) \in \{1, 2\}^n, \tag{2.24}$$

$$F_{\langle n\rangle} = \bigcup_{i_1,\dots,i_n} F_{i_1,\dots,i_n}. \tag{2.25}$$

We set $E = \mathbb{R}^F$, $E_{\langle n\rangle} = \mathbb{R}^{F_{\langle n\rangle}}$. We denote by $\mathcal{Q}$ the set of symmetric bilinear forms on $\mathbb{R}^F$, identified with $\mathbb{R}^3$ thanks to the representation by symmetric matrices of the form:

$$\begin{pmatrix} q_1 & q \\ q & q_2 \end{pmatrix} \tag{2.26}$$

and by $\mathcal{Q}^+$ the set of positive quadratic forms.

Following formula (1.26), we associate with $Q \in \mathcal{Q}$ a bilinear form $Q_{\langle n\rangle}$ on $\mathbb{R}^{F_{\langle n\rangle}}$ as follows:

$$Q_{\langle n\rangle}(f) = \sum_{i_1,\dots,i_n} \alpha^n (\alpha_{i_1} \cdots \alpha_{i_n})^{-1} Q(f_{|F_{i_1,\dots,i_n}}), \tag{2.27}$$

where $Q(f_{|F_{i_1,\dots,i_n}})$ stands for $Q(f \circ \Psi_{i_1,\dots,i_n} \circ \Psi_1^{-n})$.

We denote by $Q_{\langle n\rangle}^-$ the restriction of the bilinear form $Q_{\langle n\rangle}$ to the subspace $E_{\langle n\rangle}^- = \{f \in E_{\langle n\rangle},\ f(0) = f(\alpha^{-n}) = 0\}$ (considered as the space of functions on $F_{\langle n\rangle} \setminus \{0, \alpha^{-n}\}$).

For $Q \in \mathcal{Q}$ we set:

$$H_{\langle n\rangle}^+(Q) = \log\left|\det(\delta^{-n/2} Q_{\langle n\rangle})\right|, \tag{2.28}$$

$$H_{\langle n\rangle}^-(Q) = \log\left|\det(\delta^{-n/2} Q_{\langle n\rangle}^-)\right| \tag{2.29}$$

and by convention we put $H_{\langle 0\rangle}^-(Q) = 0$ (we see that $H_{\langle 0\rangle}^+(Q) = \log|\det(Q)|$).

N.B.: We recall that $\delta = \frac{\alpha}{1-\alpha}$. The term $\delta^{-n/2}$ is the good renormalizing constant that will ensure the convergence in Section 3.

N.B.: As in Section 2.1 $\det(\delta^{-n/2} Q_{\langle n\rangle})$ denotes the determinant of the associated symmetric matrix and we must understand that $\det(Q_{\langle n\rangle}^-)$ stands for the determinant of the symmetric matrix associated with $Q_{\langle n\rangle}^-$ when $E_{\langle n\rangle}^-$ is considered as the space of functions on $F_{\langle n\rangle} \setminus \{0, \alpha^{-n}\}$.

We see that $\det(Q_{\langle n\rangle})$ (resp. $\det(Q_{\langle n\rangle}^-)$) defines a homogeneous polynomial of degree $\#F_{\langle n\rangle}$ (resp. $\#F_{\langle n\rangle} \setminus \{0, \alpha^{-n}\}$) in (q_1, q_2, q) and so can be extended to $\mathbb{C}^3$. The functions

$H^{\pm}_{\langle n\rangle} : \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ are then plurisubharmonic (we recall that a function is plurisubharmonic if it is upper semi-continuous and if its restriction to any complex line is subharmonic, in particular the logarithm of the modulus of an entire function is plurisubharmonic, cf [13]).

The relevance of these functions comes from the fact that the counting measures $\tilde{\nu}^{\pm}_{\langle n\rangle}$ and $\nu^{\pm}_{\langle n\rangle}$ can be expressed in terms of their restriction to a complex curve.

Precisely, for the discrete case we denote by A_λ the quadratic form on $\mathbb{R}^F$ defined by $A_\lambda(f) = A(f) + \lambda \int f^2\, d\omega$ (A and ω are defined in Section 1.2.3). The map $\lambda \to A_\lambda$ defines a complex line in $\mathbb{C}^3$ (by the identification with a matrix of the form (2.26)). For the continuous case we define $A_{(\lambda)}$ as the trace of the Dirichlet form a_λ on the subset F (as in Section 2.1) and $\lambda \to A_{(\lambda)}$ defines a holomorphic curve in $B(0, |\lambda_1^-|)$ since $A_{(\lambda)}$ is holomorphic in the complement of the Dirichlet spectrum of a (cf Lemma 2.1).

We can remark that $A_{(0)} = A = (1, 1, -1)$ in the coordinates (q_1, q_2, q).

We have the following result:

PROPOSITION 2.1. – (i) *For the continuous case*: $\lambda \to H^{\pm}_{\langle n\rangle}(A_{(\lambda)})$ *defines a subharmonic function in* $B(0, |\lambda_1^-|)$ *and we have*

$$\nu^{\pm}_{\langle n\rangle} = \frac{1}{2\pi}\Delta\big(H^{\pm}_{\langle n\rangle}(A_{(\lambda)})\big), \quad \text{on } B(0, |\lambda_1^-|). \tag{2.30}$$

(ii) *For the discrete case*: $\lambda \to H^{\pm}_{\langle n\rangle}(A_\lambda)$ *defines a subharmonic function on* $\mathbb{C}$ *and we have*

$$\tilde{\nu}^{\pm}_{\langle n\rangle} = \frac{1}{2\pi}\Delta\big(H^{\pm}_{\langle n\rangle}(A_\lambda)\big). \tag{2.31}$$

N.B.: Δ denotes the distributionnal Laplacian. In particular we recall the useful formula: $\Delta \log|\lambda| = 2\pi\delta_0$.

N.B.: For the continuous case it will be enough to have a local formula since the integrated density of states have an invariance by scaling, cf formula (1.16).

Proof. – For the discrete case (ii) it is nearly a triviallity: if we set $A_{\langle n\rangle,\lambda}(f) = A_{\langle n\rangle}(f) + \lambda \int f^2\, d\omega_{\langle n\rangle}$ then $A_{\langle n\rangle,\lambda} = (A_\lambda)_{\langle n\rangle}$ defined in formula (2.27) and so we have $\det((A_\lambda)^{\pm}_{\langle n\rangle}) = \Pi(\lambda - \tilde{\lambda}^{\pm}_{\langle n\rangle,k})$ with obvious notations.

(i) We denote by $d^+_{\langle n\rangle}(\lambda)$ and $d^-_{\langle n\rangle}(\lambda)$ the "infinite" determinants associated with the spectrum of $(a_{\langle n\rangle}, \mathcal{D}^+_{\langle n\rangle})$ and $(a_{\langle n\rangle}, \mathcal{D}^-_{\langle n\rangle})$ by formulas (2.5) and (2.6) and we simply write $d^\pm$ for $d^\pm_{\langle 0\rangle}$. We prove the following formulas: there exists some constants $C^\pm_{\langle n\rangle}$ such that:

$$\log|d^+_{\langle n\rangle}(\lambda)| = H^+_{\langle n\rangle}(A_{(\lambda)}) + 2^n \log|d^-(\lambda)| + C^+_{\langle n\rangle}, \tag{2.32}$$

$$\log|d^-_{\langle n\rangle}(\lambda)| = H^-_{\langle n\rangle}(A_{(\lambda)}) + 2^n \log|d^-(\lambda)| + C^-_{\langle n\rangle}. \tag{2.33}$$

Since $\Delta \log|d^\pm_{\langle n\rangle}(\lambda)| = 2\pi\nu^\pm_{\langle n\rangle}$ and since $\mathrm{supp}(\nu^-) \cap B(0, |\lambda_1^-|) = \emptyset$ this gives the desired result. To prove these formulas we use the result of Section 2.1. In fact we apply Lemma 2.2 with Dirichlet boundary condition taken on $F_{\langle n\rangle}$. Set $\mathcal{D}^-_{F_{\langle n\rangle}} = \{f \in \mathcal{D}_{\langle n\rangle},\ f_{|F_{\langle n\rangle}} = 0\}$ and let $d^-_{F_{\langle n\rangle}}$ be the infinite determinant associated with the spectrum of $(a_{\langle n\rangle}, \mathcal{D}^-_{F_{\langle n\rangle}})$ by formula (2.6). We see that the Dirichlet condition deconnects the intervals $I_{i_1,\dots,i_n}$ and denoting $\mathcal{D}^-_{i_1,\dots,i_n} = \{f \in \mathcal{D}^-_{F_{\langle n\rangle}},\ \ f = 0 \text{ on } I_{\langle n\rangle} \setminus I_{i_1,\dots,i_n}\}$, we have:

$$\mathcal{D}^-_{F_{\langle n\rangle}} = \bigoplus_{i_1,\dots,i_n} \mathcal{D}^-_{i_1,\dots,i_n}. \tag{2.34}$$

But formula (1.20) says that $(a_{\langle n\rangle}, \mathcal{D}^-_{i_1,\dots,i_n})$ has the same spectrum as $(a, \mathcal{D}^-)$, this implies that for a constant $c_n > 0$ we have $d^-_{F_{\langle n\rangle}} = c_n (d^-)^{2^n}$.

To get formula (2.32) we apply Lemma 2.2 taking the trace of $(a_{\langle n\rangle,\lambda}, \mathcal{D}_{\langle n\rangle})$ on $F_{\langle n\rangle}$ (since we see that $(a_{\langle n\rangle,\lambda})_{F_{\langle n\rangle}} = (A_{(\lambda)})_{\langle n\rangle}$).

To get formula (2.33) we apply Lemma 2.2 taking the trace of $(a_{\langle n\rangle,\lambda}, \mathcal{D}^-_{\langle n\rangle})$ on $F_{\langle n\rangle} \setminus \{0, \alpha^{-n}\}$. $\square$

2.3. The renormalization map. The functional equation satisfied by the $H^{\pm}_{\langle n\rangle}$'s

We first set some notations. We will always identify an element $Q \in \mathcal{Q}$ with the triple (q_1, q_2, q) using the representation of Q thanks to the symmetric matrix of the form (2.26).

We set:

$$r(Q) = \det(Q) = q_1 q_2 - q^2, \tag{2.35}$$

$$p(Q) = (1-\alpha)\det(Q^-_{\langle 1\rangle}) = \alpha\big(q_1 + \delta^{-1} q_2\big). \tag{2.36}$$

N.B.: we recall that $\delta = \alpha/(1-\alpha)$ and that $Q^-_{\langle 1\rangle}$ is the restriction of $Q_{\langle 1\rangle}$ to the subspace $E^-_{\langle 1\rangle} = \{f \in E_{\langle 1\rangle},\ f(0) = f(\alpha^{-1}) = 0\}$ (considered as the space of functions on $F_{\langle 1\rangle} \setminus \{0, \alpha^{-1}\} = \{1\}$). The functions r and p are homogeneous polynomials of degrees respectively 2 and 1 in the variables (q_1, q_2, q).

We first define the renormalization map T on the set of positive quadratic forms on $\mathbb{R}^F$, denoted by $\mathcal{Q}^+$. We define $T : \mathcal{Q}^+ \to \mathcal{Q}^+$ by:

$$TQ(f) = \alpha^{-1}(Q_{\langle 1\rangle})_{\alpha^{-1}F}(f \circ \Psi_1), \quad \forall f \in \mathbb{R}^F. \tag{2.37}$$

N.B.: we recall that $(Q_{\langle 1\rangle})_{\alpha^{-1}F}$ denotes the trace of $Q_{\langle 1\rangle}$ on the subset $\alpha^{-1}F \subset F_{\langle 1\rangle}$.

Remark 2.5. – This definition of course suits with the one we gave in the introduction. Formula (2.37) is just a scaled version of formula (0.7).

It is easy to see that (cf relation (2.27)):

$$(TQ)_{\langle n\rangle}(f) = \alpha^{-1}(Q_{\langle n+1\rangle})_{\alpha^{-1}F_{\langle n\rangle}}(f \circ \Psi_1), \quad \forall f \in E_{\langle n\rangle}. \tag{2.38}$$

A computation of T gives:

$$\begin{aligned} &T\big((q_1, q_2, q)\big) \\ &\quad = \frac{\alpha^{-1}}{q_1 + \delta^{-1} q_2}\big(q_1(q_1 + \delta^{-1} q_2) - \delta^{-1} q^2, \delta q_2(q_1 + \delta^{-1} q_2) - \delta q^2, -q^2\big). \end{aligned} \tag{2.39}$$

So the map T can be extended to $\mathbb{C}^3$ minus the hyperplane $\{q_1 + \delta^{-1} q_2 = 0\}$.

We remark that the polynomial p simplifies the denominator of T and we set:

$$R(Q) = p(Q)TQ, \tag{2.40}$$

which gives:

$$R\big((q_1, q_2, q)\big) = \big(q_1(q_1 + \delta^{-1} q_2) - \delta^{-1} q^2, \delta q_2(q_1 + \delta^{-1} q_2) - \delta q^2, -q^2\big). \tag{2.41}$$

Remark 2.6. – The fact that the polynomial p simplifies the denominator of T is not circumstancial. It is a general phenomenon that $\det(Q^{-}_{\langle 1\rangle})TQ$ is given by homogeneous polynomials of same degree (this can be seen for example from Lemma 2.1). But it is not always the case that these polynomials have no common factor.

We remark that the curve $A_{(\lambda)}$ is invariant by T:

PROPOSITION 2.2. – *The following equality is true for all* $\lambda \in \mathbb{C}$ (*when the two terms are defined*):

$$T(A_{(\lambda)}) = A_{(\gamma\lambda)}. \tag{2.42}$$

Remark 2.7. – This means that, at least locally, $A_{(\lambda)}$ is a holomorphic curve invariant by the map T. We remark that $A_{(0)} = A = (1, 1, -1)$ is a fixed point of T (in general the existence of this fixed point is essential for the construction of a diffusion on the fractal, cf [25]) and that $(\frac{d}{d\lambda} A_{(\lambda)})_{\lambda=0}$ is an eigenvector of T with eigenvalue $\gamma > 1$ (we shall see later that γ is the highest eigenvalue of the differential of T at the fixed point $A_{(0)}$ and this seems to be always the case, i.e., for finitely ramified fractals). From formula (2.4) we know that $\frac{d}{d\lambda} A_{(\lambda)}$ at $\lambda = 0$ is the quadratic form defined by:

$$\left(\frac{d}{d\lambda} A_{(\lambda)}\right)_{|\lambda=0}(f) = \int (H_0 f)^2\, dm, \quad \forall f \in E. \tag{2.43}$$

Proof. – We have:

$$\begin{aligned} T(A_{(\lambda)})(f) &= \alpha^{-1}\big((A_{(\lambda)})_{\langle 1\rangle}\big)_{\alpha^{-1}F}(f \circ \Psi_1) \\ &= \alpha^{-1}\big((a_{\langle 1\rangle,\lambda})_{F_{\langle 1\rangle}}\big)_{\alpha^{-1}F}(f \circ \Psi_1) \\ &= \alpha^{-1}(a_{\langle 1\rangle,\lambda})_{\alpha^{-1}F}(f \circ \Psi_1) \\ &= (a_{\gamma\lambda})_F(f). \end{aligned}$$

The second equation follows relation (1.20), the third is a consequence of remark ? and the last one comes from the scaling relation $a_{\langle 1\rangle,\lambda}(\cdot \circ \Psi_1) = \alpha a_{\gamma\lambda}(\cdot)$. □

Next we show that $H^{\pm}_{\langle n\rangle}$ satisfies a functional equation and from this we get an expression of $H^{\pm}_{\langle n\rangle}$ in terms of the map R.

PROPOSITION 2.3. – *For any quadratic form Q on $\mathbb{R}^F$ we have*:

$$H^{\pm}_{\langle n+1\rangle}(Q) = H^{\pm}_{\langle n\rangle}(TQ) + 2^n \log|p(Q)| \pm \log\sqrt{\alpha(1-\alpha)}, \tag{2.44}$$

and

$$H^{\pm}_{\langle n\rangle}(Q) = \frac{1}{2}\big(\log\big|r\big(R^n Q\big)\big| \pm \log|r(Q)|\big). \tag{2.45}$$

N.B.: we can remark that these formulas are homogeneous since $H^{\pm}_{\langle n\rangle}(\beta Q) = H^{\pm}_{\langle n\rangle}(Q) + (2^n \pm 1)\log|\beta|$ and r is a homogeneous polynomial of degree 2 and R a homogeneous polynomial transformation of degree 2.

Remark 2.8. – We remark that if $\frac{1}{2^n} H^{\pm}_{\langle n\rangle}$ had a limit H then it should satisfy the following functional equation: $H(RQ) = 2H(Q)$. This is the functional equation satisfied by the Green function associated with the map R (cf Section 3.1) so it will not be a surprised that $\frac{1}{2^n}H^{\pm}_{\langle n\rangle}$ converges effectively to the Green function of R (even if it is not directly implied by this functional equation, cf [8, Remark 4.19]). In [22], Rammal used the functional equation (2.44) in the case of the Sierpinski gasket.

Proof. – It is enough to prove these relations for a positive quadratic form $Q \in \mathcal{Q}^+$ (they are extended to $\mathcal{Q}$ by analycity). We apply Lemma 2.3 to the trace of $Q_{\langle n+1\rangle}$ on the subset $\alpha^{-1}F_{\langle n\rangle}$. We denote by $Q^{\langle n\rangle}_{\langle n+1\rangle}$ the restriction of $Q_{\langle n+1\rangle}$ to the subset $E^{\langle n\rangle}_{\langle n+1\rangle} = \{f \in E_{\langle n+1\rangle},\ f_{|F_{\langle n\rangle}} = 0\}$. From formula (2.27) we deduce that:

$$\det\bigl(Q^{\langle n\rangle}_{\langle n+1\rangle}\bigr) = \left[\prod_{i_1,\ldots,i_n} \alpha_1^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}\right]\det(Q^-_{\langle 1\rangle})^{2^n}. \tag{2.46}$$

But $\Pi\alpha_1^n(\alpha_{i_1}\cdots\alpha_{i_n})^{-1} = \sqrt{\delta}^{n2^n}$ and, using Lemma 2.3 and formula (2.38) we get:

$$\det(Q_{\langle n+1\rangle}) = \sqrt{\delta}^{n2^n}\det(Q^-_{\langle 1\rangle})^{2^n}\det\bigl(\alpha(TQ)_{\langle n\rangle}\bigr) \tag{2.47}$$

$$= \sqrt{\delta}^{(n+1)(2^{n+1}+1)}\sqrt{\alpha(1-\alpha)}\det\left(\frac{1}{\sqrt{\delta}^n}(TQ)_{\langle n\rangle}\right)p(Q)^{2^n} \tag{2.48}$$

which gives the first formula for H^+. In the same way we get the formula for H^-. Using the fact that $H^{\pm}_{\langle n\rangle}(\beta Q) = H^{\pm}_{\langle n\rangle}(Q) + (2^n \pm 1)\log|\beta|$, we deduce from formula (2.44) that

$$H^{\pm}_{\langle n+1\rangle}(Q) = H^{\pm}_{\langle n\rangle}(RQ) \mp \bigl(\log|p(Q)| - \log\sqrt{\alpha(1-\alpha)}\bigr) \tag{2.49}$$

and since

$$H^+_{\langle 0\rangle}(Q) = \log|r(Q)|, \tag{2.50}$$

$$H^-_{\langle 0\rangle}(Q) = 0, \tag{2.51}$$

we get:

$$H^+_{\langle n\rangle}(Q) = \log\bigl|r\bigl(R^nQ\bigr)\bigr| - \sum_{k=0}^{n-1}\log\bigl|p\bigl(R^kQ\bigr)\bigr| + n\log\sqrt{\alpha(1-\alpha)} \tag{2.52}$$

and

$$H^-_{\langle n\rangle}(Q) = \sum_{k=0}^{n-1}\log\bigl|p\bigl(R^kQ\bigr)\bigr| - n\log\sqrt{\alpha(1-\alpha)}. \tag{2.53}$$

To finish the computation we remark that:

$$r\circ R(Q) = \frac{1}{\alpha(1-\alpha)}p^2(Q)r(Q), \quad \forall Q \in \mathcal{Q}, \tag{2.54}$$

and:

$$\sum_{k=0}^{n-1}\log\bigl|p\bigl(R^k(Q)\bigr)\bigr| = \frac{1}{2}\bigl(\log\bigl|r\bigl(R^n(Q)\bigr)\bigr| - \log|r(Q)|\bigr) + n\log\sqrt{\alpha(1-\alpha)}. \tag{2.55}$$

This concludes the proof. □

3. Rational dynamics in $\mathbb{P}^2$. Expression of the integrated density of states

3.1. Generalities, notations

In this section we will use some technics on iteration of rational maps of $\mathbb{P}^2$ (the complex projective plane). A good account on the subject, which deals with the case of meromorphic maps, can be found in [8] or [27] (cf also [7,6]). We will also use some technics on plurisubharmonic functions and we will refer to [13]. At the end of this text the reader will find a picture representing the geometric elements we introduce to describe the dynamics of the map.

In this section we consider the map R defined in Section 2.3 as a map on $\mathbb{C}^3$. We recall its expression:

$$R\big((q_1,q_2,q)\big)=\big(q_1(q_1+\delta^{-1}q_2)-\delta^{-1}q^2,\delta q_2(q_1+\delta^{-1}q_2)-\delta q^2,-q^2\big). \quad (3.1)$$

We denote by $\pi:\mathbb{C}^3\setminus\{0\}\to\mathbb{P}^2$ the canonical surjection from $\mathbb{C}^3$ to the complex projective space of dimension 2. The image of a point (q_1,q_2,q) by π will be denoted by $[q_1,q_2,q]$ (following the usual notations, cf [8] or [27]). We first remark that the map R is homogeneous of degree 2 since it is defined by 3 homogeneous polynomials of degree 2 (and we can note that they have no common factor). So, with R we can associate a map f on the projective space by the following formula: $f(x)=\pi(R\tilde{x})$ where $\tilde{x}\in\mathbb{C}^3\setminus\{0\}$ is such that $\pi\tilde{x}=x$. This can be done each time $R(\tilde{x})\neq 0$. We remark that $R(1,-\delta,0)=(0,0,0)$ and that $\mathbb{C}(1,-\delta,0)$ is the unique complex line on which R is null. We denote by $l=[1,-\delta,0]$ the point associated in the projective space and we say that l is a point of indeterminacy. The map f is then a map from $\mathbb{P}^2\setminus\{l\}$ to $\mathbb{P}^2$ and is holomorphic on $\mathbb{P}^2\setminus\{l\}$ (in fact the image of the point l by f can be defined as a compact Riemann surface called the blow-up of l, cf forthcoming relation (3.21)). Therefore the map f is called a meromorphic map of $\mathbb{P}^2$. Its degree is 2 in relation with the degree of the homogeneous polynomials appearing in R. The map R is the natural lift of f on $\mathbb{C}^3$ since it is represented by homogeneous polynomials with no common factor (the map T is an other lift of f but has singularities). It is interesting to note that R appears naturally in the expression of $H^{\pm}_{\langle n\rangle}$ in Proposition 2.3.

We set

$$D=\big\{[q_1,q_2,q],\ q_1+\delta^{-1}q_2=0\big\}=\pi\big(\big\{x\in\mathbb{C}^3\setminus\{0\},\ p(x)=0\big\}\big). \quad (3.2)$$

N.B.: we recall that p is the polynomial defined in Section 2.3.

The line D is sent by f to a unique point, $[-\delta^{-1},-\delta,-1]$, i.e.,

$$f(D\setminus\{l\})=\big\{\big[-\delta^{-1},-\delta,-1\big]\big\}. \quad (3.3)$$

It is called a f-constant curve (line). It is a general phenomenon that a f-constant curve contains a point of indeterminacy (cf Proposition 1.2 of [8]).

Another important property of the map f is that it has no degree lowering curve: a degree lowering curve is a f-constant curve sent by f^n to a point of indeterminacy. When such a phenomenon happens, a common factor, which can be divided out, appears in R^n and the degree of the map f^n drops. Here we remark that the orbit of D is given by $f(D\setminus\{l\})=\{[-\delta^{-1},-\delta,-1]\}$ and $f^{n+1}(D\setminus\{l\})=\{[\delta^{-2^n},\delta^{2^n},-1]\}$ so D is not a degree lowering curve. Hence degree$(f^n)=2^n$ (this means that R^n is represented by 3 homogeneous polynomials of degree 2^n with no common factor). Following the terminology of [8], f is said to be a generic meromorphic map of $\mathbb{P}^2$ (in [27], f is said to be algebraically stable, cf Definition 4.4).

We set $\tilde{I} = \bigcup_{n\geqslant 0} f^{-n}(\{l\}) = \{[1, -\delta^{-(n-1)}, 0]\}_{n\geqslant 0}$ the set of preimages of the point of indeterminacy $\{l\}$.

The Fatou set of f is defined to be the union of all open balls $U \subset \mathbb{P}^2 \setminus \tilde{I}$ on which the family $\{f^n\}_{n\geqslant 0}$ is normal. The Fatou set is denoted by $\mathcal{F}$ and its complement, the Julia set by $\mathcal{J} = \mathbb{P}^2 \setminus \mathcal{F}$. Of particular interest to us is the fact that the attractive basin of an attractive fixed point is in the Fatou set.

A useful function, to study the dynamics of f, is the Green function defined as the limit of the sequence of functions $G_n : \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$:

$$G_n(x) = \frac{1}{2^n} \log \|R^n(x)\|, \quad x \in \mathbb{C}^3, \tag{3.4}$$

where $\| \ \|$ denotes the usual norm of $\mathbb{C}^3$.

We will use the following result (cf [8, Proposition 2.11] or [27, Theorem 1.6.1]):

PROPOSITION 3.1. – (i) *The limit*

$$G(x) = \lim_{n\to\infty} G_n(x)$$

exists for all $x \in \mathbb{C}^3$. The function G is plurisubharmonic and satisfies

$$G \circ R = 2G. \tag{3.5}$$

(ii) *G is pluriharmonic on $\pi^{-1}(\mathcal{F})$.*

3.2. Construction of the holomorphic curve $\phi(\lambda)$ and description of the map f

3.2.1. Construction of $\phi(\lambda)$

We first remark that the hyperplane $P_1 = \{(q_1, q_2, -1),\ (q_1, q_2) \in \mathbb{C}^2\}$ is invariant by R. The line $\{[q_1, q_2, 0],\ [q_1, q_2] \in \mathbb{P}^1\}$ is invariant by f and classically the space $\mathbb{P}^2$ can be represented by $\mathbb{P}^2 = P_1 \cup \{q = 0\}$ ($\{q = 0\} \simeq \mathbb{P}^1$ is the line at infinity).

We see that f is a polynomial transformation on $P_1 \simeq \mathbb{C}^2$ given by

$$f(q_1, q_2) = \left(q_1\left(q_1 + \delta^{-1}q_2\right) - \delta^{-1}, \delta q_2\left(q_1 + \delta^{-1}q_2\right) - \delta\right).$$

We recall that the map $\lambda \to A_{(\lambda)}$ is meromorphic and at this point it is natural to replace it by a complex curve (with no singularities) $\lambda \to \phi(\lambda)$ which has the same projection in $\mathbb{P}^2$. The map $A_{(\lambda)}$ is holomorphic in a neighbourhood of 0 and $A_{(0)} = (1, 1, -1)$, so, locally, there exists $\phi(\lambda)$ such that $\phi(\lambda) \in P_1$ and $\pi(\phi(\lambda)) = \pi(A_{(\lambda)})$. On this neighbourhood ϕ is invariant by R, i.e., $\phi(\gamma\lambda) = R(\phi(\lambda))$ (since P_1 is invariant by R and $A_{(\gamma\lambda)} = T(A_{(\lambda)})$) and we extend it to $\mathbb{C}$ by:

$$\phi\left(\gamma^k\lambda\right) = R^k\left(\phi(\lambda)\right), \quad k \in \mathbb{N}.$$

So ϕ satisfies $\phi(\lambda) \in P_1$ for all $\lambda \in \mathbb{C}$. We also denote by ϕ its projection in $\mathbb{P}^2$ and we have:

$$R\circ\phi(\lambda)=\phi(\gamma\lambda),\quad \lambda\in\mathbb{C}, \tag{3.6}$$

$$f\circ\phi(\lambda)=\phi(\gamma\lambda),\quad \lambda\in\mathbb{C}. \tag{3.7}$$

We can give an explicit expression of $\phi(\lambda)$ (it will only be used in Section 3.4).

PROPOSITION 3.2. – *We have*:

$$\phi(\lambda)=d^-(\lambda)A_{(\lambda)}, \tag{3.8}$$

where $d^-(\lambda)$ is the infinite determinant associated with the Dirichlet spectrum of a on I as in Section 2.1.

Proof. – In a neighbourhood of 0, the function $\phi(\lambda)$ is given by

$$\phi(\lambda)=v(\lambda)A_{(\lambda)}, \tag{3.9}$$

where $v(\lambda)$ is a holomorphic function. But we have:

$$\begin{aligned}\phi(\gamma\lambda)&=R\big(\phi(\lambda)\big)=v(\lambda)^2R(A_{(\lambda)})\\&=v(\lambda)^2p(A_{(\lambda)})T(A_{(\lambda)})=v(\lambda)^2p(A_{(\lambda)})A_{(\gamma\lambda)}.\end{aligned}$$

Therefore the function v can be extended into a holomorphic function on the complex plane by the following relation:

$$v(\gamma\lambda)=v(\lambda)^2p(A_{(\lambda)})=v(\lambda)p\big(\phi(\lambda)\big). \tag{3.10}$$

It only remains to prove that $v=d^-$. Formula (2.33) with $n=1$ gives

$$d^-(\gamma\lambda)=d^-_{\langle 1\rangle}(\lambda)=C\big(d^-(\lambda)\big)^2p(A_{(\lambda)}) \tag{3.11}$$

for a positive constant C. This constant is fixed to 1 by considering that $d^-(0)=p(A_{(0)})=1$ so the functions v and d^- satisfy the same functionnal equation with same initial conditions $v(0)=d^-(0)=1$, it is then easy to prove that they are equal. For example if we consider the local expansion $\ln(p(A_{(\lambda)}))=\sum b_n\lambda^n$ then $\ln(v)$ and $\ln(d^-)$ must be equal to $\sum a_n\lambda^n$ where $a_n=b_n/(\gamma^n-2)$. This implies $v(\lambda)=d^-(\lambda)$ for λ in $\mathbb{C}$. □

To unify the notations, we will also denote, in the discrete case, by $\tilde\phi(\lambda)$ the coordinates of A_λ in $\mathbb{C}^3$, so if we chose $\omega=c\delta_0+(1-c)\delta_1$ we have $\tilde\phi(\lambda)=(1+\lambda c, 1+\lambda(1-c), -1)$.

3.2.2. Geometric elements of the dynamics of f. Convergence to the Green function

We denote by $\mathcal{C}\subset\mathbb{P}^2$ the hypersurface:

$$\mathcal{C}=\big\{[q_1,q_2,q],\ q_1q_2=q^2\big\}=\pi(\{x,\ r(x)=0\}). \tag{3.12}$$

N.B.: r is the homogeneous polynomial defined in Section 2.3; $\mathcal{C}$ is then the hypersurface of degenerate quadratic forms since $r(Q)=\det(Q)=q_1q_2-q^2$.

We set:

$$K=\big\{[q_1,q_2,-1],\ (q_1,q_2)\in\mathbb{R}^2\big\} \tag{3.13}$$

and

$$K^+ \text{ (resp. } K^-) = \{[q_1, q_2, -1] \in K,\ q_1q_2 - 1 \geqslant 0\} \quad \text{(resp. } q_1q_2 - 1 \leqslant 0). \tag{3.14}$$

The following properties are easy to check:

$$f([q_1, q_2, q]) = [q_1^2, q_2^2, -q^2], \quad \forall [q_1, q_2, q] \in \mathcal{C}, \tag{3.15}$$

$$f(\mathcal{C}) = \mathcal{C}, \qquad f^{-1}(\mathcal{C}) = \mathcal{C} \cup (D \setminus \{l\}), \tag{3.16}$$

$$f(K_\pm) \subset K_\pm, \tag{3.17}$$

$$\phi(\mathbb{R}_\pm) \subset K_\pm. \tag{3.18}$$

The second and third assertions can easily be deduced from the following relation (which has been already used in the proof of the Proposition 2.3):

$$r \circ R(x) = \frac{1}{\alpha(1-\alpha)} \big(p(x)\big)^2 r(x), \quad x \in \mathbb{C}^3. \tag{3.19}$$

The fourth assertion can be locally deduced form the expression of $A_{(\lambda)}$ (cf Lemma 2.1) and is extended to $\mathbb{R}$ by relation (3.17) and the property of invariance of the curve $\phi(\lambda)$ (relation (3.7)).

Next, we study the behaviour of f in the vicinity of the point of indeterminacy l. By f, the point l is sent to a curve called the blow-up of l. Here the blow-up is the line

$$D' = \{[q_1, q_2, q],\ q_1 + \delta^{-2} q_2 = 2\delta^{-1} q\}. \tag{3.20}$$

Precisely this means that (cf [27, Section 1.2]):

$$\bigcap_{\varepsilon > 0} \overline{f(B(l, \varepsilon) \setminus \{l\})} = D'. \tag{3.21}$$

N.B.: $B(l, \varepsilon)$ denotes the open ball with center l and radius ε.

To prove this relation we estimate f in the vicinity of $l = [1, -\delta, 0]$, for $(u, v) \in \mathbb{C}^2$ we have

$$R\big(1, -\delta(1+u), v\big) = v^2 \left(-\delta^{-1} - \frac{u}{v^2}, -\delta + \delta^2 \frac{u}{v^2}(1+u), -1\right), \tag{3.22}$$

so if $u \to 0$, $v \to 0$ and $\frac{u}{v^2} \to z$ we see that we are on the set $[-\delta^{-1} - z, -\delta + \delta^2 z, -1] \in D'$.

An important remark in the sequel is that D' meets the hypersurface $\mathcal{C}$ at the only point $[-\delta^{-1}, -\delta, -1]$ which is the image of the f-constant line D (this intersection is double, i.e., the line D' is tangent to $\mathcal{C}$).

We need now to separate the case $\delta = 1$ and $\delta \neq 1$.

The case $\delta \neq 1$

We can as well suppose $\delta > 1$.

Let $x_- = [1, 0, 0]$ and $x_+ = [0, 1, 0]$, we remark that x_+ and x_- are fixed by f and that

- x_- have one attractive direction and one repulsive (with eigenvalues 0 and δ).
- x_+ is attractive (with eigenvalues 0 and δ^{-1}).

In particular we remark that the f-constant line D is in the attractive basin of x_+ since $f^{n+1}(D \setminus \{l\}) = [\delta^{-2^n}, \delta^{2^n}, -1]$. In fact if we denote by $\mathcal{C}_+ = \{[q_1, q_2, -1] \in \mathcal{C},\ |q_1| < |q_2|\}$ (resp. $\mathcal{C}_-$ for $|q_1| > |q_2|$), then $f^n(x)$ converges to $x_\pm$ if $x \in \mathcal{C}_\pm$. We could prove that $\mathcal{C}_-$ is the stable manifold of the fixed point x_- and that the unstable manifold is the line at infinity $\{q = 0\}$.

We finally give the most important result of this section:

PROPOSITION 3.3. – *We have*:

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left|r \circ R^n\right| = 2G, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}^3), \tag{3.23}$$

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left|r \circ R^n \circ \phi\right| = 2G \circ \phi, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}), \tag{3.24}$$

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left|r \circ R^n \circ \tilde{\phi}\right| = 2G \circ \tilde{\phi}, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}). \tag{3.25}$$

Remark 3.1. – Sufficient conditions implying formula (3.23) are given in [8], Theorem 4.6 or in [27], Section 1.10, but the hypotheses do not suit our case. Here we need to get a precise estimate on the rate at which the orbite of a point of $\mathbb{P}^2$ approaches the hypersurface $\mathcal{C}$ (which is the set $\log|r| = -\infty$). This is the role of Lemma 3.1.

Proof. – We first need the following lemma.

LEMMA 3.1. – *For all* $x \in \pi^{-1}(\mathbb{P}^2 \setminus (\bigcup_{n\in\mathbb{N}} f^{-n}(\mathcal{C}) \cup \tilde{I}))$:

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left|r\left(\frac{R^n x}{\|R^n x\|}\right)\right| = 0. \tag{3.26}$$

Proof. – The function $|r|$ is clearly bounded from above on the unit ball of $\mathbb{C}^3$, so we only need to minorate $|r|$.

The key relation is formula (2.54) that we recall here:

$$r \circ R(x) = \frac{1}{\alpha(1-\alpha)} \big(p(x)\big)^2 r(x), \quad \forall x \in \mathbb{C}^3. \tag{3.27}$$

Let $C_1 > 0$ be such that $\|Rx\| \leqslant C_1\|x\|^2$ on $\mathbb{C}^3$. Let $V_1 \subset \mathbb{P}^2 \setminus \{l\}$ be a neighbourhood of $D \setminus \{l\}$ included in the attractive basin of x_+ (remind that $D \setminus \{l\}$ is sent to a unique point which is in the attractive bassin of x_+). Let V_2 be a neighbourhood of D' such that there is a neighbourhood $V_2' \subset V_2$ of $V_2 \cap \mathcal{C}$ included in the attractive basin of x_+ (this is possible since D' intersects the hypersurface $\mathcal{C}$ at a unique point $[-\delta^{-1}, -\delta, -1]$ which is in the attractive basin of x_+). So there is a constant $C_2 > 0$ such that $|r(\frac{x}{\|x\|})| \geqslant C_2$ for all $x \in \pi^{-1}(V_2 \setminus V_2')$. Using formula (3.21) we choose $\varepsilon > 0$ such that $f(B(l, \varepsilon)) \subset V_2$. Let $C_3 > 0$ be such that $|r(\frac{x}{\|x\|})| \leqslant C_3$.

Finally, we denote by $C_4 > 0$ a real such that $|p(\frac{x}{\|x\|})| \geqslant C_4$ for x in $\pi^{-1}(\mathbb{P}^2 \setminus (V_1 \cup B(l, \varepsilon)))$ (this is possible since $V_1 \cup B(l, \varepsilon)$ is a neighbourhood of $D = \pi(\{x,\ p(x) = 0\})$).

Let $C = (\frac{1}{\alpha(1-\alpha)}(\frac{C_4}{C_1})^2) \wedge (C_2/C_3)$.

Let $x \in \pi^{-1}(\mathbb{P}^2 \setminus (\bigcup_{n\geqslant 0} f^{-n}(\mathcal{C}) \cup \tilde{I}))$. We prove that for n sufficiently large we have $|r(\frac{R^{n+1}x}{\|R^{n+1}x\|})| \geqslant C|r(\frac{R^n x}{\|R^n x\|})|$. The result of Lemma 3.1 follows easily this assertion.

Let $\varepsilon' > 0$ be such that $f(B(x_+, \varepsilon')) \subset B(x_+, \varepsilon')$ and $B(x_+, \varepsilon') \cap (B(l, \varepsilon) \cup V_1) = \emptyset$. If $\pi(x)$ is in the attractive basin of x_+ we can find $N > 0$ such that $f^n(\pi(x)) \in B(x_+, \varepsilon')$ for $n \geqslant N$. It follows from relation (3.2) that

$$\left|r\left(\frac{R^{n+1}x}{\|R^{n+1}x\|}\right)\right| \geqslant \frac{1}{\alpha(1-\alpha)}\left(\frac{C_4}{C_1}\right)^2 \left|r\left(\frac{R^n x}{\|R^n x\|}\right)\right|, \tag{3.28}$$

for all $n \geqslant N$.

Assume that $\pi(x)$ is not in the attractive basin of x_+. For all $n \geqslant 0$ $f^n(\pi(x))$ is in $\mathbb{P}^2 \setminus V_1$. If $f^n(\pi(x)) \notin B(l, \varepsilon)$ then the estimate (3.28) is true. If $f^n(\pi(x)) \in B(l, \varepsilon)$ then $f^{n+1}(\pi(x)) \in V_2 \setminus V_2'$ since $\pi(x)$ is not in the attractive basin of x_+. It follows that $|r(\frac{R^{n+1}x}{\|R^{n+1}x\|})| \geqslant C_2$ and so that $|r(\frac{R^{n+1}x}{\|R^{n+1}x\|})| \geqslant \frac{C_2}{C_3}|r(\frac{R^n x}{\|R^n x\|})|$. □

We now prove Proposition 3.3. We first remark that:

$$\log|r(R^n x)| = \log\left|r\left(\frac{R^n x}{\|R^n x\|}\right)\right| + 2\log(\|R^n x\|). \tag{3.29}$$

Since r is bounded from above on the unit ball of $\mathbb{C}^3$ we have that

$$\limsup_{n\to\infty} \frac{1}{2^n} \log\left|r\left(\frac{R^n x}{\|R^n x\|}\right)\right| \leqslant 0.$$

Let G_1 be a plurisubharmonic function such that a subsequence $\frac{1}{2^{n_i}}\log|r(R^{n_i})|$ converges to G_1 in $L^1_{\mathrm{loc}}(\mathbb{C}^3)$. From Proposition 4.2.18 of [13] and Lemma 3.1 we have that $G_1(x) = 2G(x)$ on $\pi^{-1}(\mathbb{P}^2 \setminus (\bigcup_n f^{-n}(\mathcal{C}) \cup \tilde{I}))$ (since $G_1 \geqslant \limsup \frac{1}{2^{n_i}}\log|r(R^{n_i})|$). Since $\pi^{-1}(\bigcup_n f^{-n}(\mathcal{C}) \cup \tilde{I})$ has Lebesgue measure zero, it implies that $G_1 = 2G$. Since there is no subsequence n_i such that $\frac{1}{2^{n_i}}\log|r(R^{n_i})|$ converges uniformally to $-\infty$, using Proposition 3.2.12 of [13], we get that $\frac{1}{2^n}\log|r(R^n)|$ converges to $2G$ in $L^1_{\mathrm{loc}}(\mathbb{C}^3)$.

For the second and third formula we use the fact that the set $\{\lambda,\ \phi(\lambda) \in \bigcup f^{-n}(\mathcal{C})\}$ (and $\{\lambda,\ \tilde{\phi}(\lambda) \in \bigcup f^{-n}(\mathcal{C})\}$) has Lebesgue measure 0 in $\mathbb{C}$ (since there is no n such that $f^n(\phi(\lambda))$ or $f^n(\tilde{\phi}(\lambda))$ is included in $\mathcal{C}$), and the same proof gives the convergence (we can note that the second and third results are not directely implied by the first one since it may happen that $\frac{1}{2^n}\log|r \circ R^n|$ does not converges to $2G$ on a complex curve; this is for example the case on the hypersurface $\mathcal{C}$ where $\log|r \circ R^n|$ remains identically equal to $-\infty$). □

The case $\delta = 1$

We do not want to go to much into the details of this case since all the results we will deduce are well-known. However they are interesting for a sake of completeness. In fact everything can be reduced to a 1-dimensional situation. We denote by $s : \mathbb{C} \to \mathbb{C}$ the polynomial $s(z) = 2z^2 - 1$. If we denote by $[\tilde{q}_1, \tilde{q}_2, -1]$ the image $[\tilde{q}_1, \tilde{q}_2, -1] = f([q_1, q_2, -1])$ then we see that $\tilde{z} = s(z)$ if $z = \frac{q_1+q_2}{2}$ and $\tilde{z} = \frac{\tilde{q}_1+\tilde{q}_2}{2}$. In particular, the line $[z, z, -1]$ is invariant by f and on this line f equals s.

We denote by $G_s(z)$ the Green function of the map s defined by

$$G_s(z) = \lim_{n\to\infty} \frac{1}{2^n} \log\bigl(1 + \left|s^n(z)\right|^2\bigr)^{1/2} \tag{3.30}$$

(i.e., we consider s as the map $s([z,1]) = [s(z),1]$ on $\mathbb{P}^1$). The interval $[-1,1]$ is bi-invariant by s and so $G_s(z) = 0$ on $[-1,1]$ and G_s is harmonic on $\mathbb{C}\setminus[-1,1]$ since $\mathbb{C}\setminus[-1,1]$ is the Fatou set of s (cf, for example, [28]). We can prove that the Green function of f satisfies $G((q_1,q_2,-1)) = G_s(\frac{q_1+q_2}{2})$. Indeed, we see that $\tilde q_1 - \tilde q_2 = 2z(q_1-q_2)$ and that

$$\|(q_1,q_2,-1)\|^2 = \frac{1}{2}\bigl(|q_1|^2 + |q_2|^2 + 1\bigr) = \left|\frac{q_1-q_2}{2}\right|^2 + \left|\frac{q_1+q_2}{2}\right|^2 + 1.$$

So we have

$$\log\bigl\|R^n(q_1,q_2,-1)\bigr\| = \frac{1}{2}\log\bigl(1+\left|z^{(n)}\right|^2\bigr) + \frac{1}{2}\log\left(1 + \frac{|q_1^{(n)} - q_2^{(n)}|^2}{4(1+|z^{(n)}|^2}\right) \tag{3.31}$$

if $[q_1^{(n)}, q_2^{(n)}, -1]$ denotes the image by the nth iterate of f of $[q_1,q_2,-1]$, and $z^{(n)} = \frac{1}{2}(q_1^n + q_2^n)$. It is then easy to estimate the component $\frac{|q_1^{(n)}-q_2^{(n)}|^2}{1+|z^{(n)}|^2}$ to show that the second term does not contribute to the Green function. We can also show that the formulas (3.23), (3.24) and (3.25) remain valid in this case (the convergence in the one one-dimensional situation is always satisfied, cf for example [28, Theorem 6.1]).

It is also easy to see that $\phi(\lambda)$ takes its values in the line $[z,z,-1]$ (idem for the discrete case for $c=\frac{1}{2}$) and so the problem is really 1-dimensional in these situations.

3.3. Expression of the integrated density of states. First application

The main result of this paper is the following:

THEOREM 3.1. – (i) *The continuous case*: *the integrated density of states of* $\frac{d}{dm_{\langle\infty\rangle}}\frac{d}{dx}$ *on* $\mathbb{R}_+$ *exists and is*

$$\mu = \lim_{n\to\infty} \frac{1}{2^n}\nu^{\pm}_{\langle n\rangle} = \frac{1}{2\pi}\Delta(G\circ\phi). \tag{3.32}$$

(ii) *The discrete case*: *The integrated density of states of the infinitesimal generator of* $A_{\langle\infty\rangle}$ *on* $L^2(\mathbb{N},\omega_{\langle\infty\rangle})$ *exists and is given by*

$$\tilde\mu = \lim_{n\to\infty} \frac{1}{2^n}\tilde\nu^{\pm}_{\langle n\rangle} = \frac{1}{2\pi}\Delta(G\circ\tilde\phi). \tag{3.33}$$

Remark 3.2. – We recall that $\phi(\lambda)$ is a holomorphic curve with the same projection in $\mathbb{P}^2$ as $A_{(\lambda)}$ and which is invariant by R, in particular we can read the scaling relation (1.16) from this formula since $\phi(\gamma\lambda) = R(\phi(\lambda))$ and $G\circ R = 2G$. For the discrete case we have $\tilde\phi(\lambda) = (1+\lambda c, 1+\lambda(1-c), -1)$ if we chose the measure $\omega = c\delta_0 + (1-c)\delta_1$ on F and so, a priori, $\tilde\mu$ depends on c.

Proof. – (i) From Propositions 2.3 and 3.3 we deduce that

$$\lim_{n\to\infty}\frac{1}{2^n}H^{\pm}_{\langle n\rangle}\circ\phi = G\circ\phi, \quad \text{in } L^1_{\text{loc}}(\mathbb{C}).$$

Since $A_{(\lambda)}$ is non-singular on $B(0,|\lambda_1^-|)$ and $\phi(\lambda)$ is non-null we have:

$$\left(\Delta\left(H^{\pm}_{\langle n\rangle}\circ\phi(\lambda)\right)\right)_{|B(0,|\lambda_1^-|)}=\left(\Delta H^{\pm}_{\langle n\rangle}(A_{(\lambda)})\right)_{|B(0,|\lambda_1^-|)}. \tag{3.34}$$

(Indeed, we have $\phi(\lambda)=d^-(\lambda)A_{(\lambda)}$ in Proposition 3.2, and d^- is a non-null holomorphic function on this ball.) Using relation (2.30) we get that $\frac{1}{2^n}\nu^{\pm}_{\langle n\rangle}$ converges weakly to $\frac{1}{2\pi}\Delta(G\circ\phi)$ on $B(0,|\lambda_1^-|)$. Moreover we have $G\circ\phi(\gamma\lambda)=2G\circ\phi(\lambda)$ so $\tau^*(\Delta G\circ\phi)=2\Delta G\circ\phi$ where τ^* denotes the pull-back of the measure $\Delta G\circ\phi$ by the homothetie $\tau(\lambda)=\gamma\lambda$. Using the scaling relation of $\nu^{\pm}_{\langle n\rangle}$ (cf relation (1.17)) we get that $\frac{1}{2^n}\nu^{\pm}_{\langle n\rangle}$ converges weakly to $\frac{1}{2\pi}\Delta G\circ\phi$ on $\mathbb{C}$.

The discrete case (ii) is even more simple. □

We again separate the case $\delta=1$ and the case $\delta\neq 1$.

3.3.1. $\delta\neq 1$

We have:

COROLLARY 3.1. – *The measure μ (and $\tilde{\mu}$) charges no point and is supported by a Cantor subset of $\mathbb{R}_-$ (i.e., a closed subset with empty interior and no isolated point).*

Remark 3.3. – A natural question is wether the support has Lebesgue measure 0. This would imply that μ is a singular continuous measure. This seems to be a difficult question.

Proof. – We assume $\delta>1$. We prove it for the continuous case, the discrete case is similar.

We remark that if $x=(q_1,q_2,-1)$ then $\|R^n x\|\geqslant 1$ so $G(x)\geqslant 0$ on $P_1=\{[q_1,q_2,-1]\}$. So, $G\circ\phi\geqslant 0$ on $\mathbb{C}$ and this implies that $\Delta G\circ\phi$ charges no point since $G\circ\phi$ should be $-\infty$ on such a point.

We set

$$S_n=\left\{\lambda,\ f^n\left(\phi(\lambda)\right)\in\mathcal{C}\text{ and }f^{n-1}\left(\phi(\lambda)\right)\notin\mathcal{C}\right\}. \tag{3.35}$$

Since $4\pi\mu$ is the weak limit of $\frac{1}{2^n}\Delta\log|r\circ R^n\circ\phi|$ we know that

$$\operatorname{supp}(\mu)\subset\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{m\geqslant n}S_m} \tag{3.36}$$

(this comes from the fact that $\log|r\circ R^n\circ\phi(\lambda)|$ is harmonic around a point where $R^n\circ\phi(\lambda)\notin\mathcal{C}$).

Since $f^{-1}(\mathcal{C})\subset\mathcal{C}\cup D$ and $f(D)=[-\delta^{-1},-\delta,-1]$ it follows that $\lambda\in S_n$ satisfies $f^n(\phi(\lambda))=[-\delta^{-1},-\delta,-1]$ and so $f^n(\phi(\lambda))$ is in the attractive basin of x_+. This implies that the points of $\bigcup_{n\in\mathbb{N}}S_n$ are isolated since for $\lambda\in S_n$ we can find $\varepsilon>0$ such that the orbit $f^m(\phi(\lambda'))$ for $\lambda'\in B(\lambda,\varepsilon)$ and $m>n$ remains close enough to the orbit $f^m(\phi(\lambda))$ and so never meets the point $[-\delta^{-1},-\delta,-1]$ (and so $B(\lambda,\varepsilon)\cap S_m=\emptyset$ for $m>n$). Relation (3.36) implies that $\operatorname{supp}(\mu)$ has empty interior, moreover it is closed and contains no isolated point since μ charges no point. It is included in $\mathbb{R}^-$ by definition. This concludes the proof. □

3.3.2. $\delta = 1$

In this case the results are well-known but for the sake of completeness we show how they can be deduced from our method.

COROLLARY 3.2. – (i) *The continuous case*: *the repartition function* $F(\lambda) = \int_0^\lambda d\mu$ *is given by*:

$$F(\lambda) = C|\lambda|^{1/2}, \quad \forall \lambda \leqslant 0, \tag{3.37}$$

for a constant $C > 0$.

(ii) *The dicrete case*: *the integrated density of states has the following law*:

$$\tilde{\mu}(dx) = \frac{dx}{\pi\sqrt{-x(x+2)}} 1_{[-2,0]}(x). \tag{3.38}$$

N.B.: we remark here that the integrated density of states $\tilde{\mu}$ does not depend on the choice of $\omega = c\delta_0 + (1-c)\delta_1$ and it is normal since the measure $\omega_{\langle n\rangle}$ does not depend on c at the exception of the weight of the extremal points $\{0, \alpha^{-n}\}$.

Proof. – We recall that we proved that $G((q_1, q_2, -1)) = G_s(\frac{q_1+q_2}{2})$ where G_s is the Green function of the polynomial map $s(z) = 2z^2 - 1$.

(i) It is easy to see that the map ϕ takes its values in $(z, z, -1)$ and so it can be viewed as a entire map of $\mathbb{C}$. It is also clear that $\phi(\mathbb{R}_-) \subset [-1, 1]$ and that $\phi(\mathbb{C} \setminus \mathbb{R}_-) \subset \mathbb{C} \setminus [-1, 1]$. So $\psi = G \circ \phi$ is null on $\mathbb{R}_-$ and harmonic on $\mathbb{C} \setminus \mathbb{R}_-$. Then we prove that $\psi(z) = C\,\mathrm{Re}(z^{1/2})$ (where $\mathrm{Re}(z)$ denotes the real part of z). This implies the corollary using formula (3.3.42) of [13].

We conformally map $\mathbb{C} \setminus \mathbb{R}_-$ to the unit disc: we define $\hat{\psi} : B(0, 1) \to \mathbb{C}$ by $\hat{\psi}(z) = \psi((\frac{z+1}{z-1})^2)$. The function $\hat{\psi}$ is harmonic in $B(0, 1)$ and $\lim_{r\to 1} \hat{\psi}(re^{i\theta}) = 0$ for $\theta \neq 0\,[2\pi]$. From [24], exercise 8 p. 237, we get $\hat{\psi}(z) = C\,\mathrm{Re}(\frac{1+z}{1-z})$ and the result follows.

(ii) Here $G \circ \tilde{\phi}(\lambda) = G_s(1+\lambda)$ is equal to 0 on $[-2, 0]$ and is harmonic on $\mathbb{C} \setminus [-2, 0]$. The measure $\tilde{\mu}$ is then the equilibrium measure of the interval $[-2, 0]$ which is known to be given by formula (3.38). □

3.4. About the propagator of the O.D.E

For the sake of completeness we give an expression of the Lyapunov exponent associated with the propapgator of the differential equation $\frac{d}{dm_{\langle\infty\rangle}} \frac{d}{dx} = \lambda f$ on $\mathbb{R}_+$. It is well-known that in the case of 1-dimensionnal random Schrödinger operators, the integrated density of states and the Lyapunov exponent are related by the Thouless formula (cf [3] or [21]). Here it is also the case and in fact in this section we give an expression of the Lyapunov exponent in terms of the Green function of the renormalization map R.

Let us consider the ordinary differential equation

$$\frac{d}{dm_{\langle\infty\rangle}} \frac{d}{dx} f = \lambda f \tag{3.39}$$

on the half-line $\mathbb{R}_+$. The propagator $\Gamma_\lambda(s, t)$ for $0 \leqslant s < t$ is defined as the unique 2×2 matrix such that any solution f of (3.39) satisfies:

$$\Gamma_\lambda(s,t)\begin{pmatrix} f(s)\\ f'(s)\end{pmatrix}=\begin{pmatrix} f(t)\\ f'(t)\end{pmatrix}. \tag{3.40}$$

It is well-known that $\Gamma_\lambda(s,t)$ is holomorphic in λ.

With any symmetric matrix Q of the form

$$\begin{pmatrix} q_1 & q\\ q & q_2\end{pmatrix} \tag{3.41}$$

we associate the unique 2×2 matrix $\Gamma(Q)$ such that:

$$Q\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} -x'\\ y'\end{pmatrix} \quad \text{iff} \quad \Gamma(Q)\begin{pmatrix} x\\ x'\end{pmatrix}=\begin{pmatrix} y\\ y'\end{pmatrix} \tag{3.42}$$

for all $(x,y,x',y')\in\mathbb{C}^4$. An easy computation gives:

$$\Gamma(Q)=-\frac{1}{q}\begin{pmatrix} q_1 & 1\\ q_1q_2-q^2 & q_2\end{pmatrix} \tag{3.43}$$

so that $\Gamma(Q)$ can be defined when $q\neq 0$. Remark that $\Gamma(Q)$ has the following homogeneity:

$$\Gamma(Q)=D_\beta^{-1}\circ\Gamma(\beta Q)\circ D_\beta, \tag{3.44}$$

where D_β is the diagonal matrix:

$$D_\beta=\begin{pmatrix} 1 & 0\\ 0 & \beta\end{pmatrix}. \tag{3.45}$$

The interest of this definition lies in the following result.

PROPOSITION 3.4. – *When all the terms are defined then we have*:

$$\Gamma_\lambda(0,1)=\Gamma(A_{(\lambda)}) \tag{3.46}$$

and

$$\Gamma_\lambda\big(0,\alpha^{-n}\big)=\Gamma\big(\alpha^n A_{(\gamma^n\lambda)}\big). \tag{3.47}$$

Proof. – Consider a function g on F and $f=H_\lambda(g)$ its harmonic continuation. Of course the function f is solution of (3.39) on $[0,1]$. An easy integration by parts implies $A_{(\lambda)}(g)=f(1)f'(1)-f(0)f'(0)$. This immediately implies that, as a 2×2 matrix, $A_{(\lambda)}$ satisfies:

$$A_{(\lambda)}\begin{pmatrix} f(0)\\ f(1)\end{pmatrix}=\begin{pmatrix} -f'(0)\\ f'(1)\end{pmatrix} \tag{3.48}$$

and this gives formula (3.46).

By scaling, the trace of the Dirichlet form $a_{\langle n\rangle}+\lambda\int_0^{\alpha^{-n}} f^2\,dm_{\langle n\rangle}$ on the subset $\{0,\alpha^{-n}\}$ is $\alpha^n A_{(\gamma^n\lambda)}$. This gives formula (3.47). □

We remark that although Γ_λ has no singularity, the right term of (3.47) have some. The singularities of $A_{(\lambda)}$ are cancelled by the singularities of $\Gamma(Q)$ and in fact we can give a non-singular expression of the propagator in terms of the map $\phi(\lambda)$ introduced in Section 3.2.

PROPOSITION 3.5. – *The propagator* $\Gamma_\lambda(0,\alpha^{-n})$ *has the following expression*:

$$\Gamma_\lambda\big(0,\alpha^{-n}\big)=\begin{pmatrix} q_1(\phi(\gamma^n\lambda)) & \alpha^{-n}d^-(\gamma^n\lambda)\\ C\alpha^n d^+(\gamma^n\lambda) & q_2(\phi(\gamma^n\lambda))\end{pmatrix} \tag{3.49}$$

where C *is a positive constant independent of* n.

N.B.: $q_i(\phi(\gamma^n\lambda))$ denote the coordinates of the curve ϕ in $\mathbb{C}^3$. The function d^+, d^- are the infinite determinants associated with the Neuman and Dirichlet spectrum of a on I as in Section 2.1.

Proof. – Remind that $\phi(\lambda)=d^-(\lambda)A_{(\lambda)}$ is a holomorphic curve taking its values in the hyperplane $\{q=-1\}$. From formulas (3.43) and (3.44) we get

$$\begin{aligned}\Gamma_\lambda\big(0,\alpha^{-n}\big)&=\Gamma\big(\alpha^nA_{(\gamma^n\lambda)}\big)=(D_{d^-(\gamma^n\lambda)})^{-1}\circ\Gamma\big(\alpha^n\phi(\gamma^n\lambda)\big)\circ D_{d^-(\gamma^n\lambda)}\\ &=\begin{pmatrix} q_1(\phi(\gamma^n\lambda)) & \alpha^{-n}d^-(\gamma^n\lambda)\\ \alpha^n\frac{\det(\phi(\gamma^n\lambda))}{d^-(\gamma^n\lambda)} & q_2(\phi(\gamma^n\lambda))\end{pmatrix}.\end{aligned} \tag{3.50}$$

Then we can conclude using Lemma 2.2 and Proposition 3.2. □

PROPOSITION 3.6. – *The Lyapunov exponent* $\zeta(\lambda)$ *defined by*:

$$\zeta(\lambda)=\lim_{n\to\infty}\frac{1}{2^n}\ln\big\|\Gamma_\lambda\big(0,\alpha^{-n}\big)\big\| \tag{3.51}$$

exists for all $\lambda\in\mathbb{C}$ *and*

$$\zeta(\lambda)=G\circ\phi(\lambda). \tag{3.52}$$

Remark 3.4. – This definition of the Lyapunov exponent is not very satisfactory since we only considered the value of the propagator at times α^{-n}. In particular the usual deterministic Osedelec theorem (cf for example [3, Theorem IV-2-4]) cannot be applied from this definition.

Proof. – We have:

$$\begin{aligned}\ln\big\|\Gamma_\lambda\big(0,\alpha^{-n}\big)\big\|&=\frac12\ln\big(\big|q_1\big(\phi(\gamma^n\lambda)\big)\big|^2+\big|q_2\big(\phi(\gamma^n\lambda)\big)\big|^2\\ &\quad+\big|C\alpha^nd^+(\gamma^n\lambda)\big|^2+\big|\alpha^{-n}d^-(\gamma^n\lambda)\big|^2\big)\\ &\geqslant\frac12\ln\big(\big|q_1\big(\phi(\gamma^n\lambda)\big)\big|^2+\big|q_2\big(\phi(\gamma^n\lambda)\big)\big|^2\big).\end{aligned} \tag{3.53}$$

Since we know that the norm of the propagator is bounded from above (since it has determinant 1) we deduce that:

$$\liminf_{n\to\infty}\frac{1}{2^n}\ln\|\Gamma_\lambda\big(0,\alpha^{-n}\big)\|\geqslant\lim_{n\to\infty}\frac{1}{2^n}\ln\big\|R^n\circ\phi(\lambda)\big\|=G\circ\phi(\lambda). \tag{3.54}$$

Remember that we denoted by $d^\pm_{\langle n\rangle}$ the infinite determinants of $a_{\langle n\rangle}$ on $I_{\langle n\rangle}$. By scaling we have $d^+_{\langle n\rangle}(\lambda)=\gamma^{-n}d^+(\gamma^n\lambda)$ and $d^-_{\langle n\rangle}(\lambda)=d^-(\gamma^n\lambda)$. To get the inequality on the lim sup we first prove that $\frac{1}{2^n}\ln d^\pm_{\langle n\rangle}$ converges in $L^1_{\mathrm{loc}}(\mathbb{C})$ to $G\circ\Phi$. By Hartogs lemma (cf [13, Theorem 3.2.12] or [27, Theorem A.1.2]) this would imply that $\limsup\frac{1}{2^n}\ln d^\pm_{\langle n\rangle}(\lambda)\leqslant G\circ\phi(\lambda)$ for all $\lambda\in\mathbb{C}$. Considering the expression of the propagator this would give $\limsup\frac{1}{2^n}\ln\|\Gamma_\lambda(0,\alpha^{-n})\|\leqslant G\circ\phi(\lambda)$ for all λ.

From formulas (2.32) and (2.33) and Proposition 3.2 we know that:

$$\ln\left|d^{\pm}_{\langle n\rangle}(\lambda)\right| = H^{\pm}_{\langle n\rangle}\big(\phi(\lambda)\big) \mp \ln\left|d^{-}(\lambda)\right| + C^{\pm}_{\langle n\rangle}. \tag{3.55}$$

Since we already proved that $\frac{1}{2^n}H^{\pm}_{\langle n\rangle}(\phi(\lambda))$ converges in $L^1_{\text{loc}}(\mathbb{C})$ to $G\circ\phi$ we only need to get information on the constants $C^{\pm}_{\langle n\rangle}$. Since $R(A_0)=A_0$ and $d^{-}_{\langle n\rangle}(0)=1$ for all n, considering Eq. (2.45) at the point $A_{(0)}$ and Eq. (3.55) at the point $\lambda=0$, we get $H^{-}_{\langle n\rangle}(A_0)=C^{-}_{\langle n\rangle}=0$. From the definition of $d^{+}_{\langle n\rangle}(\lambda)$ we know that $d^{+}_{\langle n\rangle}(\lambda)=\ln|\lambda|+\mathrm{o}(1)$ for small λ. Considering (2.45) we have

$$\begin{aligned} H^{+}_{\langle n\rangle}(A_{(\lambda)}) &= H^{-}_{\langle n\rangle}(A_{(\lambda)}) + \ln\det(A_{(\lambda)}) \\ &= H^{-}_{\langle n\rangle}(A_{(\lambda)}) + \ln d^{+}(\lambda) - \ln\big(d^{-}(\lambda)\big) + \ln C, \end{aligned}$$

where C is the constant from Lemma 2.2. Therefore considering Eq. (3.55) for small λ gives $C^{+}_{\langle n\rangle}=-\ln C$. From these computations we deduce that $\lim_{n\to\infty}\frac{1}{2^n}C^{\pm}_{\langle n\rangle}=0$. This concludes the proof. □

3.5. Regularity of the integrated density of states and of the Lyapunov exponent

The aim of this section is to prove the local Hölder continuity of the repartition function of states $F(\lambda)$, and of the Lyapunov exponent $\zeta(\lambda)$. We restrict to the continuous case for simplicity. The key result is the following:

LEMMA 3.2. – *Let δ be in $]\frac{3+\sqrt{5}}{2},\infty[$.*
For all $z\in\mathbb{P}^2\setminus(\overline{\mathcal{C}}_-\cup\{q=0\})$ there exists $\varepsilon>0$ such that $f^n(B(z,\varepsilon))\cap B(l,\varepsilon)=\emptyset$ for all $n\in\mathbb{N}$.
The same is true for $\delta\in]0,\frac{3-\sqrt{5}}{2}[$ when $\mathcal{C}_-$ is replaced by $\mathcal{C}_+$.

N.B.: $\overline{\mathcal{C}}_-=\{[q_1,q_2,q],\ q_1q_2=q^2,\ |q_1|\geqslant|q_2|\}$ denotes the closure of $\mathcal{C}_-$.
This lemma means that $\mathbb{P}^2\setminus(\overline{\mathcal{C}}_-\cup\{q=0\})$ is included in the set of nice points for $\delta>\frac{3+\sqrt{5}}{2}$ (cf [8, Definition 2.9] or [27, Definition 1.5.1], where the nice points are called *points normaux*). We will prove this lemma later.

COROLLARY 3.3. – *For $\delta\in]0,\frac{3-\sqrt{5}}{2}[\cup]\frac{3+\sqrt{5}}{2},\infty[$, the repartition function of states $F(\lambda)=\int_0^\lambda d\mu$ and the Lyapunov exponent $\zeta(\lambda)$ are locally Hölder continuous in λ.*

N.B.: By locally Hölder continuous we mean that for any relatively compact open set $U\subset\mathbb{C}$, we can find $\alpha_0>0$ and $C>0$ such that $|F(\lambda)-F(\lambda')|\leqslant C|\lambda-\lambda'|^{\alpha_0}$ on U.

Proof. – Assume that $\delta>1$. By Proposition VI.3.9 of [3] it is enought to prove that $\zeta(\lambda)=G\circ\phi(\lambda)$ is locally Hölder continuous. But by Theorem 7.1 of [27], G is locally Hölder continuous on the set of nice points, so on $\mathbb{P}^2\setminus(\overline{C}_-\cup\{q=0\})$. Moreover, $\phi(\lambda)$ is analytic and $\phi(\lambda)\notin(\overline{C}_-\cup\{q=0\})$ for $\lambda\neq0$ so $G\circ\phi(\lambda)$ is locally Hölder continuous on $\mathbb{C}\setminus\{0\}$. Using the relation $G\circ\phi(\gamma\lambda)=2G\circ\phi(\lambda)$ we know that it is locally Hölder continuous on $\mathbb{C}$. □

We now come to the proof of Lemma 3.2. The reason why we are able to prove this only for $\delta>\frac{3+\sqrt{5}}{2}$ (or $\delta<\frac{3-\sqrt{5}}{2}$) is the following: the torus $S=\{[q_1,q_2,-1]\in\mathcal{C},\ |q_1|=|q_2|=1\}$ is bi-invariant for f but is repulsive only for $\delta>\frac{3+\sqrt{5}}{2}$ (or $\delta<\frac{3-\sqrt{5}}{2}$).

If $\frac{3-\sqrt{5}}{2} < \delta < \frac{3+\sqrt{5}}{2}$ there are cycles in S with one attractive direction. The idea of the proof is the following: for $\delta > 1$ the unstable manifold of x_- is the line at infinity $\{q = 0\}$ and the stable one is $\mathcal{C}_-$. Since $l \in \{q = 0\}$, if the orbit of a point z approaches l, it needs to approach x_- before and in fact to travel along in a neighbourhood of the curve $\mathcal{C}_-$ which is repulsive for $\delta > \frac{3+\sqrt{5}}{2}$. This implies that z must be close to $\mathcal{C}_-$.

Proof of Lemma 3.2. – Let δ be in $]\frac{3+\sqrt{5}}{2}, \infty[$.

Let $x = (q_1, q_2, q)$ be in $\pi^{-1}(\overline{\mathcal{C}}_-)$, we have $|q_1| \geqslant |q| \geqslant |q_2|$. Using the fact that $\delta(1 - \delta^{-1})^2 > 1$ for $\delta > \frac{3+\sqrt{5}}{2}$ we have:

$$\frac{1}{\alpha(1-\alpha)}|p(x)|^2\frac{\|x\|^2}{\|R(x)\|^2} \geqslant \frac{|q_1|^2(|q_1|^2 + |q_2|^2 + |q|^2)}{|q_1|^4 + |q_2|^4 + |q|^4}\delta\big(1 - \delta^{-1}\big)^2 > 1. \tag{3.56}$$

To simplify the notations we set, for $z \in \mathbb{P}^2$, $\overline{r}(z) = |r(\frac{x}{\|x\|})|$ choosing x such that $\pi(x) = z$.

Let z_0 be in $\mathbb{P}^2 \setminus (\overline{\mathcal{C}}_- \cup \{q = 0\})$. We can find a neighbourhood V_- of $\overline{\mathcal{C}}_-$ and a neighbourhood V of D such that:

$$z_0 \notin V_-, \tag{3.57}$$

$$\overline{r}(f(z)) \geqslant \overline{r}(z), \quad \forall z \in V_-, \tag{3.58}$$

$$f(V) \cap V_- = \emptyset. \tag{3.59}$$

Indeed, the second estimate comes from formulas (3.19) and (3.56), the third relation comes from the fact that $D \setminus \{l\}$ is in the attractive basin of x_+ and that the blow-up of l is the line D' that intersects $\mathcal{C}$ at the unique point $[-\delta^{-1}, -\delta, -1]$ which is also in the attractive basin of x_+ (we already used these arguments in Lemma 3.1).

Let $\varepsilon' > 0$ be such that $f(B(x_+, \varepsilon')) \subset B(x_+, \varepsilon')$. We can find V_+, a neighbourhood of $\mathcal{C} \setminus V_-$, and $N > 0$ such that:

$$V_+ \subset \bigcup_{n=0}^{N} f^{-n}\big(B(x_+, \varepsilon')\big), \tag{3.60}$$

$$l \notin \overline{\bigcup_{n\in\mathbb{N}} f^n(V_+)}. \tag{3.61}$$

Since $f^{-1}(\mathcal{C}) \subset \mathcal{C} \cup D$ we can find $C_1 > 0$ such that:

$$\overline{r}\big(f(z)\big) \geqslant C_1, \quad \forall z \in \mathbb{P}^2 \setminus (V_- \cup V_+ \cup V). \tag{3.62}$$

Since $f^{-n}(l) = \{[1, -\delta^{-(n-1)}, 0]\}$ tends to x_- we can find $\varepsilon > 0$, $N' > 0$ such that:

$$f^{-N'}\big(B(l, \varepsilon)\big) \subset V_-, \tag{3.63}$$

$$\overline{r}(z) \leqslant \frac{C_1}{2}, \quad \forall z \in f^{-N'}\big(B(l, \varepsilon)\big), \tag{3.64}$$

$$B(z_0, \varepsilon) \cap \left(\bigcup_{n=0}^{N'} f^{-n}\big(B(l, \varepsilon)\big) \cup V_-\right) = \emptyset, \tag{3.65}$$

$$\bigcup_{n\in\mathbb{N}} f^n(V_+) \cap B(l, \varepsilon) = \emptyset. \tag{3.66}$$

Suppose now that $f^n(z) \in B(l,\varepsilon)$ for $z \in B(z_0,\varepsilon)$, $n \in \mathbb{N}$. Then necessarily $n \geqslant N'$ (cf relation (3.65)) and $f^{n-N'}(z) \in f^{-N'}(B(l,\varepsilon)) \subset V_-$. Let n_0 be the last time $f^n(z)$ entered V_- before the time $n - N'$, then, using relation (3.62), $\overline{r}(f^{n_0}(z)) \geqslant C_1$ since $f^{n_0-1}(z) \notin V_- \cup V_+ \cup V$ (because of conditions (3.66), (3.59)). Finally, estimate (3.58) implies that $\overline{r}(f^{n-N'}(z)) \geqslant C_1$ and this contradicts condition (3.64). □

4. Extension to a general finitely ramified self-similar set. Remarks and conjectures

A large part of this work could be done for a general finitely ramified fractal (in the sense of [25]). It is easy to see that all the results of Section 1 and Section 2 could be extended at the exception of formula (2.45) in Proposition 2.3. Precisely, we can always define the map T and its invariant curve $A_{(\lambda)}$. We define the functions $H^{\pm}_{\langle n\rangle}$ by the same formula and they satisfy the functional equation (2.44) where 2 is replaced by the number N of similitudes involved in the property of self-similarity (the polynomial p is always defined by $p(Q) = C\det(Q^-_{\langle 1\rangle})$ for a constant $C > 0$). The difference comes in the definition of the map R. In general it is natural to define R by $R(Q) = \hat{p}(Q)T(Q)$ where $\hat{p}$ denotes the lowest common multiple of the denominators appearing in the expression of T. It is not always the case that $\hat{p} = p$. Consequently, the (eventual) limit of the sequence $\frac{1}{N^n}H^{\pm}_{\langle n\rangle}$ does not necessarily satisfy the functional equation $H(RQ) = NH(Q)$ but a second term can appear (this is for example the case for the Sierpinski gasket, as it appears in [22]). It is then hopeless to get an expression of $H^{\pm}_{\langle n\rangle}$ only in terms of $R^n(Q)$ as it is the case in formula (2.45) (indeed the term $\log|r(Q)|$ does not count when we divide by 2^n). It seems that the nature of the integrated density of states (pure point or not) depends on the functional equation the (eventual) limit of $\frac{1}{N^n}H^{\pm}_{\langle n\rangle}$ should satisfy.

We now propose a picture of the situation in the general case (some of our claims are "nearly proved" some are at the stage of conjectures). We recall that d_n the degree of the iterates f^n does not necessarely grow like d^n but that $\frac{1}{n}\log d_n$ is subadditive and we call $d = \inf \frac{1}{n}\log d_n$ the asymptotic degree of f. It seems that a dichotomy appears between the case $d < N$ (N is the number of cells) and $d = N$ (we can prove that $d \leqslant N$). The case $d < N$ seems to be the easiest case. It is known that it contains the Sierpinski gasket, the viscek set. We can prove that it contains the class of nested fractals. In these cases the sequence of plurisubharmonic functions $\frac{1}{N^n}H^{\pm}_{\langle n\rangle}$ converges to a plurisubharmonic function H of the form:

$$H = \sum c_k \log|P_k|, \tag{4.1}$$

where the P_k's are homogeneous polynomials. The expressions (3.32) and (3.33) remain valid when we replace G by H and consequently the integrated density of states appears to be pure point. So this generalizes the dichotomy that appears in [12]. We can also prove that in these cases the density of states is completely created by eigenfunctions with compact support (this type of eigenstates always appears in the case of Nested fractals, cf [2]).

The case $d = N$ seems to be the most difficult. We assume that the map f is generic so that the degrees grow like d^n (or that f^n is generic for an integer n). Here we conjecture that $\frac{1}{N^n}H^{\pm}_{\langle n\rangle}$ converges to a function H of the form:

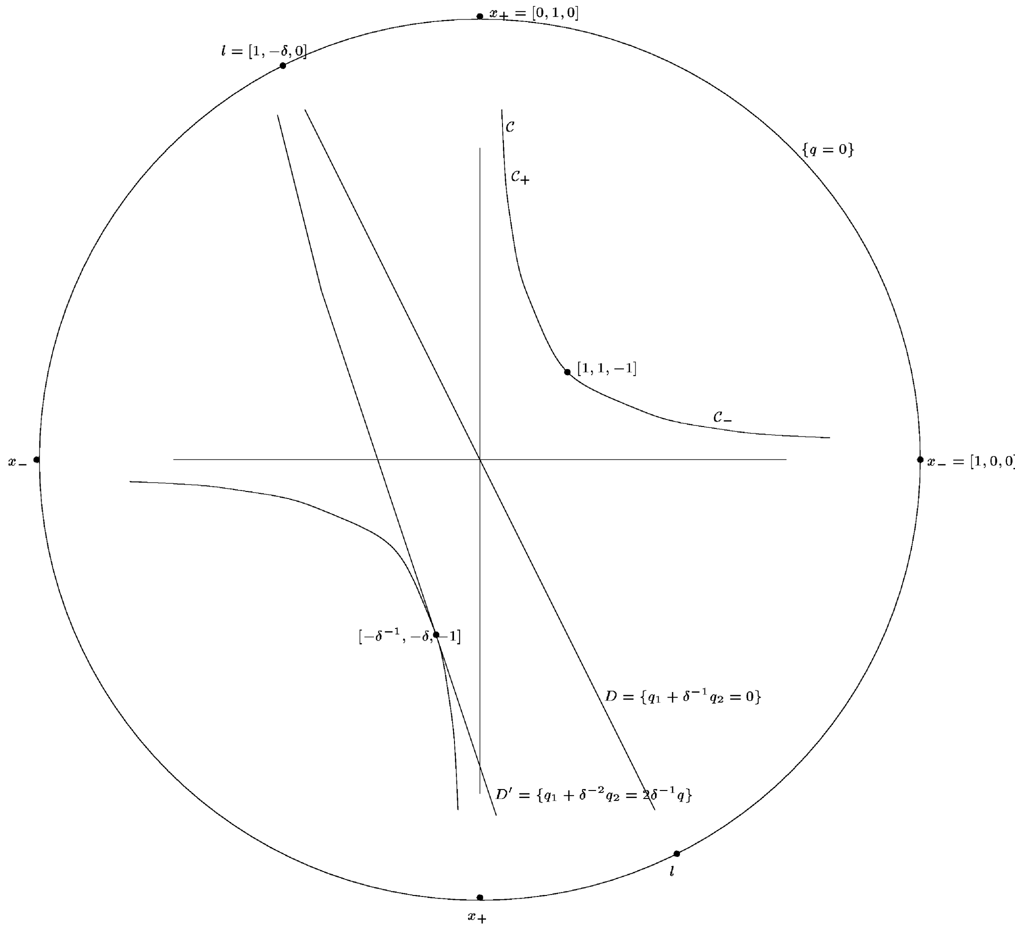

Fig. 1. Picture of $\mathbb{P}^2_{\mathbb{R}}$ for $\delta = 2$.

$$H = \beta G + (1-\beta)\sum c_k \log |P_k|, \tag{4.2}$$

where β is a real such that $0 < \beta \leqslant 1$, G is the Green function of R and the P_k's are homogeneous polynomials. The expression μ and $\tilde{\mu}$ would remain valid when we replace G by H. A point mass in the integrated density of states appears at the intersection of the curve $\phi(\lambda)$ with the curves $\{P_k = 0\}$ and with the indeterminacy points of R (which would define some "exceptional pure points"). It seems that the pure points in the integrated density of states should be associated with eigenstates with compact support. The important question is to understand if there can be eigenstates with non-compact support so pure point in the spectral decomposition that does not give pure point in the integrated density of states.

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